Coarse geometry of Polish groups
Lecture 2

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Success criteria

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The ultimate aim is to

- provide a geometric picture of topological groups as we have of say f.g. groups, Lie groups and Banach spaces,
- identify new computable isomorphic invariants of topological groups,
- show how these invariants impact other harmonic analytic features of the groups.

A universal group

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To see this, suppose that d is a compatible left-invariant metric on $\mathbb{G}$ and hence a compatible metric for the left-uniformity on $\mathbb{G}$.
Then the restriction $d|_H$ is also a compatible left-invariant metric on H and thus a compatible metric for the left-uniformity on H.

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In other words, H is coarsely embedded if the left-coarse structure on H is the restriction of the left-coarse structure on G to H.

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However, $\mathrm{Homeo}([0,1]^{\mathbb{N}})$ is coarsely equivalent to a point, so $\mathbb{Z}$ can be seen as a closed, but not coarsely embedded subgroup of $\mathrm{Homeo}([0,1]^{\mathbb{N}})$.

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Then (H, d) embeds isometrically into $\mathbb{U}$ in such a way that every isometry of (H, d) extends functorially to an isometry of $\mathbb{U}$.

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So the left-shift action $H \curvearrowright (H, d) \subseteq \mathbb{U}$ induces a group embedding

$$H \hookrightarrow \mathrm{Isom}(\mathbb{U}),$$

where the orbital map

$$h \in H \mapsto h \cdot 1_H \in \mathbb{U}$$

is coarsely proper and thus a coarse embedding.

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Theorem

Let H be a Polish group. Then H is isomorphic to a coarsely embedded closed subgroup of $\prod_{n\in\mathbb{N}} \mathrm{Isom}(\mathbb{U})$.

Locally Roelcke precompact groups

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A subset $A \subseteq G$ is Roelcke precompact if $\overline{(A, d_\wedge)}$ is compact.

Lemma

A is Roelcke precompact if and only if, for every identity neighbourhood V, there is a finite set $F \subseteq G$ with $A \subseteq VFV$.

From the diagram

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Coarsely bounded: $A \subseteq (FV)^k$

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Examples

- *Locally compact groups,*
- $\mathrm{Isom}(\mathbb{U})$,
- *Automorphisms groups of metrically homogeneous graphs, e.g.,* $\mathrm{Aut}(T_\infty)$ *and* $\mathrm{Isom}(\mathbb{Z}\mathbb{U})$,
- *all terms of* $\mathbb{Z} \to \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \to \mathrm{Homeo}_+(\mathbb{S}^1)$.

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As a consequence, one can see that, in these groups,

$$\text{Coarsely bounded} = \text{Roelcke precompact.}$$

Therefore, the coarse structure is witnessed topologically in the ambient locally compact space $\widehat{G}$.

Homeomorphism groups

The homeomorphism group $\mathrm{Homeo}(M)$ of a compact manifold is equipped with the compact-open topology, i.e., given by subbasic open sets of the form

$$O_{K,U} = \{h \in \mathrm{Homeo}(M) \mid h[K] \subseteq U\},$$

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is continuous.

Alternatively, if d is a compatible metric on M, set

$$d_\infty(h, g) = \sup_{x \in M} d(h(x), g(x)).$$

Then d_∞ is a compatible *right-invariant* metric on $\mathrm{Homeo}(M)$.

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We may thus define the corresponding fragmentation norm on the identity component $\mathrm{Homeo}_0(M)$ of isotopically trivial homeomorphisms by letting

$$\|h\|_{\mathcal{V}} = \min\left(k \;\middle|\; h = g_1 \cdots g_k \;\&\; \mathrm{supp}(g_i) \subseteq V_{m_i} \text{ for some } m_i\right).$$

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The fragmentation norm induces a left-invariant metric on $\mathrm{Homeo}_0(M)$ by

$$\rho_{\mathcal{V}}(g, f) = \|g^{-1} f\|_{\mathcal{V}}.$$

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If $\mathcal{V}$ and $\mathcal{W}$ are sufficiently fine open covers of M, then $\rho_{\mathcal{V}}$ and $\rho_{\mathcal{W}}$ are quasi-isometric.

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For compact surfaces M, E. Militon has been able to explicitly describe the fragmentation metric via maximal displacement on the universal cover $\tilde{M}$.

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Theorem (w/ M. Culler)

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Let M be a compact manifold of dimension $\geqslant 2$ with infinite fundamental group $\pi_1(M)$. Then the Banach space $C([0,1])$ is a coarsely embedded closed subgroup of $\mathrm{Homeo}_0(M)$.

Therefore, by the metric universality of $C([0,1])$, every separable metric space coarsely embeds into $\mathrm{Homeo}_0(M)$.

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In fact, with respect to a stronger quasi-metric structure defined on $\mathrm{Homeo}_0(M)$, one may speak of distorted embeddings of finitely generated groups Γ into $\mathrm{Homeo}_0(M)$.

This was previously done via ad hoc methods, by asking for an ambient f.g. group

$$\Gamma \leqslant \Delta \leqslant \mathrm{Homeo}_0(M)$$

in which Γ is distorted.

Quasimorphisms and extensions

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Indeed, let $G \leqslant \mathrm{Homeo}(\tilde{M})$ denote the closed group of lifts $\tilde{g}$ of homeomorphisms $g \in \mathrm{Homeo}_0(M)$.

Letting $\pi_1(M)$ act by deck transformations on $\tilde{M}$, we have a short exact sequence

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Let $H \leqslant G$ denote the path-connected component. Then $H \curvearrowright \pi_1(M)$ is a continuous action by conjugation of a connected group on a discrete group.

Thus, H and $\pi_1(M)$ commute, i.e., $[H, \pi_1(M)] = 1$, and $G = H \cdot \pi_1(M)$.

Letting $A = H \cap \pi_1(M)$, we obtain a central extension of $\mathrm{Homeo}_0(M)$ by A

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & A & \longrightarrow & H & \xrightarrow{\pi} & \mathrm{Homeo}_0(M) & \longrightarrow & 1 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
1 & \longrightarrow & \pi_1(M) & \longrightarrow & G & \xrightarrow{\pi} & \mathrm{Homeo}_0(M) & \longrightarrow & 1 \\
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Theorem (w/ K. Mann)

If the quotient map $\pi_1(M) \xrightarrow{\sigma} \pi_1(M)/A$ admits a bornologous section, then so does the quotient map $H \xrightarrow{\pi} H/A = \mathrm{Homeo}_0(M)$.

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Corollary (w/ K. Mann)

In this case, we have a coarse splitting of G into a direct product

$$G \quad \approx_{\mathrm{coarse}} \quad \pi_1(M)/A \times A \times \mathrm{Homeo}_0(M).$$

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Also, since $\mathrm{Homeo}_0(\mathbb{T})$ is coarsely bounded,

$$\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \quad \approx_{\mathrm{coarse}} \quad \mathbb{Z}.$$

Rephrasing the above calculation, we show that the quotient map π in the diagram below admits a bornologous lift ϕ,

$$0 \to \mathbb{Z}^n \to \mathrm{Homeo}_{\mathbb{Z}^n}(\mathbb{R}^n) \underset{\phi}{\overset{\pi}{\rightleftarrows}} \mathrm{Homeo}_0(\mathbb{T}^n) \to 1$$

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Now, for finitely generated groups, it remains an open problem by S. Gersten whether if a central extension

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That is, so that

$$\{\phi(x)\phi(y)\phi(xy)^{-1} \mid x, y \in F\}$$

is a finite subset of $\ker \pi = \mathbb{Z}^n$.

Proposition

For $n \geqslant 2$, there is no quasimorphism lifting π in the central extension

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Combined with the existence of a bornologous lift, this provides a counter-example to Gersten's question in the general setting.