FULL GROUPS, TOPOLOGICAL RANK, AND COST

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Abstract. This is the first part of lecture notes on full groups prepared for the ESI measured group theory semester. Our main goal is to relate the topological rank of the full group of a measure-preserving equivalence relation to its cost.

Contents

1. Full groups from Dye's point of view

Throughout these notes, $(X,\mathcal{B})$ denotes a standard Borel space, i.e. an uncountable measured space whose σ-algebra is the Borel σ-algebra of some complete separable metric space. The elements of $\mathcal{B}$ will be called **Borel sets**, and we will actually write a standard probability space $(X,\mathcal{B})$ simply as X.

If X is a standard Borel space and μ is a non-atomic probability measure on the Borel subsets of X, we say that (X,μ) is a **standard probability space**. We have the two following important facts on standard probability spaces (for a proof, see [Kec95]):

- They are all isomorphic to $([0,1],\lambda)$.
- Every positive measure Borel subset A of X is then itself a standard probability space for the normalised measure μ_A defined by: for all Borel $B\subseteq A$, $\mu_A(B)=\frac{\mu(B)}{\mu(A)}$.

In most of the arguments, we only need the following basic property of standard Borel spaces.

Lemma 1.1. *Let X be a standard Borel space. There is a countable family (C_n) of Borel subsets which separates points, meaning that if $x\neq y\in X$ then there is $n\in\mathbb{N}$ such that $x\in C_n$ but $y\notin C_n$.*

A bijection T of X is called Borel if $T^{-1}(B)$ is Borel for every Borel set B (this actually implies that T^{-1} is Borel, see [Kec95, 14.12]). A fundamental application of Lemma 1.1 is the following result (see Appendix A for a proof).

Theorem 1.2. *Let $T_1,...,T_q$ be Borel bijections of X and $A\subseteq X$ such that for all $x\in A$ and all $i\neq j\in\{1,...,q\}$ we have $T_i(x)\neq T_j(x)$. Then there is a Borel* *partition* (A_n) *of* A *such that for all* $i \neq j \in \{1, ..., q\}$*,* $T_i(A_n)$ *is disjoint from* $T_j(A_n)$*.*

Most of the time, we will leave it to the reader to check that the constructions we make are Borel. An important warmup exercise for this is the following.

Exercise 1.3. If A is a countable set, we let $|A| \in \mathbb{N} \cup \{\infty\}$ denote its cardinality.

1. Prove that the set of pairwise distinct n-tuples in X^n is Borel.
2. Deduce that the map $f : X^{\mathbb{N}} \to \mathbb{N} \cup \{\infty\}$ defined by $f((x_n)) = |\{x_n : n \in \mathbb{N}\}|$ is Borel.

A Borel bijection $T : X \to X$ is measure preserving if for all Borel $A \subseteq X$, one has $\mu(T(A)) = \mu(A)$ (then T^{-1} is also measure-preserving). For a Borel bijection T of X, we let its **support** be the Borel set $\operatorname{supp} T := \{x \in X : T(x) \neq x\}$.

Let us remark once and for all that two bijections with disjoint support commute.

Definition 1.4. The group $\mathrm{Aut}(X, \mu)$ is the group of measure-preserving Borel bijections of (X, μ), two such bijections T and T' being identified if $T(x) = T'(x)$ for almost every $x \in X$.

We will often abuse notation by seeing Borel measure-preserving bijections as elements of $\mathrm{Aut}(X, \mu)$ and vice-versa. Moreover, we can see every measure-preserving Borel bijection between full measure subsets as an element of $\mathrm{Aut}(X, \mu)$ via the following lemma which we will often use implicitly.

Lemma 1.5. *Let* $T : A \to B$ *be a measure-preserving Borel bijection between two full measure Borel subsets of* X*. Then there is* $\tilde{T} \in \mathrm{Aut}(X, \mu)$ *such that* $\tilde{T}(x) = T(x)$ *for almost every* $x \in X$*.*

Proof. Define $C_0 = A \cap B$ and then define by induction $C_{n+1} = T^{-1}(C_n) \cap C_n \cap T(C_n)$. Then $C := \bigcap_n C_n$ is a full measure T-invariant Borel subset of X. Define $\tilde{T}(x) = T(x)$ if $x \in C$ and $\tilde{T}(x) = x$ else. □

Since all the standard probability spaces are isomorphic, the group $\mathrm{Aut}(X, \mu)$ we have does not depend of the chosen representation of our standard probability space, in the same way that the unitary group of an infinite dimensional separable Hilbert space does not really depend of the infinite dimensional separable Hilbert space.

Example 1.6. Consider the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ equipped with its Haar measure h (which is just the pushforward of the Lebesgue measure on $[0, 1[$ via $t \mapsto e^{2i\pi t}$). Then every element z of $\mathbb{S}^1$ defines an element $T_z \in \mathrm{Aut}(\mathbb{S}^1, h)$ defined by $T_z(z') = zz'$.

A big part of ergodic theory is devoted to the study of measure-preserving transformations $T \in \mathrm{Aut}(X, \mu)$ up to conjugacy, where T is **conjugate** to T' if there is $S \in \mathrm{Aut}(X, \mu)$ such that $ST'S^{-1} = T$. Our focus will actually be on *countable groups* of measure-preserving transformations, but for now, we stick to single transformations.

Example 1.7. Consider a finite probability space $\{0, ..., n-1\}$ and let p be a non-degenerate probability measure on it; consider the standard probability space $X = \{0, ..., n-1\}^{\mathbb{Z}}$ with probability measure $\mu = p^{\otimes \mathbb{Z}}$. Define $T : X \to X$ by

$$T(f)(k) = f(k-1)$$

for all $f : \mathbb{Z} \to \{0, ..., n-1\}$ and all $k \in \mathbb{Z}$. When $n = 2$, observe that $\{0,1\}^{\mathbb{Z}}$ is the set of subsets of $\mathbb{Z}$ and that the action above corresponds to mapping a subset A to $A+1$.

Using the fact that the measure μ is completely determined by the value it assigns to basic clopen sets[1], it is not hard to show that $T \in \mathrm{Aut}(X, \mu)$. The transformation T is called a **Bernoulli shift** with parameter p. One of the greatest achievements of ergodic theory in the twentieth century is the following theorem of Kolmogorov and Ornstein: two Bernoulli shifts are conjugate if and only if their parameters have the same entropy, where the entropy of p is $-\sum_{i=1}^n p(\{i\}) \log(p(\{i\}))$.

Exercise 1.8. We reuse the notation from Example 1.6.

1. Prove that if T_z is conjugate to T'_z in $\mathrm{Aut}(\mathbb{S}^1, h)$ then z and z' have the same order as elements of $\mathbb{S}^1$.
2. Prove that if z and z' have order $n \in \mathbb{N}$, then T_z and $T_{z'}$ are conjugate (*Hint:* To simplify notation, we view T_z as an element of $\mathrm{Aut}([0,1[, \lambda)$. Show that if z has order n, then the set $A_n = [0, 1/n[$ intersects every T_z orbit in exactly one point (this is called a *fundamental domain*). Observe that A_n satisfies that $A_n, T_z(A_n), ..., T_z^{n-1}(A_n)$ are all disjoint, and use then build a conjugacy by taking $T_z^k(A_n)$ to $T_{z'}^k(A_n)$.)
3. Prove that there are uncountably many conjugacy classes in $\mathrm{Aut}(X, \mu)$ (*Hint:* Prove that in the case z and z' are of infinite order, T_z is conjugate to $T_{z'}$ if and only if $z = z'$ or $z = z'^{-1}$. To see this, consider the *Koopman representation*[2] of T_z and $T_{z'}$ on $\mathrm{L}^2(\mathbb{S}^1, h)$ and compute the eigenvalues of the associated unitaries via Fourier).

We now introduce a useful metric to study of the group $\mathrm{Aut}(X, \mu)$ and its *full* subgroups, as we will shortly see.

Definition 1.9. The **uniform metric** d_u on $\mathrm{Aut}(X, \mu)$ is defined by: for all $T, T' \in \mathrm{Aut}(X, \mu)$,

$$d_u(T, T') = \mu(\{x \in X : T(x) \neq T'(x)\}).$$

Exercise 1.10. Prove that the uniform metric is biinvariant: for every $T, U, T_1, T_2 \in \mathrm{Aut}(X, \mu)$, we have

$$d_u(T, U) = d_u(T_1 T T_2, T_1 U T_2).$$

Theorem 1.11. $(\mathrm{Aut}(X, \mu), d_u)$ *is a complete metric space.*

Proof. Let (T_n) be a Cauchy sequence. Up to taking a subsequence, we may assume that for all $n \in \mathbb{N}$, $d_u(T_n, T_{n+1}) < \frac{1}{2^n}$. Since $\sum \frac{1}{2^n} < +\infty$, the Borel-Cantelli lemma ensures us that for almost every $x \in X$ there is $N \in \mathbb{N}$ such that for all $n \geqslant N$ we have $T_n(x) = T_{n+1}(x)$. For $N \in \mathbb{N}$, let

$$A_N = \{x \in X : T_{N-1}(x) \neq T_N(x) \text{ and for all } n \geqslant N, T_n(x) = T_{n+1}(x)\}$$

[1] A subset $B \subseteq \{0,1\}^{\mathbb{Z}}$ is a basic clopen set if it is a set of functions with prescribed values on finitely many coordinates: there exists $m \in \mathbb{N}$, $k_1, ..., k_m \in \mathbb{Z}$ and $l_1, ..., l_m \in \{0, ..., n-1\}$ such that $B = \{f \in \{0, ..., n-1\}^{\mathbb{Z}} : f(k_1) = l_1, ..., f(k_n) = l_m\}$.

[2] Every $T \in \mathrm{Aut}(X, \mu)$ induces a unitary $\kappa(T)$ on $\mathrm{L}^2(X, \mu)$ defined by $\kappa(T)f(x) = f(T^{-1}x)$. As this exercise shows, such a unitary can sometimes retain a lot of information from T. However all Bernoulli shifts induce the same unitary up to unitary conjugacy, hence they cannot be distinguished by that.

Then the sets A_N form a partition of (a full measure subset of) X, and we let $T(x) = T_N(x)$ for $x \in A_N$. It is now an instructive exercise to check that such a T belongs to $\mathrm{Aut}(X, \mu)$ (see the next lemma for a hint!). Then we clearly have $d_u(T_n, T) \to 0$, hence $(\mathrm{Aut}(X, \mu), d_u)$ is complete. □

Let us now isolate the key property of $\mathrm{Aut}(X, \mu)$ which made the above argument work.

Definition 1.12. Given a sequence (T_n) of elements of $\mathrm{Aut}(X, \mu)$ and a Borel partition $(A_n)_{n\in\mathbb{N}}$ of a full measure subset of X, a map $T : \bigsqcup A_n \to X$ is obtained by **cutting and pasting** the sequence (T_n) along (A_n) if T is injective and for every $n \in \mathbb{N}$ and $x \in A_n$, we have $T(x) = T_n(x)$.

Lemma 1.13. *If T is obtained by cutting and pasting the sequence (T_n) along (A_n), then $T \in \mathrm{Aut}(X, \mu)$.*

Moreover if $T : X \to X$ is an injective Borel map such that for all $x \in X$, there is $n \in \mathbb{N}$ such that $T(x) = T_n(x)$, then there is a partition (A_n) such that T is obtained by cutting and pasting the sequence (T_n) along (A_n).

Proof. Note that the injectivity of T implies that $(T_n(A_n))$ is a partition of X. Decomposing every Borel subset A of X as $A = \bigsqcup_{n\in\mathbb{N}}(A \cap A_n)$ and using the fact that each T_n preserves the measure, we see that T preserves the measure and conclude by lemma 1.5 that $T \in \mathrm{Aut}(X, \mu)$.

For the moreover part, let $B_n = \{x \in X : T(x) = T_n(x)$ and then let $A_n = B_n \setminus \bigcup_{m<n} A_m$ (note that A_n is just the set of x such that n is the first integer satisfying $T(x) = T_n(x)$). □

Exercise 1.14. An **almost partition** of X is a sequence of Borel sets (A_n) such that $\mu(A_n \cap A_m) = 0$ for all $n \neq m$ and $\mu(\bigsqcup_n A_n) = 1$. Show that if (A_n) is an almost partition of X, and (T_n) is a sequence of elements of $\mathrm{Aut}(X, \mu)$ such that $(T_n(A_n))$ is an almost partition of X, one can still construct a $T \in \mathrm{Aut}(X, \mu)$ by cutting and pasting the sequence (T_n) along (A_n).

From now on, we will frequently drop the "almost" adjective. In particular an almost partition will be the same for us as a partition. We also allow a map to be only defined on a full measure set (see Lem. 1.5), and we will frequently write that two Borel sets are equal if they coincide up to measure zero.

Definition 1.15 (Dye)**.** A subgroup $\mathbb{G}$ of $\mathrm{Aut}(X, \mu)$ is **full** if it is stable under the cutting and pasting operation: if $T \in \mathrm{Aut}(X, \mu)$ is obtained by cutting and pasting a sequence (T_n) of elements of $\mathbb{G}$ along a partition (A_n), then $T \in \mathbb{G}$.

The exact same proof as for Lemma 1.11 now yields:

Proposition 1.16 (Dye)**.** *Every full group is a complete metric space for the uniform metric.*

So every full group is a closed subgroup of the full group $\mathrm{Aut}(X, \mu)$. However, $(\mathrm{Aut}(X, \mu), d_u)$ is not separable[3] (can you see why?). Our focus will be on those full groups which are separable for the uniform metric, i.e. those which are Polish groups for the uniform topology.

[3] $\mathrm{Aut}(X, \mu)$ is nevertheless a Polish group for the weak topology (cf. [Kec10]).

Main example. Let Γ be a countable group. A Γ-action on X is called **measure-preserving** if it is an action by Borel bijections such that for all $\gamma \in \Gamma$ and all Borel $A \subseteq X$ one has $\mu(\gamma A) = \mu(A)$. We can associate to this action a full group $[\mathcal{R}_\Gamma]$ defined by

$$[\mathcal{R}_\Gamma] = \{T \in \mathrm{Aut}(X, \mu) : T(x) \in \Gamma \cdot x \text{ for almost all } x \in X\}.$$

We will see in section 3 that $[\mathcal{R}_\Gamma]$ is separable, so that it has a countable dense subgroup. Our goal in these lectures will be to understand when it has *finitely generated* dense subgroups.

Exercise 1.17. Denote by $\alpha : \Gamma \to \mathrm{Aut}(X, \mu)$ the homomorphism associated to a measure-preserving action.

1. Prove that $[\mathcal{R}_\Gamma]$ is actually the smallest full group containing $\alpha(\Gamma)$.
2. Prove that given $S \subseteq \mathrm{Aut}(X, \mu)$, there exists a smallest full group containing S, which we will denote by $[S]$.
3. Explain why for a countable group Γ, a measure-preserving action is essentially the same as a homomorphism $\Gamma \to \mathrm{Aut}(X, \mu)$[4].

Let us now prove that full groups are big via a well-known construction: the induced transformation, also known as the first-return map. It relies on Poincaré's recurrence theorem.

Lemma 1.18. *Let $T \in \mathrm{Aut}(X, \mu)$ and A be a positive measure Borel subset of X, then for almost every $x \in A$ there exists $n \geqslant 1$ such that $T^n(x) \in A$.*

Proof. Let $B = \{x \in X : \forall n \geqslant 1, T^n(x) \notin A\}$. Then for all $n \geqslant 1$, $T^n(B)$ is disjoint from A, hence from B. But since T is injective we deduce that for all $n \geqslant 1$ and all $m \geqslant 0$, the set $T^{n+m}(B)$ is disjoint from $T^m(B)$, so that $(T^m(B))_{m\in\mathbb{N}}$ is an infinite collection of pairwise disjoint sets, all of the same positive measure, contradicting the fact that μ is a finite measure. □

So for $T \in \mathrm{Aut}(X, \mu)$ and a Borel $A \subseteq X$, we have a Borel integer-valued map $n_{A,T}$ defined for all $x \in A$ by

$$n_{A,T}(x) = \min\{n \geqslant 1 : T^n(x) \in A\}.$$

We then define the transformation induced by T on A, denoted by $T_A : X \to X$, by

$$T_A(x) = \begin{cases} T^{n_{A,T}(x)}(x) & \text{if } x \in A \\ x & \text{else.} \end{cases}$$

Note that the transformation induced by T is obtained by cutting and pasting powers of T, in particular it belongs to any full group containing T.

Theorem 1.19 (Keane)**.** *Full groups are contractible for the uniform topology.*

Proof. Let $\mathbb{G}$ be a full group. We may assume that $X = [0, 1]$ equipped with the Lebesgue measure. A homotopy H such that $H(0, T) = \mathrm{id}_X$ and $H(1, T) = T$ for all $T \in \mathbb{G}$ is given by: for all $t \in [0, 1]$ and all $T \in \mathrm{Aut}(X, \mu)$

$$H(t, T) = T_{[0,t]}$$

Exercise 1.20. Prove that H is indeed continuous. □

[4] This fact is very wrong for the much bigger group $\mathrm{Aut}(X, \mu)$: every measure-preserving Borel action of $\mathrm{Aut}(X, \mu)$ on a standard probability space has a set of fixed points of full measure! See [GTW05].

Corollary 1.21. *Every non-trivial full group is uncountable.*

Proof. If $\mathbb{G}$ were countable, then by the Baire category theorem it would have an isolated point. □

We will now see how to concretely build many elements of a non-trivial full group.

2. Aperiodic and periodic elements

Definition 2.1. A transformation $T \in \mathrm{Aut}(X,\mu)$ is **periodic** if it has only finite orbits, and **aperiodic** if it has only infinite orbits.

More generally, a measure-preserving Γ-action is called aperiodic if it has only infinite orbits and periodic if it has only finite orbits.

Exercise 2.2. Explain why if $T \in \mathrm{Aut}(X,\mu)$ satisfies that for almost every $x \in X$, the T-orbit of x is finite (respectively infinite), then one can still think of T as a periodic (resp. aperiodic) element.

Lemma 2.3. *Let $\Gamma \curvearrowright X$ be a Borel aperiodic action. Then there exists a decreasing sequence of Borel sets (A_n) such that each A_n intersects every Γ-orbit infinitely many times, but $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$.*

Proof. The proof is a kind of measurable version of the Bolzano-Weierstrass construction. We only use the fact that X has a countable family of Borel sets $(C_n)_{n\geqslant 1}$ which separates points.

We first define by recurrence a decreasing sequence of Borel set $(B_n)_{n\in\mathbb{N}}$ such that each B_n interesects every Γ-orbit infinitely many times: we let $B_0 = X$ and then

$$\begin{aligned} B_{n+1} =& \{x \in B_n : \Gamma x \cap B_n \cap C_{n+1} \text{ is infinite}\} \\ & \sqcup \{x \in B_n \setminus C_{n+1} : \Gamma x \cap B_n \cap C_{n+1} \text{ is finite}\}. \end{aligned}$$

Note that by construction for every $x \in X$ the infinite set $B_{n+1} \cap \Gamma x$ is either a subset of C_{n+1} or of its complement. Since the family $(C_n)_{n\geqslant 1}$ is moreover separating, the set $A := \bigcap_{n\in\mathbb{N}} B_n$ intersects every Γ-orbit in at most one point.

The sequence of sets (A_n) defined by $A_n = B_n \setminus A$ is now as desired.

Exercise 2.4. Prove that A_n is indeed Borel. □

Exercise 2.5. Suppose now that $\Gamma \curvearrowright X$ is a periodic Borel action. Let $B_0 = X$ and define inductively

$$\begin{aligned} B_{n+1} =& \{x \in B_n : \Gamma x \cap B_n \cap C_{n+1} \text{ is non empty}\} \\ & \sqcup \{x \in B_n \setminus C_{n+1} : \Gamma x \cap B_n \cap C_{n+1} \text{ is empty}\}. \end{aligned}$$

Show that the set $A = \bigcap_{n\in\mathbb{N}} B_n$ is a **Borel fundamental domain** for the Γ-action, meaning that it intersects every Γ-orbit in exactly one point.

Theorem 2.6. *Let $\mathbb{G}$ be a full group. Then the set of periodic elements is dense in $\mathbb{G}$.*

Proof. Let $T \in \mathbb{G}$, consider the T-invariant Borel set A of points whose T-orbit is infinite. If $\mu(A) = 0$ then T is periodic and there is nothing to prove. If not, we only have to show that the restriction of T to A can be uniformly approximated by periodic elements of the full group of the restriction of T to A.

In other words, we may and do assume that T is aperiodic. Let $\epsilon > 0$, we will show that there is a periodic $S \in [T]$ such that $d_u(T,S) < \epsilon$. By Lemma 2.3 there is a decreasing family of Borel sets (A_n) intersecting every T-orbit whose intersection is emtpy. Since μ is finite, we have $\mu(A_n) \to 0$. So we may find a Borel $A \subseteq X$ intersecting every T-orbit such that $\mu(A) < \epsilon$.

By considering the set $\bigcap_{n\in\mathbb{Z}} T_A^n(A)$ which has the same measure as A, we see that for almost every $x \in A$, there are infinitely many positive n such that $T^n(x) \in A$ as well as infinitely many negative n such that $T^n(x) \in A$. Since A intersects every T-orbit, this is actually true of almost every $x \in X$. It is then easily checked that this implies $S := TT_A^{-1}$ is periodic. But by construction $d_u(\mathrm{id}_X, T_A^{-1}) < \epsilon$ so $d_u(T, TT_A^{-1}) < \epsilon$ as desired. □

Corollary 2.7. *Let $\mathbb{G}$ be a full group. Then the involutions of $\mathbb{G}$ generate a dense subgroup of $\mathbb{G}$.*

Proof. By the above theorem, it suffices to show that every periodic element can be approximated by a product of involutions. Call $T \in \mathrm{Aut}(X, \mu)$ a **cycle** if there exists $n \in \mathbb{N}$ such that every T-orbit is either trivial or of cardinality n.

Let $T \in \mathbb{G}$, for every $n \geqslant 2$ and $x \in X$, let $T_n(x) = T(x)$ if the T-orbit of x has cardinality n, and $T_n(x) = x$ otherwise. Then the T_n's are cycles with disjoint support, and

$$T = \lim_{n\to+\infty} T_2 \cdots T_n.$$

The theorem now follows from the following:

Exercise 2.8. Show that every cycle is a product of two involutions. *Hint:* use exercise 2.5, and first prove it for a cycle in a finite symmetric group. □

We will see in section **??** how Corollary 2.7 is useful to build countable dense subgroups in full groups.

3. Measure-preserving equivalence relations and cost

From now on, we shall only be interested in full groups coming from measure-preserving actions of countable groups. Γ will always denote a countable group acting on (X, μ) by measure-preserving transformations.

Our aim in this section is to properly introduce measure-preserving equivalence relations, and to show that their full groups are separable.

Definition 3.1. Given a measure-preserving Γ-action, we associate to it an equivalence relation $\mathcal{R}_\Gamma \subseteq X \times X$ defined by

$$(x, x') \in \mathcal{R}_\Gamma \text{ whenever } x \in \Gamma x'.$$

Recall that we also associated to such an action a full group $[\mathcal{R}_\Gamma]$. The notation is justified by the fact that we ca define this full group purely in terms of the equivalence relation $\mathcal{R}_\Gamma$:

$$[\mathcal{R}_\Gamma] = \{T \in \mathrm{Aut}(X, \mu) : (T(x), x) \in \mathcal{R}_\Gamma \text{ for almost all } x \in X\}.$$

Exercise 3.2. Prove that $\mathcal{R}_\Gamma$ is Borel. Show that every injective Borel map $T : X \to X$ such that for all $x \in X$, $(T(x), x) \in \mathcal{R}_\Gamma$ is actually an element of $\mathrm{Aut}(X, \mu)$.

By the previous exercise, $\mathcal{R}_\Gamma$ satisfies the following definition.

Definition 3.3. A measure-preserving equivalence relation is a Borel subset $\mathcal{R} \subseteq X \times X$ which is an equivalence relation with countable equivalence classes, such that for every injective Borel map $T : X \to X$ with $(T(x), x) \in \mathcal{R}$ for almost all $x \in X$, we actually have $T \in \mathrm{Aut}(X, \mu)$.

For a subset $A \subseteq X \times X$, the vertical section above $x \in X$ is the set $A_x := \{y \in X : (x, y) \in A\}$. Here is a difficult theorem, followed by its main consequence for us.

Theorem 3.4 (Lusin-Novikov)**.** *Let $A \subseteq X \times X$ be a Borel set with countable vertical sections . Then the projection B of A on the first coordinate is Borel and there is a countable family of Borel functions $f_n : B \to X$ such that A is the reunion of the graphs of the functions f_n.*

Theorem 3.5 (Feldman-Moore)**.** *For every measure-preserving equivalence relation $\mathcal{R}$, there is a measure preserving action of a countable group Γ on X such that $\mathcal{R} = \mathcal{R}_\Gamma$*

Exercise 3.6. Deduce Theorem 3.5 from Theorem 3.4. *Hint:* Use the above result to first cover $\mathcal{R}$ by graphs of injective partially defined Borel maps, and then use Lemma A.1 to produce involutions.

Let $\mathcal{R}$ be a measure-preserving equivalence relation. It is useful to think of $\mathcal{R}$ as a groupoid for the multiplication $(x, y)(y, z) = (x, z)$ and inversion $(x, y)^{-1} = (y, x)$. We then have the following analogue of the counting measure on a countable group.

Definition 3.7. The **Haar measure** M is defined on the Borel subsets A of $\mathcal{R}$ by:

$$M(A) = \int_X |A_x|\, d\mu(x),$$

where $|\cdot|$ denotes the counting measure.

Exercise 3.8. Check that $x \mapsto |A_x|$ is Borel. *Hint:* see Exercise 1.3.

For any Borel function $f : X \to X$ such that $(x, f(x)) \in \mathcal{R}$ for all $x \in X$, its graph $G_f := \{(x, f(x)) : x \in X\}$ has measure 1 (in particular this applies to elements of $[\mathcal{R}]$!). By the Lusin-Novikov theorem, we thus have the following proposition.

Proposition 3.9. *$(\mathcal{R}, M)$ is a σ-finite standard measured space.*

In particular $\mathrm{L}^2(\mathcal{R}, M)$ is a separable Hilbert space. The map $[\mathcal{R}] \to \mathrm{L}^2(\mathcal{R}, M)$ which associate to $T \in [\mathcal{R}]$ the characteristic function of its graph χ_{G_T} is a homeomorphism on its image since

$$d_u(T, T') = \frac{1}{2} M(G_T \triangle G_{T'}) = \frac{1}{2} \left\|\chi_{G_T} - \chi_{G_T}\right\|^2$$

Since every subspace of a separable metric space is separable, we can conclude.

Theorem 3.10. *The full group of any measure-preserving equivalence relation is separable for the uniform metric.*

We will see later some tools which allow to prove the above theorem directly.

Exercise 3.11. Prove the converse: every separable full group is the full group of a measure-preserving equivalence relation.

We end this section with a first definition of cost and let the reader figure out what in means for a subset of $\mathcal{R}$ to generate the groupoid $\mathcal{R}$.

Definition 3.12. The cost of a measure-preserving equivalence relation is the infimum of the measures of its *generating* Borel subsets.

Exercise 3.13. Show that if $\mathcal{R} = \mathcal{R}_\Gamma$ and Γ is finitely generated, then the cost of $\mathcal{R}$ is bounded above by the rank of Γ (i.e. the minimal number of elements needed to generate Γ).

4. Orbit equivalence for countable groups

We have see than separable full groups naturally arise from measure-preserving actions of countable groups. It is now time to give some examples of these. Before we do that, let us just introduce the relevant terminology.

Definition 4.1. A measure-preserving action of a countable group Γ on (X, μ) is **free** if for all $\gamma \in \Gamma \setminus \{e\}$, we have $\mu(\{x \in X : \gamma x = x\}) = 0$. It is **ergodic** if for every Borel set $A \subseteq X$ such that $\gamma A = A$ for all $\gamma \in \Gamma$, one must have $\mu(A) = 0$ or $\mu(A) = 1$.

Example 4.2. (Bernoulli shifts, revisited) Bernoulli shifts are probably the most fundamental examples since they make sense for *any* countable group Γ. Consider the standard Borel space $X = \{1, ..., n\}^\Gamma$, fix a non-degenerate probability measure p on $\{1, ..., n\}$ and equip X with the probability measure $\mu = \bigotimes_{\gamma\in\Gamma} p$. Then we have a measure preserving Γ-action on (X, μ) given by for all $\gamma \in \Gamma$, all $f : \Gamma \to \{0, 1\}$ and all $g \in \Gamma$,

$$\gamma \cdot f(g) = f(\gamma^{-1}g).$$

Moreover, it is ergodic (see e.g. [KM04, Ex. 3.1]) as well as free (can you see why?).

One can generalize this construction as follows: start from a Γ action on a countable set I and consider the natural action of Γ on $(B^I, \nu^{\otimes I})$. Provided the measure ν is non-trivial, we always get a measure-preserving action which is ergodic when the Γ-action on I has only infinite orbits and free if every non-trivial element of Γ moves infinitely many points (see [KT08, sec. 2]).

Exercise 4.3. Given a countable dense subgroup Γ of a compact metrisable group G, show that the action of Γ by left translation on G equipped with its Haar probability measure h is an ergodic free action. *Hint:* Show that the G-action on the measure algebra[5] $\mathrm{MAlg}(G, h)$ is continuous. Deduce that every Γ-invariant Borel set is almost G-invariant ($\mu(gA \bigtriangleup A) = 0$ for all $g \in G$), and conclude using Fubini.

Actions as in the previous exercise are called **compact** actions. Here are some examples of compact actions:

(1) The action of $\mathbb{Z}$ on $\mathbb{S}^1$ by translation by some irrational rotation.
(2) The action of $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ by translation on $\prod_{n\in\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$.

[5] The measure algebra of a standard probability space (X, μ) is the algebra of Borel subsets, where two such sets A, B are identified when $\mu(A \bigtriangleup B) = 0$. We denote it by $\mathrm{MAlg}(X, \mu)$. It is a (complete separable) metric space for the distance $d_\mu(A, B) = \mu(A \bigtriangleup B)$. Here you need to show that if g is sufficiently close to g' and A is sufficiently close to B then gA is close to $g'B$. Since the G-action on $\mathrm{MAlg}(G, h)$ is isometric you actually only need that if g is sufficiently close to g' then gA is close to gA'. *Hint:* First show it for closed sets using a compatible metric.

(3) The action of a residually finite group on its profinite completion. This is a special case of profinite actions, which we won't discuss here.

Example 4.4. For $n \geqslant 2$, the action of $Sl_n(\mathbb{Z})$ on $\mathbb{R}^n/\mathbb{Z}^n$ is free and ergodic. (see [BM00, 1.4(ii)]).

Definition 4.5. Let Γ and Λ be two countable groups acting on (X,μ). Their actions are **conjugate** if there exists $T \in \mathrm{Aut}(X,\mu)$ and a group isomorphism $\gamma \mapsto \lambda_\gamma$ between Γ and Λ such that for all $\gamma \in \Gamma$ and almost all $x \in X$,

$$T(\gamma x) = \lambda_\gamma x.$$

Note that conjugacy as defined in section 1 for elements of $\mathrm{Aut}(X,\mu)$ is not exactly the same as here. Indeed, since $\mathbb{Z}$'s only non-trivial automorphism $n \mapsto -n$, two transformations $T, T' \in \mathrm{Aut}(X,\mu)$ define conjugate $\mathbb{Z}$-actions if and only if they are **flip conjugate**: either T is conjugate to T' or T is conjugate to T'^{-1}.

In orbit equivalence theory, we forget about the action and care about the partition of the space into orbits it induces.

Definition 4.6. Let Γ and Λ be two countable groups acting on (X,μ). Their actions are **orbit equivalent** if there exists $T \in \mathrm{Aut}(X,\mu)$ such that for almost all $x \in X$,

$$T(\Gamma x) = \Lambda x.$$

Since $\mathcal{R}_\Gamma$ retains the partition into orbits induced by an action, the above definition makes sense for measure-preserving equivalence relations. So we similarly say two measure-preserving equivalence relations $\mathcal{R}$ and $\mathcal{S}$ on (X,μ) are orbit equivalent if there is $T \in \mathrm{Aut}(X,\mu)$ such that for almost all $x \in X$,

$$T([x]_\mathcal{R}) = [T(x)]_\mathcal{S}.$$

The map T is then called an orbit equivalence between $\mathcal{R}$ and $\mathcal{S}$. The following exercise is of fundamental importance for us, since it tells us that *full groups are invariants of orbit equivalence as topological groups.*

Exercise 4.7. Prove that $T \in \mathrm{Aut}(X,\mu)$ is an orbit equivalence between $\mathcal{R}$ and $\mathcal{S}$ if and only if $T[\mathcal{R}]T^{-1} = [\mathcal{S}]$.

Note that ergodicity of a Γ-action is an invariant of orbit equivalence since it can be recast by saying $\mathcal{R}_\Gamma$: a measure preserving equivalence relation $\mathcal{R}$ is **ergodic** if any Borel set which is a reunion of $\mathcal{R}$-classes has measure zero or one.

The fact that $\mathcal{R}_\Gamma$ comes from a free action cannot be seen by just looking at $\mathcal{R}_\Gamma$ a priori. Nevertheless, "coming from a free action" is a non trivial invariant of orbit equivalence as per the following result.

Theorem 4.8 (Furman)**.** *There exists an ergodic measure-preserving equivalence relation $\mathcal{R}$ which is never form $\mathcal{R} = \mathcal{R}_\Gamma$ for a free Γ-action.*

Let us end this section by quoting some other famous theorems on orbit equivalence.

Theorem 4.9 (Ornstein-Weiss)**.** *Any two ergodic actions of amenable groups are orbit equivalent.*

Theorem 4.10 (Epstein)**.** *Any countable non-amenable group admits a continuum of pairwise non orbit equivalent free ergodic actions.*

Theorem 4.11 (Gaboriau). *Given a free action of $\mathbb{F}_n$, we have* $\mathrm{Cost}(\mathcal{R}_{\mathbb{F}_n}) = n$.

In particular, for all $n, m \in \mathbb{N}$, if a free action of $\mathbb{F}_n$ is orbit equivalent to free action of $\mathbb{F}_m$ then $m = n$.

5. Topological rank for full groups: the question

Definition 5.1. Let G be a separable topological group. The topological rank of G is the (possibly infinite) minimal number n such that there are $g_1, ..., g_n \in G$ which generate a dense subgroup of G.

As an example, we have $t(\mathbb{S}^1) = 1$. More generally, by a theorem of Kronecker $t(\mathbb{R}^n/\mathbb{Z}^n) = 1$. Here is an instructive example.

Lemma 5.2. *We have $t(\mathbb{R}^n) = n + 1$.*

Proof. Recall that one needs at least n elements to generate $\mathbb{R}^n$ as a vector space. Since every vector subspace of $\mathbb{R}^n$ is closed, we deduce that $t(\mathbb{R}^n) \geqslant n$. Moreover equality cannot hold: if $x_1, ..., x_n$ generate a dense subgroup of $\mathbb{R}^n$, they form a basis of $\mathbb{R}^n$. Hence up to a change of coordinates the group they generate is $\mathbb{Z}^n \leqslant \mathbb{R}^n$ which is discrete, a contradiction. So $t(\mathbb{R}^n) \geqslant n + 1$. By lifting a topological generator of $\mathbb{R}^n/\mathbb{Z}^n$ and taking the standard basis of $\mathbb{R}^n$, we conclude that $t(\mathbb{R}^n) = n + 1$. ☐

Consider a measure-preserving action of a countable group Γ. By Exercise 4.7, the topological rank of full group of $[\mathcal{R}_\Gamma]$ is an invariant of orbit equivalence

Question 1 (Kechris). *What are the possible values for $t([\mathcal{R}_\Gamma])$?*

In these notes, we will completely answer the above question and compute the topological rank of $[\mathcal{R}_\Gamma]$ in the ergodic case, building up on earlier work of Kittrel-Tsankov. The formula we will obtain is analogous to Lemma 5.2 where the vector space rank is replaced by the cost.

We now make a fundamental remark which we will reinforce in section **??**.

Exercise 5.3. Prove that if $\Lambda \leqslant [\mathcal{R}_\Gamma]$ is dense, then $\Gamma \cdot x = \Lambda \cdot x$ for almost every $x \in X$. Show that this implies that up to a restriction to a full measure Borel set, $\mathcal{R}_\Gamma = \mathcal{R}_\Lambda$.

We have seen that if an equivalence relation $\mathcal{R}$ is generated by a measure-preserving action of an n-generated group Γ, then its cost is bounded above by n (Exercise 3.13). So by Exercise 5.3, we have the inequality

$$t([\mathcal{R}]) \geqslant \mathrm{Cost}(\mathcal{R}).$$

We will see in the next section that the equality cannot hold so that $t([\mathcal{R}]) \geqslant \lfloor \mathrm{Cost}(\mathcal{R}) \rfloor + 1$ (Prop. **??**).

For now, we explore a bit further the topological rank.

Theorem 5.4 (Schreier-Ulam). *If G is a compact connected metrisable group, there is a dense set of couples $(g, g') \in G^2$ such that $\langle g, g' \rangle$ is a dense subgroup of G.*

In particular, the topological rank of a compact connected metrisable group is bounded by 2, and one can further show that it is equal to 1 iff the compact group is abelian.

Definition 5.5. A topological group is **infinitesimally finitely generated** if there exists $n \in \mathbb{N}$ such that every neighborhood of the identity contains n elements which generate a dense subgroup.

Theorem 5.4 implies that compact metrisable connected groups are infinitesimally finitely generated.

A topological group is called **non-archimedean** if it has a neighborhood basis of the identity made of subgroups. This is clearly in sharp opposition with being infinitesimally finitely generated.

The following relaxed version was introduced recently by the author and Gelander.

Definition 5.6. A topological group G is **quasi-non-archimedean** if for every neighborhood of the identity U and every $n \in \mathbb{N}$, there exists a neighborhood of the identity V such that for every $g_1, ..., g_n \in V$, the group generated by $g_1, ..., g_n$ is contained in U.

Note that every non-archimedean group is totally disconnected, but as the next exercise shows there are connected quasi-non-archimedean Polish groups!

Exercise 5.7. Prove that full groups are quasi-non-archimedean, and that the only topological group which is both quasi-non-archimedean and infinitesimally finitely generated is the trivial group.

This exercise shows that one cannot find topological generators for full groups "at random" in the sense of Baire category. However for a cost one aperiodic equivalence relation, if a first element is chosen at random among *aperiodic* elements of the full group and a second at random in the whole full group, then we almost surely get a pair which generates a dense subgroup of the full group (see [LM15, Thm. 1.7]).

The following result was obtained with Gelander, and is a very convoluted way of proving that full groups of the form $[\mathcal{R}_\Gamma]$ are never locally compact!

Theorem 5.8. *Let G be a locally compact Polish group. Then G is quasi-non-archimedean if and only if G is totally disconnected. Furthermore, G is connected if and only if G is infinitesimally finitely generated.*

Exercise 5.9. Prove directly that no non-trivial full group is locally compact. *Hint:* Take an involution and show that the full group it generates is not locally compact by using a fundamental domain for the involution (cf. Corollary 2.7 and Exercise 2.5).

Appendix A. Disjointness in standard Borel sets

Here X is simply a standard Borel space. Our aim is to show that the condition $T(x) \neq x$ can actually be "zoomed out" to the condition that $T(A)$ is disjoint from A for some sufficiently big Borel set A containing x. For a Borel map $T : X \to X$, we still denote its support by $\operatorname{supp} T = \{x \in X : T(x \neq x\}$.

Lemma A.1. *Let $T : X \to X$ be a Borel map. There is a partition (A_n) of $\operatorname{supp} T$ into Borel sets such that $T(A_n)$ is disjoint from A_n for all $n \in \mathbb{N}$.*

Proof. There is a countable family $\mathcal{C}$ of Borel sets which separates points. Then the algebra (no σ here!) $\mathcal{D}$ generated by $\bigcup_{n\in\mathbb{Z}} T^n(\mathcal{C})$ is countable and T-invariant.

We then let $\mathcal{A}$ be the set of $A \in \mathcal{D}$ such that $T(A)$ is disjoint from A. Let us prove that $\mathcal{A}$ is a cover X: for $x \in X$ since $\mathcal{D}$ is separating there is $C \in \mathcal{D}$ such that $x \in C$ but $T(x) \notin C$. Then $x \in T^{-1}(X \setminus C) \cap C$ and $T^{-1}(X \setminus C) \cap C \in \mathcal{A}$ by definition.

Enumerate $\mathcal{A} = \{B_n : n \in \mathbb{N}\}$ and finally let $A_n = B_n \setminus (\bigcup_{m<n} B_m)$. We still have $T(A_n)$ disjoint from A_n for all $n \in \mathbb{N}$, the sequence (A_n) still covers X, but it is now made of disjoint sets as desired. $\square$

By taking intersections, we deduce that for a finite family of Borel bijections $(T_i)_{i=1}^p$, there is a partition (A_n) of the intersection of their supports such that for all $i = 1, ..., p$ and all n, $T_i(A_n)$ is disjoint from A_n. We upgrade this to the following lemma.

Theorem A.2. *Let $T_1, ..., T_q$ be Borel bijections of X and $A \subseteq X$ such that for all $x \in A$ and all $i \neq j \in \{1, ..., q\}$ we have $T_i(x) \neq T_j(x)$. Then there is a Borel partition (A_n) of A such that for all $i \neq j \in \{1, ..., q\}$, $T_i(A_n)$ is disjoint from $T_j(A_n)$.*

Proof. Consider the finite family $(T_i^{-1}T_j)_{i<j}$. Then the support of every $T_i^{-1}T_j$ contains A by assumption, so by the above paragraph there is a countable Borel partition (A_n) of A such that for all $i < j$ and all n, $T_i^{-1}T_j(A_n)$ is disjoint from A_n. We conclude that for all $i < j$ and all n, $T_i(A_n)$ is disjoint from $T_j(A_n)$. $\square$

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