Université Paris Cité

École doctorale de Sciences Mathématiques de Paris Centre ED386
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On quantitative orbit equivalence between amenable groups

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Thèse de doctorat de MATHÉMATIQUES

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Présentée et soutenue publiquement le 11/12/2025

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Résumé

Titre : Sur l'équivalence orbitale quantitative entre groupes moyennables

Mots-clés : Equivalence orbitale, odomètre, système de rang un, entropie, spectre ponctuel, mélange, équivalence de Kakutani, groupe moyennable, profil isopérimétrique.

Deux actions, libres et préservant une mesure de probabilité, de groupes dénombrables sont orbitalement équivalentes si elles admettent les mêmes orbites quitte à conjuguer l'une des deux. Cependant, un théorème d'Ornstein et Weiss affirme que deux telles actions sont toujours orbitalement équivalentes lorsque les groupes sont infinis et moyennables. L'équivalence orbitale quantitative consiste à ajouter des restrictions sur les cocycles dans le but d'obtenir une théorie plus intéressante.

D'une part, ce sujet interagit avec la théorie ergodique « classique ». Premièrement, Belinskaya a démontré qu'une équivalence orbitale avec des cocycles intégrables se résume à un problème beaucoup trop proche de la conjugaison. Deuxièmement, des hypothèses plus faibles que l'intégrabilité des cocycles proposent un juste milieu entre la relation triviale qu'est l'équivalence orbitale, et le problème difficile de la conjugaison. Dans ce contexte, nous établissons plusieurs résultats de flexibilité : certaines formes quantitatives de l'équivalence orbitale ne préservent pas des comportements dynamiques reliées aux propriétés de mélange et à la notion de spectre ponctuel. Nous démontrons aussi que ces relations n'impliquent pas l'équivalence de Kakutani (un problème aussi riche que celui de la conjugaison). Enfin, nous démontrons que la préservation de l'entropie sous équivalence orbitale log-integrable (due à Kerr et Li) est optimale, en trouvant des transformations d'entropie différente qui sont orbitalement équivalentes avec des cocycles quasiment log-intégrables.

D'autre part, l'équivalence orbitale quantitative entre groupes moyennables capture la géométrie des groupes, vu que les profils isopérimétriques fournissent des seuils d'intégrabilité des cocycles. Nous faisons une étude plus fine de ces bornes en montrant qu'elle ne peuvent pas être atteintes.

Abstract

Title : On quantitative orbit equivalence between amenable groups

Keywords : Orbit equivalence, odometer, rank-one system, entropy, point spectrum, mixing, Kakutani equivalence, amenable group, isoperimetric profile.

Two free probability measure-preserving actions of countable groups are orbit equivalent if they share the same orbits up to conjugacy. However, a theorem of Ornstein and Weiss states that any two such actions are always orbit equivalent when the groups are infinite and amenable. Quantitative orbit equivalence aims at adding restrictions on the cocycles to get a more interesting theory.

On the one hand, this topic has many interactions with "classical" ergodic theory. First, Belinskaya proved that an orbit equivalence with integrable cocycles boils down to a problem very close to conjugacy. Secondly, assumptions weaker than integrability on the cocycles bridge the gap between the trivial relation of orbit equivalence and the hard problem of conjugacy. In this context, we prove many flexibility results stating that these quantitative forms of orbit equivalence do not preserve dynamical behaviours related to mixing properties and the notion of point spectrum. We also proved that these relations do not imply Kakutani equivalence (a problem as rich as the one of conjugacy). Finally we prove that the preservation of entropy under log-integrable orbit equivalence (due to Kerr and Li) is optimal, namely we find transformations with different entropies orbit equivalent with almost log-integrable cocycles.

On the other hand, quantitative orbit equivalence captures the geometry of amenable groups, since isoperimetric profiles yield integrability thresholds on the cocycles. We will deal with a finer study of these bounds: we prove that they cannot be reached.

Remerciements

Chères lectrices, chers lecteurs,

J'aimerais profiter de cette occasion pour m'adresser à vous, car j'ai conscience que vous ne serez intéressés que par cette rubrique. Et à raison ! C'est ici qu'on peut obtenir des anecdotes croustillantes sur l'auteur et son entourage. Il se peut que vous vérifiiez actuellement, avec un ctrl+F frénétique, que vous figurez bien dans ces remerciements. Peut-être que vous lisez cette thèse par curiosité, en vous demandant « c'est une bonne situation ça, doctorant ? ».

Mais, vous savez, moi je ne crois pas qu'il y ait de bonne ou de mauvaise situation. Moi, si je devais résumer ma thèse aujourd'hui avec vous, je dirais que c'est d'abord des rencontres, des gens qui m'ont tendu la main, peut-être à un moment où je ne pouvais pas, où j'étais seul chez moi. Et c'est assez curieux de se dire que les hasards, les rencontres forgent une destinée... Parce que quand on a le goût de la chose, quand on a le goût de la chose bien faite, le beau geste, parfois on ne trouve pas l'interlocuteur en face, je dirais, le miroir qui vous aide à avancer. Alors ce n'est pas mon cas, comme je le disais là, puisque moi au contraire, j'ai pu ; et je dis merci à la vie, je lui dis merci, je chante la vie, je danse la vie... Je ne suis qu'amour ! Et finalement, quand beaucoup de gens aujourd'hui me disent : « Mais comment fais-tu pour avoir cette humanité ? » Eh bien je leur réponds très simplement, je leur dis que c'est ce goût de l'amour, ce goût donc qui m'a poussé aujourd'hui à entreprendre une construction mécanique, mais demain, qui sait, peut-être simplement à me mettre au service de la communauté, à faire le don, le don de soi...

Après ce long message plein de sens et d'émotions, dont la ressemblance avec un monologue bien connu du cinéma français est fortuite, je vous le dis, chères lectrices et chers lecteurs, n'abandonnez pas, continuez, croyez en vos rêves !

Bien à vous,

Moi

C'est avec sincérité que je remercie mes deux directeurs de thèse François Le Maître et Romain Tessera. Votre enthousiasme m'aura permis de vivre ce doctorat de la meilleure des manières, pas une fois je suis entré dans votre bureau sans que nous ayons rigolé. Votre complémentarité m'a permis de découvrir des thèmes que je n'aurais jamais pensé pouvoir aborder, naïf probabiliste que j'étais. François, merci de m'avoir fait découvrir le domaine de l'équivalence orbitale lors du stage M2. J'ai toujours été admiratif devant ta dextérité à dessiner des tours de Rokhlin, à manipuler les G_δ, à toujours trouver les bonnes terminologies pour rendre mes maths un peu moins horribles. Je retiendrai ton exigence concernant la rédaction d'article. Chacune de tes relectures aura eu l'effet d'une couche de vernis sur mes papiers. Romain, ta transmission aussi bien en cours qu'en exposé ou lors d'une discussion m'aura beaucoup marqué. Tu m'as convaincu qu'on pouvait user de pédagogie dans le monde de la recherche et c'est une des raisons pour lesquelles je veux continuer dans cette voie. Merci pour tes très bons conseils et de m'avoir initié à tes maths, que dis-je, l'art martial que tu pratiques.

Je voudrais exprimer toute ma gratitude envers Thierry Giordano et David Kerr pour avoir accepté de rapporter cette thèse, ainsi que pour ces discussions enrichissantes qui

ont eu un grand impact dans mon travail. Je remercie les examinatrices et examinateurs Nathalie Aubrun, Thierry de la Rue, Amandine Escalier et Matthieu Joseph pour avoir accepté de faire partie du jury. Plus particulièrement, Matthieu et Amandine, merci pour tous vos bons conseils et tous vos exposés géniaux et inspirants.

Je salue toutes celles et ceux que j'ai rencontrés hors de l'IMJ-PRG et les remercie pour les discussions enrichissantes : Jérémie Brieussel, Fabien Durand, Damien Gaboriau, Julien Melleray, Samuel Petite, Yves Stalder, Anush Tserunyan. Je remercie Ana Cristina & Jorge Freitas de m'avoir fait découvrir la théorie ergodique lors du stage M1. Je salue toutes ces belles rencontres en conférence : Alon, Anna, Antoine, Eduardo, Hermès, Martin, Patrick, Pénélope, Petr, Rémi, Sasha, Sasha et Spyros.

Je remercie l'équipe d'algèbre d'opérateur pour l'atmosphère chaleureuse qui y règne, tout particulièrement mon tuteur Denis Perrot pour toute l'attention qu'il a portée sur moi. Je rends également hommage à toutes les personnes qui travaillent dur pour qu'on ait un bon environnement de travail, par exemple les agents d'entretien qui sont présents très tôt le matin pour que nous ayons des locaux propres, ou les gestionnaires (notamment Claire Lavollay et Marouane Abdelaziz pour leur gentillesse et leur patience).

Tout a commencé lors de mon stage de M2, avec un accueil chaleureux des doctorantes et doctorants de l'IMJ-PRG. J'en profite pour remercier Alexandre et Gabriel que j'aurai seulement connus durant le stage. Je remercie Jérôme, notre tonton cantine, pour toutes nos discussions profondes sur des sujets majeurs de société tels que le cyclisme et l'eurovision. Je salue tous les collègues que j'ai connus durant les trois ans de thèse : Agathe, Alessandro, Alexis, Armande, Bilal, Brian, Dianthe, Dorian (pour les parties d'échec palpitantes), Edward, Emmanuel, Enzo, Francesca, Francesca, Filipp, Ghadi, Hector, Hicham, Ivory, Juan Ramon, Kémo, Laura, Léo (un jour peut-être j'aurai le temps de tester « Oriflamme »), Lorenzo, Lucio, Marie-Camille, Marius, Mathieu, Michele, Quentin, Razvan, Ricardo, Robin, Skander, Tom, Yihui, Yousra.

Quant à William, mon ancien cobural, je dois le saluer à part car il n'a rien à faire dans ce labo, bien qu'on l'aime tous comme une pastel de nata ! Merci de m'avoir fait autant rire avec toutes ces vidéos qui n'ont aucun sens, et de m'avoir apporté un jour un odomètre, ma principale source d'inspiration. En tant que cobureaux, je salue également Alessandro, Alessio, Stefan, Victor, tous formidables et très sympathiques. Merci à Merlin et Vincent pour leur *Tea Seminar* très convivial. J'ai eu le plaisir d'organiser les rencontres Masters-Doctorants avec Mario, Tal et Werner, ainsi que les Bourbakettes avec Elie et Paul. Arthur, merci pour ta musique à fond dans ton bureau, c'était un plaisir de fermer ta porte plusieurs fois, et merci pour ces quelques discussions sur « Petits plats en équilibres », « Le meilleur pâtissier », ainsi que sur les groupes quantiques.

Merci à mes petits frères de thèse adorés : Anatole avec qui j'aimerais tellement implémenter Etamine-Notilus-Goelett en Lean, et Kostya, dit Koctr, pour cette rando dans les calanques et ces cours de cyrilliques. Merci à mon grand frère de thèse Juan, cette encyclopédie sur pattes qui m'a appris tellement de choses, aussi bien en maths qu'en linguistique. Merci à mon grand frère « spirituel » Antonio, l'after-Juan, pour m'avoir partagé tout son savoir, ce qui m'a permis de m'intéresser à des maths encore plus variées. Merci à mon demi-jumeau Vincent, qui m'aura contaminé au virus de la géométrie des groupes. En retour, je suis également ravi de t'avoir donné envie de faire de l'équivalence orbitale. C'est un plaisir d'organiser ce groupe de travail avec toi, d'écrire des papiers ensemble (et dire qu'on n'a pas fini, « ça me fume, gros ! ») et de révolutionner les exposés en conférence avec notre fameux *joint talk*. Enfin il est temps de remercier le big brother, que dis-je, le dieu Cronos qui aura connu tant de générations de doctorants. Merci Fabien d'avoir autant perdu ton temps à m'écouter lorsque j'étais bloqué dans mes maths, et de m'avoir partagé les tiennes. Les relectures mutuelles d'articles et de notes auront été pour moi une des activités les plus stimulantes durant ces trois ans. Tu vois, je retiens le positif

malgré ta tentative d'empoisonnement dans un restaurant sichuanais.

Je laisse ici une place privilégiée à mes amis de toujours. Le lecteur attentif remarquera un champ lexical prédominant : celui de la nourriture. Il n'y a que ça qui forge les vraies amitiés. Sériane, cette irréductible que je connais depuis la classe de 6e (et qui n'a toujours pas quitté le collège). Je lui souhaite tous mes vœux de réussite et d'épanouissement dans ses projets et auprès des élèves. Les fidèles de la 1ère S2, pour ces rendez-vous réguliers au Delim's où on se raconte nos beaux parcours professionnels. La secte Roméomorphisme : Eric, Lucas, Mario, Roméo, Sebastian, Thaddeus ; ce serait bien qu'on se refasse un resto japonais, vous ne trouvez pas ? La team loufoque : Nico, Paul et Vava ; pour ces soirées jeu de société. Mes fidèles complices pour manger pizza ou kebab, au soleil, sous la pluie, à midi ou à minuit : Baptiste (mais t'es pas neeeeeeeet), ce formidable prof de physique mais aussi ce formidable ambianceur pour des karaokés endiablés; et Céline, la « reine de la ponctualité », dont la simple évocation ravive de nombreux souvenirs, vieux délires en tout genre et références musicales (mais aussi les regards inquiets de Baptiste qu'on rendait fou pendant les cours). Les fous furieux des maths : Paul et Victor ; vivement la prochaine bronchite à Nancy, sur la plus belle place de France (askip), à parler machine learning, permutons et spaghetti. Les fous furieux de la coinche : Clara, Jad, Tristan. Cet autre fou furieux, mais cette fois-ci de théorie des nombres : Mathieu ; pour sa passion pour la pédagogie et les memes politiques.

Il est temps de rendre hommage à tous mes anciens enseignants qui ont eu un impact majeur dans mon parcours, qui m'ont tiré vers le haut et m'ont fait comprendre qu'il n'y a jamais d'objectif trop ambitieux. Je remercie tout particulièrement Tiphaine Beaslay, Catherine Bourelly, Muriel Dunlop, Fanny Durand-Raucher, Carole Jeammet, Marie-Christine Mayaud, Guillaume Montigny, Laurent Morschhauser, Pierre Schramm, Nicolas & Emmanuelle Tosel, Françoise Toullec, Cécile Villaumé, et je m'excuse de ne pas pouvoir citer tout le monde.

Je remercie mes parents, mon frère et ma grand-mère pour leur soutien indéfectible. Vous êtes les premiers artisans de mon bien-être et de ma réussite. Enfin, je dédie ce manuscrit à mes étoiles dans le ciel, notamment ma marraine qui m'a transmis la rigueur dès mon plus jeune âge en m'apprenant à colorier sans dépasser, et que j'aurai été fier de convier à ma soutenance de thèse.

Contents

Résumé long en français

Ce résumé long en français reprend les éléments essentiels de l'introduction en anglais.

Equivalence orbitale entre systèmes dynamiques

Etant donné un espace de probabilité standard et sans atome (X, μ), on note $\mathrm{Aut}(X, \mu)$ l'ensemble des bijections $X \to X$ préservant la mesure de probabilité (pmp), où on identifie deux telles bijections si elles coïncident sur une partie de mesure pleine. Deux éléments $S, T \in \mathrm{Aut}(X, \mu)$ sont orbitalement équivalents si, à conjugaison près, ils ont les mêmes orbites. Cela signifie qu'il existe un isomorphisme mesuré $\Psi\colon (X, \mu) \to (X, \mu)$ tel que pour presque tout $x \in X$,

$$\Psi\left(\{T^n x \mid n \in \mathbb{Z}\}\right) = \{S^n \Psi(x) \mid n \in \mathbb{Z}\}. \tag{1}$$

L'application Ψ est appelée une *équivalence orbitale* entre S et T. Cette notion a été introduite pour la première fois par Dye en 1959 [Dye59] dans le cadre plus général des relations d'équivalence pmp sur (X, μ). L'équivalence orbitale est un affaiblissement de la conjugaison, qui demande un isomorphisme mesuré $\Psi\colon (X, \mu) \to (X, \mu)$ tel que $\Psi(Tx) = S\Psi(x)$ pour presque tout $x \in X$. Cependant, Dye a démontré que cet affaiblissement est beaucoup trop fort : toutes les bijections pmp et ergodiques sont orbitalement équivalentes.

Pour un renforcement de l'équivalence orbitale, on a d'abord besoin de décrire plus précisément l'égalité entre orbites (1). En supposant que T et S sont apériodiques, c'est-à-dire $T^n x \neq x$ pour tout $n \neq 0$ et pour presque tout x, on peut définir des applications mesurables $c_T\colon \mathbb{Z} \times X \to \mathbb{Z}$ et $c_S\colon \mathbb{Z} \times X \to \mathbb{Z}$ avec les formules

$$\Psi T x = S^{c_T(x)} \Psi(x) \text{ et } \Psi^{-1} S x = T^{c_S(x)} \Psi^{-1}(x)$$

pour presque tout $x \in X$. Les applications c_T et c_S sont les *cocycles* associés à l'équivalence orbitale Ψ.

Etant donnée une équivalence orbitale Ψ entre S et T, les cocycles associés témoignent de la distorsion des orbites pour passer de la dynamique de T à la dynamique de S, et vice versa. L'équivalence orbitale quantitative consiste à ajouter des restrictions sur ces cocycles, pour des distorsions plus rigides. Par exemple, étant donnés $p, q \in [0, +\infty]$, nous demandons à avoir un cocycle c_T qui est L^p, et un cocycles c_S qui est L^q, c'est ce qu'on appelle une équivalence orbitale $(\mathrm{L}^p, \mathrm{L}^q)$. Etant données des applications $\varphi, \psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ (par exemple $\varphi(x) = \log x$), nous demandons plus généralement de la (φ, ψ)-integrabilité.

Une autre forme d'équivalence orbitale quantitative est l'équivalence orbitale de Shannon. Un cocycle, disons c_T, est *Shannon* si la partition $\left\{c_T^{-1}(n) \mid n \in \mathbb{Z}\right\}$ qui lui est associée est d'entropie finie, c'est-à-dire

$$-\sum_{n \in \mathbb{Z}} \mu(c_T^{-1}(n)) \log \mu(c_T^{-1}(n)) < +\infty.$$

Lorsque les deux cocycles sont Shannon, on parle d'*équivalence orbitale de Shannon.*

La question est maintenant la suivante.

Question 1. Quelles propriétés dynamiques sont préservées par les différentes formes quantitatives de l'équivalence orbitale ? Peut-on relier certaines formes quantitatives de l'équivalence orbitale à certaines notions classiques de théorie ergodique ?

Equivalence orbitale entre actions de groupes

Remarquons qu'un élément T de $\mathrm{Aut}(X,\mu)$ fournit une action pmp du groupe $\mathbb{Z}$ sur (X,μ), via $(n,x) \in \mathbb{Z} \times X \mapsto T^n x \in X$. Il est donc naturel d'étendre la notion d'équivalence orbitale au cadre des actions de groupes. Deux actions pmp de groupes Γ et Λ sur (X,μ) sont *orbitalement équivalentes* s'il existe un isomorphisme mesuré $\Psi\colon (X,\mu) \to (X,\mu)$ tel que pour presque tout $x \in X$,

$$\Psi\left(\Gamma \cdot x\right) = \Lambda \cdot \Psi(x). \tag{2}$$

Une généralisation du théorème de Dye existe pour les actions pmp, libres et ergodiques de groupes infinis moyennables, elle est due à Ornstein et Weiss. Comme précédemment, nous allons donc décrire un peu plus précisément l'égalité entre orbites (2) avec les cocycles $c_{\Gamma,\Lambda}\colon \Gamma \times X \to \Lambda$ et $c_{\Lambda,\Gamma}\colon \Lambda \times X \to \Gamma$ définis par :

$$\Psi(\gamma \cdot x) = c_{\Gamma,\Lambda}(\gamma, x) \cdot \Psi(x) \text{ et } \Psi^{-1}(\lambda \cdot x) = c_{\Lambda,\Gamma}(\lambda, x) \cdot \Psi^{-1}(x)$$

pour tout $\gamma \in \Gamma$, $\lambda \in \Lambda$ et presque tout $x \in X$ (ces applications sont bien définies par liberté des actions). Le nom « cocycle» est justifié par le fait que ces applications satisfont l'*identité de cocycle*

$$c_{\Gamma,\Lambda}(\gamma\gamma', x) = c_{\Gamma,\Lambda}(\gamma, \gamma' \cdot x) c_{\Gamma,\Lambda}(\gamma', x).$$

Si Γ est engendré par une partie finie S_Γ, on peut définir la *norme de longueur des mots* par

$$|\gamma|_{S_\Gamma} \coloneqq \min\{n \geqslant 0 \mid \exists s_1, \dots, s_n \in S_\Gamma,\ \gamma = s_1 \dots s_n\}$$

pour tout $\gamma \in \Gamma$. C'est une généralisation de la valeur absolue (dans le cas de $\mathbb{Z}$) qui permet de voir Γ comme un espace métrique, via la *métrique des mots* $(\gamma, \gamma') \mapsto |\gamma'^{-1}\gamma|$.

Etant donnés deux groupes Γ et Λ de type fini, avec des ensembles finis générateurs S_Γ et S_Λ, on dit que le cocycle $c_{\Gamma,\Lambda}\colon \Gamma \times X \to \Lambda$ est L^p si pour tout $\gamma \in \Gamma$, l'application $|c_{\Gamma,\Lambda}(\gamma, .)|_{S_\Lambda}\colon X \to \mathbb{N}$ est L^p. On définit ensuite l'équivalence orbitale $(\mathrm{L}^p, \mathrm{L}^q)$ pour les actions pmp et libres de Γ et de Λ, et plus généralement l'équivalence orbitale (φ, ψ)-intégrable pour des applications $\varphi, \psi\colon \mathbb{R}_+ \to \mathbb{R}_+$.

On s'attend à des phénomènes de rigidité pour l'équivalence orbitale quantitative entre actions de groupes de type fini, pour deux raisons:

- **Vis-à-vis de la dynamique :** comme pour l'équivalence orbitale entre éléments de $\mathrm{Aut}(X,\mu)$, les cocycles d'une équivalence orbitale entre deux actions témoignent de la distorsion des orbites pour passer de la dynamique d'une action à la dynamique de l'autre.

- **Vis-à-vis de la géométrie des groupes :** les orbites d'une action libre d'un groupe de type fini ont en quelque sorte la structure héritée du graphe de Cayley du groupe (associé à une partie finie génératrice fixée). Ainsi les cocycles d'une équivalence orbitale entre deux actions encodent également la distorsion entre les géométries des groupes.

Au vu du dernier point qui se focalise principalement sur la géométrie des groupes de type fini, il est naturel de considérer l'équivalence orbitale comme une relation entre groupes : deux groupes Γ et Λ sont *orbitalement équivalents* s'il existe des actions libres et pmp de Γ et Λ sur un espace de probabilité standard, qui ont les mêmes orbites (hors d'une

partie de mesure nulle). La donnée de ces actions est appelée un *couplage d'équivalence orbitale* entre Γ et Λ. Entre autres, on s'intéressera à la notion de couplage (φ, ψ)-intégrable de Γ vers Λ.

Pour l'équivalence orbitale, les phénomènes de flexibilité observés pour les groupes moyennables sont en net contraste avec les résultats de rigidité parmi les groupes non moyennables ([Fur99a; Gab00; Kid08; Kid10]). Les objectifs sont dans le même esprit que dans la question 1, concernant les renforcements quantitatifs.

Question 2. Quels invariants d'équivalence orbitale quantitative capturent la géométrie des groupes ?

Comportement des propriétés dynamiques sous équivalence orbitale quantitative

Historiquement, le premier résultat d'équivalence orbitale quantitative est dû à Belinskaya. Ce résultat énonce qu'étant donnée une équivalence orbitale entre S et $T \in \mathrm{Aut}(X, \mu)$, si l'un des cocycles est intégrable, alors T et S sont flip-conjugués (T et conjugué à S ou à S^{-1}). La flip-conjugaison étant trop proche du problème de la conjugaison, l'hypothèse d'intégrabilité n'est pas pertinente. Regardons maintenant ce qu'il se passe pour l'équivalence orbitale φ-intégrable avec $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ sous-linéaire, et pour l'équivalence orbitale de Shannon.

L'équivalence orbitale de Shannon n'est pas une relation triviale. En effet, Kerr et Li en ont trouvé un invariant : l'entropie.

Théorème 3 (Kerr, Li [KL24]). *S'il existe une équivalence orbitale de Shannon entre S et $T \in \mathrm{Aut}(X, \mu)$, alors $\mathrm{h}_\mu(S) = \mathrm{h}_\mu(T)$.*

Carderi, Joseph, Le Maître et Tessera ont ensuite découvert des liens entre l'équivalence orbitale de Shannon et la notion l'équivalence orbitale φ-intégrable. Ils ont également démontré que cette dernière ne se résume pas à un problème de flip-conjugaison si $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ est sous-linéaire, prouvant ainsi que le théorème de Belinskaya est optimal.

Théorème 4 (Carderi, Joseph, Le Maître, Tessera [CJLMT23, Théorème 3.16]). *Si $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ est une fonction vérifiant $\log(t) = O(\varphi(t))$ lorsque t tend vers ∞, alors toute équivalence orbitale φ-intégrable est une équivalence orbitale de Shannon.*

Théorème 5 (Carderi, Joseph, Le Maître, Tessera [CJLMT23, Théorème 1.3]). *Soit $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ une fonction sous-linéaire et soit $S \in \mathrm{Aut}(X, \mu)$ une transformation ergodique. Supposons qu'il existe un entier $n \geqslant 2$ tel que S^n est ergodique. Alors il existe une équivalence orbitale φ-intégrable entre S et une transformation qui n'est pas flip-conjuguées à S.*

On obtient donc que l'équivalence orbitale de Shannon et l'équivalence orbitale φ-intégrable (pour $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ sous-linéaire, asymptotiquement plus grand que log) ne se résument pas à de la flip-conjugaison et ne sont pas des relations triviales.

Enfin, pour une construction concrète d'équivalence orbitale entre des transformations bien connues qui ne sont pas flip-conjuguées, avec des informations quantitatives sur les cocycles, on a le résultat suivant.

Théorème 6 (Kerr, Li [KL24]). *Tout odomètre est Shannon orbitalement équivalent à l'odomètre universel.*

Propriétés géométriques des groupes capturées par l'équivalence orbitale quantitative

Concentrons-nous maintenant sur la notion d'équivalence orbitale quantitative entre groupes, et non pas entre actions de groupes prescrites. Etant donné un groupe Γ de type fini, ainsi qu'une partie finie génératrice S_Γ, la fonction de croissance de Γ est définie par

$$V_\Gamma(n) \coloneqq |\{\gamma \in \Gamma \mid |\gamma|_{S_\Gamma} \leqslant n\}|.$$

Par exemple, $V_{\mathbb{Z}^d}(n) \approx n^d$.

Dans l'appendice de [Aus16b], Bowen a démontré l'invariance de la fonction de croissance par équivalence orbitale intégrable. Par conséquent, $\mathbb{Z}$ et $\mathbb{Z}^2$ ne sont pas intégrablement orbitalement équivalents. Delabie, Koivisto, Le Maître et Tessera ont ensuite traité la question du comportement de la fonction de croissance sous équivalence orbitale (φ, ψ)-intégrable, un cadre plus large qui englobe par exemple le cas où les cocycles sont L^p pour un certain $p < 1$.

Théorème 7 ([DKLMT22, Theorem 3.1]). *Soit $\varphi\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ une fonction strictement croissante et sous-additive. S'il existe un couplage d'équivalence orbitale (φ, L^0)-intégrable de Γ vers Λ, alors*

$$V_\Gamma(n) \preccurlyeq V_\Lambda(\varphi^{-1}(n))$$

où φ^{-1} désigne la fonction inverse de φ.

Cette inégalité fournit des bornes d'intégrabilité des cocycles. Par conséquent, l'équivalence orbitale quantitative entre groupes moyennables de type fini est plus complexe que la relation triviale qu'est l'équivalence orbitale. Par exemple, pour des entiers $k > d \geqslant 1$, il n'y a pas de couplage L^p entre $\mathbb{Z}^k$ et $\mathbb{Z}^d$ si $p > \frac{d}{k}$.

Delabie, Koivisto, Le Maître et Tessera ont aussi démontré une inégalité qui utilise cette fois-ci le profil isopérimétrique, un invariant qui mesure en quelque sorte la moyennabilité d'un groupe.

Théorème 8 ([DKLMT22, Theorem 1.1]). *Soit $\varphi\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ une fonction croissante telle que $t \mapsto \frac{t}{\varphi(t)}$ est croissante. Soient Γ et Λ deux groupes de type fini. S'il existe un couplage d'équivalence orbitale (φ, L^0)-intégrable de Γ vers Λ, alors leur profils isopérimétriques vérifient l'inégalité asymptotique suivante :*

$$\varphi \circ j_{1,\Lambda}(n) \preccurlyeq j_{1,\Gamma}(n).$$

Nouveaux résultats présentés dans cette thèse

Après avoir présenté quelques résultats historiques sur l'équivalence orbitale quantitative, parlons de nos contributions.

Généralisation de la construction de Kerr et Li

Dans le chapitre I, nous présentons une généralisation du théorème 6 aux transformations de rang un. D'une part, ces systèmes admettent une construction combinatoire similaire aux odomètres, avec des tours de Rokhlin qui croissent vers la σ-algèbre de l'espace. C'est une propriété cruciale qu'utilisent Kerr et Li dans leur construction, on s'attend donc à pouvoir la généraliser. D'autre part, la famille des systèmes de rang un décrivent des propriétés dynamiques très variées, par exemple concernant les propriétés de mélange et les propriétés spectrales, elle est donc plus riche que la famille des odomètres.

On fournit ainsi des exemples concrets d'équivalences orbitales de Shannon où les propriétés de mélange et les propriétés spectrales ne sont pas préservées. En voici un résumé.

Théorème 9 (théorèmes A, C, D, E et F du chapitre I). *Soit $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ vérifiant $\varphi(t) = o\left(t^{1/3}\right)$ lorsque t tend vers $+\infty$. L'ensemble des éléments de $\mathrm{Aut}(X,\mu)$ qui sont φ-intégrablement orbitalement équivalents à l'odomètre universel contient :*

- *les odomètres;*
- *la transformation de Chacon (faiblement mélangeante mais pas fortement mélangeante);*
- *une transformation de rang un fortement mélangeante;*
- *la rotation irrationnelle d'angle θ, pour θ appartenant à une partie indénombrable et dense de $\mathbb{R}$;*
- *pour tout réel irrationnel θ, une transformation de rang un ayant $e^{2i\pi\theta}$ comme valeur propre.*

Odomutants

Dans le chapitre II, nous construisons des nouvelles transformations pmp, appelés *odomutants*, obtenus après des distorsions successives des orbites d'un odomètre. Même si un odomutant a le même spectre ponctuel que l'odomètre qui lui est associé (théorème II.3.13), il peut avoir des propriétés dynamiques beaucoup plus variées. Par exemple, il peut avoir une entropie positive et n'est pas forcément lâchement de Bernoulli[1] (les systèmes lâchement de Bernoulli forment une classe de transformations où les notions d'équivalence de Kakutani et d'équivalence de Kakutani équitable[2] sont bien comprises). C'est donc une classe de systèmes propices aux contre-exemples et aux résultats de flexibilité.

Le premier résultat montre que, même si l'équivalence orbitale de Shannon et l'équivalence de Kakutani équitable préservent l'entropie, ce ne sont pas les mêmes relations.

Théorème 10 (théorème G du chapitre II). *Il existe une bijection pmp et ergodique T qui est $\mathrm{L}^{<1/2}$ orbitalement équivalente (en particulier Shannon orbitalement équivalente) à l'odomètre dyadique mais pas équitablement Kakutani équivalente.*

Le second résultat montre que le fait que l'équivalence orbitale log-intégrable préserve l'entropie (théorèmes 3 et 4) est optimal.

Théorème 11 (théorème H du chapitre II). *Soit (X,μ) un espace de probabilité standard, α un réel strictement positif ou $+\infty$, et $S \in \mathrm{Aut}(X,\mu)$ un odomètre dont le nombre surnaturel associé $\prod_{p\in\Pi} p^{k_p}$ vérifie la propriété suivante : il existe un nombre premier $p_\star$ tel que $k_{p_\star} = +\infty$. Alors il existe $T \in \mathrm{Aut}(X,\mu)$ tel que*

1. *$\mathrm{h}_\mu(T) = \alpha$;*
2. *il existe une équivalence orbitale entre S et T, qui est φ_m-intégrable pour tout entier $m \geqslant 0$,*

avec $\varphi_m\colon t \to \frac{\log t}{\log^{(\circ m)} t}$, et où $\log^{(\circ m)}$ désigne la composition $\log \circ \ldots \circ \log$ (m fois).

[1] *Loosely Bernoulli* en anglais.

[2] Contrairement à *loosely Bernoulli*, la littérature française ne présente *a priori* aucune traduction de *even Kakutani equivalence*. Si cette information est fausse, le lecteur est chaleureusement convié à contacter l'auteur de cette thèse pour vaincre son ignorance.

Ce théorème s'applique par exemple à l'odomètre dyadique et à l'odomètre universel.

La stratégie que nous adoptons consiste à utiliser l'entropie topologique, plus facile à calculer, ainsi que le principe variationnel pour retrouver l'entropie mesurée (il faut donc que nos systèmes soient uniquement ergodiques). Derrière le théorème 11 se cache donc l'énoncé suivant qui s'inscrit dans un cadre plus topologique, avec l'entropie topologique et la notion d'*équivalence orbitale forte*.

Théorème 12 (théorème I du chapitre II)**.** *Soit α un réel strictement positif $+\infty$. Soit S un odomètre dont le nombre surnaturel associé $\prod_{p\in\Pi} p^{k_p}$ vérifie la propriété suivante : il existe un nombre premier $p_\star$ tel que $k_{p_\star} = +\infty$. Alors il existe un homéomorphisme minimal T sur l'espace de Cantor tel que*

1. $\mathrm{h_{top}}(T) = \alpha$;
2. *il existe une équivalence orbitale forte entre S et T, qui est φ_m-intégrable pour tout entier $m \geqslant 0$,*

avec $\varphi_m \colon t \to \frac{\log t}{\log^{(\circ m)} t}$, et où $\log^{(\circ m)}$ désigne la composition $\log \circ \ldots \circ \log$ (m fois).

Il est crucial de remarquer que ce théorème généralise l'énoncé suivant, dû à Boyle et Handelman.

Théorème 13 (Boyle, Handelman [BH94])**.** *Soit α un réel strictement positif ou $+\infty$. Soit S l'odomètre dyadique. Alors il existe un homéomorphisme minimal T sur l'espace de Cantor tel que*

1. $\mathrm{h_{top}}(T) = \alpha$;
2. *S et T sont fortement orbitalement équivalents.*

Il s'avère que les homéomorphismes T construits par Boyle et Handelman sont exactement les systèmes que nous construisons : des odomutants ; ce qui ne saute pas aux yeux vu qu'ils utilisent des diagrammes de Bratteli pour les décrire. De plus, ces diagrammes fournissent un moyen plus abstrait de déterminer si deux homéomorphismes minimaux sont fortement orbitalement équivalents, via un invariant complet appelé le *groupe à dimension*. En revanche, l'équivalence orbitale est plus explicite avec notre construction, nous permettant ainsi de quantifier les cocycles.

Enfin, après avoir trouvé un moyen de déterminer si un odomutant est conjugué ou non à l'odomètre qui lui est associé, nous obtenons l'extension suivante du théorème 5.

Théorème 14 (théorème J du chapitre II)**.** *Soit $\varphi \colon \mathbb{R}_+ \to \mathbb{R}_+$ une application sous-linéaire et S un odomètre. Alors il existe une bijection pmp T telle que S et T sont φ-intégrablement orbitalement équivalents mais pas flip-conjugués.*

Sur le seuil d'intégrabilié fourni par les profils isopérimétriques

Dans le contexte de l'équivalence orbitale quantitative entre groupes moyennables infinis, le théorème 8 fournit une borne d'intégrabilité des cocycles. Nous savons que cette borne est optimale dans certains cas.

Par exemple, étant donné un couplage d'équivalence orbitale entre $\mathbb{Z}^k$ et $\mathbb{Z}^{k+\ell}$, où k et ℓ sont des entiers strictement positifs, le cocycle de $\mathbb{Z}^{k+\ell}$ vers $\mathbb{Z}^k$ ne peut pas être L^p si $p > \frac{k}{k+\ell}$. De plus, on sait qu'un tel couplage couplage existe, où le cocycle de $\mathbb{Z}^{k+\ell}$ vers $\mathbb{Z}^k$ est L^p pour tout $p < \frac{k}{k+\ell}$ [DKLMT22, Theorem 1.9]. Qu'en est-il du cas $p = \frac{k}{k+\ell}$? Nous répondons à cette question dans le chapitre III. Nous montrons que le cocycle ne peut être $\mathrm{L}^{\frac{k}{k+\ell}}$, et plus généralement que la borne d'intégrabilité fournie par le profil isopérimétrique ne peut pas être atteinte pour toute paire (Γ, Λ) de groupes moyennables infinis (avec des hypothèses supplémentaires mais légères).

Théorème 15 (théorème L du chapitre III). *Soit Γ et Λ deux groupes de type fini. Supposons qu'il existe une fonction croissante h_Γ et une fonction strictement croissante h_Λ vérifiant $h_\Gamma \approx j_{1,\Gamma}$, $h_\Lambda \approx j_{1,\Lambda}$ et les hypothèses suivantes lorsque $x \to +\infty$:*

$$h_\Gamma(x) = o\left(h_\Lambda(x)\right), \tag{3}$$

$$\forall C > 0,\ h_\Gamma(Cx) = O\left(h_\Gamma(x)\right), \tag{4}$$

$$\forall C > 0,\ h_\Gamma \circ h_\Lambda^{-1}(Cx) = O\left(h_\Gamma \circ h_\Lambda^{-1}(x)\right). \tag{5}$$

Alors il n'existe pas de couplage d'équivalence orbitale $(h_\Gamma \circ h_\Lambda^{-1}, \mathrm{L}^0)$-intégrable de Γ vers Λ.

Une conséquence remarquable est la caractérisation complète pour les groupes $\mathbb{Z}^d$, $d \geqslant 1$.

Théorème 16 (théorème III.4.4 du chapitre III). *Etant donnés deux entiers strictement positifs k, ℓ, il existe un couplage d'équivalence orbitale $(\mathrm{L}^p, \mathrm{L}^0)$ de $\mathbb{Z}^{k+\ell}$ vers $\mathbb{Z}^k$ si et seulement si $p < \frac{k}{k+\ell}$.*

Introduction in english

Orbit equivalence between dynamical systems

Given a standard and atomless probability space (X, μ), we denote by $\mathrm{Aut}(X, \mu)$ the set of probability measure-preserving (pmp) bijections $X \to X$, two such maps being identified if they coincide on a subset of full measure. Two elements $S, T \in \mathrm{Aut}(X, \mu)$ are *orbit equivalent* if they share the same orbits up to conjugacy. This means that there exists a measured isomorphism $\Psi \colon (X, \mu) \to (X, \mu)$ such that for almost every $x \in X$,

$$\Psi\left(\{T^n x \mid n \in \mathbb{Z}\}\right) = \{S^n \Psi(x) \mid n \in \mathbb{Z}\}\,. \tag{6}$$

The map Ψ is called an *orbit equivalence* between S and T. This concept was first introduced by Dye in 1959 [Dye59] in the more general setting of pmp equivalence relations on (X, μ). Orbit equivalence is a weakening of the conjugacy problem, which requires a measured isomorphism $\Psi \colon (X, \mu) \to (X, \mu)$ such that $\Psi(Tx) = S\Psi(x)$ for almost every $x \in X$. However Dye proved that this weakening is trivial: any two ergodic pmp bijections are orbit equivalent.

To strengthen orbit equivalence, we first need a more precise description of the orbit equality (6). If we assume that T and S are aperiodic, namely $T^n x \neq x$ for every $n \neq 0$ and almost every x, we can define measurable maps $c_T \colon \mathbb{Z} \times X \to \mathbb{Z}$ and $c_S \colon \mathbb{Z} \times X \to \mathbb{Z}$ by the formulas

$$\Psi T x = S^{c_T(x)} \Psi(x) \text{ and } \Psi^{-1} S x = T^{c_S(x)} \Psi^{-1}(x)$$

for almost every $x \in X$. The maps c_T and c_S are called the *cocycles* associated to the orbit equivalence Ψ.

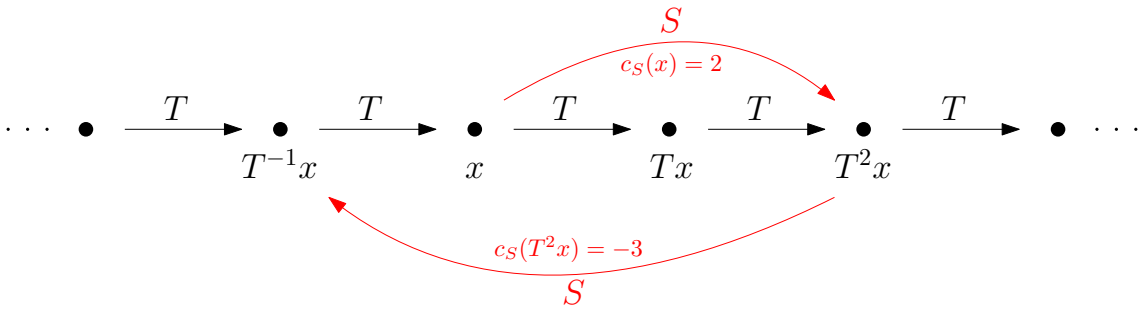


Figure 1: Orbit equivalence between aperiodic transformations in $\mathrm{Aut}(X, \mu)$, when $\Psi = id_X$.

Given an orbit equivalence Ψ between S and T, the associated cocycles tell us how much we have to distort the orbits to move from the dynamics of T to the dynamics of S, and vice versa. *Quantitative orbit equivalence* consists in adding restrictions on these cocycles for the distortions to be more rigid. For instance, given $p, q \in [0, +\infty]$, we will ask for an L^p cocycle c_T and an L^q cocycle c_S, this is what we call an $(\mathrm{L}^p, \mathrm{L}^q)$ orbit equivalence.

Given maps $\varphi, \psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ (for instance $\varphi(x) = \log x$), we more generally require (φ, ψ)-integrability. We define it with more details in Section B.2 of Appendix B, the reader can keep in mind that L^p is exactly φ-integrability when $\varphi(x) = x^p$, and that the faster the map φ goes to $+\infty$, the stronger is the restriction on the cocycle.

Another form of quantitative orbit equivalence is the Shannon property. For the reader who is not acquainted with the notion of entropy, Section A.1.h in Appendix A motivates this quantity. In the remark below, we explain intuitively what it encodes in this context of orbit equivalence.

A cocycle, let us say c_T, is *Shannon* if the entropy of its associated partition $\left\{c_T^{-1}(n) \mid n \in \mathbb{Z}\right\}$ has finite entropy, this means that

$$-\sum_{n\in\mathbb{Z}} \mu(c_T^{-1}(n)) \log \mu(c_T^{-1}(n)) < +\infty.$$

Shannon orbit equivalence asks for an orbit equivalence whose associated cocycles are both Shannon.

Remark 17. As a motivation behind the definition of entropy (see Section A.1.h in Appendix A), $-\sum_{n\in\mathbb{Z}} \mu(c_T^{-1}(n)) \log \mu(c_T^{-1}(n))$ quantifies how much the partition $\left\{c_T^{-1}(n) \mid n \in \mathbb{Z}\right\}$ is dividing the space X. In other words, it quantifies the uncertainty of the value of $c_T(x)$, where x is random with respect to μ.

It does not exactly bring the same information as the property of being in L^p. For instance, the latter gives information on the tail of $|c_T(.)|\colon X \to \mathbb{N}$, using Markov's inequality:

$$\mu\left(\{|c(.)| > n\}\right) \leqslant \frac{\int_X |c_T(x)|^p \mathrm{d}\mu(x)}{n^p}.$$

The question is now the following.

Question 18. Which dynamical properties are preserved under quantitative forms of orbit equivalence? Are there connections between quantitative forms of orbit equivalence and classical notions in ergodic theory?

As we will explain later in this introduction, Belinskaya proved that integrable orbit equivalence boils down to a relation too close to conjugacy. However, when $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a sublinear map greater than log, many results imply that the relation of φ-integrable orbit equivalence bridges the gap between the trivial relation of orbit equivalence and the hard problem of conjugacy. The same holds for Shannon orbit equivalence.

Orbit equivalence between group actions

Note that any element T of $\mathrm{Aut}(X, \mu)$ gives rise to a pmp action $\mathbb{Z} \curvearrowright (X, \mu)$, via $(n, x) \in \mathbb{Z} \times X \mapsto T^n x \in X$. It is then natural to study the following extension to more general group actions. Two pmp actions of groups Γ and Λ on (X, μ) are *orbit equivalent* if there exists a measured isomorphism $\Psi\colon (X, \mu) \to (X, \mu)$ such that for almost every $x \in X$,

$$\Psi\left(\Gamma \cdot x\right) = \Lambda \cdot \Psi(x). \tag{7}$$

However, Dye's theorem has been generalized to free ergodic pmp actions of infinite amenable groups, by Ornstein and Weiss [OW80]. As before, we more precisely describe the orbit equalities (7) with the *cocycles* $c_{\Gamma,\Lambda}\colon \Gamma \times X \to \Lambda$ and $c_{\Lambda,\Gamma}\colon \Lambda \times X \to \Gamma$ defined by:

$$\Psi(\gamma \cdot x) = c_{\Gamma,\Lambda}(\gamma, x) \cdot \Psi(x) \text{ and } \Psi^{-1}(\lambda \cdot x) = c_{\Lambda,\Gamma}(\lambda, x) \cdot \Psi^{-1}(x)$$

for every $\gamma \in \Gamma$, $\lambda \in \Lambda$ and almost every $x \in X$ (these maps are well-defined by freeness of the actions). Note that the cocycles $c_T, c_S\colon X \to \mathbb{Z}$ we defined earlier for orbit equivalent

elements of $\mathrm{Aut}(X,\mu)$ are exactly the restrictions to the generator $+1$ of the cocycles $\mathbb{Z}\times X\to\mathbb{Z}$ we define right above. The name "cocycle" comes from the fact that these maps satisfy the *cocycle identity*:

$$c_{\Gamma,\Lambda}(\gamma\gamma',x)=c_{\Gamma,\Lambda}(\gamma,\gamma'\cdot x)c_{\Gamma,\Lambda}(\gamma',x).$$

The goal is now to introduce quantitative orbit equivalence in this more general setting. For instance, we want a definition of being L^p for a cocycle $c_{\Gamma,\Lambda}\colon\Gamma\times X\to\Lambda$. We have to compose the cocycle with a real-valued function. For instance, for a $\mathbb{Z}$-valued cocycle, we use the absolute value. To this end, finitely generated groups form a natural class where we can define a norm playing the same role as the absolute value. Indeed, if Γ is generated by a finite subset S_Γ, then we can define the *word-length norm* by

$$|\gamma|_{S_\Gamma}\coloneqq\min\{n\geqslant 0\mid\exists s_1,\dots,s_n\in S_\Gamma,\ \gamma=s_1\dots s_n\}$$

for every $\gamma\in\Gamma$. This norm enables us to naturally consider finitely generated groups as metric spaces, via the *word-length metric* $(\gamma,\gamma')\mapsto|\gamma'^{-1}\gamma|$. For instance, the ℓ^1-norm of $\mathbb{R}^d$, restricted to $\mathbb{Z}^d$, is the norm of $\mathbb{Z}^d$ associated to its canonical generating subset. We give more details in Section A.2.a of Appendix A.

Now given finitely generated groups Γ and Λ, with finite generating subsets S_Γ and S_Λ, we say that a cocycle $c_{\Gamma,\Lambda}\colon\Gamma\times X\to\Lambda$ is L^p if for every $\gamma\in\Gamma$, the map $|c_{\Gamma,\Lambda}(\gamma,.)|_{S_\Lambda}\colon X\to\mathbb{N}$ is L^p. We then define the notion of $(\mathrm{L}^p,\mathrm{L}^q)$ orbit equivalence for free pmp Γ- and Λ-actions, and more generally (φ,ψ)-integrable orbit equivalence for general maps $\varphi,\psi\colon\mathbb{R}_+\to\mathbb{R}_+$ (see Section B.2 in Appendix B).

It is relevant to expect rigidity phenomena for quantitative orbit equivalence between actions of finitely generated groups, for two reasons:

- **When focusing on the dynamics:** as for orbit equivalence between elements of $\mathrm{Aut}(X,\mu)$, the cocycles of a given orbit equivalence tell us how much we have to distort the orbits to move from the dynamics of one action to the dynamics of the other orbit equivalent to it.

- **When focusing on the geometry of the groups:** the orbits of a free action of a finitely generated group somehow have the structure inherited from the Cayley graphs of the group (associated to a fixed generating subsets). This means that in every orbit, we put an edge between y and $\gamma\cdot y$ if γ lies in $S_\Gamma\cup S_\Gamma^{-1}$. Figure 2 provides an illustration for a free $\mathbb{Z}^2$-action. So the cocycles also encode the distortions to pass from the geometry of one group to the other.

  Let us give an example with $\mathbb{Z}^2$- and $\mathbb{Z}$-actions, where the groups are respectively generated by $S_{\mathbb{Z}^2}=\{e_1=(1,0),e_2=(0,1)\}$ and $S_\mathbb{Z}=\{+1\}$. The norms $|.|_{S_{\mathbb{Z}^2}}$ and $|.|_{S_\mathbb{Z}}$ quantify how far $c_{\mathbb{Z},\mathbb{Z}^2}(+1,x)$ and $c_{\mathbb{Z}^2,\mathbb{Z}}(e_i,x)$ are from being generators (or inverses of generators) of the corresponding group. So they quantify how far an orbit equivalence Ψ is from preserving the edges in the orbits. In view of the structure of $\mathbb{Z}^2$-orbits in Figure 2, while $\mathbb{Z}$-orbits are more likely to live in a "one-dimensional world" (as the T-orbits in Figure 1), it is natural to believe that an orbit equivalence distort the structures, and that $x\mapsto c_{\mathbb{Z}^2,\mathbb{Z}}(e_i,x)$ often take large values.

  To sum up, the cocycles $c_{\Gamma,\Lambda}\colon\Gamma\times X\to\Lambda$ and $c_{\Lambda,\Gamma}\colon\Lambda\times X\to\Gamma$ can be seen as measurable families $c_{\Gamma,\Lambda}(.,x)\colon\Gamma\to\Lambda$ and $c_{\Lambda,\Gamma}(.,x)\colon X\to\Gamma$. The integrability conditions on both cocycles can therefore be seen as a way to prevent too much distortion (in a way which involves the measure).

By this last item which more focuses on the geometry of finitely generated groups, it is natural to also consider orbit equivalence as a relation between groups: two groups Γ and

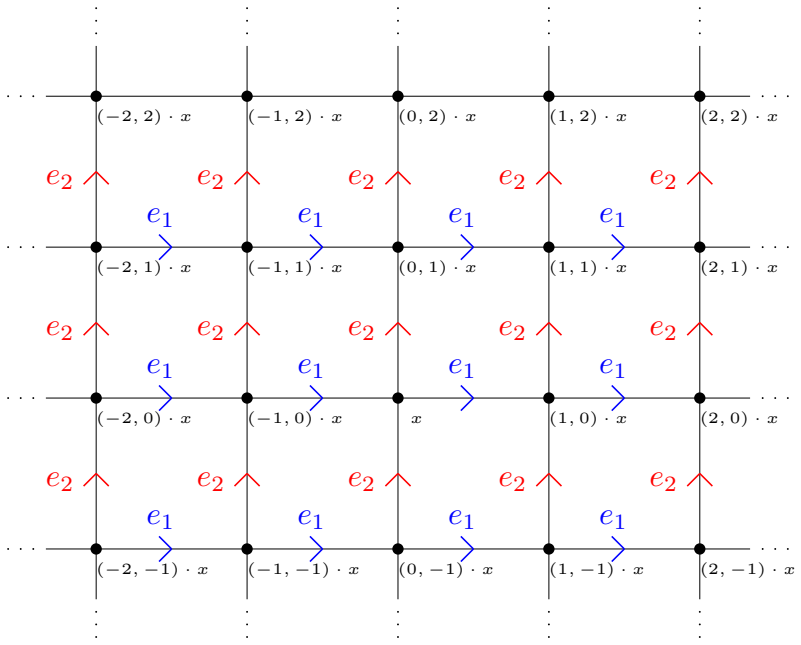


Figure 2: Given a free action of $\mathbb{Z}^2$ on X, and given $x \in X$, the $\mathbb{Z}^2$-orbit of x has the structure inherited from the Cayley graph of $\mathbb{Z}$ with respect to the finite generating set $\{e_1, e_2\}$ with $e_1 = (1,0)$ and $e_2 = (0,1)$, where x is identified with $(0,0)$.

Λ are *orbit equivalent* if there exist free pmp Γ- and Λ-actions on a standard probability space, sharing the same orbits up to a null set. An *orbit equivalence coupling* between Γ and Λ refers to the data of these actions. Then we say that Γ is (φ, ψ)-integrably orbit equivalent to Λ if there exist an orbit equivalence coupling which is (φ, ψ)-integrable.

Remark 19. It turns out that orbit equivalence between two groups is a particular instance of *measure equivalence*, a notion introduced by Gromov as a measured analogue of quasi-isometry. Conversely, measure equivalence provides a weaker notion of orbit equivalence, called *stable orbit equivalence*, roughly speaking it requires equality between portions of orbits. Coming back to actions of the group $\mathbb{Z}$, namely the data of elements of $\mathrm{Aut}(X, \mu)$, a classical instance of stable orbit equivalence is the notion of *Kakutani equivalence*: two transformations T, S are Kakutani equivalent if they admit induced transformations which are conjugate. The theory of Kakutani equivalence is as rich as the problem of conjugacy, we refer the reader to Section A.1.i in Appendix A for more details. These remarks highlight the interactions between orbit equivalence and many topics like ergodic theory and geometric group theory.

The flexibility phenomena observed for amenable groups is in sharp contrast with the rigidity results found in the non-amenable world. For instance, Gaboriau [Gab00] proved that the orbit equivalence classes among free groups are completely described by the number of generators. We also refer the reader to the work of Furman [Fur99a] on lattices in higher rank semi-simple Lie groups and the works of Kida [Kid08; Kid10] on mapping class groups, both providing rigidity phenomena under measure equivalence.

Since orbit equivalence cannot distinguish between infinite amenable groups, we can ask the following question in the same vein as Question 18.

Question 20. Which invariants of quantitative orbit equivalence capture the geometry of the groups?

Measure equivalence also comes with cocycles and analogous quantitative strengthenings (see Section B.2 in Appendix B). Since measure equivalence is also a trivial relation

among infinite amenable groups, the last question also arises in this context. In the sequel, we will only deal with orbit equivalence, but some result in fact hold more generally in the setting of quantitative measure equivalence, we refer the reader to Appendix B for a survey.

Behaviour of dynamical properties of group actions under quantitative orbit equivalence

The first result on quantitative orbit equivalence is probably Belinskaya's theorem. It states that, given an orbit equivalence between S and $T \in \mathrm{Aut}(X, \mu)$, if one of the cocycles is integrable, then T and S are flip-conjugate (T is conjugate to S or S^{-1}). Since the flip-conjugacy problem is too close to the conjugacy problem, integrability is not a relevant restriction to add on the cocycles.

Then Austin launched the study of the preservation of measure-theoretic entropy under quantitative forms of orbit equivalence. Intuitively, the entropy of $T \in \mathrm{Aut}(X, \mu)$, denoted by $\mathrm{h}_\mu(T)$, is a quantity which tells us how much iterations of T complexify the space. It is defined as the growth rate of some quantities using the entropy of partitions. We explain this notion in Section A.1.h of Appendix A.

Austin proved that integrable orbit equivalence among actions of infinite amenable groups preserves the entropy [Aus16a]. Note that this was already known for $\mathbb{Z}$-actions, by Belinskaya's theorem. In fact, Kerr and Li noticed than Austin's result also holds in the context of Shannon orbit equivalence in many cases depending on algebraic properties of the groups. In the particular case of the group $\mathbb{Z}$, the statement is the following.

Theorem 21 (Kerr, Li [KL24]). *Let $S, T \in \mathrm{Aut}(X, \mu)$. If they are Shannon orbit equivalent, then $\mathrm{h}_\mu(S) = \mathrm{h}_\mu(T)$.*

Carderi, Joseph, Le Maître and Tessera then found connections between this notion and φ-integrable orbit equivalence, and proved that the latter does not boil down to flip-conjugacy when φ is sublinear. Since integrable orbit equivalence is exactly φ-integrable orbit equivalence for any nonzero linear map φ, this implies that Belinskaya's theorem is optimal.

Theorem 22 (Carderi, Joseph, Le Maître, Tessera [CJLMT23, Theorem 3.16]). *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a map satisfying $\log(t) = O(\varphi(t))$ as t goes to ∞. If $T, S \in \mathrm{Aut}(X, \mu)$ are φ-integrably orbit equivalent, then they are Shannon orbit equivalent.*

Theorem 23 (Carderi, Joseph, Le Maître, Tessera [CJLMT23, Theorem 1.3]). *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear map and let $S \in \mathrm{Aut}(X, \mu)$ be ergodic. Assume that there exists an integer $n \geqslant 2$ such that S^n is ergodic. Then there exists $T \in \mathrm{Aut}(X, \mu)$ such that S and T are φ-integrably orbit equivalent but not flip-conjugate.*

For a concrete construction of orbit equivalence between well-known transformations that are not isomorphic, and with a quantification on the cocycles, we have the following statement of Kerr and Li. We refer the reader to Example A.1.30 in Appendix A for the definition of a universal odometer, this is related to the flip-conjugacy classification of odometers.

Theorem 24 (Kerr, Li [KL24]). *Every odometer is Shannon orbit equivalent to the universal odometer.*

Geometric properties of groups captured by quantitative orbit equivalence

Let us now focus on the notion of quantitative orbit equivalence between groups, and not between prescribed group actions. We recall that two groups are orbit equivalent if they admit free pmp actions which share the same orbits up to a null set.

Given a finitely generated group Γ with a finite generating subset S_Γ, the growth function is defined by

$$V_\Gamma(n) \coloneqq |\{\gamma \in \Gamma \mid |\gamma|_{S_\Gamma} \leqslant n\}|.$$

With a metric viewpoint, this function encodes the volume of the balls. For instance, $V_{\mathbb{Z}^d}(n) \approx n^d$.

In the appendix of [Aus16b], Bowen showed the invariance of the growth function under L^1 orbit equivalence.

Theorem 25 ([Aus16b, Theorem B.2])**.** *Let Γ and Λ be finitely generated groups. If Γ and Λ are L^1 orbit equivalent, then $V_\Gamma(n) \approx V_\Lambda(n)$.*

As an application, $\mathbb{Z}$ and $\mathbb{Z}^2$ are not integrably orbit equivalent. It is therefore natural to wonder whether these rigidity results still hold for the more general notions of (φ, ψ)-integrability which encompass for instance L^p for $p < 1$. In this wider setup, Delabie, Koivisto, Le Maître and Tessera refined Bowen's result as follows.

Theorem 26 ([DKLMT22, Theorem 3.1])**.** *Let $\varphi\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be an increasing and sub-additive map. If there is a (φ, L^0)-integrable orbit equivalence coupling from Γ to Λ, then*

$$V_\Gamma(n) \preccurlyeq V_\Lambda(\varphi^{-1}(n))$$

where φ^{-1} is the inverse function of φ.

This inequality provides explicit upper bounds on how integrable the cocycles of an orbit equivalence coupling can be. This implies that, among finitely generated amenable groups, quantitative orbit equivalence is more complex than the trivial relation of orbit equivalence. For instance, for every integers $k > d \geqslant 1$, there is no L^p orbit equivalence between $\mathbb{Z}^k$ and $\mathbb{Z}^d$ if $p > \frac{d}{k}$.

Explicit constructions show that the bound given by this theorem is almost sharp. For instance, for every positive integers $k > d$, the groups $\mathbb{Z}^k$ and $\mathbb{Z}^d$ are L^p-orbit equivalent for every $p < \frac{d}{k}$ [DKLMT22, Theorem 1.9].

Going further, Delabie, Koivisto, Le Maître and Tessera also proved in [DKLMT22] an inequality that involves rather the isoperimetric profile. For recalls about the isoperimetric profile $j_{1,\Gamma}$ of a finitely generated group Γ, we refer the reader to Section A.2.b in Appendix A. This invariant tells us how much amenable a group is. Since we want to strengthen orbit equivalence among amenable groups, this quantity is relevant.

Theorem 27 ([DKLMT22, Theorem 1.1])**.** *Let $\varphi\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a non-decreasing function such that $t \longmapsto \frac{t}{\varphi(t)}$ is non-decreasing. Let Γ and Λ be finitely generated groups. Assume that there exists a (φ, L^0)-integrable orbit equivalence coupling from Γ to Λ. Then their isoperimetric profiles satisfy the asymptotic inequality*

$$\varphi \circ j_{1,\Lambda}(n) \preccurlyeq j_{1,\Gamma}(n).$$

New results presented in this thesis

After presenting the historical results on quantitative orbit equivalence, let us move on to our contributions.

Generalization of Kerr and Li's construction

Chapter I deals with an extension of Theorem 24 to rank-one systems. We refer the reader to Section I.3.a in Chapter I for a survey on these transformations.

On the one hand, rank-one systems have a combinatorial construction similar to the odometers, with Rokhlin towers approximating the σ-algebra of the space. Kerr and Li crucially use this property in their construction, so it is relevant to extend their construction to such systems. On the other hand, rank-one systems form a richer class than odometers, favourable to counter-examples. Here are examples of dynamical properties satisfied by some of them:

- strongly mixing (for instance the rank-one systems built by Ornstein [Orn72]);
- weakly mixing but not strongly mixing (for instance the Chacon map [Cha69]);
- existence of an irrational eigenvalue (for instance irrational rotations, or rank-one systems built by Danilenko and Vieprik [DV23, Theorem 4.1]),

whereas odometers all have rational eigenvalues and none of them are weakly mixing. It is thus natural to expect that this class provides flexibility results for quantitative orbit equivalence.

We first prove that the Shannon orbit equivalence in Theorem 24 is in fact a φ-integrable orbit equivalence for any map $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\varphi(t) = o(t^{\frac{1}{3}})$ as t goes to $+\infty$. We then provide extensions to rank-one systems with this quantitative form. We do not give the definition of *BSP rank-one systems* mentioned below (see Definition I.3.5 in Chapter I), the reader can keep in mind that it contains every odometer and the Chacon map.

Theorem 28 (see Theorem A in Chapter I)**.** *Every BSP rank-one system is φ-integrably orbit equivalent to the universal odometer for any $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\varphi(t) = o\left(t^{1/3}\right)$ as t goes to $+\infty$.*

Considering the Chacon map, we deduce from this theorem that weak mixing property is not preserved under φ-integrable orbit equivalence for maps φ as in the statement. In fact, this was already known in [CJLMT23] (we refer to the comments after Theorem B.3.8 in Appendix B for more details), but we prove this flexibility result with explicit examples of transformations (the universal odometer and the Chacon map).

With the strongly mixing rank-one systems built by Ornstein [Orn72], we can even deduce that strong mixing property is not preserved.

Theorem 29 (see Theorem F in Chapter I)**.** *For every map $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\varphi(t) = o\left(t^{1/3}\right)$ as t goes to $+\infty$, there exists a strongly mixing rank-one system which is φ-integrably orbit equivalent to the universal odometer.*

We also manage to relate odometers and irrational rotations.

Theorem 30 (see Theorem C in Chapter I)**.** *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a map satisfying $\varphi(t) = o\left(t^{1/3}\right)$ as t goes to $+\infty$. The set of irrational numbers θ whose associated irrational rotation is φ-integrably orbit equivalent to the universal odometer is dense in $\mathbb{R}$.*

Theorem 31 (see Theorem D in Chapter I)**.** *For every map $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\varphi(t) = o\left(t^{1/3}\right)$ as t goes to $+\infty$, and for every non-empty open subset $\mathcal{V}$ of $\mathbb{R}$, the set of irrational numbers $\theta \in \mathcal{V}$ whose associated irrational rotation is φ-integrably orbit equivalent to the universal odometer is uncountable.*

Irrational rotations are instances of rank-one systems with irrational eigenvalues (recall that odometers only have rational eigenvalues). We do not know if every irrational rotation

is φ-integrably orbit equivalent to the universal odometer, with a map φ as in the statements. However, given any irrational number θ, we can relate the universal odometer with some rank-one systems having $e^{2i\pi\theta}$ as an eigenvalue, using constructions of Danilenko and Vieprik [DV23].

Theorem 32 (see Theorem E in Chapter I). *For every map $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\varphi(t) = o\left(t^{1/3}\right)$ as t goes to $+\infty$, and for every irrational number θ, there exists a rank-one system which has $e^{2i\pi\theta}$ as an eigenvalue and which is φ-integrably orbit equivalent to the universal odometer.*

Odomutants

In Chapter II, we build new probability measure-preserving transformations, called *odomutants*, built by distorting the orbits of an odometer. Even though an odomutant has the same point spectrum as its associated odometer (Theorem II.3.13 in Chapter II), it can have different dynamical properties. For instance, it can have positive entropy and it is not necessarily loosely Bernoulli (loosely Bernoulli systems form a class of transformations where Kakutani equivalence and even Kakutani equivalence are well understood). These explicit constructions thus provide flexibility results for quantitative orbit equivalence.

Even though Shannon orbit equivalence (or log-integrable orbit equivalence) and even Kakutani equivalence both preserve entropy, these relations are not the same, according to the following.

Theorem 33 (see Theorem G in Chapter II). *There exists an ergodic probability measure-preserving bijection T which is $\mathrm{L}^{<1/2}$ orbit equivalent (in particular Shannon orbit equivalent) to the dyadic odometer but not evenly Kakutani equivalent to it.*

In fact, the system T in this statement is built as a non loosely Bernoulli system, so it is neither Kakutani equivalent to the dyadic odometer.

Moreover the fact that log-integrable orbit equivalence preserves entropy (Theorems 21 and 22) is optimal. Indeed, we can find an odometer (of zero entropy) almost log-integrably orbit equivalent to a transformation of positive entropy with all possible values.

Theorem 34 (see Theorem H in Chapter II). *Let (X, μ) be a standard atomless probability space, let α be either a positive real number or $+\infty$, and let $S \in \mathrm{Aut}(X, \mu)$ be an odometer whose associated supernatural number $\prod_{p\in\Pi} p^{k_p}$ satisfies the following property: there exists a prime number $p_\star$ such that $k_{p_\star} = +\infty$. Then there exists a probability measure-preserving transformation $T \in \mathrm{Aut}(X, \mu)$ such that*

1. *$\mathrm{h}_\mu(T) = \alpha$;*
2. *there exists an orbit equivalence between S and T, which is φ_m-integrable for all integers $m \geqslant 0$,*

where φ_m denotes the map $t \to \frac{\log t}{\log^{(\circ m)} t}$ and $\log^{(\circ m)}$ the composition $\log \circ \ldots \circ \log$ (m times).

We refer the reader to Example A.1.30 in Appendix A for the notion of *supernatural number* associated to (the conjugacy class of) an odometer. The theorem applies to the dyadic odometer and the universal odometer for instance

It is worth noticing that this statement looks like the theorem stated below, due to Boyle and Handelman and which deals with a more topological version of orbit equivalence. Let us briefly explain the terminologies used in the statement.

Minimal homeomorphisms acting on the Cantor set are *strongly orbit equivalent* if they share the same orbits (up to topological conjugacy) and each cocycle has at most one point of discontinuity. We refer the reader to Appendix II.B in Chapter II for an overview on

this theory developed by Giordano, Putnam and Skau [GPS95]. Moreover the notation $\mathrm{h}_{\mathrm{top}}$ in the following statement refers to a topological notion of entropy (see Section A.1.h in Appendix A).

Theorem 35 (Boyle, Handelman [BH94]). *Let α be either a positive real number or $+\infty$. Let S be the dyadic odometer. Then there exists a Cantor minimal homeomorphism T such that*

1. $\mathrm{h}_{\mathrm{top}}(T) = \alpha$;
2. *S and T are strongly orbit equivalent.*

In an earlier version of our result, we noticed that odomutants are in fact the dynamical systems that they built. This is not so obvious at first glance since they deal with Bratteli diagrams to describe the systems. Moreover they use abstract invariants to get that the systems are strongly orbit equivalent, whereas our constructions are concrete, this enables us to quantify the cocycles.

The strategy to prove our result is to use topological entropy (easier to compute) and the variational principle to connect it with measure-theoretic entropy (so we have to work with uniquely ergodic transformations). A more topological version is thus hidden behind Theorem 34, its statement is the following.

Theorem 36 (see Theorem I in Chapter II). *Let α be either a positive real number or $+\infty$. Let S be an odometer whose associated supernatural number $\prod_{p\in\Pi} p^{k_p}$ satisfies the following property: there exists a prime number $p_\star$ such that $k_{p_\star} = +\infty$. Then there exists a Cantor minimal homeomorphism T such that*

1. $\mathrm{h}_{\mathrm{top}}(T) = \alpha$;
2. *there exists a strong orbit equivalence between S and T, which is φ_m-integrable for all integers $m \geqslant 0$,*

where φ_m denotes the map $t \to \frac{\log t}{\log^{(\circ m)} t}$ and $\log^{(\circ m)}$ the composition $\log \circ \ldots \circ \log$ (m times).

Finally, an odometer is a factor of its associated odomutants, and odometers also have this stunning property of being *coalescent*: if an odometer is a factor of a transformation isomorphic to it, then every factor map is an isomorphism. Since we know an explicit factor map from an odomutant to its associated odometer, this property provides a way of finding odomutants not isomorphic to their associated odometers. In fact, it enables us to extend Theorem 23 to odometers.

Theorem 37 (see Theorem J in Chapter II). *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear map and S an odometer. There exists a probability measure-preserving transformation T such that S and T are φ-integrably orbit equivalent but not flip-conjugate.*

On integrability threshold provided by isoperimetric profiles

In the context of quantitative orbit equivalence between infinite amenable groups, Theorem 27 provides an upper bound on how integrable a cocycle can be. More precisely, if the group Γ is a much "bigger" than Λ, there are obstructions for the cocycle from Γ to Λ to be φ-integrable. Moreover this bound is sharp in many examples.

For instance, given an orbit equivalence coupling between $\mathbb{Z}^k$ and $\mathbb{Z}^{k+\ell}$, with positive integers k, ℓ, the cocycle from $\mathbb{Z}^{k+\ell}$ to $\mathbb{Z}^k$ cannot be L^p for any $p > \frac{k}{k+\ell}$. On the other hand, there exists a coupling where this cocycle is L^p for every $p < \frac{k}{k+\ell}$. What about the case $p = \frac{k}{k+\ell}$? This is the question we answer in Chapter III. We proved that such a cocycle

cannot be $\mathrm{L}^{\frac{k}{k+\ell}}$, and more generally that the upper bound provided by the isoperimetric profiles cannot be reached for every pairs (Γ, Λ) of infinite amenable groups, with mild assumptions.

Theorem 38 (see Theorem L in Chapter III)**.** *Let Γ and Λ be finitely generated groups. Assume that there exist a non-decreasing function h_Γ and an increasing function h_Λ satisfying $h_\Gamma \approx j_{1,\Gamma}$, $h_\Lambda \approx j_{1,\Lambda}$ and the following assumptions as $x \to +\infty$:*

$$h_\Gamma(x) = o\left(h_\Lambda(x)\right), \tag{8}$$

$$\forall C > 0,\ h_\Gamma(Cx) = O\left(h_\Gamma(x)\right), \tag{9}$$

$$\forall C > 0,\ h_\Gamma \circ h_\Lambda^{-1}(Cx) = O\left(h_\Gamma \circ h_\Lambda^{-1}(x)\right). \tag{10}$$

Then there is no $(h_\Gamma \circ h_\Lambda^{-1}, \mathrm{L}^0)$-integrable orbit equivalence coupling from Γ to Λ.

Assumption (8) simply means that Γ is a "bigger" group than Λ, and the idea behind Assumptions (9) and (10) is to get rid of the constants appearing inside the maps in the definition of being asymptotically equivalent.

As consequences, we have the following complete characterisations.

Theorem 39 (see Theorem III.4.4 in Chapter III)**.** *Given positive integers k, ℓ, there exists an $(\mathrm{L}^p, \mathrm{L}^0)$ orbit equivalence coupling from $\mathbb{Z}^{k+\ell}$ to $\mathbb{Z}^k$ if and only if $p < \frac{k}{k+\ell}$.*

Plan of the thesis

The three chapters respectively correspond to the articles [Cor25a; Cor25b; Cor25c]. Chapters I and II are devoted to flexibility results. In the first one, we generalize Kerr and Li's construction to rank-one systems. In the second one, we introduce new systems called odomutants which form a richer class in the dynamical viewpoint and which are naturally orbit equivalent to odometers. In Chapter III, we deal with the integrability thresholds provided by the isoperimetric profiles. For the reader who is not acquainted with some basics of ergodic theory or geometric group theory, Appendix A offers an introduction to these topics. Finally, a more detailed state of the art on orbit equivalence and related notions is provided in Appendix B.

Chapter I

Rank-one systems, flexible classes and Shannon orbit equivalence

This chapter corresponds to the article [Cor25a].

Abstract

We build a Shannon orbit equivalence between the universal odometer and a variety of rank-one systems. This is done in a unified manner, using what we call flexible classes of rank-one transformations. Our main result is that every flexible class contains an element which is Shannon orbit equivalent to the universal odometer. Since a typical example of flexible class is $\{T\}$ when T is an odometer, our work generalizes a recent result by Kerr and Li, stating that every odometer is Shannon orbit equivalent to the universal odometer.

When the flexible class is a singleton, the rank-one transformation given by the main result is explicit. This applies to odometers and Chacon's map. We also prove that strongly mixing systems, systems with a given eigenvalue, or irrational rotations whose angle belongs to any fixed nonempty open subset of the real line form flexible classes. In particular, strong mixing, rationality or irrationality of the eigenvalues are not preserved under Shannon orbit equivalence.

Contents

I.1 Introduction

At the level of ergodic probability measure-preserving bijections, *quantitative orbit equivalence* aims at bridging the gap between the well-studied but very complicated relation of *conjugacy*, and the trivial relation of *orbit equivalence*, which is equality of orbits up to conjugacy.

To be more precise, given two ergodic probability measure-preserving bijections S and T on a standard atomless probability space $(X, \mathcal{A}, \mu)$, if S and some system $\Psi^{-1}T\Psi$ conjugate to T have the same orbits, then S and T are said to be *orbit equivalent* and the probability measure-preserving bijection $\Psi\colon X \to X$ is called an *orbit equivalence* between T and S. Dye's theorem [Dye59] states that, if S and T are ergodic, then they are orbit equivalent.

To get an interesting theory, let us define the *cocycles* associated to Ψ, these are the integer-valued functions c_S and c_T defined by $Sx = \Psi^{-1}T^{c_S(x)}\Psi(x)$ and $Tx = \Psi S^{c_T(x)}\Psi^{-1}(x)$. *Shannon orbit equivalence* requires that there exists an orbit equivalence whose cocycles are Shannon, meaning that the partitions associated to c_S and c_T are both of finite entropy. For *φ-integrable orbit equivalence* we ask that both integrals $\int_X \varphi(|c_S(x)|)\mathrm{d}\mu(x)$ and $\int_X \varphi(|c_T(x)|)\mathrm{d}\mu(x)$ are finite. In the particular case of a linear map φ, φ-integrable orbit equivalence exactly requires the integrability of the cocycles, and is simply called integrable orbit equivalence.

Belinskaya's theorem [Bel69] implies that integrable orbit equivalence is exactly flip-conjugacy (S and T are flip-conjugate if S is conjugate to T or T^{-1}). In fact it only requires that one of the two cocycles is integrable. Carderi, Joseph, Le Maître and Tessera [CJLMT23] proved that this result is optimal, meaning that φ-integrable orbit equivalence never implies flip-conjugacy for a sublinear map φ. Moreover, φ-integrable orbit equivalence implies Shannon orbit equivalence when φ is asymptotically greater than log. An impressive result of Kerr and Li [KL24] guarantees that these relations are not trivial: entropy is preserved under Shannon orbit equivalence (and this is the only invariant that we know of). As a consequence, two transformations with different entropies can neither be Shannon orbit equivalent nor φ-integrably orbit equivalent for any φ greater than log.

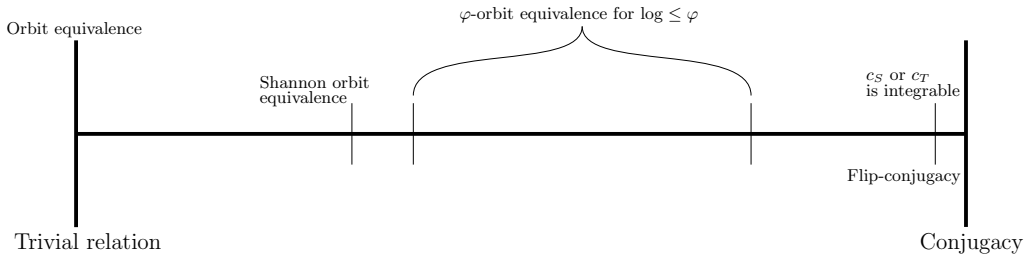


Figure I.1: Here is a schematic view of the interplay between the relations on ergodic bijections we have seen so far.

Historically, the question of preservation of entropy in quantitative orbit equivalence was asked in the more general setting of group actions. We do not give any definition in this setting, as the paper is only about probability measure-preserving bijections S, which can be seen as $\mathbb{Z}$-actions via $(n, x) \in \mathbb{Z} \times X \mapsto S^n x$. Austin [Aus16a] showed that integrable orbit equivalence between actions of infinite finitely generated amenable groups preserves entropy. Kerr and Li [KL21; KL24] then generalized this result, replacing integrable orbit equivalence by Shannon orbit equivalence, and going beyond the amenable case using sofic entropy.

The universal odometer and a theorem of Kerr and Li [KL24]. In [CJLMT23], the statement about φ-integrable orbit equivalence in the sublinear case is the following.

This gives a result on Shannon orbit equivalence since this is implied by φ-integrable orbit equivalence for φ greater than log.

Theorem (Carderi, Joseph, Le Maître, Tessera [CJLMT23]). *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear function. Let S be an ergodic probability measure-preserving transformation and assume that S^n is ergodic for some $n \geqslant 2$. Then there is another ergodic probability measure-preserving transformation T such that S and T are φ-integrably orbit equivalent but not flip-conjugate.*

Corollary (Carderi, Joseph, Le Maître, Tessera [CJLMT23]). *Let S be an ergodic probability measure-preserving transformation and assume that S^n is ergodic for some $n \geqslant 2$. Then there is another ergodic probability measure-preserving transformation T such that S and T are Shannon orbit equivalent but not flip-conjugate.*

The proof is constructive and the resulting transformation T is built so that T^n is not ergodic. It is natural to wonder whether this statement holds for systems T without ergodic non-trivial powers. A well-known example of such a system is the universal odometer.

Question I.1.1. Which systems are Shannon orbit equivalent to the universal odometer?

A first answer is given by Kerr and Li.

Theorem (Kerr, Li [KL24]). *Every odometer is Shannon orbit equivalent to the universal odometer.*

Odometers are exactly probability measure-preserving bijections admitting a nested sequence of partitions of the space, each of them being a Rokhlin tower, and increasing to the σ-algebra $\mathcal{A}$, see Figure I.2 (we refer the reader to the end of Section I.3.a for concrete examples with adding machines). Kerr and Li use this combinatorial specificity of these bijections to build an orbit equivalence between them.

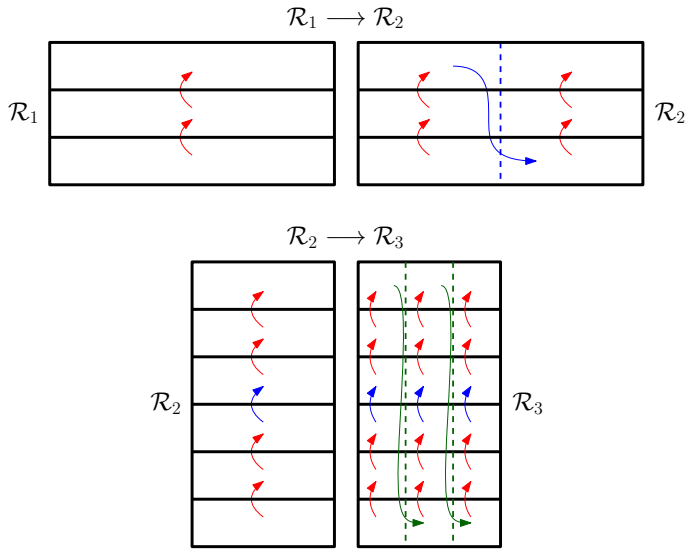


Figure I.2: In this example, $(\mathcal{R}_n)$ denotes the nested sequence of Rokhlin towers defining an odometer. Dividing $\mathcal{R}_1$ in two sub-towers and stacking them, this gives the next tower $\mathcal{R}_2$. From $\mathcal{R}_2$, $\mathcal{R}_3$ is defined by dividing in three sub-towers and stacking them.

Rank-one systems. The aim of the paper is to extend Kerr and Li's result to *rank-one bijections*. These are more general transformations admitting a nested sequence of Rokhlin towers increasing to the σ-algebra $\mathcal{A}$ but the towers do not necessarily partition the space. This means that from a tower to the next one, we need to add some parts of the space which

are not covered by the previous tower, called *spacers*, so that the measure of the subset covered by the n-th tower tends to 1 as n goes to $+\infty$. As illustrated in Figure I.3, to get the next tower, the current one is subdivided in sub-towers which are stacked with optional spacers between them. The number of sub-towers is called the cutting parameter and the number of consecutive spacers between these sub-towers are the spacing parameters (see Definition I.3.2). For example, an odometer admits a cutting-and-stacking construction with spacing parameters equal to zero at each step.

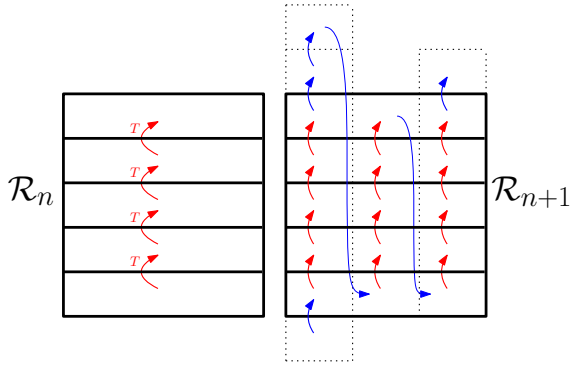


Figure I.3: In this example, there are four spacers and the cutting parameter is three.

Rank-one systems all have entropy zero. They include systems with *discrete spectrum* ([Jun76]), also called *compact* systems. Such systems are not weakly mixing and are completely classified up to conjugacy by their point spectrum ([HVN42]). Examples include odometers and irrational rotations.

The family of rank-one systems is much richer than its subclass of discrete spectrum systems. Indeed, the latter are not weakly mixing whereas there exist strongly mixing systems of rank one, and also rank-one systems which are weakly mixing but not strongly mixing (Chacon's map was the first example of such a system and opened the study of rank-one systems). Rank-one systems can have irrational eigenvalues (i.e. of the form $\exp(2i\pi\theta)$ with irrational numbers θ), it is the case of irrational rotations, whereas odometers only have rational eigenvalues. The reader may refer to the complete survey of Ferenczi [Fer97] about rank-one systems and more generally systems of finite rank.

The combinatorial structure of a general rank-one system does not differ too much from the structure of an odometer but the systems can have completely different properties, thus this class may extend the result of Kerr and Li and provide interesting flexibility results about Shannon orbit equivalence.

A first extension of Kerr and Li's theorem. The construction of an orbit equivalence between the universal odometer S and any rank-one system T is a natural generalization of Kerr and Li's method for the universal odometer and any odometer (see Remark I.5.20). The difficulty is to quantify the cocycles.

At the beginning of our work, we first proved that the Shannon orbit equivalence established by Kerr and Li in [KL24] is actually a φ-integrable orbit equivalence for any $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(t) = o(t^{1/3})$. We then generalized this to rank-one systems called BSP, for "bounded-spacing-parameter", see Definition I.3.5. This notion of BSP systems was already introduced by Gao and Ziegler in [GZ19], using the symbolic definition of rank-one systems (in this paper we will only consider the cutting-and-stacking definition of rank-one systems, which is often more appropriate for constructions in a measure-theoretic setting).

Theorem A. *Every BSP rank-one system is φ-integrably orbit equivalent to the universal odometer for any $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\varphi(t) \underset{t\to+\infty}{=} o\left(t^{1/3}\right)$.*

Therefore φ-integrable orbit equivalence, for a φ as in the above theorem, and Shannon orbit equivalence do not preserve weak mixing since Chacon's map is a BSP rank-one system.

Now the goal is to get a result for systems of rank one outside the class of BSP systems. For this purpose, we find a more general framework with the notion of *flexible classes*, and a general statement (Theorem B) implying Theorem A and other flexibility results (Theorems C, E, F). Theorem D is a refinement of Theorem C.

A modified strategy. We first have to understand why the quantification of the cocycles is more difficult to determine for general rank-one systems than for odometers (or even for BSP systems in Theorem A). In [KL24], the quantification of the cocycles relies on a series whose terms vanish to zero as the cutting parameters get larger and larger. The key is then to get quickly increasing cutting parameters for the series to converge. In order to do so, it suffices to skip steps in the cutting-and-stacking process, i.e. from the n-th Rokhlin tower, we can directly build the $(n+k)$-th Rokhlin for k so big that the new cutting parameter is large enough. In other words, we can recursively choose the cutting parameters so that they increase quickly enough.

When the rank-one system is not an odometer, we need an asymptotic control on the spacing parameters (recall that they are zero for an odometer) for the cocycles to be well quantified. When skipping steps in the cutting-and-stacking method, the spacing parameters may increase too quickly, preventing us from quantifying the cocycles. As we will see in Lemma I.3.6, we do not have this problem with BSP rank-one systems.

When the rank-one system is not BSP, skipping steps in the cutting-and-stacking construction is not relevant as it may improperly change the spacing parameters. In Section I.5.c (see Lemma I.5.11), we will notice that the construction of Kerr and Li enables us to build the universal odometer S while we are building the rank-one system T, focusing only on the combinatorics behind the systems, whereas for Kerr and Li T and its cutting-and-stacking settings are fixed and S is built from these data. This new strategy will enable us to have a result for systems of rank one outside the class of BSP systems, with the notion of flexible class.

Flexible classes. A flexible class (see Definition I.3.7) is basically a class of rank-one systems satisfying a common property (e.g. the set of strongly mixing rank-one systems), with the following two requirements. We first ask for a sufficient condition, given by a set $\mathcal{F}_{\mathcal{C}}$, on the first n cutting and spacing parameters (for all integers $n \geqslant 0$) for the underlying rank-one system to be in this class. Secondly, given a sequence of n cutting and spacing parameters in $\mathcal{F}_{\mathcal{C}}$ (they will be the first n parameters of a cutting-and-stacking construction), we require that it can be completed in a sequence of $n+1$ parameters in $\mathcal{F}_{\mathcal{C}}$, with infinitely many choices for the $(n+1)$-th cutting parameters, and with the appropriate asymptotic control on the $(n+1)$-th spacing parameters.

The idea is to inductively choose the parameters so that the cutting parameters increase fastly enough, with the appropriate asymptotics on the spacing parameters, and the underlying rank-one system has the desired property, namely the system is in the flexible class that we consider.

The general statement on flexible classes is the following.

Theorem B (see Theorem I.3.9). *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a map satisfying $\varphi(t) \underset{t\to+\infty}{=} o\left(t^{1/3}\right)$. If $\mathcal{C}$ is a flexible class, then there exists T in $\mathcal{C}$ which is φ-integrably orbit equivalent to the universal odometer.*

A very interesting phenomenon is when a rank-one system T is flexible, meaning that $\{T\}$ is a flexible class. This first means that given the parameters of a cutting-and-stacking

construction of T, it is possible to change the $(n+1)$-th parameters so that they have the desired asymptotic control, and to inductively do so for every n so that the underlying rank-one system is again T. We do not know if every rank-one system is flexible. Secondly, Theorem B is an existence result and when a flexible class is a singleton $\{T\}$, this statement provides a concrete example of rank-one system which is φ-integrably orbit equivalent to the universal odometer.

The following proposition gives examples of flexible classes.

Proposition I.1.2 (see Proposition I.3.8). *1. Every BSP rank-one system is flexible.*

2. *For every nonempty open subset $\mathcal{V}$ of $\mathbb{R}$, the set $\{R_\theta \mid \theta \in \mathcal{V} \cap (\mathbb{R}\backslash\mathbb{Q})\}$ is a flexible class.*

3. *For every irrational number θ, the class of rank-one systems which have $e^{2i\pi\theta}$ as an eigenvalue is flexible.*

4. *The class of strongly mixing rank-one systems is flexible.*

Proving that a BSP system is flexible is not difficult and we rely on the fact that bounded spacing parameters already have the desired asymptotics even though we skip steps in the cutting-and-stacking process for the cutting parameters to increase quickly enough (see Section I.4.a). We use a construction by Drillick, Espinosa-Dominguez, Jones-Baro, Leng, Mandelshtam and Silva [DEJLMS23] to prove Proposition I.1.2 for irrational rotations (see Section I.4.b). We also consider a construction by Danilenko and Vieprik [DV23] for the rank-one systems with a given eigenvalue (see Section I.4.c). Finally, Ornstein [Orn72] gives the first example of strongly mixing rank-one systems and the fact that these systems form a flexible class follows from his construction (see Section I.4.d).

Combined with Proposition I.1.2, Theorem B provides four flexibility results. The first one is Theorem A stated above, this is a generalization of Kerr and Li's theorem. The second one is another result with almost explicit examples of systems which are φ-integrably orbit equivalent to the universal odometer.

Theorem C. *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a map satisfying $\varphi(t) \underset{t\to+\infty}{=} o\left(t^{1/3}\right)$. The set of irrational numbers θ whose associated irrational rotation is φ-integrably orbit equivalent to the universal odometer is dense in $\mathbb{R}$.*

The point spectrum of R_θ is exactly the circle subgroup generated by $\exp(2i\pi\theta)$ and the eigenvalues of the universal odometer are rational, so Theorem C implies that there exist two Shannon orbit equivalent systems (more specifically φ-integrably orbit equivalent with $\varphi(t) \underset{t\to+\infty}{=} o\left(t^{1/3}\right)$), with non-trivial point spectrums and such that 1 is the only common eigenvalue.

The way we prove Theorem B will enable us to get the following refinement, its proof is written at the end of the paper.

Theorem D. *For every map $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\varphi(t) \underset{t\to+\infty}{=} o\left(t^{1/3}\right)$, and for every nonempty open subset $\mathcal{V}$ of $\mathbb{R}$, the set of irrational numbers $\theta \in \mathcal{V}$ whose associated irrational rotation is φ-integrably orbit equivalent to the universal odometer is uncountable.*

Question I.1.3. Let us consider the set of irrational numbers θ whose associated irrational rotation is φ-integrably orbit equivalent to the universal odometer. Is this set conull with respect to the Lebesgue measure? equal to the set of irrational numbers?

Finally we get the following corollaries, providing implicit examples.

Theorem E. *For every map $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\varphi(t) \underset{t\to+\infty}{=} o\left(t^{1/3}\right)$, and for every irrational number θ, there exists a rank-one system which has $e^{2i\pi\theta}$ as an eigenvalue and which is φ-integrably orbit equivalent to the universal odometer.*

Theorem F. *For every map $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\varphi(t) \underset{t\to+\infty}{=} o\left(t^{1/3}\right)$, there exists a strongly mixing rank-one system which is φ-integrably orbit equivalent to the universal odometer.*

As $\exp(2i\pi\theta)$ is an eigenvalue of the irrational rotation of angle θ, and as we do not know if Theorem C holds for every irrational number θ, Theorem E then completes this statement with a weaker result for the remaining θ.

Theorem F implies that φ-integrable orbit equivalence, with $\varphi(t) \underset{t\to+\infty}{=} o\left(t^{1/3}\right)$, and Shannon orbit equivalence do not preserve strong mixing. This is also a consequence of the result from [CJLMT23]. Indeed if S is strongly mixing, then all its non-trivial powers are ergodic and the statements give some T with a non-trivial power which is not ergodic, so T is not strongly mixing. Here Theorem F gives an example starting from a very non-strongly mixing system S (the universal odometer). Finally, note that strongly mixing systems are not BSP. This is a consequence of Theorem 1.3 in [GZ19]: BSP rank-one systems are not topologically mixing, therefore they are not measure-theoretically strongly mixing.

Further comments. As they both preserve entropy, we may wonder whether there is a connection between Shannon orbit equivalence (or more generally φ-integrable orbit equivalence for φ greater than log) and *even Kakutani equivalence*. Two ergodic probability measure-preserving bijections S and T, respectively acting on (X,μ) and (Y,ν), are evenly Kakutani equivalent if there exist measurable subsets $A \subseteq X$ and $B \subseteq Y$ with equal measure, i.e. $\mu(A) = \nu(B)$, such that the induced maps S_A and T_B are conjugate. Even Kakutani equivalence is an equivalence relation, contrarily to Shannon orbit equivalence and φ-integrable orbit equivalence a priori (except for linear maps φ, by Belinskaya's theorem). The theory of Ornstein, Rudolph and Weiss [ORW82] gives a complete classification up to even Kakutani equivalence among loosely Bernoulli (LB) systems and entropy is a complete invariant. Moreover the class of LB systems is closed by even Kakutani equivalence, meaning that if S is LB and equivalent to T, then T is also LB.

Rank-one systems are zero-entropy and LB, and by Theorems A, C, E and F, some of them are Shannon orbit equivalent to the universal odometer.

Question I.1.4. Does even Kakutani equivalence imply Shannon orbit equivalence or φ-integrable orbit equivalence for some φ?

In a forthcoming paper we will provide a new construction of orbit equivalence in order to prove that the converse is false: for every $\varepsilon > 0$, there exists a non-LB system which is $(x \mapsto x^{\frac{1}{2}-\varepsilon})$-integrably orbit equivalent to the dyadic odometer. So $(x \mapsto x^{\frac{1}{2}-\varepsilon})$-integrable orbit equivalence and Shannon orbit equivalence do not imply even Kakutani equivalence.

Outline of the paper. After a few preliminaries in Section I.2, rank-one systems are defined in Section I.3 using the cutting-and-stacking method. We also define the central notion of flexible classes of rank-one transformations. In Section I.4, we prove Proposition I.1.2 (Proposition I.3.8 in Section I.3), i.e. we show that the classes mentioned in Theorem B (Theorem I.3.9 in Section I.3) are flexible. It remains to show that every flexible class admits an element which is φ-integrably orbit equivalent to the universal odometer (Theorem I.3.9). In Section I.5, we will describe the construction of Kerr and Li, generalized to rank-one systems, and establish that this is an orbit equivalence with

some important properties preparing for the proof of Theorem I.3.9. Theorems A, C, E and F directly follows from Proposition I.3.8 and Theorem I.3.9. We prove Theorem D at the end of the paper.

I.2 Preliminaries

Basics of ergodic theory. The probability space $(X, \mathcal{A}, \mu)$ is assumed to be standard and atomless. Such a space is isomorphic to $([0,1], \mathcal{B}([0,1]), \mathrm{Leb})$, i.e. there exists a bimeasurable bijection $\Psi\colon X \to [0,1]$ (defined almost everywhere) such that $\Psi_\star\mu = \mathrm{Leb}$, where $\Psi_\star\mu$ is defined by $\Psi_\star\mu(A) = \mu(\Psi^{-1}(A))$ for every measurable set A. We consider maps $T\colon X \to X$ acting on this space and which are bijective, bimeasurable and **probability measure-preserving** (**p.m.p.**), meaning that $\mu(T^{-1}(A)) = \mu(A)$ for all measurable sets $A \subseteq X$, and the set of these transformations is denoted by $\mathrm{Aut}(X, \mathcal{A}, \mu)$, or simply $\mathrm{Aut}(X, \mu)$, two such maps being identified if they coincide on a measurable set of full measure. In this paper, elements of $\mathrm{Aut}(X, \mu)$ are called **transformations** or (**dynamical**) **systems**.

A measurable set $A \subseteq X$ is **T-invariant** if $\mu(T^{-1}(A)\Delta A) = 0$, where Δ denotes the symmetric difference. A transformation $T \in \mathrm{Aut}(X, \mu)$ is said to be **ergodic** if every T-invariant set is of measure 0 or 1. If T is ergodic, then T is **aperiodic**, i.e. $T^n(x) \neq x$ for almost every $x \in X$ and for every $n \in \mathbb{Z}\backslash\{0\}$, or equivalently the **$T$-orbit** of x, denoted by $\mathrm{Orb}_T(x) \coloneqq \{T^n(x) \mid n \in \mathbb{Z}\}$, is infinite for almost every $x \in X$.

T is **weakly mixing** if

$$\frac{1}{n}\sum_{k=0}^{n} \left|\mu(A \cap T^{-n}(B)) - \mu(A)\mu(B)\right| \underset{n\to+\infty}{\to} 0$$

for every measurable sets A, B. T is **strongly mixing** if

$$\left|\mu(A \cap T^{-n}(B)) - \mu(A)\mu(B)\right| \underset{n\to+\infty}{\to} 0$$

for every measurable sets A, B. It is not difficult to prove that strong mixing implies weak mixing and that the latter implies ergodicity.

The notions of weak mixing and ergodicity can be translated in terms of eigenvalues. Denoting by $\mathrm{L}^2(X, \mathcal{A}, \mu)$ the space of complex-valued and square-integrable functions defined on X, a complex number λ is an **eigenvalue** of T if there exists $f \in \mathrm{L}^2(X, \mathcal{A}, \mu)\backslash\{0\}$ such that $f \circ T = \lambda f$ almost everywhere (f is then called an eigenfunction). An eigenvalue λ is automatically an element of the unit circle $\mathbb{T} \coloneqq \{z \in \mathbb{C} \mid |z| = 1\}$. The **point spectrum** of T is then the set of all its eigenvalues. Notice that $\lambda = 1$ is always an eigenvalue since the constant functions are in its eigenspace. Finally T is ergodic if and only if the constant functions are the only eigenfunctions with eigenvalue one, in other words the eigenspace of $\lambda = 1$ is the line of constant functions (we say that it is a simple eigenvalue). If T is ergodic, it is weakly mixing if and only if the only eigenvalue of T is 1. For a complete survey on spectral theory for dynamical systems, the reader may refer to [VO16].

All the properties that we have introduced are preserved under conjugacy. Two transformations $S \in \mathrm{Aut}(X, \mu)$ and $T \in \mathrm{Aut}(Y, \nu)$ are **conjugate** if there exists a bimeasurable bijection $\Psi\colon X \to Y$ such that $\Psi_\star\mu = \nu$ and $\Psi \circ S = T \circ \Psi$ almost everywhere. Some classes of transformations have been classified up to conjugacy, the two examples to keep in mind are the following. By Ornstein [Orn70], entropy is a total invariant of conjugacy among Bernoulli shifts, and Ornstein and Weiss [OW87] generalized this result for Bernoulli shifts of amenable groups. For more details about entropy, see [Dow11] for non necessarily invertible transformations $T\colon X \to X$, and [KL16] more generally for actions of amenable groups. Finally Halmos and von Neumann [HVN42] showed that two ergodic systems

with discrete spectrums are conjugate if and only if they have equal point spectrums (a system has discrete spectrum if the span of all its eigenfunctions is dense in $\mathrm{L}^2(X, \mathcal{A}, \mu)$).

Quantitative orbit equivalence. The conjugacy problem in full generality is very complicated (see [FRW11]). We now give the formal definition of orbit equivalence, which is a weakening of the conjugacy problem.

Definition I.2.1. Two aperiodic transformations $S \in \mathrm{Aut}(X, \mu)$ and $T \in \mathrm{Aut}(Y, \nu)$ are **orbit equivalent** if there exist a bimeasurable bijection $\Psi\colon X \to Y$ satisfying $\Psi_\star \mu = \nu$, such that $\mathrm{Orb}_S(x) = \mathrm{Orb}_{\Psi^{-1}T\Psi}(x)$ for almost every $x \in X$. The map Ψ is called an **orbit equivalence** between S and T.

We can then define the **cocycles** associated to this orbit equivalence. These are measurable functions $c_S\colon X \to \mathbb{Z}$ and $c_T\colon Y \to \mathbb{Z}$ defined almost everywhere by

$$Sx = \Psi^{-1}T^{c_S(x)}\Psi(x) \text{ and } Ty = \Psi S^{c_T(y)}\Psi^{-1}(y)$$

($c_S(x)$ and $c_T(y)$ are uniquely defined by aperiodicity).

Given a function $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$, a measurable function $f\colon X \to \mathbb{Z}$ is said to be **φ-integrable** if

$$\int_X \varphi(|f(x)|)\mathrm{d}\mu < +\infty.$$

For example integrability is exactly φ-integrability when φ is non-zero and linear. Then a weaker quantification on cocycles is the notion of φ-integrability for a *sublinear* map φ, meaning that $\lim_{t\to+\infty} \varphi(t)/t = 0$. Two transformations in $\mathrm{Aut}(X, \mu)$ are said to be **φ-integrably orbit equivalent** if there exists an orbit equivalence between them whose associated cocycles are φ-integrable. Another form of quantitative orbit equivalence is Shannon orbit equivalence. We say that a measurable function $f\colon X \to \mathbb{Z}$ is **Shannon** if the associated partition $\{f^{-1}(n) \mid n \in \mathbb{Z}\}$ of X has finite entropy. Two transformations in $\mathrm{Aut}(X, \mu)$ are **Shannon orbit equivalent** if there exists an orbit equivalence between them whose associated cocycles are Shannon.

I.3 Rank-one systems

I.3.a The cutting-and-stacking method

Before the definition of a rank-one system (Definition I.3.2), and for the definition of flexible classes (Definition I.3.7), we need to define sequences of integers which will provide the combinatorial data of a rank-one system, namely the cutting and spacing parameters.

Definition I.3.1. By a **cutting and spacing parameter**, we mean a tuple of the form

$$(q, (\sigma_{\cdot,0}, \dots, \sigma_{\cdot,q}))$$

with an integer $q \geqslant 2$ (the **cutting parameter**) and non-negative integers $\sigma_{\cdot,0}, \dots, \sigma_{\cdot,q}$ (the **spacing parameters**), and we denote by $\mathcal{P}$ the set of all cutting and spacing parameters. We also define the set of finite sequences of cutting and spacing parameters:

$$\mathcal{P}^* \coloneqq \bigcup_{n\in\mathbb{N}} \mathcal{P}^n.$$

Given a sequence of cutting and spacing parameters $\boldsymbol{p} = (q_k, (\sigma_{k,0}, \dots, \sigma_{k,q_k}))_{k\geqslant 0} \in \mathcal{P}^{\mathbb{N}}$ and an integer $n \geqslant 0$, the tuple $(q_n, (\sigma_{n,0}, \dots, \sigma_{n,q_n}))$ in $\mathcal{P}$ is the **n-th cutting and spacing parameter** of $\boldsymbol{p}$, and the tuple $(q_k, (\sigma_{k,0}, \dots, \sigma_{k,q_k}))_{0\leqslant k\leqslant n}$ is the **projection** of $\boldsymbol{p}$ on $\mathcal{P}^{n+1}$ (it gives the first $n+1$ cutting and spacing parameters). From $\boldsymbol{p}$, we also define three sequences:

- $(h_n)_{n\geqslant 0}$ the **height sequence** of $\boldsymbol{p}$, inductively defined by $\begin{cases} h_0 = 1, \\ h_{n+1} = q_n h_n + \sigma_n \end{cases}$, h_n is called the **height of the n-th tower**;
- $(\sigma_n)_{n\geqslant 0}$, with $\sigma_n \coloneqq \sum_{i=0}^{q_n} \sigma_{n,i}$ (the number of new spacers at step n);
- $(Z_n)_{n\geqslant 0}$, with $Z_n \coloneqq \max\{\sigma_{j,i} \mid 0 \leqslant j \leqslant n, 0 \leqslant i \leqslant q_j\}$,

and it is also possible to consider the finite sequences $(h_k)_{0\leqslant k\leqslant n+1}$, $(\sigma_k)_{0\leqslant k\leqslant n}$ and $(Z_k)_{0\leqslant k\leqslant n}$ associated to a finite sequence of cutting and spacing parameters in $\mathcal{P}^{n+1}$.

The terminology "cutting", "spacing", "tower", "height", etc, is justified by Definition I.3.2 and Figure I.4. There are many definitions of rank-one systems (see [Fer97] for a complete survey and various facts in this section). In this paper the goal is to use the combinatorial structure given by the cutting-and-stacking method (see Figure I.4).

Definition I.3.2. A transformation $T \in \mathrm{Aut}(X, \mu)$ is of **rank one** if there exist

1. a sequence of cutting and spacing parameters $\boldsymbol{p} = (q_n, (\sigma_{n,0}, \ldots, \sigma_{n,q_n}))_{n\geqslant 0} \in \mathcal{P}^{\mathbb{N}}$ satisfying
$$\sum_{n=0}^{+\infty} \frac{\sigma_n}{h_{n+1}} < +\infty, \tag{F}$$
where (h_n) and (σ_n) are the sequences associated to $\boldsymbol{p}$, as described in Definition I.3.1;
2. measurable subsets of X, denoted by B_n (for every $n \geqslant 0$), $B_{n,i}$ (for every $n \geqslant 0$ and $0 \leqslant i \leqslant q_n - 1$), and $\Sigma_{n,i,j}$ (for every $n \geqslant 0$, $0 \leqslant i \leqslant q_n$ and $1 \leqslant j \leqslant \sigma_{n,i}$; if $\sigma_{n,i} = 0$, then there are no $\Sigma_{n,i,j}$) such that for all $n \geqslant 0$
   (a) $B_n, \ldots, T^{h_n-1}(B_n)$ are pairwise disjoint;
   (b) $(B_{n,0}, B_{n,1}, \ldots, B_{n,q_n-1})$ is a partition of B_n;
   (c) $T^{h_n}(B_{n,i}) = \begin{cases} \Sigma_{n,i+1,1} & \text{if } \sigma_{n,i} > 0 \\ B_{n,i+1} & \text{if } \sigma_{n,i} = 0 \text{ and } i < q_n - 1 \end{cases}$;
   (d) if $\sigma_{n,i} > 0$, then $T(\Sigma_{n,i,j}) = \begin{cases} \Sigma_{n,i,j+1} & \text{if } j < \sigma_{n,i} \\ B_{n,i} & \text{if } j = \sigma_{n,i} \text{ and } i \leqslant q_n - 1 \end{cases}$;
   (e) $B_{n+1} = \begin{cases} \Sigma_{n,0,1} & \text{if } \sigma_{n,0} > 0 \\ B_{n,1} & \text{if } \sigma_{n,0} = 0 \end{cases}$;

and if the Rokhlin towers $\mathcal{R}_n \coloneqq (T^k(B_n))_{0\leqslant k\leqslant h_n-1}$ are increasing to the σ-algebra $\mathcal{A}$, meaning that the σ-algebra generated by $\{T^k(B_n) \mid n \in \mathbb{N},\ 0 \leqslant k \leqslant h_n - 1\}$ is $\mathcal{A}$ up to null sets (since $\mathcal{A}$ is standard, this also means that $\{T^k(B_n) \mid n \in \mathbb{N},\ 0 \leqslant k \leqslant h_n - 1\}$ separates the points). Note that $\mathcal{R}_0$ is the tower with only one level B_0. The sets $\Sigma_{n,i,j}$ are called the **spacers**. In this paper we will usually write

- $X_n \coloneqq B_n \sqcup \ldots \sqcup T^{h_n-1}(B_n)$ the subset covered by the n-th tower $\mathcal{R}_n$;
- $\varepsilon_n \coloneqq \mu((X_n)^c)$ where $(X_n)^c$ denotes the complement of the subset X_n of X.

Since X_n is increasing and $\mathcal{R}_n$ increases to the atomless σ-algebra $\mathcal{A}$, we have $\mu(X_n) \underset{n\to+\infty}{\to} 1$. In other words ε_n tends to 0.

Before giving examples, the following lemmas give some easy properties on these systems in order to understand their combinatorial structure and the hypotheses required in the definition.

Lemma I.3.3. *Let (h_n) and (σ_n) be the sequences associated to $(q_n, (\sigma_{n,0}, \ldots, \sigma_{n,q_n}))_n \in \mathcal{P}^{\mathbb{N}}$ (see Definition I.3.1). The following assertions are equivalent:*

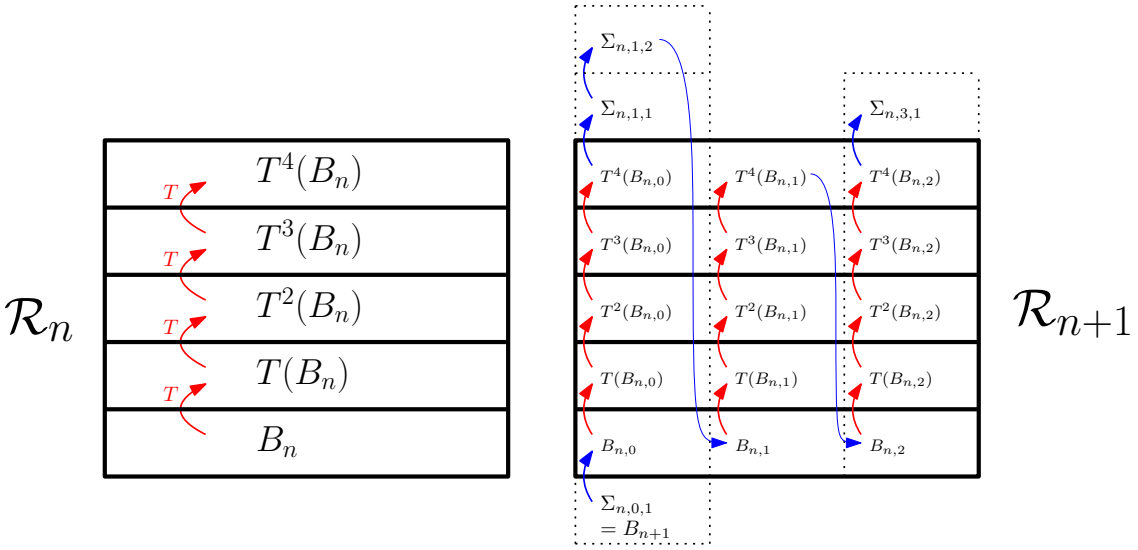


Figure I.4: An example of cutting-and-stacking construction with $h_n = 5$, $q_n = 3$, $\sigma_{n,0} = 1$, $\sigma_{n,1} = 2$, $\sigma_{n,2} = 0$, $\sigma_{n,3} = 1$. We then have $h_{n+1} = 19$.

1. *the series* $\sum \dfrac{\sigma_n}{h_{n+1}}$ *converges (condition* (F) *in Definition I.3.2);*
2. *the series* $\sum \dfrac{\sigma_n}{q_0 \dots q_n}$ *converges;*
3. *there exists a constant* $M_0 \leqslant 1$ *such that* $h_{n+1} \underset{n\to+\infty}{\sim} \dfrac{q_0 \dots q_n}{M_0}$,

and if one of these equivalent assertions is true, then $\sum_{n\geqslant 0} \frac{\sigma_n}{q_0 \dots q_n} = \frac{1}{M_0} - 1$.

Proof of Lemma I.3.3. If the series $\sum \frac{\sigma_n}{q_0 \dots q_n}$ converges, so does the series $\sum \frac{\sigma_n}{h_{n+1}}$ since h_{n+1} is greater or equal to $q_0 \dots q_n$. Now assume that the series $\sum \frac{\sigma_n}{h_{n+1}}$ converges. Notice that we have

$$\frac{\sigma_n}{h_{n+1}} = \frac{h_{n+1} - q_n h_n}{h_{n+1}} = 1 - q_n \frac{h_n}{h_{n+1}}$$

and since the series is convergent, the product $\prod q_n \frac{h_n}{h_{n+1}}$ converges to some $M_0 > 0$, i.e. $q_0 \dots q_n / h_{n+1} \to M_0$. The constant M_0 is less than or equal to 1 since we have $h_{n+1} \geqslant q_n h_n$ for every $n \geqslant 0$. Finally let us assume $q_0 \dots q_n / h_{n+1} \to M_0$. Notice that we have

$$\frac{\sigma_n}{q_0 \dots q_n} = \frac{h_{n+1} - q_n h_n}{q_0 \dots q_n} = \frac{h_{n+1}}{q_0 \dots q_n} - \frac{h_n}{q_0 \dots q_{n-1}},$$

so by telescoping consecutive terms, we get $\sum_{n\geqslant 0} \frac{\sigma_n}{q_0 \dots q_n} = \lim_{n\to\infty} \frac{h_{n+1}}{q_0 \dots q_n} - h_0 = \frac{1}{M_0} - 1$ and we are done for the equivalence between the three assumptions. □

Lemma I.3.4. *Let* $T\colon X \to X$ *be a bimeasurable bijection. Assume that* T *preserves a non-zero measure* μ *and it admits a sequence of Rokhlin towers as in Definition I.3.2. The following hold:*

1. *the levels* $T^k(B_n)$ *of the* n*-th Rokhlin tower* $\mathcal{R}_n$ *have* μ*-measure* $\dfrac{\mu(B_0)}{q_0 \dots q_{n-1}}$;
2. μ *is finite if and only if the condition* (F) *is satisfied. Furthermore, if* μ *is a probability measure (this implies that* T *is a rank-one system), then* $\mu(B_0) = M_0$ *and* $h_{n+1} \leqslant \dfrac{q_0 \dots q_n}{M_0}$, *where* M_0 *is given by Lemma I.3.3.*

Proof of Lemma I.3.4. For a fixed n, the levels of $\mathcal{R}_n$ have the same measure by T-invariance of the measure μ. Moreover the first level B_n is a disjoint union of q_n levels $B_{n,0},\dots,B_{n,q_n-1}$ of $\mathcal{R}_{n+1}$. Then it is clear by induction that a level of $\mathcal{R}_n$ has measure $\frac{\mu(B_0)}{q_0\dots q_{n-1}}$. Since the sequence $(X_n)_{n\geqslant 0}$ is increasing to X, and X_{n+1} is obtained from X_n by adding σ_n spacers, which are levels of $\mathcal{R}_{n+1}$, we get

$$\mu(X)=\mu(X_0)+\sum_{n\geqslant 0}\mu(X_{n+1}\backslash X_n)=\mu(B_0)+\sum_{n\geqslant 0}\frac{\mu(B_0)\sigma_n}{q_0\dots q_n}, \tag{I.1}$$

so $\mu(B_0)$ is non-zero, and $\mu(X)$ is finite if and only if the sum $\sum_{n\geqslant 0}\frac{\sigma_n}{q_0\dots q_n}$ is finite. Finally, let us assume that μ is a probability measure. This implies $\sum_{n\geqslant 0}\frac{\sigma_n}{q_0\dots q_n}=\frac{1}{M_0}-1$ and, using (I.1), we get $\mu(B_0)=M_0$. The measurable set X_n is the disjoint union of h_n levels of $\mathcal{R}_n$, so the inequality $h_n\leqslant\frac{q_0\dots q_{n-1}}{M_1}$ follows from the fact that μ is a probability measure. □

It is possible to build a finite measure-preserving transformation T of rank one with a given combinatorial setting $(q_n,(\sigma_{n,0},\dots,\sigma_{n,q_n}))_{n\geqslant 0}\in\mathcal{P}^{\mathbb{N}}$ satisfying the hypothesis (F). For instance it suffices to build (X_n) as an increasing sequence of intervals of $\mathbb{R}_+$, with $B_{n,i}$ and $\Sigma_{n,i,j}$ being subintervals of equal length and disjoint (for a fixed n), each on which T is defined as an affine map, and with $B_0=[0,M_0]$. The convergence of the series $\sum\frac{\sigma_n}{h_{n+1}}$ and Lemma I.3.3 ensure that $X\coloneqq\bigcup X_n$ is equal to $[0,1]$ (up to a null set), so the Lebesgue measure on $[0,1]$ is a probability measure preserved by T. Notice that if the series is divergent, we can set $B_0=[0,1]$ and this defines T on the set of positive real numbers endowed with the Lebesgue measure, so this is an infinite measure-preserving transformation.

Therefore for every $(q_n,(\sigma_{n,0},\dots,\sigma_{n,q_n}))_{n\geqslant 0}\in\mathcal{P}^{\mathbb{N}}$ satisfying the condition (F), there exists a rank-one system having a cutting-and-stacking construction with these cutting and spacing parameters, this fact will be used in this paper since it enables us to only take into account the combinatorics behind the systems.

The hypothesis on the Rokhlin towers $\mathcal{R}_n$ aims not only to have $\varepsilon_n\to 0$ but also to define two isomorphic systems when they admit cutting-and-stacking constructions with the same cutting and spacing parameters. Moreover if T admits such a construction with Rokhlin towers increasing to a sub-σ-algebra $\mathcal{B}$ of $\mathcal{A}$, then T, seen as an element of $\mathrm{Aut}(X,\mathcal{A},\mu)$, is not necessarily a rank-one system but admits a rank-one system (T on the sub-σ-algebra $\mathcal{B}$) as a factor.

Two different families of cutting and spacing parameters do not necessarily define non-isomorphic systems. Indeed in a construction of a rank-one system with parameters q_n and $\sigma_{n,i}$, one can decide to only consider a subsequence $\mathcal{R}_{n_k}$ of Rokhlin towers. For example, the new cutting parameters will be $q_{n_k}q_{n_k+1}\dots q_{n_{k+1}-1}$ for $k\geqslant 0$.

The rank-one systems form a class of ergodic and zero entropy systems. The easiest examples of rank-one systems are the **irrational rotations**

$$R_\theta\colon z\in\mathbb{T}\mapsto e^{2i\pi\theta}z\in\mathbb{T}$$

for every irrational numbers θ, where $\mathbb{T}$ is the unit circle endowed with its Haar measure. These systems are not weakly mixing. Moreover they have discrete spectrum and the point spectrum of R_θ is $\{e^{in\theta}\mid n\in\mathbb{Z}\}$, so by the Halmos-von Neumann Theorem [HVN42], R_θ and $R_{\theta'}$ are isomorphic if and only if $\theta=\theta'$ mod $\mathbb{Z}$ or $\theta=-\theta'$ mod $\mathbb{Z}$.

The **odometers** are rank-one. These are exactly the rank-one systems without spacers (i.e. $\sigma_{n,i}=0$), so the Rokhlin towers are partitions of the space. Such a system is isomorphic to the adding machine S in the space $\prod_{n\geqslant 0}\{0,1,\dots,q_n-1\}$, namely the addition by $(1,0,0,0,\dots)$ with carry over to the right, defined for every $x\in\prod_{n\geqslant 0}\{0,1,\dots,q_n-1\}$ by

$$Sx=\begin{cases}(0,\dots,0,x_i+1,x_{i+1},\dots) & \text{if } i\coloneqq\min\{j\geqslant 0\mid x_j\neq q_j-1\}\text{ is finite}\\ (0,0,0,\dots) & \text{if } x=(q_0-1,q_1-1,q_2-1,\dots)\end{cases}$$

and it preserves the product of uniform probability measures on each finite set $\{0, 1, \dots, q_n - 1\}$. Denote the cylinders of length k by

$$[x_0, \dots, x_{k-1}]_k := \left\{ y \in \prod_{n \geqslant 0} \{0, 1, \dots, q_n - 1\} \mid y_0 = x_0, \dots, y_{k-1} = x_{k-1} \right\}.$$

If S is the odometer on the space $\prod_{n \geqslant 0} \{0, 1, \dots, q_n - 1\}$, we can also set a partially defined map

$$\zeta_n \colon X \backslash [\bullet, \dots, \bullet, q_{n-1} - 1]_n \to X \backslash [\bullet, \dots, \bullet, , 0]_n$$

(the symbol $\bullet$ means that there is no requirement on the value at some coordinate) which is the addition by

$$(\underbrace{0, \dots, 0}_{n-1 \text{ times}}, 1, 0, 0, \dots)$$

(so S and ζ_1 coincide on $X \backslash [q_0 - 1]_1$). Then we have

$$B_n = [\underbrace{0, \dots, 0}_{n \text{ times}}]_n,$$

$$B_{n,i} = [\underbrace{0, \dots, 0}_{n \text{ times}}, i]_{n+1}$$

and $B_{n,i} = \zeta_{n+1}^i(B_{n,0})$ for every $0 \leqslant i \leqslant q_n - 1$, so it provides a scale in B_n. Note that it is possible to recover the odometer S from these partially defined maps ζ_n (see Figure I.5). In Section I.5.a, the strategy will be to build S from partially defined maps ζ_n.

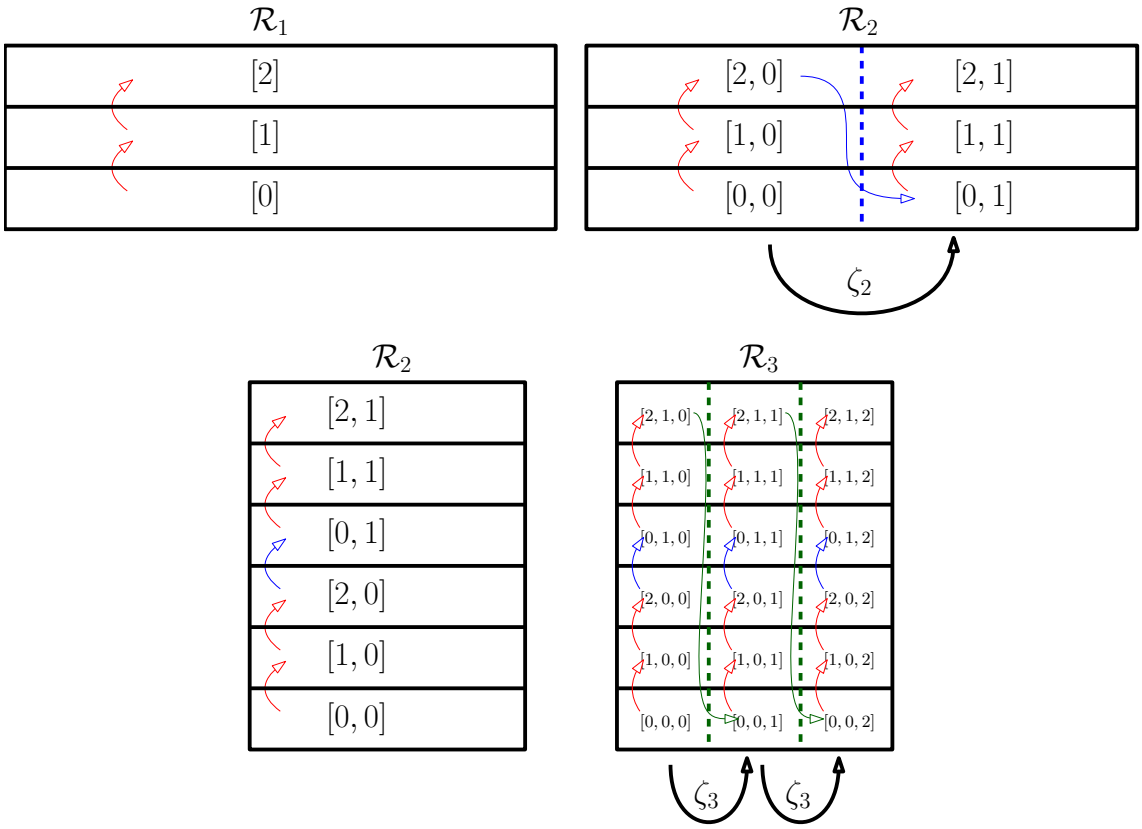


Figure I.5: Example of odometer with $q_0 = 3$, $q_1 = 2$, $q_2 = 3$.

In the class of odometers, the number of occurrences of every prime factors in the set $\{q_n \mid n \geqslant 0\}$ form a total invariant of conjugacy. As for irrational rotations, it is a consequence of the Halmos-von Neumann Theorem since odometers have discrete spectrum and

their eigenvalues are given by these occurrences. In particular odometers have eigenvalues non-equal to 1 and are not weakly mixing, moreover odometers and irrational rotations are not isomorphic. Notice that the Halmos-von Neumann Theorem implies that the conjugacy classes among ergodic systems with discrete spectrum coincide with the flip-conjugacy classes since the point spectrum of a system is a subgroup of $\mathbb{T}$. If every prime number has infinite multiplicity in the set $\{q_n \mid n \geqslant 0\}$, then the odometer is said to be **universal**. An odometer is **dyadic** if 2 is the only prime factor.

Chacon's map is the first example of weakly mixing system which is not strongly mixing [Cha69] and was the starting point for the notion of rank-one systems. It is a rank-one transformation defined with cutting and spacing parameters $q_n = 3$, $\sigma_{n,0} = \sigma_{n,1} = \sigma_{n,3} = 0$, $\sigma_{n,2} = 1$.

I.3.b Flexible classes

Now we introduce classes of rank-one systems to which the main result of this paper applies. First let us consider cutting-and-stacking constructions whose spacing parameters have controlled asymptotics. Recall that $\mathcal{P}^{\mathbb{N}}$ is the set of sequences of cutting and spacing parameters. As introduced in Definition I.3.1, (h_n), (σ_n) and (Z_n) denotes the sequences associated to a sequence in $\mathcal{P}^{\mathbb{N}}$: h_n is the height of the n-th tower, σ_n the number of new spacers at step n and Z_n is the maximum number of spacers between two consecutive towers, over the first n steps.

Definition I.3.5. A construction by cutting and stacking with cutting and spacing parameters $(q_n, (\sigma_{n,0}, \dots, \sigma_{n,q_n}))_{n\geqslant 0} \in \mathcal{P}^{\mathbb{N}}$ is said **CSP** ("**controlled-spacing-parameter**") if there exists a constant $C > 0$ such that $Z_n \leqslant Ch_n$ for all n. It is furthermore **BSP** ("**bounded-spacing-parameter**") if $Z_n \leqslant C$ and $\sigma_{n,0} = \sigma_{n,q_n} = 0$ for all n. A rank-one system T is **BSP** if it admits a BSP cutting-and-stacking construction.

Odometers and Chacon's map are examples of BSP rank-one systems. Moreover BSP implies CSP. The interest in the BSP property is its stability after skipping steps in the cutting-and-stacking process, as stated in the following lemma.

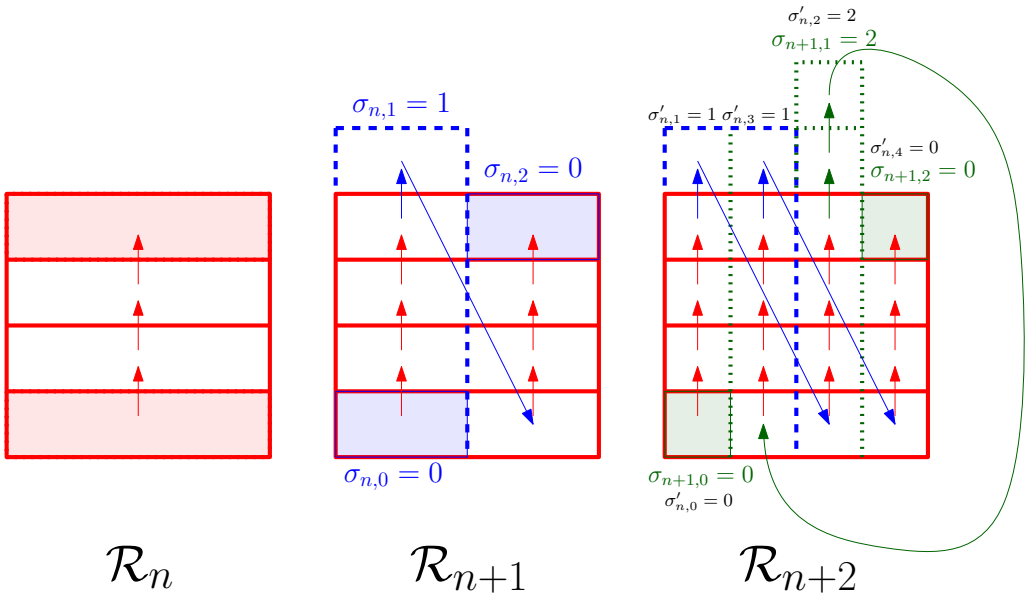


Figure I.6: Illustration of the proof of Lemma I.3.6, spacing parameters from $\mathcal{R}_n$ to $\mathcal{R}_{n+2}$ with $q_n = q_{n+1} = 2$ (the coloured levels are the base and the roof of the towers).

Lemma I.3.6. *Given a BSP cutting-and-stacking construction, any subsequence of its Rokhlin towers still provides a BSP construction with the same constant C.*

Proof of Lemma I.3.6. Let q_n and $\sigma_{n,i}$ be the cutting and spacing parameters of the BSP construction, C the bound for the spacing parameters $\sigma_{n,i}$, $\mathcal{R}_n$ the associated towers and $\mathcal{R}_{n_k}$ a subsequence. Let k be an integer and assume $n_{k+1} = n_k + 2$. Denote by q'_{n_k} and $\sigma'_{n_k,i}$ the new cutting and spacing parameters from $\mathcal{R}_{n_k}$ to $\mathcal{R}_{n_{k+1}}$. It is easy to show that $q'_{n_k} = q_{n_k} q_{n_k+1}$, $\sigma'_{n_k,0} = \sigma'_{n_k,q'_{n_k}} = 0$ and for every $1 \leqslant j \leqslant q_{n_k+1}$, $\sigma'_{n_k,(j-1)q_{n_k}+i}$ is equal to $\sigma_{n_k,i}$ if $1 \leqslant i \leqslant q_{n_k} - 1$, $\sigma_{n_k+1,j}$ if $i = q_{n_k}$ (see Figure I.6). Thus the non-zero spacing parameters from $\mathcal{R}_{n_k}$ to $\mathcal{R}_{n_{k+1}}$ are of the form $\sigma_{n_k,i}$ or $\sigma_{n_k+1,i}$ and they are all bounded above by C. For n_{k+1} bigger than $n_k + 2$, the result is now clear by induction. □

If the parameters σ_{n,q_n} are non-zero, then skipping steps in the cutting-and-stacking process will cause an accumulation of spacers above the last columns and the new spacing parameters will not be bounded if the subsequence is properly chosen so that the jumps $n_{k+1} - n_k$ increase quickly enough. We have the same problem for $\sigma_{n,0}$ (accumulation of spacers at the bottom of the first columns), hence the conditions $\sigma_{n,0} = \sigma_{n,q_n} = 0$ in the definition of BSP.

Lemma I.3.6 has no reason to hold for CSP construction that are not BSP. Indeed the spacing parameters from $\mathcal{R}_{n_k}$ to $\mathcal{R}_{n_{k+1}}$ have to be compared with h_{n_k}, the height of $\mathcal{R}_{n_k}$. The comparison is easily obtained for the spacing parameters $\sigma_{n_k,i}$, $0 \leqslant i \leqslant q_{n_k}$, but for the other spacing parameters, we only know that they are bounded above by $Ch_{n_k+1}, Ch_{n_k+2}, \ldots, Ch_{n_{k+1}-1}$.

In the sequel we will see other important CSP examples by considering classes containing "nice" cutting-and-stacking constructions, meaning that we will be able to properly choose the parameters in order to have the desired quantification of the cocycles for the orbit equivalence built in Section I.5.a. By definition, every flexible class $\mathcal{C}$ will be associated to some subset $\mathcal{F}_\mathcal{C}$ of $\mathcal{P}^*$, which can be considered as sufficient conditions that the cutting and spacing parameters have to satisfy at each step for the underlying transformation to belong to $\mathcal{C}$. Recall that $\mathcal{P}^*$ denotes the set of all finite sequences of cutting and stacking parameters.

Definition I.3.7. A class $\mathcal{C}$ of rank-one systems is said to be **flexible** if there exists a subset $\mathcal{F}_\mathcal{C}$ of $\mathcal{P}^*$ satisfying the following properties:

1. there exists a constant $C > 0$ such that for all $(q_n, (\sigma_{n,0}, \ldots, \sigma_{n,q_n}))_{n\geqslant 0} \in \mathcal{P}^{\mathbb{N}}$ satisfying the condition (F) in Definition I.3.2, if $\mathcal{F}_\mathcal{C}$ contains every projection $(q_k, (\sigma_{k,0}, \ldots, \sigma_{k,q_k}))_{0\leqslant k\leqslant n} \in \mathcal{P}^{n+1}$ for $n \geqslant 0$, then these parameters define a CSP construction (with the constant C) and the underlying rank-one transformation is in $\mathcal{C}$;

2. there exists a cutting and spacing parameter $(q_0, (\sigma_{0,0} \ldots, \sigma_{0,q_0}))$ in $\mathcal{F}_\mathcal{C}$ with $q_0 \geqslant 3$;

3. there is a constant $C' > 0$ such that for all $n \geqslant 1$, if $(q_k, (\sigma_{k,0}, \ldots, \sigma_{k,q_k}))_{0\leqslant k\leqslant n-1}$ is in $\mathcal{F}_\mathcal{C}$, then there are infinitely many integers q_n such that $(q_k, (\sigma_{k,0}, \ldots, \sigma_{k,q_k}))_{0\leqslant k\leqslant n}$ is in $\mathcal{F}_\mathcal{C}$ for some $\sigma_{n,0}, \ldots, \sigma_{n,q_n}$ satisfying the inequality

$$\sigma_n \leqslant C' q_n h_{n-1}$$

where $(h_k)_{0\leqslant k\leqslant n+1}$ and $(\sigma_k)_{0\leqslant k\leqslant n}$ denote the finite sequences associated to the finite sequence $(q_k, (\sigma_{k,0}, \ldots, \sigma_{k,q_k}))_{0\leqslant k\leqslant n}$ of cutting and stacking parameters.

A rank-one system T is **flexible** if $\{T\}$ is a flexible class.

The third point of the definition aims to recursively choose the cutting parameters (and we want them to increase quickly enough) with an asymptotic control on $(\sigma_n)_n$, while the first point guarantees that it is possible to do so for the underlying system to be in the class $\mathcal{C}$. The second point is minor, but it is required for the initialization of the recursive construction of an odometer orbit equivalent to an element of our flexible class (see Lemma I.5.12 and Remark I.5.13). It also ensures that $\mathcal{F}_{\mathcal{C}}$ is not an empty set.

Notice that if a construction satisfies $Z_n \leqslant Ch_{n-1}$ for all n, then it is in particular CSP and we get $\sigma_n \leqslant C(q_n+1)h_{n-1} \leqslant 2Cq_nh_{n-1}$ as in the third point of Definition I.3.7.

We now give examples of flexible classes. The proof is given in Section I.4.

Proposition I.3.8.

1. *Every BSP rank-one system is flexible.*
2. *For every nonempty open subset $\mathcal{V}$ of $\mathbb{R}$, the set $\{R_\theta \mid \theta \in \mathcal{V} \cap (\mathbb{R}\backslash\mathbb{Q})\}$ is a flexible class.*
3. *For every irrational number θ, the class of rank-one systems which have $e^{2i\pi\theta}$ as an eigenvalue is flexible.*
4. *The class of strongly mixing rank-one systems is flexible.*

Theorems A, C, E and F follow from Proposition I.3.8 and the following theorem which is the main result.

Theorem I.3.9. *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a map satisfying $\varphi(t) \underset{t\to+\infty}{=} o\left(t^{1/3}\right)$. If $\mathcal{C}$ is a flexible class, then there exists T in $\mathcal{C}$ which is φ-integrably orbit equivalent to the universal odometer.*

I.4 Proof of Proposition I.3.8

In this section we prove the four statements in Proposition I.3.8.

I.4.a BSP systems

Let T be a BSP rank-one system, $\mathcal{C} \coloneqq \{T\}$ and q_n, $\sigma_{n,i}$ the parameters of a BSP construction of T, with a constant $C > 0$. For every $n \geqslant 0$ and $j \geqslant 1$, let $\sigma_0^{(n,n+j)}, \ldots, \sigma_{q_n\ldots q_{n+j-1}}^{(n,n+j)}$ be the spacing parameters from $\mathcal{R}_n$ to $\mathcal{R}_{n+j}$, assuming that the steps for $\mathcal{R}_{n+1}, \ldots, \mathcal{R}_{n+j-1}$ are skipped during the construction (we then have $\sigma_i^{(n,n+1)} = \sigma_{n,i}$ and also $\sigma_i^{(n,n+j)} = 0$ for i equal to 0 and $q_n \ldots q_{n+j-1}$ by Lemma I.3.6). The new cutting parameters are $q^{(n,n+j)} \coloneqq q_n \ldots q_{n+j-1}$ and are large enough with huge jumps j. Now define

$$\mathcal{F}_{\mathcal{C}} \coloneqq \left\{ \left(q^{(n_k,n_{k+1})}, \left(\sigma_0^{(n_k,n_{k+1})}, \ldots, \sigma^{(n_k,n_{k+1})}_{\left(q^{(n_k,n_{k+1})}\right)} \right) \right)_{0\leqslant k\leqslant m} \;\middle|\; m \geqslant 0,\ 0 = n_0 < n_1 < \ldots < n_{m+1} \right\}.$$

Using Lemma I.3.6, the new spacing parameters $\sigma_j^{(n_k,n_{k+1})}$ are bounded by C and we get

$$\sum_{1\leqslant j\leqslant q^{(n_k,n_{k+1})}} \sigma_j^{(n_k,n_{k+1})} \leqslant Cq^{(n_k,n_{k+1})}.$$

The set of parameters $\mathcal{F}_{\mathcal{C}}$ thus witnesses that $\{T\}$ is flexible.

I.4.b Irrational rotations

We now consider a construction from [DEJLMS23]. For every irrational number θ, Drillick, Espinosa-Dominguez, Jones-Baro, Leng, Mandelshtam and Silva give an explicit cutting-and-stacking construction of a transformation T which is the irrational rotation of angle θ when the construction yields a finite measure-preserving system.

The construction in [DEJLMS23]. Let θ be an irrational number and $[q_{-1}; q_0, q_1, \ldots]$ its continued fraction expansion, with $q_{-1} \coloneqq \lfloor \theta \rfloor$ and positive integers $q_0, q_1, \ldots$. Let us assume that there is no integer n such that $q_k = 1$ for every $k \geqslant n$. We consider the sequence $(h_n)_{n \geqslant 0}$ defined by $h_{-1} \coloneqq 0$, $h_0 \coloneqq 1$ and $h_{k+1} \coloneqq q_k h_k + h_{k-1}$ for every $k \geqslant 0$ (the integer h_k is the denominator of the k-th convergent of the irrational number θ). Finally, for every $k \geqslant 0$, we set $\sigma_{k,i} = 0$ for every $i \in \{0, 1, \ldots, q_k - 1\}$, and $\sigma_{k,q_k} = h_{k-1}$. Then, the sequence of cutting and stacking parameters $(q_k, (\sigma_{k,0}, \ldots, \sigma_{k,q_k}))_{k \geqslant 0}$ provides a rank-one system. This system is the irrational rotation of angle θ if and only if Condition (F) is satisfied, and this last condition holds if and only if the series $\sum_{k \geqslant 0} \frac{1}{q_k q_{k+1}}$ converges (see Theorems 3.1 and 5.1 in [DEJLMS23]).

Remark I.4.1. Equivalently, we can define rank-one systems with cutting parameters potentially equal to 1, provided that there are infinitely many cutting parameters greater than or equal to 2, but our construction of orbit equivalence is not well-defined with this weaker assumption. Therefore, in the proof of Proposition I.3.8 for irrational rotations, one of the main goals is to avoid cutting parameters equal to 1.

Remark I.4.2. It is proven in [DEJLMS23] that the set of irrational numbers θ such that the associated series $\sum_{k \geqslant 0} \frac{1}{q_k q_{k+1}}$ converges has measure zero.

Proof of Proposition I.3.8 for these systems. Let $\mathcal{V}$ be a nonempty open subset of $\mathbb{R}$ and

$$\mathcal{C} \coloneqq \{R_\theta \mid \theta \in \mathcal{V} \cap (\mathbb{R} \backslash \mathbb{Q})\}.$$

We now prove that $\mathcal{C}$ is a flexible class.

We first use the following basic fact from the theory of continued fractions: if A denotes the set of sequences $(q_i)_{i \geqslant -1}$ of integers such that $q_0, q_1, \ldots$ are positive, and if A is equipped with the induced product topology, then the map

$$(q_i)_{i \geqslant -1} \in A \mapsto [q_{-1}; q_0, q_1, \ldots] \in \mathbb{R} \backslash \mathbb{Q},$$

is a homeomorphism (see [EW11, Lemma 3.4] for instance). We can then fix integers $n_0 \geqslant 0$ and $Q_{-1}, Q_0, \ldots, Q_{n_0}$ (where $Q_0, \ldots, Q_{n_0}$ are positive) such that $Q_0 \ldots Q_{n_0}$ is greater than or equal to 3 and the following holds: for every irrational number θ, if the first coefficients of its continued fraction expansion are $Q_{-1}, Q_0, \ldots, Q_{n_0}$, then θ is in $\mathcal{V}$.

We write $\boldsymbol{Q} \coloneqq (Q_0, \ldots, Q_{n_0})$ and we consider the set $\tilde{\mathcal{F}}(\boldsymbol{Q})$ of finite sequences $(\tilde{q}_k, (\tilde{\sigma}_{k,0}, \ldots, \tilde{\sigma}_{k,\tilde{q}_k}))_{0 \leqslant k \leqslant n}$ such that $n \geqslant n_0$ and for all $k \in \{0, \ldots, n\}$,

$$\begin{aligned} \tilde{q}_k &= Q_k \quad \text{if } k \leqslant n_0, \\ \tilde{q}_k &\geqslant 2 \quad \text{if } k > n_0, \end{aligned}$$

and

$$\begin{aligned} \tilde{\sigma}_{k,i} &= 0 \qquad \text{for } i \in \{0, \ldots, \tilde{q}_k - 1\}, \\ \tilde{\sigma}_{k,\tilde{q}_k} &= \tilde{h}_{k-1} \end{aligned}$$

(where $(\tilde{h}_k)_{0 \leqslant k \leqslant n}$ is the associated height sequence and $\tilde{h}_{-1} \coloneqq 0$). The finite sequences of $\tilde{\mathcal{F}}(\boldsymbol{Q})$ may not be finite sequences of cutting and stacking parameters in the sense of Definition I.3.1, since the integers $Q_0, \ldots, Q_{n_0}$ may be equal to 1. Moreover, even if the integers

$Q_0, \dots, Q_{n_0}$ were greater than or equal to 2, we could not prove that $\mathcal{C}$ is a flexible class with $\mathcal{F}_{\mathcal{C}} = \tilde{\mathcal{F}}(\boldsymbol{Q})$, since the first cutting parameters $\tilde{q}_1, \dots, \tilde{q}_{n_0}$ cannot be chosen large enough. Notice that, although we may have $\tilde{q}_k = 1$ for some $k \in \{0, 1, \dots, n_0\}$, a finite sequence $(\tilde{q}_k, (\tilde{\sigma}_{k,0}, \dots, \tilde{\sigma}_{k,q_k}))_{0 \leqslant k \leqslant n}$ can still define the first $(n+1)$ steps of a cutting-and-stacking construction, and we associate to it another finite sequence $(q_k, (\sigma_{k,0}, \dots, \sigma_{k,q_k}))_{0 \leqslant k \leqslant n-n_0}$ corresponding to the cutting-and-stacking construction obtained from the previous one by skipping the steps $1, 2, \dots, n_0$. We get $h_0 = \tilde{h}_0 = 1$, $q_0 = Q_0 \dots Q_{n_0} \geqslant 3$ and for all $\forall k \in \{1, \dots, n - n_0\}$,

$$\begin{aligned}
&q_k = \tilde{q}_{n_0+k} \geqslant 2, \\
&h_k = \tilde{h}_{n_0+k}, \\
&\sigma_{k,i} = \tilde{\sigma}_{n_0+k,i} = 0 && \text{for } i \in \{0, \dots, q_k - 1\}, \\
&\sigma_{k,q_k} = \tilde{\sigma}_{n_0+k,\tilde{q}_{n_0+k}} = \tilde{h}_{n_0+k-1} = h_{k-1} && \text{if } k \geqslant 2.
\end{aligned}$$

For $k = 1$, we have $\sigma_{1,q_1} = \tilde{h}_{n_0}$, where $\tilde{h}_{n_0}$ is not equal to h_0. Setting $C = C' \coloneqq \tilde{h}_{n_0}$ (this constant only depends on $Q_0, \dots, Q_{n_0}$), we have $Z_1 \leqslant Ch_1$ and $\sigma_1 \leqslant C'h_0$. We immediately get the inequalities $Z_k \leqslant Ch_k$ and $\sigma_k \leqslant C'h_{k-1}$ for $k \in \{2, \dots, n - n_0\}$.

Let $\mathcal{F}(\boldsymbol{Q})$ be the set of finite sequences $(q_k, (\sigma_{k,0}, \dots, \sigma_{k,q_k}))_{0 \leqslant k \leqslant n-n_0}$ obtained from finite sequences $(\tilde{q}_k, (\tilde{\sigma}_{k,0}, \dots, \tilde{\sigma}_{k,q_k}))_{0 \leqslant k \leqslant n} \in \tilde{\mathcal{F}}(\boldsymbol{Q})$. It is now easy to check that $\mathcal{C}$ is a flexible class, with the set of parameters $\mathcal{F}_{\mathcal{C}} \coloneqq \mathcal{F}(\boldsymbol{Q})$ and the constants C and C'.

I.4.c Systems with a given eigenvalue

Let θ be an irrational number and $\mathcal{C}$ the class of rank-one systems which has $\lambda \coloneqq e^{2i\pi\theta}$ as an eigenvalue. In [DV23], Danilenko and Vieprik present an explicit cutting-and-stacking construction of a system in $\mathcal{C}$. The parameters are chosen in the following way (see the proof of Theorem 4.1 in [DV23]).

The construction of Danilenko and Vieprik. For every $n \geqslant 1$, we fix a number $j_n \in \{1, \dots, n\}$ such that $\delta_n \coloneqq |1 - \lambda^{j_n}|$ is less than $2\pi/n$. Fix $n \geqslant 1$, assume that $(q_k, (\sigma_{k,0}, \dots, \sigma_{k,q_k}))_{0 \leqslant k \leqslant n-1}$ has already been constructed with an auxiliary condition

$$h_n > \frac{n^4}{\delta_{n^2}}. \tag{I.2}$$

Danilenko and Vieprik show the existence of a sequence $(\ell_m^{(n)})_{m \geqslant 1}$ of positive integers less than or equal to $2\pi/\delta_{n^2}$, such that for every $m \geqslant 1$,

$$|1 - \lambda^{mh_n + (\ell_1^{(n)} + \dots + \ell_m^{(n)})j_{n^2}}| < \frac{2\pi}{n^2}. \tag{I.3}$$

Next, let q_n be an integer large enough so that the auxiliary condition (I.2) holds at the next step, namely

$$h_{n+1} \coloneqq q_n h_n + (\ell_1^{(n)} + \dots + \ell_{q_n-1}^{(n)})j_{n^2} > \frac{(n+1)^4}{\delta_{(n+1)^2}}$$

(in [DV23], q_n is chosen as the smallest integer satisfying the property but it is not needed, so there are infinitely many choices). Finally the spacing parameters at this step are defined by $\sigma_{n,0} = \sigma_{n,q_n} = 0$ and $\sigma_{n,m} = \ell_m^{(n)} j_{n^2}$ for $1 \leqslant m \leqslant q_n - 1$.

With these parameters satisfying (I.2) and (I.3), λ is an eigenvalue of the underlying rank-one system (see [DV23], proof of Theorem 4.1, for details).

Proof of Proposition I.3.8 for these systems. Let us consider the same construction as above, but with the following auxiliary condition:

$$h_n > \max\left(\frac{n^4}{\delta_{n^2}}, \frac{(n+1)^4}{\delta_{(n+1)^2}}\right), \tag{I.4}$$

which is stronger than the previous auxiliary condition (I.2). Note that the real numbers δ_i have been fixed before setting the parameters.

The subset $\mathcal{F}_\mathcal{C}$ of $\mathcal{P}^*$ is defined to be the set of finite sequences $(q_k, (\sigma_{k,0}, \dots, \sigma_{k,q_k}))_{0\leqslant k\leqslant n}$ constructed in a recursive way. Any cutting and spacing parameter $(q_0, (\sigma_{0,0}, \dots, \sigma_{q_0,0}))$ is in $\mathcal{F}_\mathcal{C}$, and if $(q_k, (\sigma_{k,0}, \dots, \sigma_{k,q_k}))_{0\leqslant k\leqslant n-1}$ is in $\mathcal{F}_\mathcal{C}$, then so is $(q_k, (\sigma_{k,0}, \dots, \sigma_{k,q_k}))_{0\leqslant k\leqslant n}$ for every $(q_n, \sigma_{n,0}, \dots, \sigma_{n,q_n})$ that we can obtain at the next step, as described above but with the stronger auxiliary condition (I.4).

Let $\boldsymbol{p} \coloneqq (q_n, (\sigma_{n,0}, \dots, \sigma_{n,q_n}))_{n\geqslant 0}$ be a sequence of cutting and spacing parameters. If all its projections are in $\mathcal{F}_\mathcal{C}$, then $\boldsymbol{p}$ provides a CSP construction with $C = 2\pi$. Indeed, we have $\sigma_{n,m} = \ell_m^{(n)} j_{n^2} \leqslant 2\pi n^2/\delta_{n^2} < 2\pi h_n$. As mentioned above, Conditions (I.2) and (I.3) imply that the sequence $\boldsymbol{p}$ provides rank-one systems which have λ as an eigenvalue.

Finally, if $(q_k, (\sigma_{k,0}, \dots, \sigma_{k,q_k}))_{0\leqslant k\leqslant n-1}$ is a finite sequence in $\mathcal{F}_\mathcal{C}$, we can choose a large enough integer q_n so that the following holds at the next step:

$$h_{n+1} > \max\left(\frac{(n+1)^4}{\delta_{(n+1)^2}}, \frac{(n+2)^4}{\delta_{(n+2)^2}}\right)$$

(in particular, the new auxiliary condition (I.4) is satisfied). We use the same spacing parameters as before, namely $\sigma_{n,m} = \ell_m^{(n)} j_{n^2}$. Using $j_{n^2} \leqslant n^2$ and $\ell_m^{(n)} \leqslant \frac{2\pi}{\delta_{n^2}}$, this gives

$$\sigma_n = (\ell_1^{(n)} + \dots + \ell_{q_n-1}^{(n)}) j_{n^2} \leqslant q_n \frac{2\pi n^2}{\delta_{n^2}} \leqslant q_n h_{n-1},$$

so the third point of Definition I.3.7 is satisfied for $C' = 1$.

I.4.d Strongly mixing systems

Let $\mathcal{C}$ be the class of strongly mixing rank-one systems. We consider the construction of Ornstein in [Orn72]. The property the parameters have to satisfy at each step is given by the following lemma (Lemma 3.2 in [Orn72]), proven with a probabilistic method.

Lemma I.4.3. *Let N and K be positive integers and $\varepsilon > 0$, $\alpha > 0$. Then there exist integers $m > N$ and $a_1, \dots, a_m$ such that*

- $\left|\sum_{i=j}^{j+k} a_i\right| \leqslant K$ *for all* $1 \leqslant j \leqslant j+k \leqslant m$;
- *denoting by $H(\ell, k)$ the number of j such that $\sum_{i=j}^{j+k} a_i = \ell$, for $1 \leqslant j \leqslant j+k \leqslant m$, then $H(\ell, k) < \alpha \dfrac{(m-k)}{K}$ for every $k < (1-\varepsilon)m$.*

The set of parameters $\mathcal{F}_\mathcal{C}$ is defined in a recursive way, as in Section I.4.c: any cutting and spacing parameter $(q_0, (\sigma_{0,0}, \dots, \sigma_{q_0,0}))$ is in $\mathcal{F}_\mathcal{C}$, and from a finite sequence of parameters $(q_k, (\sigma_{k,0}, \dots, \sigma_{k,q_k}))_{0\leqslant k\leqslant n-1}$ in $\mathcal{F}_\mathcal{C}$, $(q_k, (\sigma_{k,0}, \dots, \sigma_{k,q_k}))_{0\leqslant k\leqslant n}$ is also in $\mathcal{F}_\mathcal{C}$ if the new parameters can be written as $q_n = m$ and $\sigma_{n,i} = a_i + h_{n-1}$ where $m, a_1, \dots, a_m$ are integers whose existence is granted by Lemma I.4.3 with $N > 10^n$, $K = h_{n-1}$, $\varepsilon = 10^{n-3}$ and $\alpha = 5/4$. There are infinitely many possibilities for q_n as N can be arbitrarily large. It is shown in [Orn72] that cutting-and-stacking constructions with these parameters give strongly mixing systems, it is clear that they are CSP with $C = 2$ and the third point of Definition I.3.7 is satisfied for $C' = 2$.

I.5 From flexible classes to the universal odometer

The goal of this section is to prove Theorem I.3.9, namely that for every $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\varphi(x) = o(t^{1/3})$, any flexible class contains a rank-one system which is φ-integrably orbit equivalent to the universal odometer.

I.5.a The construction

Overview of the construction. We first present a natural adaptation to the case of rank-one system of Kerr and Li's construction of an explicit orbit equivalence between the universal odometer and any other odometers. We will then see that the quantification of the cocycles becomes more complicated due to the presence of non-zero spacing parameters.

Let $T \in \mathrm{Aut}(X,\mu)$ be a rank-one system and consider a cutting-and-stacking construction of this transformation with the same notations $q_n, \sigma_{n,i}, \sigma_n, h_n, \mathcal{R}_n, \varepsilon_n, X_n$ as in Definition I.3.2. From the sequence of Rokhlin towers $\mathcal{R}_n$, new towers $\mathcal{R}'_n$ will be built as Rokhlin towers for a new system S. These towers $\mathcal{R}'_n$ will have no spacers, i.e. $\sigma'_{n,i} = 0$, so they will be partitions of X. The construction will ensure that $\mathcal{R}'_n$ increases to the σ-algebra $\mathcal{A}$ using the fact that it is the case for $\mathcal{R}_n$, so S will be an odometer. For the odometer S to be universal, we fix a sequence of prime numbers $(p_n)_{n\geqslant 0}$ such that every prime number appears infinitely many times, and every cutting parameter q'_n will be a multiple of p_n.

We will recursively define S on subsets increasing to X up to a null set. More precisely if the n-th tower $\mathcal{R}'_n$ has been built and its base and its height are denoted by B'_n and h'_n, then S is provisionally defined on all the levels of the tower except the highest one and maps the i-th level to the $(i+1)$-th one. So $\mathcal{R}'_n$ is exactly $(B'_n, S(B'_n), \dots, S^{h'_n-1}(B'_n))$ and S is defined on $X\backslash S^{h'_n-1}(B'_n)$. To refine S, i.e. to define it on a greater set, we have to build the next tower $\mathcal{R}'_{n+1}$ and define S as for $\mathcal{R}'_n$. In order to do so and according to Definition I.3.2, we have to determine a subdivision of the base B'_n into q'_n subsets $B'_{n,0}, \dots, B'_{n,q'_n-1}$. We find a function ζ_{n+1} mapping bimeasurably each $B'_{n,i}$ to $B'_{n,i+1}$ for $0 \leqslant i \leqslant q'_n - 2$. On the subset $D_{n+1} \coloneqq \bigsqcup_{0\leqslant i\leqslant q'_n-2} S^{h'_n-1}(B'_{n,i})$ of the roof $S^{h'_n-1}(B_n)$ of $\mathcal{R}'_n$, S will coincide with $\zeta_{n+1}S^{-h'_n}$ and will be defined on $X\backslash S^{h'_{n+1}}(B'_{n+1}) = D_1 \sqcup \ldots \sqcup D_{n+1}$ where $B'_{n+1} = B'_{n,0}$ is the base of the new Rokhlin tower $\mathcal{R}_{n+1}$ for S. To sum up, S is successively defined by the finite approximations obtained from the maps ζ_n. Up to conjugacy, ζ_n is exactly the addition by $(\underbrace{0,\dots,0}_{n-1 \text{ times}},1,0,0,\dots)$ with carry over to the right (as defined in Section I.3.a), restricted to $[0,\dots,0]_{n-1}\backslash[0,\dots,0,q'_{n-1}-1]_n$.

The construction of the maps ζ_n is by induction on $n \geqslant 0$. At step n we will actually define

$$\zeta_{n+1}\colon B'_{n,0} \sqcup \ldots \sqcup B'_{n,q'_n-2} \to B'_{n,1} \sqcup \ldots \sqcup B'_{n,q'_n-1}.$$

In order to build ζ_{n+1}, a second induction on a parameter $m \geqslant n$ is required. Actually, $B'_{n,i}$ will be the disjoint union of the $B'_{n,i}(m)$ for $m \geqslant n$, and this inner recursion consists in choosing m-bricks to define $B'_{n,i}(m)$. By definition, the **m-bricks** will be the **m-levels** (i.e. the levels of $\mathcal{R}_m$) explicitly chosen to constitute $B'_{n,i}(m)$. Using powers of T, m-bricks of $B'_{n,i}$ are mapped to the ones of $B'_{n,i+1}$ (there will be as many in $B_{n,i}$ as in $B'_{n,i+1}$) and this gives ζ_{n+1} whose orbits are included in those of T, implying immediately that the orbits of S satisfy the same property. The reverse inclusion between the orbits will be more difficult to prove and will be due to the choice of the bricks (see Remark I.5.3 after the construction).

The construction. T is a rank-one system in $\mathrm{Aut}(X,\mu)$. We fix one of its cutting-and-stacking construction whose parameters are denoted as in Definition I.3.2. Let $(p_n)_{n\geqslant 0}$ be a sequence of prime numbers such that every prime number appears infinitely many times.

In the sequel, we will assume that, given the cutting parameters of T, some positive integers q'_n and $t_{n,m}$ that we will introduce are well-defined. In Section I.5.c (see Lemma I.5.12), we will give conditions on the parameters of T for these quantities (and so the construction) to be well-defined.

- $\boldsymbol{n=1}$ **:** We first build $\mathcal{R}'_1$ and ζ_1 by an induction over $m \geqslant 1$. We could denote by $\mathcal{R}'_0$ the trivial tower (X) with its base $B'_0 \coloneqq X$. At the end of step $n=1$, S is not yet defined on the roof of the tower $\mathcal{R}'_1$, i.e. on its highest level, which is a Rokhlin tower of S.

  - $\boldsymbol{m=1}$ **:** Let $q'_0 > 0$ be the largest multiple of p_0 such that $q'_0 \leqslant q_0 - 1$.

    Remark I.5.1. At this step, we simply have to assume $q_0 > p_0$ for the integer q'_0 to be non-zero. However, for the well definition of other quantities at other steps, the conditions on the cutting parameters of T get more and more technical, this is the reason why we first assume that the parameters of T are chosen so that the positive quantities are well-defined and we will then state the conditions in Lemma I.5.12 (as mentioned before the beginning of the construction).

    For every $0 \leqslant i \leqslant q'_0 - 1$, we define

    $$B'_{0,i}(1) \coloneqq T^i(B_1)$$

    and

    $$\zeta_1(1)\colon \bigsqcup_{0\leqslant i\leqslant q'_0-2} B'_{0,i}(1) \to \bigsqcup_{0\leqslant i\leqslant q'_0-2} B'_{0,i+1}(1)$$

    coinciding with T on its domain (hence every subset $B'_{0,i}(1)$ is composed of a unique 1-brick $T^i(B_1)$).

  - $\boldsymbol{m>1}$ **:** Assume that the subsets $B'_{0,i}(M)$ have been built for every $1 \leqslant M \leqslant m-1$ and $0 \leqslant i \leqslant q'_0 - 1$. Let

    $$W_{1,m} \coloneqq X \backslash \bigsqcup_{1\leqslant M\leqslant m-1} \bigsqcup_{0\leqslant i\leqslant q'_0-1} B'_{0,i}(M)$$

    be the remaining piece of X at the end of the $(m-1)$-th step (we could also define $W_{1,1} \coloneqq X$).

    Remark I.5.2. Notice that m-levels are either contained in $W_{1,m}$ or disjoint from it since $X \backslash W_{1,m}$ is composed of M-levels for $1 \leqslant M \leqslant m-1$ and the Rokhlin towers are nested. This will more generally hold true for $W_{n,m}$ with $n \geqslant 2$.

    Let $r_{1,m}$ be the number of integers $j \in [\![0, h_m - 1]\!]$ such that $T^j(B_m) \subseteq W_{1,m}$, denoted by

    $$0 \leqslant j_1^{(1,m)} < j_2^{(1,m)} < \ldots < j_{r_{1,m}}^{(1,m)} < h_m.$$

    Let $t_{1,m}$ be the positive integer such that $q'_0 t_{1,m}$ is the largest multiple of q'_0 such that $q'_0 t_{1,m} < r_{1,m}$ (we assume that we can choose the cutting parameters of T for this integer to be positive, see Lemma I.5.12). The first $q'_0 t_{1,m}$ m-levels contained in $W_{1,m}$ are now used as m-bricks, they are split in q'_0 groups of $t_{1,m}$ m-bricks of the subsets $B'_{0,i}$ in the following way and the same will be done at steps $n > 1$ (the fact that the inequality $q'_0 t_{1,m} < r_{1,m}$ is strict, and the way we make the q'_0 groups will guarantee an easy control of the cocycles, see Lemma I.5.6 used for Lemmas I.5.18 and I.5.21). For every $0 \leqslant i \leqslant q'_0 - 1$, we define

$$B'_{0,i}(m) \coloneqq \bigsqcup_{0\leqslant t\leqslant t_{1,m}-1} T^{\left(j^{(1,m)}_{i+1+tq'_0}\right)}(B_m)$$

and $\zeta_1(m)$ coinciding with $T^{\left(j^{(1,m)}_{i+2+tq'_0}\right)-\left(j^{(1,m)}_{i+1+tq'_0}\right)}$ on $T^{\left(j^{(1,m)}_{i+1+tq'_0}\right)}(B_m)$ for every $0 \leqslant i \leqslant q'_0-2$ and $0 \leqslant t \leqslant t_{1,m}-1$, so that each brick $T^{\left(j^{(1,m)}_{i+1+tq'_0}\right)}(B_m)$ is mapped on another $T^{\left(j^{(1,m)}_{i+2+tq'_0}\right)}(B_m)$. Thus $\zeta_1(m)$ maps each $B'_{0,i}(m)$ on $B'_{0,i+1}(m)$ and this gives

$$\zeta_1(m)\colon \bigsqcup_{0\leqslant i\leqslant q'_0-2} B'_{0,i}(m) \to \bigsqcup_{0\leqslant i\leqslant q'_0-2} B'_{0,i+1}(m).$$

End of Step $n = 1$: For every $0 \leqslant i \leqslant q'_0 - 1$, we define

$$B'_{0,i} \coloneqq \bigsqcup_{m\geqslant 1} B'_{0,i}(m)$$

(the set of its m-bricks for all $m \geqslant 1$), $B'_1 \coloneqq B'_{0,0}$ and

$$\zeta_1\colon \bigsqcup_{0\leqslant i\leqslant q'_0-2} B'_{0,i} \to \bigsqcup_{0\leqslant i\leqslant q'_0-2} B'_{0,i+1}$$

coinciding with the maps $\zeta_1(m)$ on their respective domain (see Figure I.7).

The universal odometer S we want to build is partially defined on X. More precisely we define it on the domain $D_1 \coloneqq \bigsqcup_{0\leqslant i\leqslant q'_0-2} B'_{0,i}$ of ζ_1 so that it coincides with ζ_1. This gives the first Rokhlin tower $\mathcal{R}'_1 \coloneqq (B'_{0,0}, \dots, B'_{0,q'_1-1}) = (B'_1, S(B'_1), \dots, S^{q'_0-1}(B'_1))$. The next step will provide us a refinement $\mathcal{R}'_2$ of the tower $\mathcal{R}'_1$, allowing us to extend partially S on the highest level of the $\mathcal{R}'_1$.

- **$n > 1$:** Assume that steps $1, \dots, n-1$ have been achieved. There are nested towers $\mathcal{R}'_1, \dots, \mathcal{R}'_{n-1}$. The k-th tower $\mathcal{R}'_k$ has $h'_k \coloneqq q'_0 \dots q'_{k-1}$ levels and its base B'_k is partitioned in q'_k levels $B'_{k,0}, \dots, B'_{k,q'_k-1}$. These levels belong to $\mathcal{R}'_{k+1}$, whose base is $B'_{k+1} \coloneqq B'_{k,0}$, with ζ_{k+1} mapping $B'_{k,i}$ to $B'_{k,i+1}$. The map S is defined on $D_1 \sqcup \dots \sqcup D_{n-1}$ using the maps $\zeta_1, \dots, \zeta_{n-1}$, where $D_1 \sqcup \dots \sqcup D_{n-1}$ corresponds to the union of all the levels of $\mathcal{R}'_{n-1}$ except its roof.

  The map S is not yet defined on the roof of $\mathcal{R}'_{n-1}$. By partitioning B'_{n-1} in subsets $B'_{n-1,0}, \dots, B'_{n-1,q'_{n-1}-1}$, we will define $\mathcal{R}'_n$ which refines $\mathcal{R}'_{n-1}$ and a function ζ_n mapping $B'_{n-1,i}$ to $B'_{n-1,i+1}$. The extension of S will be defined on all the levels of $\mathcal{R}'_n$, except its roof (which is contained in the one of $\mathcal{R}'_{n-1}$). We will construct the subsets $B'_{n-1,i}$ as was done for the subsets $B'_{0,i}$, except that we only use the "material" in B'_{n-1} to form the m-bricks of each $B'_{n-1,i}$. In order to do so, notice that the base B'_{n-1} is exactly $B'_{n-2,0}$ (the first subset in the subdivision of B'_{n-2}) which is the disjoint union of subsets of the form $B'_{n-2,0}(m)$ for $m \geqslant n-1$. Moreover for all $n-1 \leqslant M \leqslant m$, every m-level is contained in an M-level, we will then pick the new m-bricks in $B'_{n-2,0}(n), \dots, B'_{n-2,0}(m)$. This motivates the definition of each set $W_{n,m}$ (the set of the remaining material to form m-bricks with an incremented integer m). We now discuss separately the following cases.

  - **$m = n$:** Set

$$W_{n,n} \coloneqq B'_{n-2,0}(n-1) \sqcup B'_{n-2,0}(n) \tag{I.5}$$

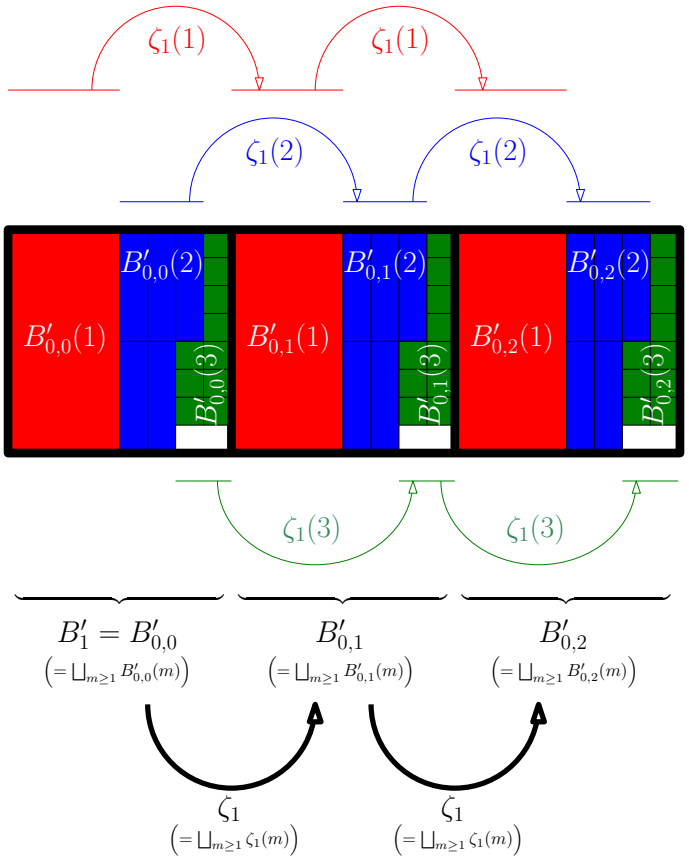


Figure I.7: First step of the construction (i.e. $n = 1$).
In Section I.5.d, we will define sets $E_{n,m}$ for every pair of integers (n, m) satisfying $m \geqslant n \geqslant 1$. The set $E_{1,1}$ (resp. $E_{1,2}$; $E_{1,3}$) is the union of the red areas (resp. red and blue areas; red, blue and green areas).

and let $r_{n,n}$ be the number of integers $j \in [\![0, h_n - 1]\!]$ such that $T^j(B_n) \subseteq W_{n,n}$ (note that we could have defined $r_{1,1} = q_0$), denoted by

$$0 \leqslant j_1^{(n,n)} < j_2^{(n,n)} < \ldots < j_{r_{n,n}}^{(n,n)} < h_n.$$

Let q'_{n-1} be the largest multiple of p_{n-1} such that $q'_{n-1} < r_{n,n}$ (we assume that we can choose the cutting parameters of T for this integer to be positive, see Lemma I.5.12). We then define for every $0 \leqslant i \leqslant q'_{n-1} - 1$,

$$B'_{n-1,i}(n) \coloneqq T^{\left(j_{i+1}^{(n,n)}\right)}(B_n),$$

meaning that among the n-levels in $W_{n,n}$, the n-bricks at step (n, n) are exactly the first q'_{n-1} ones (and set $t_{n,n} = 1$ for consistency later on). Let

$$\zeta_n(n)\colon \bigsqcup_{0\leqslant i\leqslant q'_{n-1}-2} B'_{n-1,i}(n) \to \bigsqcup_{0\leqslant i\leqslant q'_{n-1}-2} B'_{n-1,i+1}(n)$$

be the map coinciding with $T^{\left(j_{i+2}^{(n,n)}\right)-\left(j_{i+1}^{(n,n)}\right)}$ on each $B'_{n-1,i}(n)$, so that $B'_{n-1,i}(n)$ is mapped to $B'_{n-1,i+1}(n)$.

– ***m > n :*** Set

$$W_{n,m} \coloneqq \left(\bigsqcup_{n-1\leqslant M\leqslant m} B'_{n-2,0}(M)\right) \setminus \left(\bigsqcup_{n\leqslant M\leqslant m-1} \bigsqcup_{0\leqslant i\leqslant q'_{n-1}-1} B'_{n-1,i}(M)\right) \quad \text{(I.6)}$$

and let $r_{n,m}$ be the number of integers $j \in [\![0, h_m - 1]\!]$ such that $T^j(B_m) \subseteq W_{n,m}$, denoted by

$$0 \leqslant j_1^{(n,m)} < j_2^{(n,m)} < \ldots < j_{r_{n,m}}^{(n,m)} < h_m.$$

Let $t_{n,m}$ be the positive integer such that $q'_{n-1}t_{n,m}$ is the largest multiple of q'_{n-1} such that $q'_{n-1}t_{n,m} < r_{n,m}$ (we assume that we can choose the cutting parameters of T for this integer to be positive, see Lemma I.5.12). The first $q'_{n-1}t_{n,m}$ m-levels contained in $W_{n,m}$ are now used as m-bricks at step (n,m), they are split in q'_{n-1} groups of $t_{n,m}$ m-bricks of the subsets $B'_{n-1,i}$ in the following way. For every $0 \leqslant i \leqslant q'_{n-1} - 1$, we define

$$B'_{n-1,i}(m) \coloneqq \bigsqcup_{0 \leqslant t \leqslant t_{n,m}-1} T^{\left(j_{i+1+tq'_{n-1}}^{(n,m)}\right)}(B_m)$$

and $\zeta_n(m)$ coinciding with $T^{\left(j_{i+2+tq'_{n-1}}^{(n,m)}\right)-\left(j_{i+1+tq'_{n-1}}^{(n,m)}\right)}$ on $T^{\left(j_{i+1+tq'_{n-1}}^{(n,m)}\right)}(B_m)$ for every $0 \leqslant i \leqslant q'_{n-1}-2$, $0 \leqslant t \leqslant t_{n,m}-1$, so that each m-brick $T^{\left(j_{i+1+tq'_{n-1}}^{(n,m)}\right)}(B_m)$ is mapped on another $T^{\left(j_{i+2+tq'_{n-1}}^{(n,m)}\right)}(B_m)$. Thus $\zeta_n(m)$ maps each $B'_{n-1,i}(m)$ on $B'_{n-1,i+1}(m)$ and this gives

$$\zeta_n(m)\colon \bigsqcup_{0 \leqslant i \leqslant q'_{n-1}-2} B'_{n-1,i}(m) \to \bigsqcup_{0 \leqslant i \leqslant q'_{n-1}-2} B'_{n-1,i+1}(m).$$

End of Step n: We define for every $0 \leqslant i \leqslant q'_{n-1} - 1$,

$$B'_{n-1,i} \coloneqq \bigsqcup_{m \geqslant n} B'_{n-1,i}(m)$$

(the set of its m-bricks for $m \geqslant n$), $B'_n = B'_{n-1,0}$ and

$$\zeta_n\colon \bigsqcup_{0 \leqslant i \leqslant q'_{n-1}-2} B'_{n-1,i} \to \bigsqcup_{0 \leqslant i \leqslant q'_{n-1}-2} B'_{n-1,i+1}$$

coinciding with the maps $\zeta_n(m)$ on their respective domain (see Figure I.8 for step $n = 2$, after the first step illustrated in Figure I.7).

As the base B'_{n-1} of $\mathcal{R}'_{n-1}$ is partitioned in $B'_{n-1,0} \sqcup \ldots \sqcup B'_{n-1,q'_{n-1}-1}$, its highest level $S^{h'_{n-1}-1}(B'_{n-1})$ is partitioned in $S^{h'_{n-1}-1}(B'_{n-1,0}) \sqcup \ldots \sqcup S^{h'_{n-1}-1}(B'_{n-1,q'_{n-1}-1})$. The map S is extended in the following way. On

$$D_n \coloneqq S^{h'_{n-1}-1}(B'_{n-1,0}) \sqcup \ldots \sqcup S^{h'_{n-1}-1}(B'_{n-1,q'_{n-1}-2}),$$

it coincides with $\zeta_n S^{-(h'_{n-1}-1)}$. So S maps $S^{h'_{n-1}-1}(B'_{n-1,i})$ on $B'_{n-1,i+1}$. This gives a Rokhlin tower $\mathcal{R}'_n$ for S, nested in the previous one, of base $B'_n \coloneqq B'_{n-1,0}$ and height $h'_n \coloneqq q'_0 \ldots q'_{n-1}$. Now S is defined on $(D_1 \sqcup \ldots \sqcup D_{n-1}) \sqcup D_n$. The set D_n consists in the levels of $\mathcal{R}_n$, except the highest one, which are contained in the highest level of $\mathcal{R}'_{n-1}$.

Remark I.5.3. Notice that the inclusion of the S-orbits in the T-orbits is easy since S is defined from maps $\zeta_n(m)$ which are "piecewise powers of T".

The reverse inclusion will follow from the fact that we have $t_{n,n} = 1$ for every $n \geqslant 1$ (at step (n,n) we form groups of only one n-level). Indeed, uniqueness implies that these chosen blocks are linked by $\zeta_n(n)$ and hence clearly by S (on the contrary, an m-level, for $m > n$, of $B'_{n-1,i}$ is mapped by $\zeta_n(m)$ to only one of the $t_{n,m}$ m-levels of $B'_{n-1,i+1}$, and not to the other). Thus ensuring that the unique n-brick of each $B'_{n-1,i}$ is a large part of it enables the system S to capture most of the T-orbits.

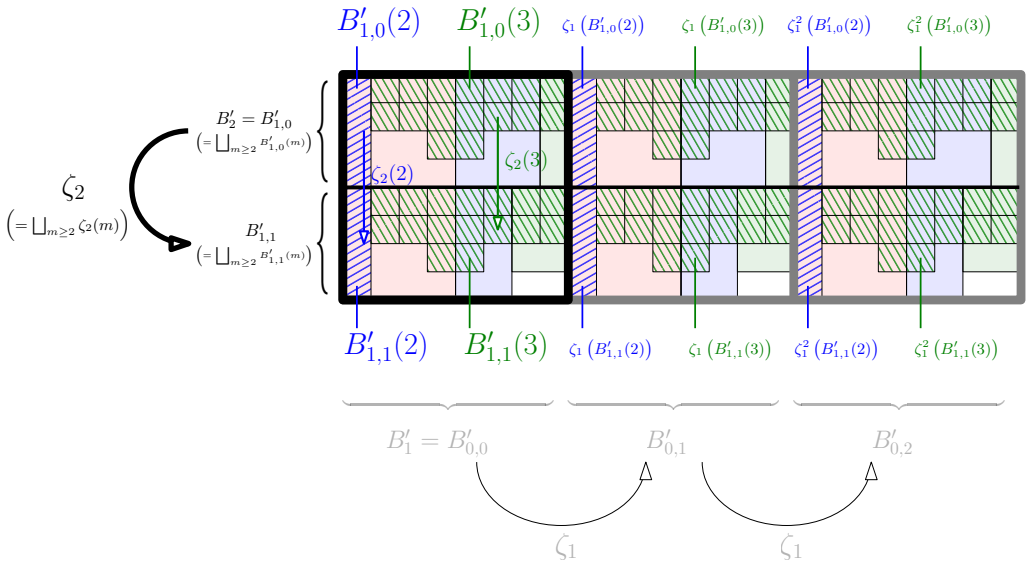


Figure I.8: From the first step (illustrated in Figure I.7) to the second one.
In B'_1, we inductively build $B'_{1,i}(2), B'_{1,i}(2), B'_{1,i}(3), \dots$ for every $0 \leqslant i \leqslant q'_1 - 1$ (in this example, we have $q'_1 = 2$). Each set $B'_{1,i}(2)$ is composed of a unique 2-level in $B'_{0,0}(1) \sqcup B'_{0,0}(2)$ (i.e. in pale red and pale blue areas). Then each set $B'_{1,i}(3)$ is composed of 3-levels in $B'_{0,0}(1) \sqcup B'_{0,0}(2) \sqcup B'_{0,0}(3)$ (i.e. in pale red, pale blue and pale green areas) and so on. The structure that we build in $B'_1 = B'_{0,0}$ can be translated in $B'_{0,1}$ and $B'_{0,2}$ using the map ζ_1.
In Section I.5.d, we will define sets $E_{n,m}$ for every pair of integers (n, m) satisfying $m \geqslant n \geqslant 1$. The set $E_{2,1}$ (resp. $E_{2,2}$) is the union of the areas hatched in blue (resp. in blue or green).

I.5.b First properties of this construction

Recall that we consider a cutting-and-stacking construction of T with the same notations as in Definition I.3.2 (q_n, $\sigma_{n,i}$, $\mathcal{R}_n$, X_n, ε_n, ...), and the sequences (h_n), (σ_n) and (Z_n) associated to the cutting and spacing parameters, and the notations q'_n, $\mathcal{R}'_n$, ... refer to the construction of S.

We state some important properties preparing for further results in Section I.5.d. Many of them enable us to only take into account the combinatorics behind a cutting-and-stacking construction. We assume that all the "largest multiples" (for every $n < m$, the largest multiple q'_{n-1} of p_{n-1} such that $q'_{n-1} < r_{n,n}$, and the largest multiple $q'_{n-1}t_{n,m}$ of q'_{n-1} such that $q'_{n-1}t_{n,m} < r_{n,m}$) are non-zero. In Section I.5.c (see Lemma I.5.12), we will see how to choose the parameters for the construction to be well-defined.

Lemma I.5.4. *Every tower $\mathcal{R}'_n$ is a partition of X.*

Proof of Lemma I.5.4. Let $n \geqslant 1$. The levels of $\mathcal{R}'_n$ are pairwise disjoint by the definition of $(W_{n,m})_{m \geqslant n}$. It remains to show that $\mathcal{R}'_n$ covers the whole space. Recall that X_n denotes the subset covered by the tower $\mathcal{R}_n$, and ε_n the measure of its complement.

The result holds for $n = 1$ since $\mu(W_{1,m}) \underset{m \to +\infty}{\to} 0$. Indeed $W_{1,m+1} \cap X_m$ is the union of the m-levels which are not chosen at step $(1, m)$. By the definition of $t_{1,m}$, there are at most q'_0. So we have $W_{1,m+1} \leqslant \varepsilon_m + q'_0\mu(B_m) \to 0$.

For $n > 1$, it suffices to show that the levels $B'_{n-1,0}, \dots, B'_{n-1,q'_{n-1}-1}$ of $\mathcal{R}'_n$ form a

partition of the base B'_{n-1} of $\mathcal{R}'_{n-1}$. We have to show that the measure of

$$\tilde{W}_{n,m} \coloneqq B'_{n-1} \setminus \left(\bigsqcup_{n \leqslant M \leqslant m-1} \bigsqcup_{0 \leqslant i \leqslant q'_{n-1}-1} B'_{n-1,i}(M) \right)$$

tends 0 as $m \to +\infty$. But this set $\tilde{W}_{n,m}$ is the disjoint union of $\bigsqcup_{M \geqslant m+1} B'_{n-2,0}(M)$ and $W_{n,m}$. It is clear that

$$\mu \left(\bigsqcup_{M \geqslant m+1} B'_{n-2,0}(M) \right) \underset{m \to +\infty}{\to} 0,$$

since μ is a finite measure. The set $W_{n,m}$ is obtained from $W_{n,m-1}$ by adding $B'_{n-2,0}(m)$ and removing $q'_{n-1}t_{n,m-1}$ $(m-1)$-levels. Thus we have $\mu(W_{n,m}) \underset{m \to +\infty}{\to} 0$ by the definition of $(t_{n,m})_{m \geqslant n}$. Hence we have $\mu\left(\tilde{W}_{n,m}\right) \underset{m \to +\infty}{\to} 0$ and we are done. □

As a consequence, if $(\mathcal{R}'_n)_n$ increases to the σ-algebra $\mathcal{A}$ (this will be proved in Corollary I.5.16), then S is a rank-one system without spacer, so this is an odometer.

Lemma I.5.5. *Let $n \geqslant 1$. On the base B'_n of the n-th S-Rokhlin tower $\mathcal{R}'_n$, S is defined as follows. For every $0 \leqslant i \leqslant h'_n - 1$, we have*

$$S^i = \zeta_1^{i_0} \dots \zeta_n^{i_{n-1}} \text{ on } B'_n$$

with $i_0 \in [\![0, q'_0 - 1]\!], \dots, i_{n-1} \in [\![0, q'_{n-1} - 1]\!]$ such that $i = \sum_{\ell=0}^{n-1} q'_0 \dots q'_{\ell-1} i_\ell = \sum_{\ell=0}^{n-1} h'_\ell i_\ell$.

Proof of Lemma I.5.5. By induction over $n \geqslant 1$. It is clear for $n = 1$ since S coincides with ζ_1 on the levels of $\mathcal{R}'_1$ except its roof. Assume that the result holds for $n \geqslant 1$. The tower $\mathcal{R}'_n$ is divided in q'_n subcolumns whose levels are exactly the ones of $\mathcal{R}'_{n+1}$, and the i_n-th subcolumn ($0 \leqslant i_n \leqslant h'_n - 1$) is the S-Rokhlin tower of height h'_n and base B'_{n,i_n}. Let $0 \leqslant i \leqslant h'_{n+1} - 1$. By the equality $B'_{n+1} = B'_{n,0}$ and by the definition of S from ζ_{n+1} (at the end of step $n+1$ of the construction), $S^i = S^j \zeta_{n+1}^{i_n}$ on B'_{n+1} for non-negative integers i_n and j such that $i = i_n h'_n + j$ and $j < h'_n$. The set $\zeta_{n+1}^{i_n}(B'_{n,0})$ is equal to B'_{n,i_n}, so this is a subset of B'_n, hence the result by the induction hypothesis. □

Therefore the subset D_n defined in the construction can be written as follows:

$$\begin{aligned} D_n &= \zeta_1^{q'_0-1} \dots \zeta_{n-1}^{q'_{n-2}-1} \left(\bigsqcup_{0 \leqslant i_n \leqslant q'_{n-1}-2} B'_{n-1,i_n} \right) \\ &= \zeta_1^{q'_0-1} \dots \zeta_{n-1}^{q'_{n-2}-1} \left(\bigsqcup_{0 \leqslant i_n \leqslant q'_{n-1}-2} \zeta_n^{i_n}(B'_{n-1,0}) \right) \end{aligned} \tag{I.7}$$

and S coincides with $\zeta_n \zeta_{n-1}^{-(q'_{n-2}-1)} \dots \zeta_1^{-(q'_0-1)}$ on D_n.

By the **cocycle** of $\zeta_n(m)$, we mean the integer-valued map defined on the domain of $\zeta_n(m)$ and which maps x to the unique integer k satisfying $\zeta_n(m)x = T^k x$.

Lemma I.5.6. *The cocycle of $\zeta_n(m)$ is positive and bounded above by $h_{m-1} + Z_{m-1}$.*

Proof of Lemma I.5.6. By definition, for fixed integers $0 \leqslant i \leqslant q'_{n-1}-2$ and $0 \leqslant t \leqslant t_{n,m}-1$, the cocycle on $\mathbb{B} \coloneqq T^{\left(j^{(n,m)}_{i+1+tq'_{n-1}}\right)}(B_m)$ takes the value $\Delta j \coloneqq j^{(n,m)}_{i+2+tq'_{n-1}} - j^{(n,m)}_{i+1+tq'_{n-1}}$. Let us recall that the integers

$$0 \leqslant j^{(n,m)}_1 < j^{(n,m)}_2 < \ldots < j^{(n,m)}_{r_{n,m}} < h_m$$

are the set of indices $j \in [\![0, h_m - 1]\!]$ such that $T^j(B_m) \subseteq W_{n,m}$. Thus Δj is obviously positive. Let us fix an $(m-1)$-level $\mathbb{B}^\star$ which is not chosen at step $(n, m-1)$, so it is contained in $W_{n,m}$. If m is equal to n, we can choose $\mathbb{B}^\star = B'_{n-2,0}(n-1)$. For $m > n$, the existence of $\mathbb{B}^\star$ is granted by the fact that we have $q'_{n-1}t_{n,m-1} < r_{n,m-1}$. We write it as $\mathbb{B}^\star = T^{k_0}(B_{m-1})$, where k_0 is an integer in $[\![0, h_{m-1} - 1]\!]$.

By definition, Δj is the least positive integer j such that $T^j(\mathbb{B})$ is in $W_{n,m}$. Moreover the m-levels of $\mathbb{B}^\star$ are in $W_{n,m}$. Therefore the consecutive m-levels $T(\mathbb{B}), \ldots, T^{\Delta j-1}(\mathbb{B})$ are not in $\mathbb{B}^\star$.

First case. In the tower $\mathcal{R}_m$, assume that the m-levels $T(\mathbb{B}), \ldots, T^{\Delta j-1}(\mathbb{B})$ are before $T^{k_0}(B_{m-1,0})$, i.e. before the first m-level of $\mathbb{B}^\star$. Therefore the enumeration $\mathbb{B}, \ldots, T^{\Delta j}(\mathbb{B})$ is included in the enumeration

$$\Sigma_{m-1,0,1}, \ldots, \Sigma_{m-1,0,\sigma_{m-1,0}}, B_{m-1,0}, \ldots, T^{k_0}(B_{m-1,0}),$$

implying that $\Delta j \leqslant \sigma_{m-1,0} + k_0 \leqslant Z_{m-1} + h_{m-1}$.

Second case. Now assume that $T(\mathbb{B}), \ldots, T^{\Delta j-1}(\mathbb{B})$ are after $T^{k_0}(B_{m-1,q_{m-1}-1})$, i.e. after the last m-level of $\mathbb{B}^\star$. Therefore the enumeration $\mathbb{B}, \ldots, T^{\Delta j}(\mathbb{B})$ is included in the enumeration

$$T^{k_0}(B_{m-1,q_{m-1}-1}), \ldots, T^{h_{m-1}-1}(B_{m-1,q_{m-1}-1}), \Sigma_{m-1,q_{m-1},1}, \ldots, \Sigma_{m-1,q_{m-1},\sigma_{m-1,q_{m-1}}},$$

and we get $\Delta j \leqslant (h_{m-1} - k_0 - 1) + \sigma_{m-1,q_{m-1}} \leqslant h_{m-1} + Z_{m-1}$.

Third case. Finally if $T(\mathbb{B}), \ldots, T^{\Delta j-1}(\mathbb{B})$ are between $T^{k_0}(B_{m-1,i})$ and $T^{k_0}(B_{m-1,i+1})$ for some $0 \leqslant i \leqslant q_{m-1}-2$, i.e. between two consecutive m-levels of $\mathbb{B}^\star$, then the enumeration $\mathbb{B}, \ldots, T^{\Delta j}(\mathbb{B})$ is included in the enumeration

$$T^{k_0}(B_{m-1,i}), \ldots, T^{h_{m-1}-1}(B_{m-1,i}), \Sigma_{m-1,i,1}, \ldots, \Sigma_{m-1,i,\sigma_{m-1,i}}, B_{m-1,i+1}, \ldots, T^{k_0}(B_{m-1,i+1}),$$

this gives $\Delta j \leqslant (h_{m-1} - k_0 - 1) + \sigma_{m-1,i} + (k_0 + 1) \leqslant h_{m-1} + Z_{m-1}$. □

Lemma I.5.7. *An m-brick at step n is included in an M-brick at step $n-1$ for some $n-1 \leqslant M \leqslant m$.*

Proof of Lemma I.5.7. This follows directly from the definition of $W_{n,m}$ in the construction (see Section I.5.a). Indeed the "(M)" in "$B'_{n-2,0}(M)$" means that we only consider the M-bricks, at step $n-1$, composing $B'_{n-2,0}$. □

We now present a combinatorial description of the construction.

Lemma I.5.8. *The quantities $r_{n,m}, q_n, q'_n, t_{n,m}, \sigma_n$ satisfy the following recurrence relation:*

$$t_{0,1} \coloneqq 0;$$

$$\text{for } m \geqslant 2, \qquad t_{0,m} \coloneqq \sigma_{m-1};$$

$$\text{for } m = n \geqslant 1, \quad \begin{cases} r_{n,n} = q_{n-1} + t_{n-1,n}, \\ q'_{n-1} = \left\lfloor \dfrac{r_{n,n} - 1}{p_{n-1}} \right\rfloor p_{n-1}, \\ t_{n,n} = 1; \end{cases}$$

$$\text{for } m > n \geqslant 1, \quad \begin{cases} r_{n,m} = q_{m-1}(r_{n,m-1} - q'_{n-1}t_{n,m-1}) + t_{n-1,m}, \\ t_{n,m} = \left\lfloor \dfrac{r_{n,m} - 1}{q'_{n-1}} \right\rfloor. \end{cases}$$

During the construction, some integers have been defined for consistency ($r_{1,1} \coloneqq q_0$, $t_{n,n} \coloneqq 1$). Note that in this lemma, we also define the integers $t_{n,m}$ for $n = 0$. This enables us to extend the relations

$$r_{n,n} = q_{n-1} + t_{n-1,n} \text{ and } r_{n,m} = q_{m-1}(r_{n,m-1} - q'_{n-1}t_{n,m-1}) + t_{n-1,m}$$

for $n = 1$.

Proof of Lemma I.5.8. Case $n = 1$. For $m = 1$, the $r_{1,1}$ 1-levels potentially chosen to be 1-bricks are exactly the levels of $\mathcal{R}_1$, so we have $r_{1,1} = q_0 + t_{0,1}$ since $t_{0,1} \coloneqq 0$. We choose q'_0 of them, where q'_0 is the largest multiple of p_0 such that $q'_0 < r_{1,1}$, so q'_0 is equal to $\lfloor (r_{1,1} - 1)/p_0 \rfloor p_0$. Finally q'_0 is obviously equal to $q'_0 t_{1,1}$ since $t_{1,1} \coloneqq 1$. For $m > 1$, there are $r_{n,m}$ m-levels in $W_{1,m}$: some of them are in the $r_{1,m-1} - q'_0 t_{1,m-1}$ $(m-1)$-levels which are not chosen at step $(1, m-1)$ and the other are the spacers from $\mathcal{R}_{m-1}$ to $\mathcal{R}_m$. So we have

$$r_{1,m} = q_{m-1}(r_{1,m-1} - q'_0 t_{1,m-1}) + \sigma_{m-1}$$

and we set $t_{0,m} \coloneqq \sigma_{m-1}$. We choose $q'_0 t_{1,m}$ of them as m-bricks, where $q'_0 t_{1,m}$ is the largest multiple of q'_0 such that $q'_0 t_{1,m} < r_{1,m}$, i.e. $t_{1,m} \coloneqq \lfloor (r_{1,m} - 1)/q'_0 \rfloor$.

Case $n > 1$. For $m = n$, there are $r_{n,n}$ n-levels in $W_{n,n} = B'_{n-2,0}(n-1) \sqcup B'_{n-2,0}(n)$. First, since we have $t_{n-1,n-1} = 1$, the set $B'_{n-2,0}(n-1)$ is an $(n-1)$-brick at step $n-1$ and it contains q_{n-1} n-levels. Secondly $B'_{n-2,0}(n)$ is the union of $t_{n-1,n}$ n-bricks. Hence we have $r_{n,n} = q_{n-1} + t_{n-1,n}$. By definition, q'_{n-1} is equal to $\lfloor (r_{n,n} - 1)/p_{n-1} \rfloor p_{n-1}$ and obviously to $q'_{n-1} t_{n,n}$ with $t_{n,n} \coloneqq 1$. For $m > n$, there are $r_{n,m}$ m-levels in $W_{n,m}$. This set is composed of

$$\left(\bigsqcup_{n-1 \leqslant M \leqslant m-1} B'_{n-2,0}(M) \right) \setminus \left(\bigsqcup_{n \leqslant M \leqslant m-1} \bigsqcup_{0 \leqslant i \leqslant q'_{n-1}-1} B'_{n-1,i}(M) \right)$$

and

$$B'_{n-2,0}(m).$$

The first one is the union of the $r_{n,m-1} - q'_{n-1}t_{n,m-1}$ $(m-1)$-levels which are not chosen at step $(n, m-1)$, and the second one is built at step $(n-1, m)$ from its $t_{n-1,m}$ m-bricks. So we have

$$r_{n,m} = q_{m-1}(r_{n,m-1} - q'_{n-1}t_{n,m-1}) + t_{n-1,m}.$$

We choose $q'_{n-1}t_{n,m}$ of these m-levels as m-bricks at this step, where $q'_{n-1}t_{n,m}$ is the largest multiple of q'_{n-1} such that $q'_{n-1}t_{n,m} < r_{n,m}$, i.e. $t_{n,m} \coloneqq \lfloor (r_{n,m} - 1)/q'_{n-1} \rfloor$. □

It will be more convenient to use the following slight modification of Lemma I.5.8:

$$t_{0,1} = 0;$$

$$\text{for } m \geqslant 2, \qquad t_{0,m} = \sigma_{m-1};$$

$$\text{for } m = n \geqslant 1, \quad \begin{cases} r_{n,n} = q_{n-1} + t_{n-1,n}, \\ q'_{n-1} \leqslant r_{n,n} - 1, \\ t_{n,n} = 1; \end{cases} \tag{I.8}$$

$$\text{for } m > n \geqslant 1, \quad \begin{cases} r_{n,m} \leqslant q_{m-1} q'_{n-1} + t_{n-1,m}, \\ t_{n,m} \leqslant \dfrac{r_{n,m} - 1}{q'_{n-1}}. \end{cases}$$

This is a consequence of the inequalities $\lfloor x \rfloor \leqslant x$ and $r_{n,m-1} - q'_{n-1} t_{n,m-1} \leqslant q'_{n-1}$ (by the definition of $t_{n,m-1}$).

As the strategy will be to recursively choose large enough cutting parameters q_n for T, we would like to understand the asymptotic behaviour of q'_n as q_n increases. Then the goal is to find bounds for q'_n/q_n.

Lemma I.5.9. *For every $n \geqslant 0$, we have*

$$q'_n \geqslant q_n - (1 + p_n).$$

Proof of Lemma I.5.9. Using the equalities $q'_n = \left\lfloor \frac{r_{n+1,n+1}-1}{p_n} \right\rfloor p_n$ and $r_{n+1,n+1} = q_n + t_{n,n+1}$ in Lemma I.5.8, where the integer $t_{n,n+1}$ is non-negative, we get

$$q'_n \geqslant \left(\frac{r_{n+1,n+1} - 1}{p_n} - 1 \right) p_n \geqslant q_n - 1 - p_n$$

and we are done. □

We have found a lower bound for q'_n/q_n (up to some additional term $-(1 + p_n)$). Let us find an upper bound.

Lemma I.5.10. *For every $n \geqslant 1$, we have*

$$q'_n \leqslant 3q_n + \frac{\sigma_n}{q'_0 \cdots q'_{n-1}}.$$

With an asymptotic control on σ_n, using flexible classes, we will be able to get $q'_n \leqslant 4q_n$ (see Lemma I.5.14).

Proof of Lemma I.5.10. By induction over $i \in [\![0, n-1]\!]$ (with $n \geqslant 1$) and using (I.8), we show that

$$q'_n \leqslant q_n \left(2 + \sum_{j=1}^{i} \prod_{k=1}^{j} \frac{1}{q'_{n-k}} \right) + t_{n-1-i,n+1} \prod_{k=1}^{i+1} \frac{1}{q'_{n-k}}.$$

For $i = 0$, we have $q'_n < r_{n+1,n+1} = q_n + t_{n,n+1}$ and

$$t_{n,n+1} \leqslant \frac{r_{n,n+1} - 1}{q'_{n-1}} \leqslant \frac{1}{q'_{n-1}} \left(q_n q'_{n-1} + t_{n-1,n+1} \right) = q_n + \frac{t_{n-1,n+1}}{q'_{n-1}},$$

so we get $q'_n \leqslant 2q_n + \frac{t_{n-1,n+1}}{q'_{n-1}}$. For $0 \leqslant i \leqslant n-2$, we have

$$t_{n-1-i,n+1} \leqslant \frac{r_{n-1-i,n+1} - 1}{q'_{n-2-i}} \leqslant \frac{1}{q'_{n-2-i}} \left(q_n q'_{n-2-i} + t_{n-2-i,n+1} \right) = q_n + \frac{t_{n-2-i,n+1}}{q'_{n-2-i}}.$$

If the result holds true for i, we get

$$\begin{aligned} q'_n &\leqslant q_n\left(2+\sum_{j=1}^{i}\prod_{k=1}^{j}\frac{1}{q'_{n-k}}\right)+\left(q_n+\frac{t_{n-2-i,n+1}}{q'_{n-2-i}}\right)\prod_{k=1}^{i+1}\frac{1}{q'_{n-k}} \\ &= q_n\left(2+\sum_{j=1}^{i+1}\prod_{k=1}^{j}\frac{1}{q'_{n-k}}\right)+t_{n-1-(i+1),n}\prod_{k=1}^{i+2}\frac{1}{q'_{n-k}}, \end{aligned}$$

so the result is also true for $i+1$.

Taking $i=n-1$, this gives the lemma since $q'_\ell \geqslant 2$ for every integer $\ell \geqslant 1$. □

I.5.c Towards flexible classes

We now explain why flexible classes fit in this construction.

First a condition for the construction to be well-defined needs an inductive choice of the cutting parameters $(q_n)_{n\geqslant 0}$ of T (see Lemma I.5.11). Secondly, a control on the spacing parameters will imply useful asymptotic controls for the quantification of the cocycles (see Lemma I.5.14). Note that, in the proof of Theorem I.3.9 (see Section I.5.e), we will need other estimates to quantify the cocycles. It will be possible, again using the definition of a flexible class, to inductively build large enough cutting parameters in order to have these estimates.

If the parameters are chosen according to a set $\mathcal{F}_{\mathcal{C}} \subseteq \mathcal{P}^*$ associated to a flexible class $\mathcal{C}$, the underlying rank-one system has the desired property, i.e. it is in $\mathcal{C}$, and is orbit equivalent to the universal odometer, with some quantification guaranteed by the control of the spacing parameters and by the fact that the cutting parameters q_n have been recursively chosen and large enough.

Lemma I.5.11. *Let T be a rank-one system with cutting and spacing parameters*

$$(q_n,(\sigma_{n,0},\dots,\sigma_{n,q_n}))_{n\geqslant 0}$$

such that the construction in Section I.5.a is well-defined. Then, for every $n\in\mathbb{N}$, q'_n only depends on $(q_k,(\sigma_{k,0},\dots,\sigma_{k,q_k}))_{0\leqslant k\leqslant n}$.

Proof of Lemma I.5.11. This directly follows from Lemma I.5.8. □

Then the main novelty in this paper is to build the rank-one system T while we are building the universal odometer S. Once $(q'_0,\dots,q'_n)$ has been built from $(q_k,(\sigma_{k,0},\dots,\sigma_{k,q_k}))_{0\leqslant k\leqslant n}$, we are free to choose $(q_{n+1},(\sigma_{n+1,0},\dots,\sigma_{n+1,q_{n+1}}))$ for the definition of q'_{n+1}. The recursive definition of the cutting parameters is one of the main ideas behind the definition of a flexible class, and it allows to find cutting parameters satisfying some assumptions, for example the assumptions of the following lemma.

Lemma I.5.12. *Assume that for every $n\in\mathbb{N}$,*

$$q_n > \max(p_n, q'_0,\dots,q'_{n-1}). \tag{I.9}$$

Then the construction is well-defined, i.e. all the "largest multiples" are non-zero (that is, the largest multiple q'_{n-1} of p_{n-1} such that $q'_{n-1}<r_{n,n}$, and the largest multiple $q'_{n-1}t_{n,m}$ of q'_{n-1} such that $q'_{n-1}t_{n,m}<r_{n,m}$).

Remark I.5.13. Without loss of generality, we can assume that p_0 is equal to 2. Therefore the assumption of Lemma I.5.12 for $n=0$ requires that q_0 is greater than 2, which explains the second item of Definition I.3.7.

Proof of Lemma I.5.12. First, let us prove this result at step $n = 1$ of the outer recursion. At step $m = 1$ of the inner recursion, q_0 is greater than p_0, so q'_0 (the largest multiple of p_0 such that $q'_0 \leqslant q_0 - 1$) is positive. For a step $m > 1$, notice that there exists an $(m-1)$-level which is not chosen at the previous step (as we have $r_{1,m-1}$ $(m-1)$-levels in $W_{1,m-1}$ and we choose $q'_0 t_{1,m-1}$ of them, with $q'_0 t_{1,m-1} < r_{1,m-1}$) so its q_{m-1} m-levels are in $W_{1,m}$ and this gives $r_{1,m} \geqslant q_{m-1}$. Therefore we have $r_{1,m} > q'_0$ and $t_{1,m}$ is non-zero.

Now consider a step $n > 1$ of the outer recursion. For $m = n$, $B'_{n-2,0}(n-1)$ is an $(n-1)$-level in $W_{n,n}$, so we have $r_{n,n} \geqslant q_{n-1} > p_{n-1}$, hence the positivity of q'_{n-1}. For $m > n$, we have $r_{n,m} \geqslant q_{m-1}$ (same argument as for $n = 1$), this implies $r_{n,m} > q'_{n-1}$ and $t_{n,m}$ is positive. □

The next lemma refines the estimate given by Lemma I.5.10, with assumptions which will be satisfied in the context of flexible classes.

Lemma I.5.14. *Let $(q_n, (\sigma_{n,0}, \ldots, \sigma_{n,q_n}))_{n\geqslant 0}$ be the parameters of a CSP construction of T with associated constant $C > 0$. Assume that there exists a constant $C' > 0$ such that*

$$\forall n \geqslant 1,\ \sigma_n \leqslant C' q_n h_{n-1} \text{ and } q_n - (1 + p_n) \geqslant C' h_n$$

(for instance, if the third point of Definition I.3.7 holds and if, given $(q_k, (\sigma_{0,k}, \ldots, \sigma_{q_k,k}))_{0\leqslant k\leqslant n-1}$, q_n is chosen large enough). Then we get the following bound:

$$\forall n \in \mathbb{N},\ \frac{q'_n}{q_n} \leqslant 4.$$

Proof of Lemma I.5.14. For $n = 0$, this is a consequence of the inequality $q'_0 \leqslant q_0 - 1$. Now let us prove the result for $n \geqslant 1$. Using Lemma I.5.10, it suffices to get

$$\forall n \geqslant 1,\ \frac{\sigma_n}{q'_0 \cdots q'_{n-1}} \leqslant q_n.$$

But we have

$$\sigma_n \leqslant C' q_n h_{n-1} \leqslant q_n \left(q_{n-1} - (1 + p_{n-1})\right),$$

and the right hand side is bounded above by $q_n q'_{n-1}$ (by Lemma I.5.9), so the result follows from the inequality $q'_{n-1} \leqslant q'_0 \ldots q'_{n-1}$. □

I.5.d Equality of the orbits, universal odometer and quantitative control of the cocycles

Recall the notations for the construction of T by cutting and stacking, $(q_n)_n$ and $(\sigma_{n,i})_{n,i}$ are respectively the cutting and spacing parameters. The tower $\mathcal{R}_n$ is the n-th T-Rokhlin tower, its height is h_n, it covers the subset X_n of X, ε_n is the measure of its complement, Z_n is the maximum of the spacing parameters over the first n steps and M_0 is the measure of the unique 0-level B_0.

We use similar notations q'_n, h'_n and $\mathcal{R}'_n$ for S. We also set

$$H'_n \coloneqq h'_1 + \ldots + h'_n$$

for all $n \geqslant 1$, and $H'_0 \coloneqq 0$.

The construction is assumed to be well-defined, considering a cutting-and-stacking definition of T with parameters satisfying the criterion (I.9) (see Lemma I.5.12). Since S is piecewise given by powers of T, the S-orbits are included in the T-orbits. It remains to show the reverse inclusion, to prove that $(\mathcal{R}'_n)_{n\geqslant 0}$ is increasing to the σ-algebra $\mathcal{A}$ and to quantify the cocycles.

As in [KL24], we set

$$E_{n,m} \coloneqq \bigsqcup_{i=0}^{h'_n-1} S^i\left(\bigsqcup_{n\leqslant M\leqslant m} B'_{n-1,0}(M)\right) = \bigsqcup_{\substack{0\leqslant i_0\leqslant q'_0-1\\ \vdots \\ 0\leqslant i_{n-1}\leqslant q'_{n-1}-1}} \zeta_1^{i_0}\dots\zeta_n^{i_{n-1}}\left(\bigsqcup_{n\leqslant M\leqslant m} B'_{n-1,0}(M)\right)$$

and

$$K_n \coloneqq \bigsqcup_{\substack{0<i\leqslant h_n-1\\ T^{i-1}(B_n)\sqcup T^i(B_n)\subseteq E_{n,n}}} T^i(B_n).$$

Since $B'_{n-1,0}$ is exactly the base B'_n of $\mathcal{R}'_n$, the subsets $S^i(B'_{n-1,0})$, for $0\leqslant i\leqslant h'_n-1$, are exactly the levels of $\mathcal{R}'_n$ which is a partition of X. So the motivation behind the definition of $E_{n,m}$ is first to approximate $B'_{n-1,0}$ by its M-bricks for $n\leqslant M\leqslant m$, and then the set $E_{n,m}$ is actually the union of the M-bricks, for $n\leqslant M\leqslant m$, of step n of the outer recursion, and their translates by S in the other levels in $\mathcal{R}'_n$ (the sets $E_{1,1}$, $E_{1,2}$, $E_{1,3}$, $E_{2,1}$ and $E_{2,2}$ are illustrated in Figures I.7 and I.8). We get a better approximation of X as m increases and notice that $E_{n,m}$ is a subset of X_m since every M-brick, for $n\leqslant M\leqslant m$, is a union of m-levels. Finally the sets K_n, for $n\geqslant 1$, are introduced in order to show that the system S captures the T-orbits (recall Remark I.5.3).

Lemma I.5.15. *The following holds:*

$$\mu\left(X_m\backslash E_{n,m}\right)\leqslant\begin{cases}\dfrac{H'_n}{h_m} & \text{for } n<m\\[2ex] \dfrac{H'_{n-1}+p_{n-1}h'_{n-1}}{h_n} & \text{for } n=m\end{cases}.$$

Proof of Lemma I.5.15. We prove the inclusions

$$E_{n,m}\subset E_{n-1,m}\subset\ldots\subset E_{2,m}\subset E_{1,m}\subset X_m$$

and we bound the measures of $X_m\backslash E_{1,m}$ and each set $E_{k,m}\backslash E_{k-1,m}$. The result follows from the decomposition

$$X_m\backslash E_{n,m} = \left(X_m\backslash E_{1,m}\right)\sqcup\bigsqcup_{2\leqslant k\leqslant n}\left(E_{k-1,m}\backslash E_{k,m}\right) \tag{I.10}$$

and σ-additivity of μ.

The set $E_{1,m}$ is composed of m-levels, so it is contained in X_m. If $m=1$, then $X_m\backslash E_{1,m}$ is the disjoint union of $r_{1,1}-q'_0$ 1-levels (see step $(1,1)$ of the construction). If $m>1$, then $X_m\backslash E_{1,m}$ is the disjoint union of $r_{1,m}-q'_0t_{1,m}$ m-levels (see step $(1,m)$ of the construction). By definition of q'_0 (if $m=1$) or $t_{1,m}$ (if $m>1$), we thus have

$$\mu\left(X_m\backslash E_{1,m}\right)\leqslant\begin{cases}\dfrac{p_0}{h_m} & \text{if } m=1\\[2ex] \dfrac{h'_1}{h_m} & \text{if } m>1\end{cases}$$

(recall that $h'_1=q'_0$).

Let $k\in[\![2,n]\!]$. The function ζ_k has been built in order to map each M-brick ($M\geqslant k$) at step k to another. But such a brick is contained in an M'-brick ($k-1\leqslant M'\leqslant M$) from the previous step $k-1$ (see Lemma I.5.7). We then have

$$\bigsqcup_{0\leqslant i_{k-1}\leqslant q'_{k-1}-1}\zeta_k^{i_{k-1}}\left(\bigsqcup_{k\leqslant M\leqslant m} B'_{k-1,0}(M)\right)\subseteq\bigsqcup_{k-1\leqslant M\leqslant m} B'_{k-2,0}(M).$$

Applying $\zeta_1^{i_0}\dots\zeta_{k-1}^{i_{k-2}}$ and considering the union over $i_0,\dots,i_{k-2}$, we get the inclusion $E_{k,m}\subseteq E_{k-1,m}$ and the equality

$$E_{k-1,m}\backslash E_{k,m} = \bigsqcup_{\substack{0\leqslant i_0\leqslant q'_0-1\\ \vdots \\ 0\leqslant i_{k-2}\leqslant q'_{k-2}-1}} \zeta_1^{i_0}\dots\zeta_{k-1}^{i_{k-2}}\left(\underbrace{\left(\bigsqcup_{k-1\leqslant M\leqslant m} B'_{k-2,0}(M)\right)\setminus\left(\bigsqcup_{k\leqslant M\leqslant m}\bigsqcup_{0\leqslant i_{k-1}\leqslant q'_{k-1}-1} B'_{k-1,i_k}(M)\right)}_{=:\ [*]}\right).$$

So the measure of $E_{k-1,m}\backslash E_{k,m}$ is $q'_0\dots q'_{k-2}\mu([*]) = h'_{k-1}\mu([*])$ by T-invariance. The set $[*]$ is obtained from $W_{k,m}$ (see (I.5) and (I.6)) by removing the m-bricks that have been chosen at step (k,m). If $m=k$, then $[*]$ is the disjoint union of $r_{k,k}-q'_{k-1}$ m-levels (see step (k,k) of the construction). If $m>k$, then $[*]$ is the disjoint union of $r_{k,m}-q'_{k-1}t_{k,m}$ m-levels (see step (k,m) of the construction). By definition of q'_{k-1} (if $m=k$) or $t_{k,m}$ (if $m>k$), we thus have

$$\mu([*]) \leqslant \begin{cases} \dfrac{p_{k-1}}{h_m} & \text{if } m=k \\ \\ \dfrac{q'_{k-1}}{h_m} & \text{if } m>k \end{cases}$$

and

$$\mu(E_{k-1,m}\backslash E_{k,m}) \leqslant \begin{cases} \dfrac{h'_{k-1}p_{k-1}}{h_m} & \text{if } m=k \\ \\ \dfrac{h'_k}{h_m} & \text{if } m>k \end{cases}.$$

Using (I.10) and σ-additivity of μ, we get the following inequalities. If $m>n$, we get

$$\mu(X_m\backslash E_{n,m}) = \mu(X_m\backslash E_{1,m}) + \sum_{2\leqslant k\leqslant n}\mu(E_{k-1,m}\backslash E_{k,m}) \leqslant \sum_{1\leqslant k\leqslant n}\frac{h'_k}{h_m} = \frac{H'_n}{h_m}.$$

If $m=n$, we get

$$\begin{aligned}\mu(X_m\backslash E_{n,m}) &= \left(\mu(X_m\backslash E_{1,m}) + \sum_{2\leqslant k\leqslant m-1}\mu(E_{k-1,m}\backslash E_{k,m})\right) + \mu(E_{n-1,n}\backslash E_{n,n}) \\ &\leqslant \sum_{1\leqslant k\leqslant m-1}\frac{h'_k}{h_m} + \frac{p_{n-1}h'_{n-1}}{h_n} \\ &= \frac{H'_{n-1}}{h_n} + \frac{p_{n-1}h'_{n-1}}{h_m}\end{aligned}$$

and we are done. □

The quantity $H'_{n-1}+p_{n-1}h'_{n-1}$ only depends on $q'_1,\dots,q'_{n-2}$ which only depend on $(q_i,(\sigma_{i,j})_{0\leqslant j\leqslant q_i})_{0\leqslant i\leqslant n-2}$ (see Lemma I.5.11), and h_n is larger than $q_1\dots q_{n-1}/M_0$ with q_{n-1} appearing at step $n-1$. Then the strategy will be to recursively choose the cutting parameters q_{n-1} so that

$$\frac{H'_{n-1}+p_{n-1}h'_{n-1}}{h_n} \underset{n\to+\infty}{\to} 0. \tag{I.11}$$

As $\mu(X_n)\underset{n\to+\infty}{\to}1$, this gives $\mu(E_{n,n})\underset{n\to+\infty}{\to}1$ by Lemma I.5.15.

Corollary I.5.16. *If* $\mu(E_{n,n}) \underset{n\to+\infty}{\to} 1$, *then* S *is the universal odometer.*

Proof of Corollary I.5.16. By the definition of q'_n at step (n,n) and by choice of the sequence (p_n), every prime number appears infinitely many time as a prime factor among the integers $q'_0, q'_1, q'_2, \ldots$. If S is an odometer, then it is clearly universal. It remains to show that $(\mathcal{R}'_n)_{n\in\mathbb{N}}$ increases to the σ-algebra $\mathcal{A}$. Then S is a rank-one system with zero spacing parameters by Lemma I.5.4, so this is an odometer.

Consider a subsequence $(n_k)_{k\geqslant 0}$ such that the series $\sum_{k\geqslant 0}\mu((E_{n_k,n_k})^c)$ is convergent. By the Borel-Cantelli lemma, the set $X_0 \coloneqq \bigcup_{j\geqslant 0}\bigcap_{k\geqslant j} E_{n_k,n_k}$ is of full measure. Let $x, y \in X_0$. Assume that they belong to the same level of $\mathcal{R}'_n$ for every n larger that some threshold N_0. The goal is to show that x and y are equal, so that $(\mathcal{R}'_n)_{n\in\mathbb{N}}$ separates the points of a set of full measure and hence it increases to $\mathcal{A}$.

By the definition of X_0, there exists an infinite subset I of $\mathbb{N}$, bounded below by N_0, such that $E_{n,n}$ contains x and y for every $n \in I$. Let us fix an integer $n \in I$. By the definition of $E_{n,n}$, x is in some $S^i(B'_{n-1,0}(n))$ and y in some $S^j(B'_{n-1,0}(n))$, for $0 \leqslant i, j \leqslant q'_0 \ldots q'_{n-1} - 1$. But x and y are in the same level of $\mathcal{R}'_n$, furthermore $S^i(B'_{n-1,0}(n))$ is included in the level $S^i(B'_n)$ and $S^j(B'_{n-1,0}(n))$ in the level $S^j(B'_n)$, so we have $i = j$. Moreover, since we have $t_{n,n} = 1$, all the sets $S^k(B'_{n-1,0}(n))$ are n-levels, i.e. levels of the n-th T-Rokhlin tower $\mathcal{R}_n$, so x and y are in the same n-level. This holds for every $n \in I$, so for infinitely many n. Moreover $(\mathcal{R}_n)_{n\in\mathbb{N}}$ separates the points up to a null set, since T is rank-one, hence the result. □

Lemma I.5.17. *For every* $n \in \mathbb{N}$, *we have*

$$\mu(K_n) \geqslant \mu(X_n) - \mu\left(B_n\right) - 2\mu\left(X_n \backslash E_{n,n}\right).$$

Moreover, $\mu\left(K_n\right) \underset{n\to+\infty}{\to} 1$ *if* $\mu(E_{n,n}) \underset{n\to+\infty}{\to} 1$.

Proof of Lemma I.5.17. The set K_n is equal to $\left(E_{n,n}\backslash B_n\right) \setminus T\left(X_n \backslash E_{n,n}\right)$, so we get

$$\begin{aligned}
\mu\left(K_n\right) &\geqslant \mu\left(E_{n,n}\backslash B_n\right) - \mu\left(T\left(X_n\backslash E_{n,n}\right)\right) \\
&\geqslant \mu\left(E_{n,n}\right) - \mu\left(B_n\right) - \mu\left(X_n\backslash E_{n,n}\right) \\
&= \mu(X_n) - \mu\left(B_n\right) - 2\mu\left(X_n\backslash E_{n,n}\right).
\end{aligned}$$

The second result follows from the fact that $\mu(X_n) \underset{n\to+\infty}{\to} 1$ and $\mu(B_n) \underset{n\to+\infty}{\to} 0$. □

Lemma I.5.18. *For every* $x \in K_n$, *there exists* $k \in \mathbb{Z}$ *such that*

$$|k| \leqslant 4(h_{n-1} + Z_{n-1})(h'_{n-1})^2$$

and $T^{-1}x = S^k x$.

Proof of Lemma I.5.18. Let $x \in K_n$. By the definition of K_n, the points x and $T^{-1}x$ are in $E_{n,n}$ and there exists $1 \leqslant i \leqslant h_n - 1$ such that $x \in T^i(B_n)$. Writing $E_{n,n}$ this way:

$$E_{n,n} = \bigsqcup_{\substack{0\leqslant i\leqslant h'_{n-1}-1 \\ 0\leqslant i_{n-1}\leqslant q'_{n-1}-1}} S^i\zeta_n^{i_{n-1}}\left(B'_{n-1,0}(n)\right) = \bigsqcup_{\substack{0\leqslant i\leqslant h'_{n-1}-1 \\ 0\leqslant i_{n-1}\leqslant q'_{n-1}-1}} S^i\left(B'_{n-1,i_{n-1}}(n)\right),$$

it is clear that there exist $0 \leqslant k_0, k_1 \leqslant h'_{n-1} - 1$ such that $y \coloneqq S^{-k_0}x$ and $z \coloneqq S^{-k_1}T^{-1}x$ are in $\bigsqcup_{0\leqslant i_{n-1}\leqslant q'_{n-1}-1} B'_{n-1,i_{n-1}}(n)$.

We first show that we can write $y = \zeta_n^{k_2} z$ for some k_2, using the fact that ζ_n connects the n-bricks of step (n,n) of the construction (since $t_{n,n} = 1$). Secondly ζ_n can be written as a power of S and the equality $y = S^{k_3}z$ holds for some k_3 that we will be able to

bound by Lemma I.5.6. Finally the result follows from the bound for each integer k_0, k_1, k_3.

Step 1: Finding k_2 such that $y = \zeta_n^{k_2} z$. Using Lemma I.5.5, we can write

$$x = \zeta_1^{i_0} \dots \zeta_{n-1}^{i_{n-2}} y \text{ and } T^{-1}x = \zeta_1^{j_0} \dots \zeta_{n-1}^{j_{n-2}} z$$

for some integers $0 \leqslant i_0, j_0 \leqslant q'_0 - 1, \dots, 0 \leqslant i_{n-2}, j_{n-2} \leqslant q'_{n-2} - 1$, and there exist $0 \leqslant i_{n-1}, j_{n-1} \leqslant q'_{n-1} - 1$ such that

$$y \in B'_{n-1,i_{n-1}}(n) \text{ and } z \in B'_{n-1,j_{n-1}}(n).$$

More precisely, by Lemma I.5.7 and the fact that y and z are in n-bricks at step (n, n), we have

$$x = \zeta_1(M_1)^{i_0} \dots \zeta_{n-1}(M_{n-1})^{i_{n-2}} y \text{ and } T^{-1}x = \zeta_1(L_1)^{j_0} \dots \zeta_{n-1}(L_{n-1})^{j_{n-2}} z$$

with $k \leqslant L_k, M_k \leqslant n$ for every $1 \leqslant k \leqslant n-1$. By construction, T and the maps $\zeta_k(m)$, for $1 \leqslant k \leqslant n-1$ and $k \leqslant m \leqslant n$, satisfy the following property: for every n-level $T^k(B_n)$, with $0 \leqslant k \leqslant h_n - 1$, contained in the domain of the map, if it is mapped to another n-level $T^{k+\ell}(B_n)$, with $0 \leqslant k + \ell \leqslant h_n - 1$, then the application coincides with T^ℓ on $T^k(B_n)$. In other word it consists in going up or down $|\ell|$ floors in the tower $\mathcal{R}_n$, without going above its roof or below its base. Therefore, from $B'_{n-1,i_{n-1}}(n)$ to $B'_{n-1,j_{n-1}}(n)$, the map

$$\tilde{S} \coloneqq \left(\zeta_1(L_1)^{j_0} \dots \zeta_{n-1}(L_{n-1})^{j_{n-2}}\right)^{-1} T^{-1} \zeta_1(M_1)^{i_0} \dots \zeta_{n-1}(M_{n-1})^{i_{n-2}}$$

consists in successively going up or down in the tower, so this is a power of T given by the difference between the floor of $B'_{n-1,i_{n-1}}(n)$ and the one of $B'_{n-1,j_{n-1}}(n)$. The map $\zeta_n^{j_{n-1}-i_{n-1}}$ also satisfies this property, thus $\zeta_n^{j_{n-1}-i_{n-1}}$ and $\tilde{S}$ coincide on $B'_{n-1,i_{n-1}}(n)$ and $y = \zeta_n^{k_2} z$ with $k_2 \coloneqq j_{n-1} - i_{n-1}$.

Step 2: Finding k_3 such that $y = S^{k_3} z$. Using the Lemma I.5.5 and the equality $\zeta_n^i(B'_n) = B'_{n-1,i}$, we have $S^{h'_{n-1}(j_{n-1}-i_{n-1})} y = z$, we set $k_3 \coloneqq h'_{n-1}(j_{n-1} - i_{n-1})$ and it remains to find a bound for $j_{n-1} - i_{n-1}$. We need to get more information on the power of T, denoted by T^ℓ, which coincides with $\tilde{S}$ on $B'_{n-1,i_{n-1}}(n)$. By Lemma I.5.6 and the definition of $\tilde{S}$, we get

$$\begin{aligned} |\ell| &\leqslant (h_{n-1} + Z_{n-1})(i_0 + \dots + i_{n-2}) + 1 + (h_{n-1} + Z_{n-1})(j_0 + \dots + j_{n-2}) \\ &\leqslant 2(h_{n-1} + Z_{n-1})(q'_0 + \dots + q'_{n-2}) + 1 \\ &\leqslant 3(h_{n-1} + Z_{n-1})(q'_0 + \dots + q'_{n-2}) \end{aligned}$$

where "+1" comes from "T^{-1}" in the expression of $\tilde{S}$ and has been bounded by $(h_{n-1} + Z_{n-1})(q'_0 + \dots + q'_{n-2})$. The sum $q'_0 + \dots + q'_{n-2}$ is less than the product $q'_0 \dots q'_{n-2} = h'_{n-1}$, this gives

$$|\ell| \leqslant 3(h_{n-1} + Z_{n-1})h'_{n-1}.$$

Since ζ_n has a positive cocycle (by Lemma I.5.6), the equality $\zeta_n^{(j_{n-1}-i_{n-1})} = T^\ell$ implies $|\ell| \geqslant |j_{n-1} - i_{n-1}|$. Therefore we find the bound

$$|k_3| \leqslant 3(h_{n-1} + Z_{n-1})(h'_{n-1})^2.$$

Step 3: Bounding the integer k such that $T^{-1}x = S^k x$. By the definition of k_0, k_1 and k_3, $T^{-1}x$ is equal to $S^k x$ with $k \coloneqq k_1 - k_3 - k_0$ which is thus bounded as follows:

$$\begin{aligned} |k| &\leqslant |k_0| + |k_1| + |k_3| \\ &\leqslant 2(h'_{n-1} - 1) + 3(h_{n-1} + Z_{n-1})(h'_{n-1})^2 \\ &\leqslant 4(h_{n-1} + Z_{n-1})(h'_{n-1})^2, \end{aligned}$$

hence the result. □

Corollary I.5.19. *If $\mu(E_{n,n}) \underset{n\to+\infty}{\to} 1$, then T and S have the same orbits.*

Proof of Corollary I.5.19. It is clear that the S-orbits are contained in the T-orbits. By Lemma I.5.17, $\bigcup_{n\in\mathbb{N}} K_n$ is of full measure, so the reverse inclusion follows from Lemma I.5.18. □

Remark I.5.20. Corollary I.5.19 holds for every rank-one system T. Indeed skipping steps in the cutting-and-stacking process of T recursively increases the cutting parameters q_n, it enables us to get criteria (I.9) and (I.11) (the first one implies that the construction in Section I.5.a is well-defined, the second one that $\mu(E_{n,n}) \to 1$).

However the quantification of the cocycles will not necessarily hold for all the rank-one systems, since we will need to control the quantities Z_n depending on the spacing parameters (see Section I.5.e).

Note that by Dye's theorem, it was already known that every rank-one system is orbit equivalent to the universal odometer, but the proof of this theorem does not provide an explicit orbit equivalence, thus preventing us from quantifying the cocycles.

Now the goal is to control the cocycle c_S. The equalities (I.7) in Section I.5.b and the decomposition of $B_{n-1,i}$ in bricks motivate the following definition:

$$\begin{aligned} \forall m \geqslant n \geqslant 1, \quad D_n(m) &:= \zeta_1^{q_0'-1} \dots \zeta_{n-1}^{q_{n-2}'-1} \left(\bigsqcup_{0\leqslant i_n \leqslant q_{n-1}'-2} B'_{n-1,i_n}(m) \right) \\ &= \zeta_1^{q_0'-1} \dots \zeta_{n-1}^{q_{n-2}'-1} \left(\bigsqcup_{0\leqslant i_n \leqslant q_{n-1}'-2} \zeta_n^{i_n}(m)(B'_{n-1,0}) \right). \end{aligned} \tag{I.12}$$

It is the union of all the translates of the m-bricks at step (n,m) composing D_n. Note that S coincides with $\zeta_n(m)\zeta_{n-1}^{-(q_{n-2}'-1)} \dots \zeta_1^{-(q_0'-1)}$ on $D_n(m)$ (since it coincides with $\zeta_n\zeta_{n-1}^{-(q_{n-2}'-1)} \dots \zeta_1^{-(q_0'-1)}$ and ζ_n coincides with $\zeta_n(m)$ on the m-bricks at step n). The partition of D_n into such subsets $D_n(m)$, for $m \geqslant n$, gives a fine control of the cocycle c_S.

Lemma I.5.21. *For $1 \leqslant n < m$, $D_n(m)$ is contained in $X_m \backslash E_{n,m-1}$ and we have*

$$\mu(D_n(m)) \leqslant \begin{cases} \varepsilon_{m-1} - \varepsilon_m + \dfrac{H_n'}{h_{m-1}} & \text{if } m > n+1 \\ \varepsilon_{m-1} - \varepsilon_m + \dfrac{H_{n-1}' + p_{n-1}h_{n-1}'}{h_n} & \text{if } m = n+1 \end{cases}.$$

For all $n \geqslant 1$, we have

$$\mu(D_n(n)) \leqslant \frac{q_{n-1}'}{h_n}.$$

Moreover for every $x \in D_n(m)$,

$$|c_S(x)| \leqslant (h_{m-1} + Z_{m-1})h_{n-1}'.$$

Proof of Lemma I.5.21. For $1 \leqslant n < m$, $D_n(m)$ is composed of translates of the m-bricks used at step (n,m), so it is disjoint from the translates of the M-bricks used at step (n,M) for $n \leqslant M \leqslant m-1$, hence the inclusion $D_{n,m} \subseteq X_m \backslash E_{n,m-1}$. The bound for $\mu(D_n(m))$ follows from the decomposition $X_m \backslash E_{n,m-1} = (X_m \backslash X_{m-1}) \sqcup (X_{m-1} \backslash E_{n,m-1})$ and Lemma I.5.15.

For $n \geqslant 1$, by the definition of $D_n(n)$ and the ζ_i-invariance of the measure, we get

$$\mu(D_n(n)) = (q_{n-1}' - 1)\mu\left(B'_{n-1,0}(n)\right) \leqslant q_{n-1}'\mu\left(B'_{n-1,0}(n)\right),$$

hence the result, since $B'_{n-1,0}(n)$ is an n-level, so it has measure less than $1/h_n$.

For the cocycle c_S, we first decompose $D_n(m)$ in the following way:

$$D_n(m) = \bigsqcup_\ell \underbrace{\zeta_1^{q_0'-1} \dots \zeta_{n-1}^{q_{n-2}'-1}(\beta_\ell))}_{=:D^\ell}$$

where $(\beta_\ell)_\ell$ is the family of m-bricks, at step (n,m), which constitute the subset $\bigsqcup_{0\leqslant i_n\leqslant q_{n-1}'-2} B'_{n-1,i_n}(m)$. For a fixed ℓ, by Lemma I.5.7 there exist $1 \leqslant L_1 \leqslant m, \dots, n-1 \leqslant L_{n-1} \leqslant m$ such that

$$D^\ell = \zeta_1^{q_0'-1}(L_1) \dots \zeta_{n-1}^{q_{n-2}'-1}(L_{n-1})\,(\beta_\ell)$$

and, on this subset, S coincides with $\zeta_n(m)\zeta_{n-1}^{-(q_{n-2}'-1)}(L_{n-1}) \dots \zeta_1^{-(q_0'-1)}(L_1)$. Then using Lemma I.5.6, we get

$$\begin{aligned} |(c_S)_{|D^\ell}| &\leqslant (h_{m-1}+Z_{m-1})((q_0'-1)+\dots+(q_{n-2}'-1)+1) \\ &\leqslant (h_{m-1}+Z_{m-1})q_0' \dots q_{n-2}' \\ &= (h_{m-1}+Z_{m-1})h_{n-1}', \end{aligned}$$

hence the result. □

I.5.e Proof of Theorem I.3.9

Let T be a rank-one system whose parameters satisfy the criteria (I.9) and (I.11). The first one ensures that the construction is well-defined (Lemma I.5.12), the second one implies $\mu(E_{n,n}) \to 1$ (Lemma I.5.15), so we have an orbit equivalence between T and S (Lemma I.5.19). We can then define the cocycles $c_T, c_S \colon X \to \mathbb{Z}$ by

$$\forall x \in X,\ Tx = S^{c_T(x)}x \text{ and } Sx = T^{c_S(x)}x.$$

In Lemmas I.5.18 and I.5.21, we obtained bounds for the cocycles on precise subsets covering X: $(K_n)_n$ for c_T, $(D_n(m))_{n,m}$ for c_S. This will provide a bound for the φ-integral of each cocycle. But first, we need to change φ via the following lemma inspired by Lemma 2.12 in [CJLMT23]. Without loss of generality, φ has the properties given by the lemma and this will simplify the bound for each φ-integral.

Lemma I.5.22. *Let $0 < \alpha \leqslant 1$ and $\varphi \colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\varphi(t) = o(t^\alpha)$. Then there exists $\Phi \colon \mathbb{R}_+ \to \mathbb{R}_+$ with the following properties:*

- *Φ is increasing;*
- *Φ is subadditive: $\forall t, s \in \mathbb{R}_+,\ \Phi(t+s) \leqslant \Phi(t) + \Phi(s)$;*
- *$\Phi(t) = o\,(t^\alpha)$;*
- *$\varphi(t) = O\,(\Phi(t))$.*

Proof of Lemma I.5.22. Set

$$\begin{aligned} \theta \ :\ \mathbb{R}_+^* &\to \mathbb{R}_+ \\ t &\mapsto \min\left(1,\ \sup_{s\geqslant t} \frac{\varphi(s)+1}{s}\right) \end{aligned}$$

and

$$\begin{aligned} \Phi \ :\ \mathbb{R}_+ &\to \mathbb{R}_+ \\ t &\mapsto \int_0^t \theta(s)\mathrm{d}s \end{aligned}.$$

The map θ is positive-valued and non-increasing, so Φ is an increasing and subadditive function satisfying $\Phi(t) \geqslant t\theta(t)$ for every $t \in \mathbb{R}_+$. The assumption $\varphi(t) = o(t^\alpha)$ implies that $\theta(t) = \sup_{s\geqslant t} \frac{\varphi(s)+1}{s}$ for $t > 0$ large enough, so we have

$$\Phi(t) \geqslant t\theta(t) \geqslant t \sup_{s\geqslant t} \frac{\varphi(s)+1}{s} \geqslant \varphi(t).$$

Finally, for a fixed $\varepsilon > 0$, there exists $t_0 > 0$ such that $\varphi(s) \leqslant \varepsilon s^\alpha$ for every $s \geqslant t_0$. For every $t \geqslant t_0$, this gives

$$\sup_{s\geqslant t} \frac{\varphi(s)+1}{s} \leqslant \sup_{s\geqslant t} \left(\frac{\varepsilon}{s^{1-\alpha}} + \frac{1}{s} \right) = \frac{\varepsilon}{t^{1-\alpha}} + \frac{1}{t}$$

and for every $t \geqslant t_0$, we have

$$\int_{t_0}^{t} \theta(s)\mathrm{d}s \leqslant \int_{t_0}^{t} \left(\frac{\varepsilon}{s^{1-\alpha}} + \frac{1}{s} \right) \mathrm{d}s = \frac{\varepsilon}{\alpha} t^\alpha + \ln t - \frac{\varepsilon}{\alpha} t_0^\alpha - \ln t_0,$$

hence $\Phi(t) = o(t^\alpha)$. □

Lemma I.5.23. *Assume that criteria* (I.9) *(in Lemma I.5.12) and* (I.11) *(after Lemma I.5.15) are satisfied. Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing and subadditive map. Then, setting*

$$\begin{aligned}
\Delta(n) &\coloneqq \left(1 + 2(H'_n + p_n h'_n)\right) (h'_n)^2 \left(\frac{\varphi(h_{n+1}^3)}{h_{n+1}} + \frac{\varphi(Z_{n+1} h_{n+1}^2)}{h_{n+1}} \right); \\
\Delta_\varepsilon(n) &\coloneqq \varepsilon_{n+1} (h'_n)^2 \left(\varphi(h_{n+1}^3) + \varphi(Z_{n+1} h_{n+1}^2) \right),
\end{aligned}$$

we have the following bound:

$$\int_X \varphi(|c_T(x)|)\mathrm{d}\mu \leqslant \varphi(4(h_0 + Z_0)(h'_0)^2) + 4 \sum_{n=0}^{+\infty} \Delta(n) + 4 \sum_{n=0}^{+\infty} \Delta_\varepsilon(n). \tag{I.13}$$

Proof of Lemma I.5.23. Motivated by Lemma I.5.18, we will rather quantify the cocycle $c_{T^{-1}}$ defined on X (up to a null set) by

$$T^{-1}x = S^{c_{T^{-1}}(x)} x.$$

It is equivalent to quantifying c_T since we have

$$\forall x \in X,\ c_{T^{-1}}(x) = -c_T(T^{-1}x)$$

and μ is T-invariant.

Let $(K'_n)_{n>0}$ be the partition of X inductively defined by

$$\begin{cases} K'_1 \coloneqq K_1, \\ \forall n > 0,\ K'_{n+1} \coloneqq K_{n+1} \backslash (K_1 \cup \ldots \cup K_n). \end{cases}$$

The subsets K'_n are pairwise disjoint and cover the whole space since we have

$$K'_1 \cup \ldots \cup K'_n = K_1 \cup \ldots \cup K_n$$

and $\mu(K_n) \to 1$ (using Lemma I.5.17). By the fact that K_n is included in X_n, and by Lemmas I.5.17 and I.5.15, we have

$$\begin{aligned}
\mu(K'_{n+1}) &\leqslant \mu(X \backslash K_n) \\
&= \mu(X \backslash X_n) + \mu(X_n \backslash K_n) \\
&\leqslant \varepsilon_n + \mu(B_n) + 2\mu(X_n \backslash E_{n,n}) \\
&\leqslant \varepsilon_n + \frac{1 + 2(H'_{n-1} + p_{n-1} h'_{n-1})}{h_n}.
\end{aligned}$$

Since K'_{n+1} is contained in K_{n+1}, Lemma I.5.18 implies

$$\forall x \in K'_{n+1},\ |c_{T^{-1}}(x)| \leqslant 4(h_n + Z_n)(h'_n)^2.$$

We then get

$$\begin{aligned}
\int_X \varphi(|c_T(x)|)\mathrm{d}\mu &= \int_X \varphi(|c_{T^{-1}}(x)|)\mathrm{d}\mu \\
&= \sum_{n=0}^{+\infty} \int_{K'_{n+1}} \varphi(|c_{T^{-1}}(x)|)\mathrm{d}\mu \\
&\leqslant \sum_{n=0}^{+\infty} \mu(K'_{n+1})\varphi(4(h_n + Z_n)(h'_n)^2) \\
&\leqslant \varphi(4(h_0 + Z_0)(h'_0)^2) \\
&\quad + \sum_{n=1}^{+\infty} \left(\varepsilon_n + \frac{1 + 2(H'_{n-1} + p_{n-1}h'_{n-1})}{h_n}\right) \varphi(4(h_n + Z_n)(h'_n)^2).
\end{aligned}$$

Now we use the assumptions on φ to simplify the previous bound. We have $h'_n = h'_{n-1}q'_{n-1} \leqslant h'_{n-1}h_n$ (by construction we have $q'_{n-1} \leqslant r_{n,n} \leqslant h_n$). By monotonicity and subadditivity, this yields

$$\begin{aligned}
\varphi(4(h_n + Z_n)(h'_n)^2) &\leqslant \varphi(4(h_n + Z_n)(h'_{n-1}h_n)^2) \\
&\leqslant 4(h'_{n-1})^2 \left(\varphi(h_n^3) + \varphi(Z_n h_n^2)\right)
\end{aligned}$$

and we get the bound (I.13). □

Lemma I.5.24. *Assume that criteria* (I.9) *(in Lemma I.5.12) and* (I.11) *(after Lemma I.5.15) are satisfied and that the following holds:*

$$\forall n \geqslant 0,\ \frac{q'_n}{q_n} \leqslant 4$$

(this is an assumption that we will be able to get by Lemma I.5.14, using flexible classes). Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing and subadditive map. Then, setting

$$\begin{aligned}
\Gamma_1(n) &\coloneqq 4h'_n \left(\frac{\varphi(h_{n+1}^2)}{h_{n+1}} + \frac{\varphi(Z_{n+1}h_{n+1})}{h_{n+1}}\right); \\
\Gamma_2(n) &\coloneqq \left(H'_n + p_n h'_n\right) h'_n \left(\frac{\varphi(h_{n+1})}{h_{n+1}} + \frac{\varphi(Z_{n+1})}{h_{n+1}}\right); \\
\Gamma_3(n,m) &\coloneqq H'_n h'_{n-1} \left(\frac{\varphi(h_m)}{h_m} + \frac{\varphi(Z_m)}{h_m}\right); \\
\Gamma_\varepsilon(n,m) &\coloneqq \varepsilon_m h'_n(\varphi(h_m) + \varphi(Z_m)),
\end{aligned}$$

we have the following bound:

$$\begin{aligned}
\int_X \varphi(|c_S|)\mathrm{d}\mu \leqslant\ & \mu(D_1(1))\varphi((h_0 + Z_0)h'_0) \\
& + \sum_{n\geqslant 0} \Gamma_1(n) + \sum_{n\geqslant 0} \Gamma_2(n) + \sum_{n\geqslant 1}\sum_{m\geqslant n+1} \Gamma_3(n,m) \\
& + \sum_{n\geqslant 0}\sum_{m\geqslant n+1} \Gamma_\varepsilon(n,m)
\end{aligned} \quad . \tag{I.14}$$

Proof of Lemma I.5.24. By Lemma I.5.21, for each subset $D_n(m)$, we found a bound for the cocycle c_S on it, we then get

$$\begin{aligned}\int_X \varphi(|c_S|)\mathrm{d}\mu &= \sum_{m\geqslant n\geqslant 1}\int_{D_n(m)}\varphi(|c_S|)\mathrm{d}\mu\\ &\leqslant \sum_{m\geqslant n\geqslant 1}\mu(D_n(m))\varphi((h_{m-1}+Z_{m-1})h'_{n-1})\\ &\leqslant \mu(D_1(1))\varphi((h_0+Z_0)h'_0)\\ &\quad+\sum_{n\geqslant 2}\gamma_1(n)+\sum_{n\geqslant 1}\gamma_2(n)+\sum_{n\geqslant 1}\sum_{m\geqslant n+2}\gamma_3(n,m)\end{aligned}$$

where

$$\gamma_1(n) \coloneqq \mu(D_n(n))\varphi((h_{n-1}+Z_{n-1})h'_{n-1}),$$

$$\gamma_2(n) \coloneqq \mu(D_n(n+1))\varphi((h_n+Z_n)h'_{n-1}),$$

$$\gamma_3(n,m) \coloneqq \mu(D_n(m))\varphi((h_{m-1}+Z_{m-1})h'_{n-1}).$$

Lemma I.5.21 also yields a bound for the measure of each set $D_n(m)$, this implies:

$$\gamma_1(n) \leqslant \frac{q'_{n-1}}{h_n}\varphi((h_{n-1}+Z_{n-1})h'_{n-1}),$$

$$\gamma_2(n) \leqslant \left(\varepsilon_n+\frac{H'_{n-1}+p_{n-1}h'_{n-1}}{h_n}\right)\varphi((h_n+Z_n)h'_{n-1}),$$

$$\gamma_3(n,m) \leqslant \left(\varepsilon_{m-1}+\frac{H'_n}{h_{m-1}}\right)\varphi((h_{m-1}+Z_{m-1})h'_{n-1}).$$

For all $n\geqslant 2$, note that we have

$$\begin{aligned}\varphi((h_{n-1}+Z_{n-1})h'_{n-1}) &\leqslant \varphi((h_{n-1}+Z_{n-1})h'_{n-2}h_{n-1})\\ &\leqslant h'_{n-2}\left(\varphi(h_{n-1}^2)+\varphi(Z_{n-1}h_{n-1})\right)\end{aligned}$$

and

$$\frac{q'_{n-1}}{h_n} \leqslant \frac{q'_{n-1}}{h_{n-1}q_{n-1}} \leqslant \frac{4}{h_{n-1}},$$

so wet get

$$\gamma_1(n) \leqslant 4h'_{n-2}\left(\frac{\varphi(h_{n-1}^2)}{h_{n-1}}+\frac{\varphi(Z_{n-1}h_{n-1})}{h_{n-1}}\right) = \Gamma_1(n-2).$$

For $\gamma_2(n)$ and $\gamma_3(n,m)$, note that we have

$$\forall n\geqslant 1, \forall m\geqslant n+1,\ \varphi((h_{m-1}+Z_{m-1})h'_{n-1}) \leqslant h'_{n-1}(\varphi(h_{m-1})+\varphi(Z_{m-1})),$$

so we get

$$\begin{aligned}\gamma_2(n) &\leqslant \left(\varepsilon_n+\frac{H'_{n-1}+p_{n-1}h'_{n-1}}{h_n}\right)h'_{n-1}(\varphi(h_n)+\varphi(Z_n))\\ &= \varepsilon_n h'_{n-1}(\varphi(h_n)+\varphi(Z_n))+\left(H'_{n-1}+p_{n-1}h'_{n-1}\right)h'_{n-1}\left(\frac{\varphi(h_n)}{h_n}+\frac{\varphi(Z_n)}{h_n}\right)\\ &= \Gamma_\varepsilon(n-1,n)+\Gamma_2(n-1)\end{aligned}$$

and

$$\begin{aligned}\gamma_3(n,m) &\leqslant \left(\varepsilon_{m-1}+\frac{H'_n}{h_{m-1}}\right)h'_{n-1}(\varphi(h_{m-1})+\varphi(Z_{m-1}))\\ &= \varepsilon_{m-1}h'_{n-1}(\varphi(h_{m-1})+\varphi(Z_{m-1}))+H'_nh'_{n-1}\left(\frac{\varphi(h_{m-1})}{h_{m-1}}+\frac{\varphi(Z_{m-1})}{h_{m-1}}\right).\\ &= \Gamma_\varepsilon(n-1,m-1)+\Gamma_3(n,m-1)\end{aligned}$$

The bound (I.14) now follows immediately. □

Proof of Theorem I.3.9. Let $\mathcal{C}$ be a flexible class and $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ a map satisfying $\varphi(t) \underset{t\to+\infty}{=} o\left(t^{1/3}\right)$. If $\Phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is another map satisfying $\varphi(t) = O\left(\Phi(t)\right)$, then Φ-integrability implies φ-integrability. Therefore, without loss of generality, we assume that φ satisfies the assumptions of Lemma I.5.22, i.e. φ is increasing and subadditive.

Using the definition of a flexible class, we will build T with large enough and inductively chosen cutting parameters q_n. Let $\mathcal{F}_{\mathcal{C}}$ be an associated set of parameters, and fix the associated constants C and C' given in Definition I.3.7. First choose any cutting and spacing parameter $(q_0, (\sigma_{0,0}, \dots, \sigma_{0,q_0}))$ in $\mathcal{F}_{\mathcal{C}}$ such that $q_0 \geqslant 3$. Without loss of generality, we assume $p_0 = 2$ and we get $q_0 > p_0$, as required in the assumption of Lemma I.5.12 for $n = 0$. For a fixed $n \geqslant 1$, assume that $(q_k, (\sigma_{k,0}, \dots, \sigma_{k,q_k}))_{0\leqslant k\leqslant n-1}$ has already been determined in $\mathcal{F}_{\mathcal{C}}$, this immediately gives $q'_0, \dots, q'_{n-1}$ (see Lemma I.5.11). The goal is to find the next parameters with q_n large enough. Consider $\kappa_n > 0$ such that for every $t \geqslant h_n\kappa_n$ the following hold:

$$\kappa_n > \max\left(p_n, q'_0, \dots, q'_{n-1}\right); \tag{I.15}$$

$$\frac{H'_n + p_n h'_n}{t} \leqslant \frac{1}{n}. \tag{I.16}$$

The assumption $\varphi(t) = o(t^{1/3})$ also implies the following inequations for a large enough κ_n:

$$\left(1 + 2(H'_n + p_n h'_n)\right)(h'_n)^2 \left(\frac{\varphi(t^3)}{t} + \frac{\varphi(Ct^3)}{t}\right) \leqslant \frac{1}{2^n}; \tag{I.17}$$

$$(h'_n)^2 \left(\varphi(t^3) + \varphi(Ct^3)\right) \leqslant \frac{t}{2^n q_0 \dots q_{n-1}}; \tag{I.18}$$

$$4h'_n \left(\frac{\varphi(t^2)}{t} + \frac{\varphi(Ct^2)}{t}\right) \leqslant \frac{1}{2^n}; \tag{I.19}$$

$$\left(H'_n + p_n h'_n\right) h'_n \left(\frac{\varphi(t)}{t} + \frac{\varphi(Ct)}{t}\right) \leqslant \frac{1}{2^{n+1}}; \tag{I.20}$$

$$\forall 1 \leqslant \ell \leqslant n,\ H'_\ell h'_{\ell-1} \left(\frac{\varphi(t)}{t} + \frac{\varphi(Ct)}{t}\right) \leqslant \frac{1}{2^{n+2}}; \tag{I.21}$$

$$\forall 0 \leqslant \ell \leqslant n,\ h'_\ell \left(\varphi(t) + \varphi(Ct)\right) \leqslant \frac{t}{2^n q_0 \dots q_{n-1}}, \tag{I.22}$$

for every $t \geqslant h_n\kappa_n$. With Inequations (I.17), (I.18), (I.19), (I.20), (I.21) and (I.22), we will respectively find bounds for the quantities $\Delta(n)$, $\Delta_\varepsilon(n)$, $\Gamma_1(n)$, $\Gamma_2(n)$, $\Gamma_3(n,m)$ and $\Gamma_\varepsilon(n,m)$ (see Lemmas I.5.23 and I.5.24).

We then set a new cutting parameter $q_n \geqslant \kappa_n$ large enough with associated spacing parameters $\sigma_{n,0}, \dots, \sigma_{n,q_n}$ so that $(q_k, (\sigma_{k,0}, \dots, \sigma_{k,q_k}))_{0\leqslant k\leqslant n} \in \mathcal{F}_{\mathcal{C}}$, $\sigma_n \leqslant C'q_n h_{n-1}$ and the following additional assumptions are satisfied:

$$q_n - (1 + p_n) \geqslant C'h_n \tag{I.23}$$

and

$$\forall 0 \leqslant k \leqslant n-1,\ q_n \geqslant C'2^{n-k}q_k. \tag{I.24}$$

Let (h_n), (σ_n) and (Z_n) be the sequences associated to $\boldsymbol{p} \coloneqq (q_n, (\sigma_{n,0}, \dots, \sigma_{n,q_n}))_{n\geqslant 0} \in \mathcal{P}^{\mathbb{N}}$ (as described in Definition I.3.1), (h'_n) the height sequence of the cutting sequence $(q'_n)_{n\geqslant 0}$ for the universal odometer that we build. We first check that the underlying system is finite measure-preserving, i.e. the condition (F) in Definition I.3.2 is satisfied. But we have

$$\frac{\sigma_n}{h_{n+1}} \leqslant \frac{C'q_n h_{n-1}}{q_n q_{n-1} h_{n-1}} = \frac{C'}{q_{n-1}},$$

so the summability easily follows from Inequality (I.24). The underlying system preserves a probability measure, so it is rank-one. Moreover it belongs to $\mathcal{C}$ by the definition of a flexible class.

Inequality (I.15) ensures that the criterion (I.9) holds and that the construction in Section I.5.a is well-defined (see Lemma I.5.12). Using $h_{n+1} \geqslant h_n q_n$, the limit in (I.11) is a consequence of Inequality (I.16) and implies $\mu(E_{n,n}) \to 1$. Inequality (I.23) implies

$$\forall n \in \mathbb{N},\ \frac{q'_n}{q_n} \leqslant 4$$

(see Lemma I.5.14).

Then Lemmas I.5.23 and I.5.24 imply that the bounds (I.13) for the φ-integral of c_T and (I.14) for the φ-integral of c_S hold. It remains to prove that these bounds are finite, namely that the series

$$\sum_{n\geqslant 0} \Delta(n),\ \sum_{n\geqslant 0} \Delta_\varepsilon(n),\ \sum_{n\geqslant 0} \Gamma_1(n),\ \sum_{n\geqslant 0} \Gamma_2(n),\ \sum_{n\geqslant 1}\sum_{m\geqslant n+1} \Gamma_3(n,m) \text{ and } \sum_{n\geqslant 0}\sum_{m\geqslant n+1} \Gamma_\varepsilon(n,m)$$

converge.

Using the monotonicity of φ and the inequalities $Z_{n+1} \leqslant C h_{n+1}$ and (I.17) for $t = h_{n+1}$ (which is greater or equal to $h_n\kappa_n$), we get $\Delta(n) \leqslant \frac{1}{2^n}$, so the series $\sum_{n\geqslant 0}\Delta(n)$ converges. It is also straightforward to see that the series $\sum_{n\geqslant 0}\Gamma_1(n)$ and $\sum_{n\geqslant 0}\Gamma_2(n)$ are convergent, using Inequalities (I.19) and (I.20). Inequality (I.21) implies $\Gamma_3(n,m) \leqslant \frac{1}{2^{m+1}}$, so we get

$$\sum_{m\geqslant n+1} \Gamma_3(n,m) \leqslant \frac{1}{2^{n+1}}$$

for every $n \geqslant 0$, and the series $\sum_{n\geqslant 1}\sum_{m\geqslant n+1}\Gamma_3(n,m)$ converges.

For the other series $\sum_{n\geqslant 0}\Delta_\varepsilon(n)$ and $\sum_{n\geqslant 0}\sum_{m\geqslant n+1}\Gamma_\varepsilon(n,m)$, we have to control the sequence (ε_n) (recall that $\varepsilon_n \coloneqq \mu((X_n)^c)$). Denote by M_0 the measure of B_0 (the unique level of the T-Rokhlin tower $\mathcal{R}_0$). For every $n \geqslant 1$, we have

$$\varepsilon_n = \sum_{k\geqslant n} \frac{M_0}{q_0\dots q_k}\sigma_k \leqslant \sum_{k\geqslant n} \frac{M_0 C' h_{k-1}}{q_0\dots q_{k-1}} \leqslant \sum_{k\geqslant n} \frac{C'}{q_{k-1}} \leqslant \frac{1}{q_{n-1}}\sum_{k\geqslant n}\frac{1}{2^{k-n}} \leqslant \frac{2}{q_{n-1}},$$

using Lemma I.3.4 and Inequation (I.24).

Given $n \geqslant 0$, Inequation (I.22) and Lemma I.3.4 imply

$$(h'_n)^2(\varphi(h_{n+1}^3) + \varphi(Z_{n+1}h_{n+1}^2)) \leqslant \frac{h_{n+1}}{2^n q_0\dots q_{n-1}} \leqslant \frac{q_n}{2^n M_0}.$$

Combining this with the inequality $\varepsilon_{n+1} \leqslant 2/q_n$, we then get

$$\Delta_\varepsilon(n) = \varepsilon_{n+1}(h'_n)^2(\varphi(h_{n+1}^3) + \varphi(Z_{n+1}h_{n+1}^2)) \leqslant \frac{1}{2^{n-1}M_0},$$

so the series $\sum_{n\geqslant 0}\Delta_\varepsilon(n)$ converges.

For fixed integers $n \geqslant 0$ and $m \geqslant n+1$, Inequation (I.22) and Lemma I.3.4 imply

$$h'_n(\varphi(h_m) + \varphi(Z_m)) \leqslant \frac{h_m}{2^{m-1}q_0\dots q_{m-2}} \leqslant \frac{q_{m-1}}{2^{m-1}M_0}.$$

Combining this with the inequality $\varepsilon_m \leqslant 2/q_{m-1}$, we then get

$$\Gamma_\varepsilon(n,m) = \varepsilon_m h'_n(\varphi(h_m) + \varphi(Z_m)) \leqslant \frac{1}{2^{m-2}M_0}.$$

This gives

$$\sum_{m\geqslant n+1}\Gamma_\varepsilon(n,m)\leqslant\frac{1}{2^{n-2}M_0}$$

for every $n\geqslant 0$, so the series $\sum_{n\geqslant 0}\sum_{m\geqslant n+1}\Gamma_\varepsilon(n,m)$ converges.

Therefore the cocycles c_T and c_S are φ-integrable as wanted, which concludes the proof. □

Remark I.5.25. For φ-integrability of c_S, we only need to control quantities of the form $\varphi(u^2)/u$ and $\varphi(u)/u$ ($\varphi(u^3)/u$ does not appear). Therefore Theorem I.3.9 can be stated with a stronger quantification on the cocycle c_S, namely ψ-integrability with $\psi(t)=o(t^{1/2})$ (it suffices to replace $t^{1/3}$ by $t^{1/2}$ in Inequation (I.22)).

We are now able to prove Theorem D.

Proof of Theorem D. Let $\varphi\colon\mathbb{R}_+\to\mathbb{R}_+$ be a map satisfying $\varphi(t)\underset{t\to+\infty}{=}o\left(t^{1/3}\right)$. By Lemma I.5.22, we may and do assume that φ is increasing and subadditive.

Given a flexible class $\mathcal{C}$, an associated set of parameters $\mathcal{F}_\mathcal{C}$ and constants C and C', the last proof shows that we can choose the parameters in the following way. First, we choose any cutting and spacing parameter $(q_0,(\sigma_{0,0},\dots,\sigma_{0,q_0}))$ in $\mathcal{F}_\mathcal{C}$, with $q_0\geqslant 3$. Then, if $\boldsymbol{p_n}\coloneqq(q_k,(\sigma_{k,0},\dots,\sigma_{0,q_k}))_{0\leqslant k\leqslant n-1}$ has been set, there exists a constant depending on φ, $\mathcal{F}_\mathcal{C}$, C, C' and $\boldsymbol{p_n}$, denoted by $\mathrm{K}_\varphi(\mathcal{F}_\mathcal{C},C,C',\boldsymbol{p_n})$, such that Conditions (I.15), (I.16), (I.17), (I.18), (I.19), (I.20), (I.21), (I.22), (I.23) and (I.24) hold for every $q_n\geqslant\mathrm{K}_\varphi(\mathcal{F}_\mathcal{C},C,C',\boldsymbol{p_n})$, and it remains to find such an integer q_n and spacing parameters $\sigma_{n,0},\dots,\sigma_{n,q_n}$ such that $\boldsymbol{p_{n+1}}\coloneqq(q_k,(\sigma_{k,0},\dots,\sigma_{0,q_k}))_{0\leqslant k\leqslant n}$ is in $\mathcal{F}_\mathcal{C}$ and the inequality $\sigma_n\leqslant C'q_nh_{n-1}$ holds.

Let $\boldsymbol{Q}\coloneqq(Q_{-1},\dots,Q_{n_0})$ be a sequence of integers, where $n_0,Q_0,\dots,Q_{n_0}$ are positive and $Q_0\dots Q_{n_0}\geqslant 3$, and let us consider the set of parameters $\mathcal{F}(\boldsymbol{Q})$ built in Section I.4.b, and the associated constants $C_{\boldsymbol{Q}}$ and $C'_{\boldsymbol{Q}}$. In this case, the spacing parameters $\sigma_{k,i}$ at step k are equal to 0 or h_{k-1}, so they are determined by the previous cutting parameters. Moreover, the first cutting parameter q_0 is equal to $Q_0\dots Q_{n_0}$. Therefore, for every finite sequence $\boldsymbol{p_n}\coloneqq(q_k,(\sigma_{k,0},\dots,\sigma_{0,q_k}))_{0\leqslant k\leqslant n-1}$ in $\mathcal{F}(\boldsymbol{Q})$, we write $\mathrm{K}_\varphi(\boldsymbol{Q},q_1,\dots,q_{n-1})$ instead of $\mathrm{K}_\varphi(\mathcal{F}(\boldsymbol{Q}),C_{\boldsymbol{Q}},C'_{\boldsymbol{Q}},\boldsymbol{p_n})$.

Recall that A denotes the set of sequences $(q_i)_{i\geqslant -1}$ of integers such that $q_0,q_1,\dots$ are positive. To every sequence $\boldsymbol{\varepsilon}=(\varepsilon_i)_{i\geqslant 0}\in\{0,1\}^{\mathbb{N}}$, we associate a sequence $q(\boldsymbol{\varepsilon})\in A$ inductively defined by:

$$\begin{array}{ll} & q(\boldsymbol{\varepsilon})_0=q_0,\\ \forall i\geqslant 0, & q(\boldsymbol{\varepsilon})_{i+1}=\mathrm{K}_\varphi(\boldsymbol{Q},q(\boldsymbol{\varepsilon})_1,\dots,q(\boldsymbol{\varepsilon})_i)+\varepsilon_i\end{array}.$$

Every sequence $\boldsymbol{\varepsilon}=(\varepsilon_i)_{i\geqslant 0}\in\{0,1\}^{\mathbb{N}}$ provides a sequence of parameters in $\mathcal{F}(\boldsymbol{Q})$, whose cutting parameters are $q(\boldsymbol{\varepsilon})_0,q(\boldsymbol{\varepsilon})_1,\dots$, and which gives rise to the irrational rotation of angle $\theta(\boldsymbol{\varepsilon})\coloneqq[Q_{-1},\dots,Q_{n_0},q(\boldsymbol{\varepsilon})_1,q(\boldsymbol{\varepsilon})_2,\dots]$.

Let us now consider a nonempty open subset $\mathcal{V}$ of $\mathbb{R}$ and a finite sequence $\boldsymbol{Q}$ so that $\theta(\boldsymbol{\varepsilon})$ is in $\mathcal{V}$ for every $\boldsymbol{\varepsilon}\in\{0,1\}^{\mathbb{N}}$. We get that the set of irrational numbers θ in $\mathcal{V}$ such that the irrational rotation of angle θ is φ-integrably orbit equivalent to the universal odometer contains the set $\{q(\boldsymbol{\varepsilon})\mid\boldsymbol{\varepsilon}\in\{0,1\}^{\mathbb{N}}\}$, so it is uncountable using the facts that the map $\boldsymbol{\varepsilon}\in\{0,1\}^{\mathbb{N}}\mapsto q(\boldsymbol{\varepsilon})\in A$ is injective and the continued fraction expansion is unique for every irrational number. □

Chapter II

Odomutants and flexibility results for quantitative orbit equivalence

This chapter corresponds to the article [Cor25b].

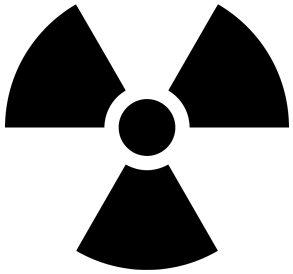
Abstract

We introduce new systems that we call odomutants, built by distorting the orbits of an odometer. We use these transformations for flexibility results in quantitative orbit equivalence.

It follows from the work of Kerr and Li that if the cocycles of an orbit equivalence are log-integrable, the entropy is preserved. Although entropy is also an invariant of even Kakutani equivalence, we prove that this relation and $L^{<1/2}$ orbit equivalence are not the same, using a non-loosely Bernoulli system of Feldman which is an odomutant.

We also show that Kerr and Li's result on preservation of entropy is optimal, namely we find odomutants of all positive entropies orbit equivalent to an odometer, with almost log-integrable cocycles. We actually build a strong orbit equivalence between uniquely ergodic Cantor minimal homeomorphisms, so our result is a refinement of a famous theorem of Boyle and Handelman.

We finally prove that Belinskaya's theorem is optimal for all the odometers, namely for every odometer, we find a odomutant which is almost-integrably orbit equivalent to it but not flip-conjugate. This yields an extension of a theorem by Carderi, Joseph, Le Maître and Tessera.



No odometers were harmed during this work.

Contents

II.1 Introduction

Two ergodic probability measure-preserving bijections S and T on a standard atomless probability space $(X, \mathcal{A}, \mu)$, are *orbit equivalent* if S and some system $\Psi^{-1}T\Psi$ conjugate to T have the same orbits up to measure zero. The isomorphism Ψ is called an *orbit equivalence* between T and S.

A stunning theorem of Dye [Dye59] states that all ergodic measure-preserving bijections of a standard probability space are orbit equivalent. To get a more interesting theory, *quantitative orbit equivalence* proposes to add quantitative restrictions on the *cocycles* associated to orbit equivalence Ψ. These are integer-valued functions c_S and c_T defined by

$$Sx = \Psi^{-1}T^{c_S(x)}\Psi(x) \text{ and } Tx = \Psi S^{c_T(x)}\Psi^{-1}(x),$$

they are well-defined in the ergodic case. In this paper, we consider two quantitative forms of orbit equivalence: *Shannon orbit equivalence* and *φ-integrably orbit equivalence*, for maps $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$. Shannon orbit equivalence requires that there exists an orbit equivalence whose cocycles are Shannon, meaning that the partitions associated to c_S and c_T are both of finite entropy. For φ-integrable orbit equivalence, we ask that both integrals

$$\int_X \varphi(|c_S(x)|)\mathrm{d}\mu(x) \text{ and } \int_X \varphi(|c_T(x)|)\mathrm{d}\mu(x)$$

are finite.

In this paper, when $\varphi(x) = x^p$, we are asking that both cocycles c_S and c_T are in L^p, and thus call it an L^p orbit equivalence. Also when c_S and c_T are in L^q for every $q < p$, we say that we have an $\mathrm{L}^{<p}$ orbit equivalence. The notion of L^p orbit equivalence can be traced back to the work of Bader, Furman and Sauer [BFS13] in the more general context of measure equivalence, while Shannon orbit equivalence was defined by Kerr and Li. Finally, φ-integrable orbit equivalence was first defined and studied by [DKLMT22].

The main goal is to understand which probability measure-preserving bijections are φ-integrably orbit equivalent or Shannon orbit equivalent. However the construction in the proof of Dye's theorem is not explicit and does not give any quantitative information on the cocycles. Then a more tractable question is the preservation of dynamical properties under these forms of quantitative orbit equivalence. In order to get flexibility results and then partially answer these questions, we introduce in this paper an explicit construction of orbit equivalence between odometers and systems with completely different properties, that we call *odomutants*.

In recent years, odometers have been a central class of systems for explicit constructions, thanks to their combinatorial structure. For example, Kerr and Li [KL24] prove that every odometer is Shannon orbit equivalent to the universal odometer, providing concrete examples of Shannon orbit equivalent systems which are non conjugate. This result was generalized: we show in [Cor25a] that many rank-one systems (including the odometers and many irrational rotations) with various spectral and mixing properties are φ-integrably orbit equivalent to the universal odometer, with $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\varphi(x) \underset{x\to+\infty}{=} o(x^{1/3})$. Finally, in order to show that the main result of [DKLMT22, Theorem 1.1] is optimal in many examples, Delabie, Koivisto, Le Maître and Tessera provide concrete orbit equivalences between group actions[1] built with Følner tilings (see [DKLMT22, Section 6]). It turns out that we get a $\mathbb{Z}^k$-odometer in the case of the group $\mathbb{Z}^k$, thus highlighting how useful the combinatorial structures of such systems are.

[1] We do not give any definition in this setting, as the paper is only about probability measure-preserving bijections S, which can be seen as $\mathbb{Z}$-actions via $(n, x) \in \mathbb{Z} \times X \mapsto S^n x$.

In our paper, the construction is also based on odometers, it is motivated by a construction by Feldman [Fel76]. The odomutants associated to the same odometer are explicitely built from successive distortions of its orbits, have the same point spectrum (Theorem II.3.13) but they can be completely different. They provide flexibility and optimality results: Theorems G, H, I and J that we explain with more details in the following paragraphs.

A theorem of preservation of entropy proved by Kerr and Li. We may wonder whether Shannon or φ-integrable orbit equivalence are trivial or not. Kerr and Li proved that a well-known invariant of conjugacy, the measure-theoretic entropy, is an invariant of Shannon orbit equivalence.

Theorem ([KL24, Theorem A]). *Entropy is preserved under Shannon orbit equivalence.*

A connection between φ-integrable orbit equivalence and Shannon orbit equivalence is given by the following statement which is a consequence of [CJLMT23, Lemma 3.15].

Lemma. *Let $f\colon X \to \mathbb{Z}$ be a measurable map. If it is* log*-integrable, then it is Shannon.*

As a consequence, φ-integrable orbit equivalence implies Shannon orbit equivalence when φ is greater than log and, combined with Kerr and Li's theorem, we get the following result.

Theorem. *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a map satisfying $\log t \underset{t\to+\infty}{=} O(\varphi(t))$. Then entropy is preserved under φ-integrable orbit equivalence.*

On non-preservation of even Kakutani equivalence. Entropy is also preserved under *even Kakutani equivalence* (see Section II.2.d). We may wonder whether there is a connection between this equivalence relation and Shannon orbit equivalence or φ-integrable orbit equivalence for a map $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\log t \underset{t\to+\infty}{=} O(\varphi(t))$. Note that these quantitative forms of orbit equivalence are not equivalence relations *a priori*. In the result below, $\mathrm{L}^{<1/2}$ orbit equivalence means that the cocycles are in L^p for every $p < \frac{1}{2}$.

Theorem G (See Theorem II.4.1). *There exists an ergodic probability measure-preserving bijection T which is $\mathrm{L}^{<1/2}$ orbit equivalent (in particular Shannon orbit equivalent) to the dyadic odometer but not evenly Kakutani equivalent to it.*

We actually prove that $\mathrm{L}^{<1/2}$ orbit equivalence does not preserve loose Bernoullicity[2], so it does not imply Kakutani equivalence (weaker than even Kakutani equivalence). In [Fel76], Feldman builds a zero-entropy ergodic system which is not loosely Bernoulli. This system, denoted by T, is actually an odomutant built from the dyadic odometer S (this is the first example of odomutant and the starting point of our work). We prove that S and T are $\mathrm{L}^{<1/2}$ orbit equivalent and Theorem G follows from the fact that every odometer is loosely Bernoulli.

Remark II.1.1. In [Fel76], Feldman did not consider the question of the point spectrum of his non loosely Bernoulli system. As a corollary of Theorem II.3.13, we get that it has the same point spectrum as the dyadic odometer.

Question II.1.2. Does there exists a sublinear map $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that φ-integrable orbit equivalence implies Kakutani equivalence or even Kakutani equivalence? Such a map would be at least $x \mapsto x^{1/2}$. We may also wonder whether loose Bernoullicity is preserved under φ-integrable orbit equivalence for some sublinear map φ. Note that the case of a linear map φ is straightforward, as a consequence of Belinskaya's theorem.

[2]Loosely Bernoulli systems form a class of ergodic systems, which is closed under Kakutani equivalence (see Section II.2.d).

Optimality result for the preservation of entropy. As stated above, φ-integrable orbit equivalence preserves entropy when the map φ satisfies $\log t \underset{t\to+\infty}{=} O(\varphi(t))$. Theorem H shows that this result is almost sharp.

Theorem H. *Let (X,μ) be a standard atomless probability space, let α be either a positive real number or $+\infty$, and let $S \in \mathrm{Aut}(X,\mu)$ be an odometer whose associated supernatural number $\prod_{p\in\Pi} p^{k_p}$ satisfies the following property: there exists a prime number $p_\star$ such that $k_{p_\star} = +\infty$. Then there exists a probability measure-preserving transformation $T \in \mathrm{Aut}(X,\mu)$ such that*

1. *$\mathrm{h}_\mu(T) = \alpha$;*
2. *there exists an orbit equivalence between S and T, which is φ_m-integrable for all integers $m \geqslant 0$,*

where φ_m denotes the map $t \to \frac{\log t}{\log^{(\circ m)} t}$ and $\log^{(\circ m)}$ the composition $\log \circ \ldots \circ \log$ (m times).

The notion of supernatural number associated to an odometer is defined after Definition II.2.12, it totally describes its conjugacy class. Examples of odometers S to which this theorem applies are the dyadic odometer, more generally the p-odometer for every prime number p, or the universal odometer. In our proof, the transformation T is an odomutant associated to S, we now explain how to build such a system.

Theorem H is actually a corollary of Theorem I which is stated in a topological framework. Indeed, to prove this corollary, the main idea was to use topological entropy instead, simpler than measure-theoretic entropy in this context, and connected to it via the variational principle. Moreover, for the topological entropy to be well-defined, we have to consider odomutants that can be extended as homeomorphisms on the Cantor set. We notice that we build a *strong orbit equivalence*, namely an orbit equivalence between homeomorphisms on the Cantor set such that the equality of the orbits holds at every point of the space (and not up to measure zero), and whose associated cocycles each have at most one point of discontinuity.

Theorem I (See Theorem II.5.1)**.** *Let α be either a positive real number or $+\infty$. Let S be an odometer whose associated supernatural number $\prod_{p\in\Pi} p^{k_p}$ satisfies the following property: there exists a prime number $p_\star$ such that $k_{p_\star} = +\infty$. Then there exists a Cantor minimal homeomorphism T such that*

1. *$\mathrm{h}_{\mathrm{top}}(T) = \alpha$;*
2. *there exists a strong orbit equivalence between S and T, which is φ_m-integrable for all integers $m \geqslant 0$,*

where φ_m denotes the map $t \to \frac{\log t}{\log^{(\circ m)} t}$ and $\log^{(\circ m)}$ the composition $\log \circ \ldots \circ \log$ (m times).

In order to create topological entropy, we build an odomutant T from the odometer S in such a way that the dynamics of T describes more words $\{\mathcal{P}(T^i(x))_{0\leqslant i\leqslant n-1} \mid x \in X\}$ than S does, for partitions $\mathcal{P}$ in clopen sets that we will define[3]. Note that this is more or less the strategy applied by Feldman for the construction of a non loosely Bernoulli system, since loose Bernoullicity property also deals with the words produced by a system. Then Theorem H follows from Theorem I and the variational principle since such a transformation T is necessarily uniquely ergodic (see Proposition II.2.20).

[3] $\mathcal{P}(y)$ denotes the atom of the partition $\mathcal{P}$ which contains $y \in X$.

For the study of strong orbit equivalence, *Bratteli diagrams* have played a crucial role. Every properly ordered Bratteli diagram provides a Cantor minimal homeomorphism, called a Bratteli-Vershik system. Conversely, Herman, Putnam and Skau proved in [HPS92] that every Cantor minimal homeomorphism is topologically conjugate to a *Bratteli-Vershik system.* Moreover, using this characterization, Giordano, Putnam and Skau completely classified the Cantor minimal homeomorphisms up to strong orbit equivalence, using the *dimension group* which turns out to be a complete invariant (see [GPS95]). We refer the reader to Appendix II.B for a brief overview.

An earlier version of Theorem H (and more generally Theorem I) stated that there exists an odomutant with positive entropy which is orbit equivalent to an odometer with almost log-integrable cocycles. Thanks to a suggestion of Thierry Giordano, we noticed that odomutants have already appeared in [BH94]. Indeed Boyle and Handelman stated a result similar to Theorem I, without any quantitative information on the cocycles.

Theorem ([BH94, Theorem 2.8 and Section 3]). *Let S be the dyadic odometer. If α is a positive real number or $\alpha = +\infty$, then there exists a Cantor minimal homeomorphism T such that:*

1. $\mathrm{h_{top}}(T) = \alpha$;
2. *S and T are strongly orbit equivalent.*

Their proof exactly consists in building a Bratteli diagram of an odomutant associated to the dyadic odometer. We thus manage to give a similar statement but with quantitative information on the cocycles (Theorem I). The case of the finite entropy is an improvement of our earlier proof, and the case of the infinite entropy is a translation of Boyle and Handelman's proof in our formalism.

Another crucial point is that the orbit equivalence we build in our paper is explicit, whereas Boyle and Handelman use the dimension group and so establish the strong orbit equivalence in a more abstract way. The comparison between Boyle and Handelman's construction and our formalism will be detailed in Appendix II.B.

Optimality of Belinskaya's theorem. Belinskaya's theorem [Bel69] states that if S and T are orbit equivalent and one of the two associated cocycles is integrable, then S and T are flip-conjugate, meaning that S is conjugate to T or T^{-1}. As a consequence, L^1 orbit equivalence is exactly flip-conjugacy. Since integrability exactly means φ-integrability for linear maps φ, it is interesting to study the sublinear case, as was done in [CJLMT23].

Theorem ([CJLMT23, Theorem 1.3]). *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear function*[4]*. Let S be an ergodic probability measure-preserving transformation and assume that S^n is ergodic for some $n \geqslant 2$. Then there is another ergodic probability measure-preserving transformation T such that S and T are φ-integrably orbit equivalent but not flip-conjugate.*

The authors asked whether this holds for a system S such that S^n is non-ergodic for all $n \geqslant 2$. The following statement provides an answer for the odometers which satisfy this property.

Theorem J (See Theorem II.6.1). *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear map and S an odometer. There exists a probability measure-preserving transformation T such that S and T are φ-integrably orbit equivalent but not flip-conjugate.*

As in the proofs of Theorems G and I, the counter-example T for Theorem J is again an odomutant associated to S. To ensure that S and T are not flip-conjugate, we notice

[4] This means that $\lim\limits_{t\to+\infty} \frac{\varphi(t)}{t} = 0$.

that an odometer is a factor of its associated odomutants, and we use the property of *coalescence* for the odometers, which states that an extension of an odometer is conjugate to it if and only if every factor map associated to this extension is an isomorphism.

Remark II.1.3. Note that a probability measure-preserving transformation S such that S^n is non-ergodic for every $n \geqslant 2$ factors onto some odometer. It would be interesting to combine the proof of Theorem J with this remark so as to completely remove the assumption that S^n is ergodic for some $n \geqslant 2$ in [CJLMT23, Theorem 1.3].

Outline of the paper. After a few preliminaries in Section II.2, we introduce the notion of odomutants in Section II.3, we study its measure-theoretic and topological properties, and the orbit equivalence with their associated odometers. Theorems G, I and J are respectively proven in Sections II.4, II.5 and II.6. Appendix II.A deals with combinatorial results preparing for the proof of Theorem I. In Appendix II.B, we describe odomutants as Bratteli-Vershik systems and compare our proof of Theorem I with the proof of Boyle and Handelman's theorem in [BH94]. Finally Appendix II.C is devoted to prove the well-known (but left unproved in the literature) equivalence between definitions of loose Bernoullicity in the zero-entropy case.

II.2 Preliminaries

II.2.a Basic definitions in ergodic theory

In a measure-theoretic framework. The author may refer to [KL16] and [VO16] for complete surveys about the notions introduced in this section.

The probability space $(X, \mathcal{A}, \mu)$ is assumed to be standard and atomless. Such a space is isomorphic to $([0,1], \mathcal{B}([0,1]), \mathrm{Leb})$, i.e. there exists a bimeasurable bijection $\Psi\colon X \to [0,1]$ (defined almost everywhere) such that $\Psi_\star\mu = \mathrm{Leb}$, where $\Psi_\star\mu$ is defined by $\Psi_\star\mu(A) = \mu(\Psi^{-1}(A))$ for every measurable set A. We consider maps $T\colon X \to X$ acting on this space and which are bijective, bimeasurable and **probability measure-preserving** (**p.m.p.**), meaning that $\mu(T^{-1}(A)) = \mu(A)$ for all measurable sets $A \subset X$, and the set of these transformations is denoted by $\mathrm{Aut}(X, \mathcal{A}, \mu)$, or simply $\mathrm{Aut}(X, \mu)$, two such maps being identified if they coincide on a measurable set of full measure. In this paper, elements of $\mathrm{Aut}(X, \mu)$ are called **transformations** or (**dynamical**) **systems**.

A measurable set $A \subset X$ is **T-invariant** if $\mu(T^{-1}(A)\Delta A) = 0$, where Δ denotes the symmetric difference. The system $T \in \mathrm{Aut}(X, \mu)$ is **(μ-)ergodic**, or μ is **T-ergodic**, if every T-invariant set is of measure 0 or 1. If T is ergodic, then T is **aperiodic**, i.e. $T^n(x) \neq x$ for almost every $x \in X$ and for every $n \in \mathbb{Z}\backslash\{0\}$, or equivalently the **$T$-orbit** of x, denoted by $\mathrm{Orb}_T(x) \coloneqq \{T^n(x) \mid n \in \mathbb{Z}\}$, is infinite for almost every $x \in X$. A transformation T is **uniquely ergodic** on X if it admits a unique T-invariant probability measure μ. In this case, μ is T-ergodic since in full generality the extremal points of the convex set of T-invariant probability measures are exactly the ergodic ones.

Denoting by $\mathrm{L}^2(X, \mathcal{A}, \mu)$ the space of complex-valued and square-integrable functions defined on X, a complex number λ is an **eigenvalue** of T if there exists $f \in \mathrm{L}^2(X, \mathcal{A}, \mu)\backslash\{0\}$ such that $f \circ T = \lambda f$ almost everywhere (f is then called an **eigenfunction**). An eigenvalue λ is automatically an element of the unit circle $\mathbb{T} \coloneqq \{z \in \mathbb{C} \mid |z| = 1\}$. The **point spectrum** of T, denoted by $\mathrm{Sp}(T)$, is then the set of all its eigenvalues. Notice that $\lambda = 1$ is always an eigenvalue since the constant functions are in its eigenspace. Moreover T is ergodic if and only if the constant functions are the only eigenfunctions with eigenvalue one, in other words the eigenspace of $\lambda = 1$ is the line of constant functions (we say that it is a simple eigenvalue). Finally, a system has **discrete spectrum** if the span of all its eigenfunctions is dense in $\mathrm{L}^2(X, \mathcal{A}, \mu)$.

All the properties that we have introduced are preserved under conjugacy. Two transformations $T \in \mathrm{Aut}(X,\mu)$ and $S \in \mathrm{Aut}(Y,\nu)$ are **conjugate** if there exists a bimeasurable bijection $\Psi\colon X \to Y$ such that $\Psi_\star\mu = \nu$ and $\Psi \circ T = S \circ \Psi$ almost everywhere. Some classes of transformations have been classified up to conjugacy, the two examples to keep in mind are the following. By Ornstein [Orn70], entropy is a total invariant of conjugacy among Bernoulli shifts (entropy will be introduced in Section II.2.c). Moreover Halmos and von Neumann [HVN42] prove that two ergodic systems with discrete spectrums are conjugate if and only if they have equal point spectrums. For example, the odometers (introduced in Section II.2.e) have discrete spectrum and this theorem enables us to classify them up to conjugacy.

Transformations T and S are said to be **flip-conjugate** if T is conjugate to S or to S^{-1}. Since the point spectrum forms a circle subgroup, the Halmos-von Neumann theorem actually states that the point spectrum is a total invariant of flip-conjugacy in the class of ergodic discrete spectrum systems. Therefore we are able to classify the odometers up to flip-conjugacy.

We say that S is a **factor** of T, or T is an **extension** of S, if there exists a measurable map $\Psi\colon X \to Y$ which is onto and such that $\Psi_\star\nu = \mu$ and $S \circ \Psi = \Psi \circ T$ almost everywhere. The map Ψ is called a **factor map** from T to S.

In a topological framework. The notions that we have introduced are part of a measure-theoretic setting. On the topological side, a **topological (dynamical) system** is a continuous map $T\colon X \to X$ on a topological space X (usually X is assumed to be compact). Two topological systems T and S, respectively on topological spaces X and Y, are **topologically conjugate** if there exists a homeomorphism $\Psi\colon X \to Y$ such that $\Psi \circ T = S \circ \Psi$ on X. A topological system is **minimal** if every orbit is dense. In this paper, we will only consider **Cantor minimal homeomorphisms**, namely minimal invertible topological systems on the Cantor set.

In this paper, "systems", "conjugacy", "entropy" will always refer to the measure-theoretic setting. For the topological setting, we will always specify "topological system", "topological conjugacy", "topological entropy".

II.2.b Measurable partitions

A set $\mathcal{P}$ of measurable subsets of X is a **measurable partition** of X if:

- for every $P_1, P_2 \in \mathcal{P}$, we have $\mu(P_1 \cap P_2) = 0$;
- the union $\bigcup_{P\in\mathcal{P}} P$ has full measure.

The elements of $\mathcal{P}$ are called the **atoms**. If $\mathcal{P}$ and $\mathcal{Q}$ are measurable partitions of (X,μ), we say that $\mathcal{P}$ **refines** (or is a refinement of, or is finer than) $\mathcal{Q}$, denoted by $\mathcal{P} \geqslant \mathcal{Q}$, if every atom of $\mathcal{Q}$ is a union of atoms of $\mathcal{P}$ (up to a null set). More generally, their **joint partition** is

$$\mathcal{P} \vee \mathcal{Q} \coloneqq \{P \cap Q \mid P \in \mathcal{P}, Q \in \mathcal{Q}\},$$

namely the coarsest partition which refines $\mathcal{P}$ and $\mathcal{Q}$.

A measurable partition $\mathcal{P}$ defines almost everywhere a map $\mathcal{P}(.)\colon X \to \mathcal{P}$ where $\mathcal{P}(x)$ is the atom of $\mathcal{P}$ which contains x. Given a measurable map $T\colon X \to X$, $\mathcal{P}$ provides **coding maps**

$$[\mathcal{P}]_{i,n}\colon x \in X \mapsto (\mathcal{P}(T^j x))_{i\leqslant j\leqslant n} \in \mathcal{P}^{\{i,\dots,n\}}.$$

In particular, $[\mathcal{P}]_n(x) \coloneqq [\mathcal{P}]_{0,n-1}(x)$ is the n-**word** of x.

Given atoms $P_i, P_{i+1}, \dots, P_n$ of $\mathcal{P}$, the equality $[\mathcal{P}]_{i,n}(x) = (P_i, \dots, P_n)$ exactly means that x is an element of $T^{-i}(P_i) \cap T^{-(i+1)}(P_{i+1}) \cap \dots \cap T^{-n}(P_n)$. Therefore the partition

which gives the values of $[\mathcal{P}]_{i,n}$ is the following joint partition

$$\mathcal{P}_i^n \coloneqq \bigvee_{j=i}^{n} T^{-j}(\mathcal{P})$$

with $T^{-j}(\mathcal{P}) \coloneqq \{T^{-j}(P) \mid P \in \mathcal{P}\}$, this is a division of the space given by the dynamics of T, over the timeline $\{i, \dots, n\}$ and with respect to $\mathcal{P}$.

II.2.c Measure-theoretic entropy, topological entropy

Here we present two notions of entropy. For more details, the reader may refer to [Dow11] and [KL16].

Measure-theoretic entropy. Entropy, or measure-theoretic entropy, or metric entropy, of a measurable transformation is an invariant of conjugacy. To define it, we first define the entropy of a partition, which then enables us to quantify how much a transformation complexifies the partitions.

Let T be a system on (X, μ), not necessarily invertible, and $\mathcal{P}$ a finite measurable partition of X. Let us define the **entropy of** $\mathcal{P}$ by

$$\mathrm{H}_\mu(\mathcal{P}) \coloneqq -\sum_{P \in \mathcal{P}} \mu(P) \log \mu(P),$$

where $\mu(P) \log \mu(P) = 0$ if P is a null set. This is a positive real number. The following quantity

$$\mathrm{h}_\mu(T, \mathcal{P}) \coloneqq \lim_{n \to +\infty} \frac{\mathrm{H}_\mu(\mathcal{P}_0^{n-1})}{n}$$

is well-defined, this is the **entropy of** T **with respect to** $\mathcal{P}$, and it tells us how quickly the dynamics of T is dividing the space X with the partition $\mathcal{P}$. Finally, let us define the **entropy of** T by

$$\mathrm{h}_\mu(T) \coloneqq \sup_{\mathcal{P}} \mathrm{h}_\mu(T, \mathcal{P}),$$

where the supremum is over all the finite measurable partitions $\mathcal{P}$ of X. This quantity is non-negative and can be infinite.

The following result, due to Kolmogorov and Sinaï, enables us to prove the well-known fact that the odometers have zero entropy (see Section II.2.e).

Theorem II.2.1 ([Dow11, after Definition 4.1.1]). *Let $(\mathcal{P}_k)_{k \geqslant 0}$ be an increasing sequence of partitions which generates the σ-algebra of X (up to restriction to full-measure sets). Then we have*

$$\mathrm{h}_\mu(T, \mathcal{P}_k) \underset{k \to +\infty}{\to} \mathrm{h}_\mu(T).$$

Topological entropy. In the topological setting, topological entropy is an invariant of topological conjugacy and is defined with similar ideas.

The topological space X has to be compact. We define the **joint cover** of two open covers $\mathcal{U}$ and $\mathcal{V}$ by

$$\mathcal{U} \vee \mathcal{V} \coloneqq \{U \cap V \mid U \in \mathcal{U}, V \in \mathcal{V}\}.$$

Let T be a topological system on X and $\mathcal{U}$ an open cover of X. Let us define

$$\mathcal{U}_0^{n-1} \coloneqq \bigvee_{i=0}^{n-1} T^{-i}(\mathcal{U}),$$

where $T^{-i}(\mathcal{U}) \coloneqq \{T^{-i}(U) \mid U \in \mathcal{U}\}$, and

$$\mathcal{N}(\mathcal{U}) \coloneqq \min\{|\mathcal{U}'| \mid \mathcal{U}' \text{ is a subcover of } \mathcal{U}\},$$

where $|\mathcal{U}'|$ denotes the cardinality of $\mathcal{U}'$. The quantity $\mathcal{N}(\mathcal{U})$ is finite since X is compact.

The **topological entropy of T with respect to the open cover $\mathcal{U}$** is the well-defined limit

$$\mathrm{h}_{\mathrm{top}}(T,\mathcal{U}) \coloneqq \lim_{n\to+\infty} \frac{\log \mathcal{N}(\mathcal{U}_0^{n-1})}{n},$$

it tells us how quickly the dynamics of T is shrinking the open sets of $\mathcal{U}$.

Finally, let us define the **topological entropy of** T by

$$\mathrm{h}_{\mathrm{top}}(T) \coloneqq \sup_{\mathcal{U}} \mathrm{h}_{\mathrm{top}}(T,\mathcal{U}),$$

where the supremum is over all the open covers $\mathcal{U}$ of X. This quantity is non-negative and can be infinite.

The following result will enables us to build an odomutant with a prescribed topological entropy (see Lemma II.5.3). We say that a sequence $(\mathcal{U}_n)_{n\geqslant 0}$ of open covers generates the topology on X if for every $\varepsilon > 0$, there exists $N \geqslant 0$ such that for every $n \geqslant N$, the open sets of $\mathcal{U}_n$ have a diameter less than ε.

Theorem II.2.2 ([Dow11, Remark 6.1.7]). *Let T be a topological system on X and $(\mathcal{U}_n)_{n\geqslant 0}$ a generating sequence of open covers. Then we have*

$$\mathrm{h}_{\mathrm{top}}(T) = \lim_{n\to+\infty} \mathrm{h}_{\mathrm{top}}(T,\mathcal{U}_n).$$

Example II.2.3. The compact space X that we consider in this paper is of the form

$$X \coloneqq \prod_{n\geqslant 0} \{0, 1, \ldots, q_n - 1\},$$

with integers q_n greater or equal to 2. It admits open covers which are partitions in clopen sets. If $\mathcal{U}$ is such an open cover, then $\mathcal{U}_0^{n-1}$ denotes both joint of open covers and joint of partitions. We have $\mathcal{N}(\mathcal{U}_0^{n-1}) = |\mathcal{U}_0^{n-1}\backslash\{\varnothing\}|$ and this is exactly the number of words of the form $[\mathcal{U}]_n(x)$, for $x \in X$, where $[\mathcal{U}]_n$ is the coding map associated to the partition $\mathcal{U}$ (see Section II.2.b). Therefore, in the proof of Theorem I, a method to create topological entropy consists in building a system T whose number of n-words (with respect to some partition in clopen sets) increases quickly enough as n goes to ∞.

More precisely, the open covers $\mathcal{U}$ that we will consider are

$$\mathcal{P}(\ell) \coloneqq \{[i_0, \ldots, i_{\ell-1}]_\ell \mid 0 \leqslant i_0 < q_0, \ldots, 0 \leqslant i_{\ell-1} < q_{\ell-1}\},$$

for $\ell \geqslant 1$, where $[i_0, \ldots, i_{\ell-1}]_\ell$ denotes the ℓ-cylinder

$$\{x = (x_n)_{n\geqslant 0} \mid x_0 = i_0, \ldots, x_{\ell-1} = i_{\ell-1}\}.$$

Note that $(\mathcal{P}(\ell))_{\ell\geqslant 1}$ is a generating sequence of open covers. In Definition II.3.3, we will also consider other partitions $\tilde{\mathcal{P}}(\ell)$, for some reasons explained in the paragraph following this definition.

The variational principle. In Example II.2.3, we explain the method that we will apply in this paper to create topological entropy and then prove Theorem I. However we also would like to create measure-theoretic entropy to prove Theorem H. The variational principle enables us to connect these notions.

Theorem II.2.4 (Variational principle [Dow11, Theorem 6.8.1]). *Let $T\colon X \to X$ be a topological system on a metric compact set X. Then we have*

$$\mathrm{h}_{\mathrm{top}}(T) = \sup_{\mu} \mathrm{h}_{\mu}(T),$$

where the supremum is over all the T-invariant Borel probability measures μ on X.

As a consequence, if T is uniquely ergodic, then we have

$$\mathrm{h}_{\mathrm{top}}(T) = \mathrm{h}_{\mu}(T),$$

where μ denotes the only T-invariant Borel probability measure.

II.2.d Even Kakutani equivalence, loose Bernoullicity

The notions introduced in this section can be found in [Fel76] and [ORW82].

Let $T \in \mathrm{Aut}(X, \mu)$. Given a measurable set A, the return time $r_A\colon A \to \mathbb{N}^* \cup \{\infty\}$ is defined by:

$$\forall x \in A,\ r_A(x) \coloneqq \inf\{k \geqslant 1 \mid T^k x \in A\}.$$

It follows from Poincaré recurrence theorem that, if A has positive measure, then the set $\{k \in \mathbb{N}^* \mid T^k x \in A\}$ is infinite for almost every $x \in A$. In particular, $r_A(x)$ is finite for almost every $x \in A$.

Then we can define a transformation T_A on the set $\{x \in A \mid r_A(x) < \infty\}$, namely on A up to a null set, called the **induced tranformation** on A:

$$T_A x \coloneqq T^{r_A(x)} x.$$

The map T_A is an element of $\mathrm{Aut}(A, \mu_A)$, where $\mu_A \coloneqq \mu(.)/\mu(A)$ is the conditional probability measure. Its entropy is given by Abramov's formula:

$$\mathrm{h}_{\mu_A}(T_A) = \frac{\mathrm{h}_{\mu}(T)}{\mu(A)}.$$

Definition II.2.5. Let $S \in \mathrm{Aut}(X, \mu)$, $T \in \mathrm{Aut}(Y, \nu)$ be two ergodic transformations.

1. T and S are said to be **Kakutani equivalent** if T_A and S_B are isomorphic for some measurable sets $A \subset X$ and $B \subset Y$.

2. Moreover they are **evenly Kakutani equivalent** if in addition two such measurable sets have the same measure: $\mu(A) = \nu(B)$.

It is well-known that Kakutani equivalence and even Kakutani equivalence are equivalence relations. It follows from Abramov's formula that entropy is preserved under even Kakutani equivalence.

Similarly to Ornstein's theory [Orn70] for the conjugacy problem, Ornstein, Rudolph and Weiss [ORW82] found a class of systems, called loosely Bernoulli system, where Kakutani and even Kakutani equivalences are well understood. These systems were first introduced by Feldman [Fel76].

Definition II.2.6 (see [Fel76]).

- The f-metric between words of same length is defined by:

$$f_n((a_i)_{1\leqslant i\leqslant n}, (b_i)_{1\leqslant i\leqslant n}) = 1 - \frac{k}{n}$$

  where k is the greatest integer for which we can find equal subsequences $(a_{i_\ell})_{1\leqslant \ell\leqslant k}$ and $(b_{j_\ell})_{1\leqslant \ell\leqslant k}$, with $i_1 < \ldots < i_k$ and $j_1 < \ldots < j_k$.

- Let $T \in \mathrm{Aut}(X, \mu)$ and $\mathcal{P}$ be a partition of X. The couple $(T, \mathcal{P})$, called a process, is **loosely Bernoulli** if for every $\varepsilon > 0$, for every sufficiently large integer N and for each $M > 0$, there exists a collection $\mathcal{G}$ of "good" atoms in $\mathcal{P}_{-M}^{0}$ whose union has measure greater than or equal to $1 - \varepsilon$, and so that for each pair A, B of atoms in $\mathcal{G}$, the following holds: there is a probability measure $n_{A,B}$ on $\mathcal{P}^N \times \mathcal{P}^N$ satisfying

  1. $n_{A,B}(\{w\} \times \mathcal{P}^N) = \mu_A(\{[\mathcal{P}]_{1,N}(.) = w\})$ for every $w \in \mathcal{P}^N$;
  2. $n_{A,B}(\mathcal{P}^N \times \{w'\}) = \mu_B(\{[\mathcal{P}]_{1,N}(.) = w'\})$ for every $w' \in \mathcal{P}^N$;
  3. $n_{A,B}(\{(w, w') \in \mathcal{P}^N \times \mathcal{P}^N \mid f_N(w, w') > \varepsilon\}) < \varepsilon$.

- T is **loosely Bernoulli** if $(T, \mathcal{P})$ is loosely Bernoulli for all finite partitions $\mathcal{P}$ of X.

Loose Bernoullicity for a process $(T, \mathcal{P})$ expresses the fact that, conditionally to two pasts A and B, the laws for the future are close, meaning that there exists a good coupling between them, with close words for the f-metric.

Example II.2.7. The Bernoulli shift on $\{1, \dots, k\}^{\mathbb{Z}}$ is loosely Bernoulli with respect to the partition $\{[1]_1, \dots, [k]_1\}$. Indeed, conditionally to every past, the law for the N-word is always the uniform distribution on $\{1, \dots, k\}^N$, so it suffices to define $n_{A,B}$ as the uniform distribution on the diagonal of $\mathcal{P}^N \times \mathcal{P}^N$, with the notations of the previous definition. This system is more generally loosely Bernoulli since $\{[1]_1, \dots, [k]_1\}$ is a generating partition[5].

We will also prove that odometers are loosely Bernoulli (see Proposition II.2.15 in the next section), using the following equivalent definition of loose Bernoullicity for zero-entropy systems.

Theorem II.2.8. *Let $T \in \mathrm{Aut}(X, \mu)$ and $\mathcal{P}$ be a partition of X and assume that $\mathrm{h}_\mu(T, \mathcal{P}) = 0$. Then $(T, \mathcal{P})$ is loosely Bernoulli if and only if for every $\varepsilon > 0$ and for every sufficiently large integer N, there exists a collection $\mathcal{H}$ of "good" atoms in $\mathcal{P}_1^N$ whose union has measure greater than or equal to $1-\varepsilon$ and so that we have $f_N(w, w') \leqslant \varepsilon$ for every $w, w' \in [\mathcal{P}]_{1,N}(\mathcal{H})$.*

This has been stated by Feldman [Fel76, Remark in p. 22] and Ornstein, Rudolph and Weiss [ORW82, after Definition 6.1] for instance. However, to our knowledge, there is no justification of this statement in the literature. This is the reason why we provide a proof in Appendix II.C.

The choice of the metric is very important. Indeed, with the d-metric:

$$d_n((a_i)_{1 \leqslant i \leqslant n}, (b_i)_{1 \leqslant i \leqslant n}) = |\{1 \leqslant i \leqslant n \mid a_i \neq b_i\}|,$$

also called the Hamming distance, we get the notion of very weakly Bernoulli systems and this is exactly the class considered in Ornstein's theory for the conjugacy problem.

As mentioned above, Kakutani equivalence and even Kakutani equivalence are well understood in the class of loosely Bernoulli systems.

Theorem II.2.9 ([ORW82, Theorems 5.1 and 5.2])**.** *Let $S \in \mathrm{Aut}(X, \mu)$, $T \in \mathrm{Aut}(Y, \nu)$ be two ergodic transformations.*

1. *If S is loosely Bernoulli and is Kakutani equivalent to T, then T is also loosely Bernoulli.*

2. *If S and T are loosely Bernoulli, then they are evenly Kakutani equivalent if and only if they have the same entropy.*

[5] To prove that a system is loosely Bernoulli, it is enough to prove it with respect to a generating partition (see [ORW82] and the equivalent notion of finitely fixed process).

II.2.e Odometers

Given integers $q_0, q_1, q_2, \ldots$ greater than or equal to 2, let us consider the Cantor space

$$X \coloneqq \prod_{n\geqslant 0} \{0, 1, \ldots, q_n - 1\},$$

endowed with the infinite product topology and the associated Borel σ-algebra. The **odometer** on X is the adding machine $S\colon X \to X$, defined for every $x \in X$ by

$$Sx = \begin{cases} (\underbrace{0, \ldots, 0}_{i \text{ times}}, 1 + x_i, x_{i+1}, \ldots) & \text{if } i \coloneqq \min\{j \geqslant 0 \mid x_j \neq q_j - 1\} \text{ is finite} \\ (0, 0, 0, \ldots) & \text{if } x = (q_0 - 1, q_1 - 1, q_2 - 1, \ldots) \end{cases}.$$

In other words, S is the addition by $(1, 0, 0, \ldots)$ with carry over to the right.

An odometer is more generally a system which is conjugate to S for some choice of integers q_n. In this paper, we only consider this concrete example with the adding machine and we refer to it as "the odometer on $\prod_{n\geqslant 0}\{0, 1, \ldots, q_n - 1\}$".

Let us introduce the **cylinders of length** k, or **k-cylinders**,

$$[x_0, \ldots, x_{k-1}]_k \coloneqq \left\{ (y_n)_{n\geqslant 0} \in \prod_{n\geqslant 0} \{0, 1, \ldots, q_n - 1\} \;\middle|\; y_0 = x_0, \ldots, y_{k-1} = x_{k-1} \right\}.$$

We can define a cylinder with a subset I_j of $\{0, 1, \ldots, q_j - 1\}$ instead of x_j. For instance, $[x_0, I_1, x_2]_3$ denotes the set of sequences $(y_n)_{n\geqslant 0}$ satisfying $y_0 = x_0$, $y_1 \in I_1$ and $y_2 = x_2$. We also use the symbol $\bullet$ when we do not want to fix the value at some coordinate. For instance, $[x_0, \bullet, x_2]_3$ denotes the set of sequences $(y_n)_{n\geqslant 0}$ satisfying $y_0 = x_0$ and $y_2 = x_2$. By convention, the 0-cylinder is X. For any $n \geqslant 1$, we also set a partially defined map

$$\zeta_n\colon X\backslash[\bullet, \ldots, \bullet, q_{n-1} - 1]_n \to X\backslash[\bullet, \ldots, \bullet, 0]_n$$

which is the addition by

$$(\underbrace{0, \ldots, 0}_{n-1 \text{ times}}, 1, 0, 0, \ldots)$$

with carry over to the right, and which coincides with $S^{q_0 \cdots q_{n-2}}$ on $X\backslash[\bullet, \ldots, \bullet, q_{n-1} - 1]_n$. As illustrated in Figure II.1, the cylinders and the maps ζ_n offer a very interesting combinatorial structure with successive nested towers $\mathcal{R}_1, \mathcal{R}_2, \ldots$.[6]

From $(q_n)_{n\geqslant 0}$, a new sequence $(h_n)_{n\geqslant 1}$ is defined by

$$\forall n \geqslant 1,\ h_n \coloneqq q_0 q_1 \ldots q_{n-1}.$$

The integer h_n is the height of the tower $\mathcal{R}_n$ (see Figure II.1). By convention, we set $h_0 \coloneqq 1$, the height of the tower $\mathcal{R}_0 \coloneqq (X)$ with a single level.

As a topological system, S is a Cantor minimal homeomorphism. As a measure-theoretic system, S is uniquely ergodic and its only invariant measure is the product $\mu \coloneqq \bigotimes_{n\geqslant 0} \mu_n$ where μ_n is the uniform distribution on $\{0, 1, \ldots, q_n - 1\}$. For the sake of completeness, we give a proof of the following well-known fact on odometers, which shows that the point spectrum is also fully understood.

Proposition II.2.10. *Let S be the odometer on $\prod_{n\geqslant 0}\{0, 1, \ldots, q_n - 1\}$. Its point spectrum is*

$$\mathrm{Sp}(S) = \left\{ \exp\left(\frac{2i\pi k}{h_n}\right) \mid n \geqslant 1, 0 \leqslant k \leqslant h_n - 1 \right\}$$

[6] This kind of construction that we see in Figure II.1 is called a cutting-and-stacking construction.

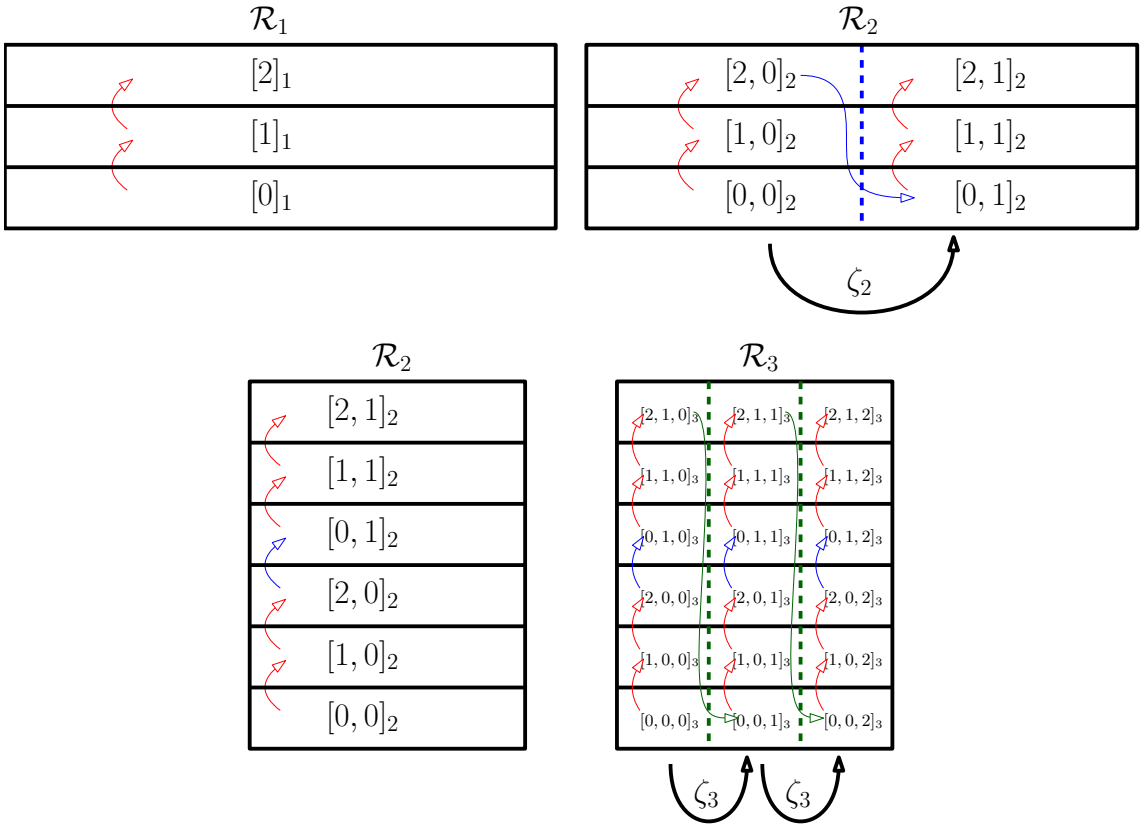


Figure II.1: Example of odometer with $q_0 = 3$, $q_1 = 2$, $q_2 = 3$ (so $h_1 = 3$, $h_2 = 6$, $h_3 = 18$).

and for every $\lambda = \exp\left(\frac{2i\pi k}{h_n}\right) \in \mathrm{Sp}(S)$, *the map*

$$f_\lambda \colon x \in X \mapsto \sum_{j=0}^{h_n-1} \lambda^j \mathbb{1}_{S^j([0,\dots,0]_n)}(x)$$

is an eigenfunction associated to λ. *Moreover* S *has discrete spectrum.*

Remark II.2.11. The definition of f_λ does not depend on the choice of k and n such that $\lambda = \exp\left(\frac{2i\pi k}{h_n}\right)$. Moreover, for $n = 0$, we have $f_1 = \mathbb{1}_X$ (by convention, the 0-cylinder is X).

Proof of Proposition II.2.10. Let us set $\Lambda := \left\{\exp\left(\frac{2i\pi k}{h_n}\right) \mid n \geqslant 1, 0 \leqslant k \leqslant h_n - 1\right\}$. It is straightforward to check that f_λ is an eigenfunction associated to λ, for every $\lambda \in \Lambda$. Let us show that the span of $\{f_\lambda \mid \lambda \in \Lambda\}$ is dense in $\mathrm{L}^2(X, \mu)$. It will implies that S has discrete spectrum and that $\Lambda = \mathrm{Sp}(S)$.

Let $n \geqslant 1$ and $\lambda = \exp\left(\frac{2i\pi}{h_n}\right)$. Given $a_0, \dots, a_{h_n-1} \in \mathbb{C}$, we have

$$\sum_{\ell=0}^{h_n-1} a_\ell f_{\lambda^\ell} = \sum_{j=0}^{h_n-1} P(\lambda^j) \mathbb{1}_{S^j([0,\dots,0]_n)}$$

with the polynomial $P = a_0 + a_1 Y + \ldots + a_{h_n-1} Y^{h_n-1}$. For every $j \in \{0, \dots, h_n - 1\}$, there exists a polynomial P_j of degree less than h_n, satisfying $P_j(\lambda^j) = 1$ and $P(\lambda^k) = 0$ for all $k \in \{0, \dots, h_n - 1\} \backslash \{j\}$. This implies that the characteristic functions of cylinders are linear combinations of the eigenfunctions f_λ for $\lambda \in \Lambda$, hence the result. □

Let us now explain the classification of odometers up to conjugacy (and even flip-conjugacy). Let Π denote the set of prime numbers.

Definition II.2.12. A **supernatural number** is a formal product of the form $\prod_{p\in\Pi} p^{k_p}$, with $k_p \in \mathbb{N} \cup \{+\infty\}$.

Given a prime number $p \in \Pi$, denote by $\nu_p(k)$ the p-adic valuation of a positive integer k. To every odometer defined with integers $q_0, q_1, \ldots$, we associate a supernatural number $\prod_{p\in\Pi} p^{k_p}$ defined by

$$k_p := \sum_{n\geqslant 0} \nu_p(q_n).$$

As a consequence of Proposition II.2.10 and the Halmos-von Neumann theorem, the supernatural number $\prod_{p\in\Pi} p^{k_p}$ forms a total invariant of measure-theoretic conjugacy in the class of odometers. If $k_p = \infty$ for every prime number p, then the odometer is said to be **universal**. Given a prime number p, the **p-odometer** is the odometer such that $k_p = \infty$ and $k_q = 0$ for every $q \in \Pi\backslash\{p\}$. In the case $p = 2$, it is also called the **dyadic** odometer.

Proposition II.2.10 also implies that every odometer is coalescent.

Definition II.2.13. A transformation $S \in \mathrm{Aut}(X, \mu)$ is coalescent if every system $T \in \mathrm{Aut}(X, \mu)$ which is isomorphic to S satisfies the following: every factor map from T to S is an isomorphism.

The fact that odometers are coalescent is proven in [HP68] and [New71]. In these articles, one proves that more general systems are coalescent and the phenomenon can be generalized in the context of group actions (see [IT16]). Here we give a short proof for ergodic systems with discrete spectrum.

Theorem II.2.14. *Every ergodic system with discrete spectrum is coalescent.*

Proof of Theorem II.2.14. Let $S \in \mathrm{Aut}(X, \mu)$ be an ergodic system with discrete spectrum, $T \in \mathrm{Aut}(X, \mu)$ isomorphic to S, and $\Psi\colon X \to X$ a factor map from T to S. Given $\lambda \in \mathbb{T}$, let us denote by $E_S(\lambda)$ (resp. $E_T(\lambda)$) the eigenspace of S (resp. T) associated to λ. First, ergodicity implies that non-zero eigenspaces have dimension 1 (see Proper Value Theorem in [Hal56, page 34]). Secondly, since Ψ is a factor map, every eigenfunction f of S gives rise to the eigenfunction $f \circ \Psi$ of T, and more precisely $f \circ \Psi$ lies in $E_T(\lambda)$ if f lies in $E_S(\lambda)$. Hence, since S and T are isomorphic, these two remarks imply that $E_T(\lambda) = \{f \circ \Psi \mid f \in E_S(\lambda)\}$ for every λ in the point spectrum of S (or equivalently the point spectrum of T). This implies

$$\mathrm{L}^2(X, \mu) = \{f \circ \Psi \mid f \in \mathrm{L}^2(X, \mu)\}$$

since they have discrete spectrum. Hence Ψ is an isomorphism. □

For the proof of Theorem J, the systems that we will consider will be an odometer S and an associated odomutant T (the odomutants are introduced in Section II.3.a). Since the odomutants are extensions of their associated odometer and since we explicitely know a factor map ψ between them (see Proposition II.3.4), Theorem II.2.14 will ensure that we will not build an orbit equivalence between flip-conjugate systems if ψ is not invertible.

Finally, odometers have the following properties.

Proposition II.2.15. *Odometers have zero measure-theoretic and topological entropies.*

Proposition II.2.16. *Odometers are loosely Bernoulli.*[7]

[7] More generally, rank-one systems are loosely Bernoulli, this is proven by Ornstein, Rudolph and Weiss [ORW82] (see Lemma 8.1) and we present their proof in the special case of odometers.

Remark II.2.17. In the case of odometers, we can notice in the following proofs that zero entropy and loose Bernoullicity follow from a poor dynamics of these systems. Indeed, given concrete partitions (for instance the partitions $\mathcal{P}(k)$ given by the cylinders of length k, which increase to the σ-algebra), the dynamics of an odometer does not generate a lot of words and the different futures are close (in the sense of the definition of loose Bernoullicity). The idea behind the definition of odomutants will be to get systems with a less "laconic" dynamics.

Proof of Proposition II.2.15. Let S be an odometer. The equality $\mathrm{h}_\mu(S) = \mathrm{h}_{\mathrm{top}}(S)$ follows from unique ergodicity and the variational principle (Theorem II.2.4). Let $\mathcal{P}(k)$ be the partition given by the cylinders of length k. The odometer S acts as a cyclic permutation on the elements of $\mathcal{P}(k)$, so the sequence $((\mathcal{P}(k))_0^{n-1})_{n\geqslant 1}$ of partitions is stationary and we have $\mathrm{h}_\mu(S, \mathcal{P}(k)) = 0$. The sequence $(\mathcal{P}(k))_{k\geqslant 0}$ increases to the σ-algebra of X, so we have $\mathrm{h}_\mu(S, \mathcal{P}(k)) \underset{k\to+\infty}{\to} \mathrm{h}_\mu(S)$ by Theorem II.2.1, and we get $\mathrm{h}_\mu(S) = 0$. □

Proof of Proposition II.2.16. Let S be an odometer, associated to the integers $q_0, q_1, \ldots$, let $\mathcal{P}(k)$ be the partition given by the cylinders of length k. We prove that $(S, \mathcal{P}(k))$ is loosely Bernoulli for every $k \geqslant 1$, and we deduce from this that $(S, \mathcal{P})$ is loosely Bernoulli for any finite partition $\mathcal{P}$. We use the characterization provided by Theorem II.2.8.

Let us prove that $(S, \mathcal{P}(k))$ is loosely Bernoulli. Let $\varepsilon > 0$, $N \geqslant 2h_k/\varepsilon$ and $\mathcal{H} = \mathcal{P}_1^N$. Let us denote by W the word $\left(S^i([0,\ldots,0]_k)\right)_{0\leqslant i\leqslant h_k-1} \in (\mathcal{P}(k))^{\{0,\ldots,h_k-1\}}$ of length h_k, this is the enumeration of the k-cylinders, with the order given by the dynamics of S. For every $x \in X$, the word $[\mathcal{P}(k)]_{1,N}(x)$ consists of the tail of the word W, followed by many concatenations of W, and the beginning of W. So any two words $w = [\mathcal{P}(k)]_{1,N}(x)$ and $w' = [\mathcal{P}(k)]_{1,N}(x')$ satisfy $f_N(w, w') \leqslant 2h_k/N \leqslant \varepsilon$. This proves that $(S, \mathcal{P}(k))$ is loosely Bernoulli.

Now let $\mathcal{P}$ be a finite measurable partition and let us show that $(S, \mathcal{P})$ is loosely Bernoulli. The sequence $(\mathcal{P}(k))_{k\geqslant 0}$ increases to the σ-algebra of X, so for a given $\varepsilon > 0$, there exists $k \geqslant 0$ such that $\mathcal{P}$ and $\mathcal{P}(k)$ are close, meaning that there exists a $\mathcal{P}(k)$-measurable partition $\mathcal{Q}$, with $|\mathcal{Q}| = |\mathcal{P}| =: n$, and a good enumeration of the atoms of $\mathcal{Q}$ and $\mathcal{P}$ such that $\sum_{i=j}^n \mu(P_j \Delta Q_j) < \varepsilon$. Since $\mathcal{P}(k)$ refines $\mathcal{Q}$, words with respect to $\mathcal{P}(k)$ completely determine words with respect to $\mathcal{Q}$, so $(S, \mathcal{Q})$ is immediately loosely Bernoulli. Then, if N is sufficiently large, there exists $\mathcal{H} \subset \mathcal{Q}_1^N$ covering at least $1 - \varepsilon$ of the space and such that any two words $w, w' \in [\mathcal{Q}]_{1,N}(\mathcal{H})$ satisfy $f_N(w, w') \leqslant \varepsilon$ (the f-metric with respect to $\mathcal{Q}$). By the ergodic theorem, for every sufficiently large integer $N > 0$, there exists a subset X_0 of X such that $\mu(X_0) \geqslant 1 - \varepsilon$ and every $x \in X_0$ satisfies

$$\frac{1}{N}\left|\left\{i \in \{1, 2, \ldots, N\} \mid S^i x \in \bigcup_{j=1}^n (P_j \cap Q_j)\right\}\right| \geqslant 1 - 2\varepsilon.$$

This implies that for every $x \in X_0$, the word $[\mathcal{Q}]_{1,N}(x)$ determines at least a fraction $1-2\varepsilon$ of the word $[\mathcal{P}]_{1,N}(x)$. Therefore, given $x, x' \in X_0 \cap (\bigcup_{C\in\mathcal{H}} C)$, the words $w = [\mathcal{P}]_{1,N}(x)$ and $w' = [\mathcal{P}]_{1,N}(x')$ satisfy $f_N(w, w') \leqslant 5\varepsilon$ (the f-metric with respect to $\mathcal{P}$). It remains to define $\mathcal{H}' \subset \mathcal{P}_1^N$ as the set of atoms with non trivial intersection with $X_0 \cap \bigcup_{C\in\mathcal{H}} C$. It covers at least $1 - 3\varepsilon$ of the space and, with respect to $\mathcal{P}$, every two N-words w and w' produced in $\mathcal{H}'$ satisfy $f_N(w, w') \leqslant 5\varepsilon$, so we are done. □

II.2.f Orbit equivalence

The conjugacy problem in full generality is very complicated (see [FRW11]). We now give the formal definition of orbit equivalence, which is a weakening of the conjugacy problem.

Definition II.2.18. Two aperiodic transformations $S \in \mathrm{Aut}(X,\mu)$ and $T \in \mathrm{Aut}(Y,\nu)$ are **orbit equivalent** if there exists a bimeasurable bijection $\Psi\colon X \to Y$ satisfying $\Psi_\star\mu = \nu$, such that $\mathrm{Orb}_S(x) = \mathrm{Orb}_{\Psi^{-1}T\Psi}(x)$ for almost every $x \in X$. The map Ψ is called an **orbit equivalence** between S and T.

We can then define the **cocycles** associated to this orbit equivalence. These are measurable functions $c_S\colon X \to \mathbb{Z}$ and $c_T\colon Y \to \mathbb{Z}$ defined almost everywhere by

$$Sx = \Psi^{-1}T^{c_S(x)}\Psi(x) \text{ and } Ty = \Psi S^{c_T(y)}\Psi^{-1}(y)$$

($c_S(x)$ and $c_T(y)$ are uniquely defined by aperiodicity).

Remark II.2.19. Conversely, the existence of a cocycle, let us say c_T, implies the inclusion of the $(\Psi^{-1}T\Psi)$-orbits in the S-orbits. So the existence of both cocycles c_S and c_T implies equality of orbits. This well-known characterization of orbit equivalence will be used in the proof of Theorem II.3.16.

Given a map $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$, a measurable function $f\colon X \to \mathbb{Z}$ is said to be **φ-integrable** if

$$\int_X \varphi(|f(x)|)\mathrm{d}\mu < +\infty.$$

For example, integrability is exactly φ-integrability when φ is non-zero and linear, and a weaker quantification on cocycles is the notion of φ-integrability for a sublinear map φ, meaning that $\lim_{t\to+\infty}\varphi(t)/t = 0$. Two transformations in $\mathrm{Aut}(X,\mu)$ are said to be **φ-integrably orbit equivalent** if there exists an orbit equivalence between them whose associated cocycles are φ-integrable. The notion of L^p **orbit equivalence** refers to the map $\varphi\colon x \to x^p$, and a $\mathrm{L}^{<p}$ **orbit equivalence** is by definition an orbit equivalence which is L^q for all $q < p$.

Another form of quantitative orbit equivalence is Shannon orbit equivalence. We say that a measurable function $f\colon X \to \mathbb{Z}$ is **Shannon** if the associated partition $\{f^{-1}(n) \mid n \in \mathbb{Z}\}$ of X has finite entropy, namely

$$-\sum_{n\in\mathbb{Z}} \mu(f^{-1}(n))\log\mu(f^{-1}(n)) < +\infty.$$

Two transformations in $\mathrm{Aut}(X,\mu)$ are **Shannon orbit equivalent** if there exists an orbit equivalence between them whose associated cocycles are Shannon.

Note that orbit equivalence preserves ergodicity. The next statement specifically connects orbit equivalence and unique ergodicity. Theorem I and this proposition together with the variational principle directly imply Theorem H.

Proposition II.2.20. *Assume that two aperiodic measurable bijections S and T on a Borel space X are orbit equivalent in the following stronger way: S and T are defined on the whole X and the equality $\mathrm{Orb}_S(x) = \mathrm{Orb}_T(x)$ holds for every $x \in X$.*[8] *Then S is uniquely ergodic if and only if T is uniquely ergodic. In this case, S and T have the same invariant probability measure.*

Proof of Proposition II.2.20. Assume that S is uniquely ergodic and denote by μ its only invariant probability measure. The cocycle $c_S\colon X \to \mathbb{Z}$ is defined on the whole X and is measurable. Let ν be a T-invariant probability measure. For every measurable set A, we

[8] This is stronger than asking this property up to a null set.

have

$$\begin{aligned}\nu(S(A)) &= \sum_{k\in\mathbb{Z}} \nu(S(A\cap\{c_S=k\})) \\ &= \sum_{k\in\mathbb{Z}} \nu(T^k(A\cap\{c_S=k\})) \\ &= \sum_{k\in\mathbb{Z}} \nu(A\cap\{c_S=k\}) \\ &= \nu(A),\end{aligned}$$

so ν is S-invariant and is equal to μ. Therefore T is uniquely ergodic and μ is its only invariant probability measure. □

For instance, strong orbit equivalence is a form of orbit equivalence, introduced in a topological framework by Giordano, Putnam and Skau [GPS95], to which Proposition II.2.20 applies. The definition is the following.

Definition II.2.21. Two Cantor minimal homeomorphisms (X,S) and (Y,T) are strongly orbit equivalent if there exists a homeomorphism $\Psi\colon X\to Y$ such that S and $\Psi^{-1}T\Psi$ have the same orbits on X and the associated cocycles each have at most one point of discontinuity.

Boyle proved in his thesis [Boy83] that strong orbit equivalence with continuous cocycles boils down to topological flip-conjugacy, namely S is topologically conjugate to T or to T^{-1}. As mentioned in the introduction, the classification of Cantor minimal homeomorphisms up to strong orbit equivalence is fully understood, with complete invariants such as the dimension group (see [GPS95], and Appendix II.B for a brief overview).

II.3 Odomutants

II.3.a Definitions

Let $X \coloneqq \prod_{n\geqslant 0}\{0,1,\dots,q_n-1\}$ with integers $q_n\geqslant 2$, and let us recall the notation $h_n\coloneqq q_0\dots q_{n-1}$. The space X is endowed with the infinite product topology and we denote by μ the product of the uniform distributions on each $\{0,1,\dots,q_n-1\}$. We consider the odometer $S\colon X\to X$ on this space. Recall that it is defined by

$$Sx=\begin{cases}(\underbrace{0,\dots,0}_{i\text{ times}},x_i+1,x_{i+1},\dots) & \text{if } i\coloneqq\min\{j\geqslant 0\mid x_j\neq q_j-1\}\text{ is finite}\\ (0,0,0,\dots) & \text{if } x=(q_0-1,q_1-1,q_2-1,\dots)\end{cases},$$

and it is a μ-preserving homeomorphism.

In this section, we introduce new systems that we call odomutants, defined from S with successive distortions of its orbits, encoded by the following maps ψ and ψ_n (for $n\geqslant 0$).

For every $n\geqslant 0$, we fix a finite sequence $\left(\sigma_i^{(n)}\right)_{0\leqslant i<q_{n+1}}$ of permutations of the set $\{0,1,\dots,q_n-1\}$, and we introduce

$$\psi_n\colon\begin{cases}X & \to & X\\ x=(x_0,x_1,\dots) & \mapsto & (\sigma_{x_1}^{(0)}(x_0),\sigma_{x_2}^{(1)}(x_1),\sigma_{x_3}^{(2)}(x_2),\dots,\sigma_{x_{n+1}}^{(n)}(x_n),x_{n+1},x_{n+2},\dots)\end{cases}.$$

It is not difficult to see that ψ_n is a homeomorphism and preserves the measure μ, its inverse is given by

$$\psi_n^{-1}\colon\begin{cases}X & \to & X\\ x=(x_0,x_1,\dots) & \mapsto & (z_0(x),z_1(x),\dots,z_n(x),x_{n+1},x_{n+2},\dots)\end{cases}$$

with $z_i(x)$ defined by backwards induction as follows:

$$\begin{aligned} z_n(x) &\coloneqq \left(\sigma^{(n)}_{x_{n+1}}\right)^{-1}(x_n), \\ z_i(x) &\coloneqq \left(\sigma^{(i)}_{z_{i+1}(x)}\right)^{-1}(x_i) \text{ for every } i \in \{0,1,\dots,n-1\}. \end{aligned} \tag{II.1}$$

Let us also introduce

$$\psi\colon \begin{cases} X & \to & X \\ x=(x_0,x_1,\dots) & \mapsto & \left(\sigma^{(n)}_{x_{n+1}}(x_n)\right)_{n\geqslant 0} \end{cases}.$$

The map ψ is continuous but is not invertible in full generality. It is not difficult to see that $\psi_n(x) \underset{n\to+\infty}{\to} \psi(x)$ for every $x \in X$. The map ψ also have the following properties.

Proposition II.3.1. *$\psi\colon X \to X$ preserves the probability measure μ and is onto.*

Proof of Proposition II.3.1. To prove that μ is ψ-invariant, it suffices to prove the equality $\mu(\psi^{-1}(A)) = \mu(A)$ when A is a cylinder. If A is an $(n+1)$-cylinder, then $\psi^{-1}(A) = \psi_n^{-1}(A)$, so the ψ-invariance follows from the ψ_n-invariance for all $n \geqslant 0$.

Given $y \in X$, let us find $x \in X$ such that $\psi(x) = y$. By definition, for every $n \geqslant 0$, $\psi(\psi_n^{-1}(y))$ is in the cylinder $[y_0,\dots,y_n]_{n+1}$, so $\psi(\psi_n^{-1}(y)) \underset{n\to+\infty}{\to} y$. By compactness, there exists a convergent subsequence of $\left(\psi_n^{-1}(y)\right)_{n\geqslant 0}$, of limit $x \in X$, and we have $\psi(x) = y$ since ψ is continuous. □

The following computations motivate the definition of odomutants. Let us respectively set the **minimal** and **maximal** points of X:

$$x^- \coloneqq (0,0,0,\dots) \text{ and } x^+ \coloneqq (q_0-1, q_1-1, q_2-1, \dots).$$

We define the following sets

$$\begin{aligned} X_n^- &\coloneqq \{x \in X \mid (x_0,\dots,x_n) \neq (x_0^-,\dots,x_n^-)\}, \\ X_n^+ &\coloneqq \{x \in X \mid (x_0,\dots,x_n) \neq (x_0^+,\dots,x_n^+)\}, \\ X_\infty^- &\coloneqq X\backslash\{x^-\} \text{ and } X_\infty^+ \coloneqq X\backslash\{x^+\}. \end{aligned}$$

It is not difficult to see that X_∞^+ is the increasing union of the sets X_n^+, so for every $x \in X_\infty^+$, we denote by $N^+(x)$ the least integer $n \geqslant 0$ satisfying $x \in X_n^+$. This also holds for X_∞^- and X_n^-, and $N^-(x)$ is defined similarly.

Let $x \in \psi^{-1}(X_\infty^+)$ and $N \coloneqq N^+(\psi(x))$. By definition of N, for every $n \geqslant N$, $S\psi_n(x)$ is equal to

$$(\underbrace{\mathbf{0},\dots,\mathbf{0}}_{N \text{ times}},\ \sigma^N_{x_{N+1}}(x_N)+\mathbf{1},\ \sigma^{(N+1)}_{x_{N+2}}(x_{N+1}),\ \dots,\ \sigma^{(n)}_{x_{n+1}}(x_n),\ x_{n+1},\ x_{n+2},\ \dots).$$

Using (II.1), we get

$$\psi_n^{-1}S\psi_n(x) = (y_0^{(n)}(x),\dots,y_n^{(n)}(x),x_{n+1},x_{n+2},\dots)$$

with $y_i^{(n)}(x)$ defined by backwards induction as follows:

$$\begin{aligned} y_n^{(n)}(x) &\coloneqq \left(\sigma^{(n)}_{x_{n+1}}\right)^{-1}(\sigma^{(n)}_{x_{n+1}}(x_n)) = x_n, \\ \forall\ n > i > N,\ y_i^{(n)}(x) &\coloneqq \left(\sigma^{(i)}_{y_{i+1}^{(n)}(x)}\right)^{-1}(\sigma^{(i)}_{x_{i+1}}(x_i)), \\ y_N^{(n)}(x) &\coloneqq \left(\sigma^{(N)}_{y_{N+1}^{(n)}(x)}\right)^{-1}(\sigma^{(N)}_{x_{N+1}}(x_N)+1), \\ \forall\ N > i \geqslant 0,\ y_i^{(n)}(x) &\coloneqq \left(\sigma^{(i)}_{y_{i+1}^{(n)}(x)}\right)^{-1}(0). \end{aligned}$$

By induction, it is easy to get $(y_{N+1}^{(n)}(x),\dots,y_n^{(n)}(x)) = (x_{N+1},\dots,x_n)$ and this implies the following simplification: $\psi_n^{-1}S\psi_n(x)$ is equal to $(y_0^{(n)}(x),\dots,y_N^{(n)}(x),x_{N+1},x_{N+2},\dots)$ with $y_i^{(n)}(x)$ inductively defined by

$$y_N^{(n)}(x) \coloneqq \left(\sigma_{x_{N+1}}^{(N)}\right)^{-1}(\sigma_{x_{N+1}}^{(N)}(x_N)+1),$$
$$\forall\ 0\leqslant i\leqslant N-1,\ y_i^{(n)}(x) \coloneqq \left(\sigma^{(i)}_{y_{i+1}^{(n)}(x)}\right)^{-1}(0).$$

Finally, $(y_0^{(n)}(x),\dots,y_N^{(n)}(x))$ does not depend on the integer $n\geqslant N^+(\psi(x))$.

Definition II.3.2. For every $x\in\psi^{-1}(X_\infty^+)$, let us define

$$Tx \coloneqq \psi_n^{-1}S\psi_n(x)$$

for any $n\geqslant N^+(\psi(x))$. The map T is called the **odomutant** associated to the odometer S and the sequences of permutations $\left(\sigma_i^{(n)}\right)_{0\leqslant i<q_{n+1}}$ for $n\geqslant 0$.

As illustrated in Figure II.2, an odomutant T is a probability measure-preserving bijection that we build step by step. At step n, T is well-defined on $\{N^+(x)=n\}$. This is a cutting-and-stacking method very similar to the odometer, but at every step the way we connect the subcolumns of the tower depend on the next coordinates.

II.3.b Odomutants with multiplicities

At first view, when looking at Figure II.2, we can think that an odomutant is encoded by a cutting-and-stacking construction where the new towers at each step are built by stacking only *one* copy of the dynamics of each subcolumn. Actually, with some redondancies in the permutations of a same step, it is possible to encode a cutting-and-stacking construction where, at every step and for every subcolumn, many copies of its dynamics could appear in each new tower (as illustrated in Figure II.3). In this case, the partitions in cylinder of the same length are not the information we want to keep in mind, since they also remember that we divide the subcolumns to get many copies of its dynamics. This motivates the following definition that we explain with more details after.

Definition II.3.3. Let $(q_n)_{n\geqslant 0}$ be a sequence of integers greater than or equal to 2. Let $\boldsymbol{c}=(c_{n,0},\dots,c_{n,\tilde{q}_n-1})_{n\geqslant 1}$ be a sequence where $\tilde{q}_n$ and $c_{n,i}$ are positive integers satisfying $q_n = c_{n,1}+\ldots+c_{n,\tilde{q}_n}$, and $(\tau_j^{(n)})_{j\in\{0,\dots,\tilde{q}_{n+1}-1\}}$ be a sequence of permutations of the set $\{0,\dots,q_n-1\}$ for every $n\geqslant 0$. For every $n\geqslant 1$ and every $j\in\{0,\dots,\tilde{q}_n-1\}$, we set

$$I_j^{(n)} \coloneqq \left(\sum_{i=0}^{j-1} c_{n,i}\right) + \{0,1,\dots,c_{n,j}-1\}.^{9}$$

Then we say that T is the odomutant built with **$\boldsymbol{c}$-multiple permutations** $\tau_j^{(n)}$, if T is the odomutant associated to the odometer on the space $\prod_{n\geqslant 0}\{0,\dots,q_n-1\}$ and families of permutations $(\sigma_i^{(n)})_{0\leqslant i<q_{n+1}}$, where for every $n\geqslant 0$ and every $j\in\{0,\dots,\tilde{q}_{n+1}\}$, we have $\sigma_i^{(n)} \coloneqq \tau_j^{(n)}$ for all integers $i\in I_j^{(n+1)}$.

In this case, we associate partitions $\tilde{\mathcal{P}}(\ell)$ for every $\ell\geqslant 1$, defined by

$$\tilde{\mathcal{P}}(\ell) \coloneqq \left\{[i_0,\dots,i_{\ell-2},I_j^{(\ell-1)}]_\ell \mid 0\leqslant i_0<q_0,\dots,0\leqslant i_{\ell-2}<q_{\ell-2},0\leqslant j\leqslant\tilde{q}_{\ell-1}-1\right\}.$$

We say that the odomutant is built with **uniformly $\boldsymbol{c}$-multiple permutations** if we have $c_{n,0}=\ldots=c_{n,\tilde{q}_n-1}=:c_n$ for every $n\geqslant 1$, and we simply write $\boldsymbol{c}\coloneqq(c_n,\tilde{q}_n)_{n\geqslant 0}$.

[9] We write $s+\{0,1,\dots,k\} \coloneqq \{s,s+1,\dots,s+k\}$. The family $(I_0^{(n)},\dots,I_{\tilde{q}_n-1}^{(n)})$ forms a partition of $\{0,1,\dots,q_n-1\}$.

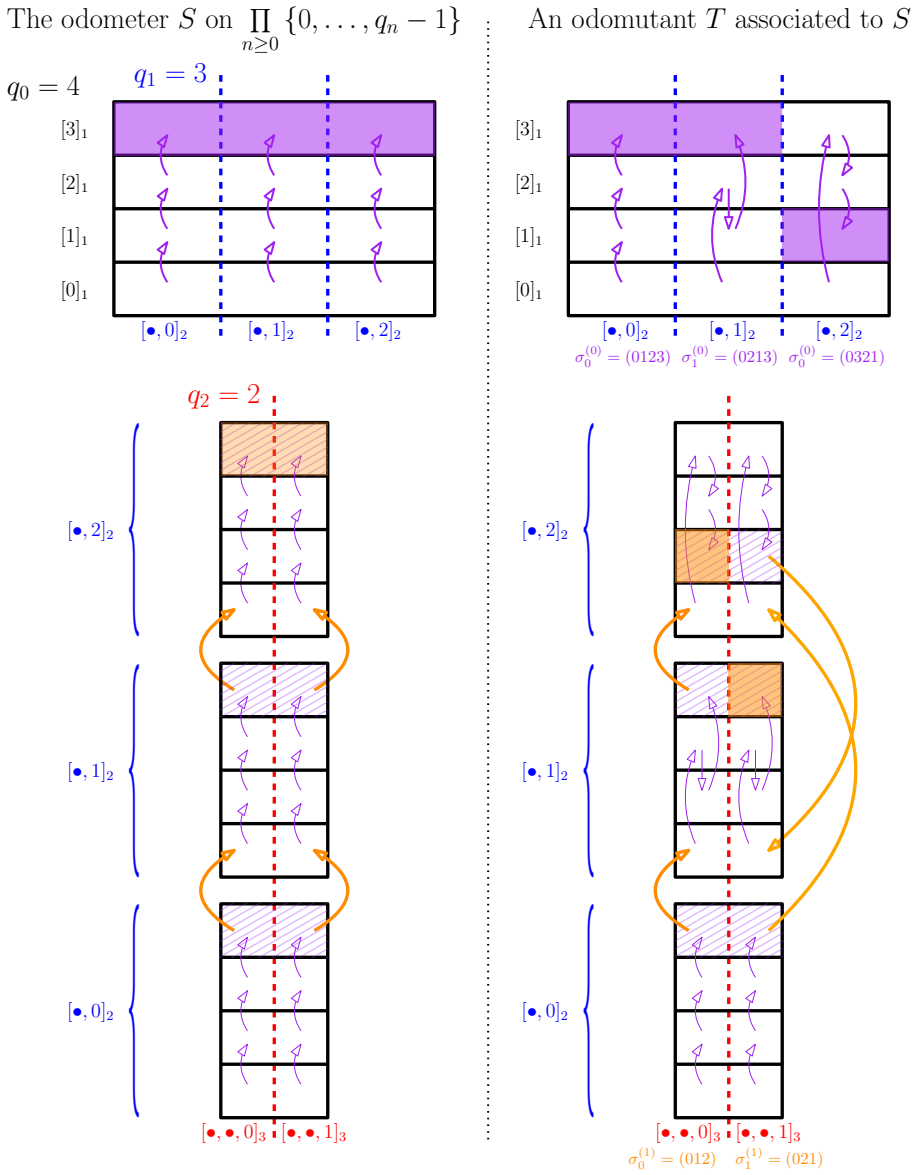


Figure II.2: Example of the first two steps in the construction of an odometer (on the left) and an associated odomutant (on the right). For a permutation σ of the set $\{0,\dots,k-1\}$, the notation $\sigma=(i_0\dots i_{k-1})$ means that σ is defined by $\sigma(j)=i_j$ for every $j\in\{0,\dots,k-1\}$. The area coloured in purple (resp. orange) is the subset on which S and T are not yet defined at the end of the first step (resp. second step), it is equal to $\{N^+=1\}$ (resp. $\{N^+=2\}$) for the odometer, $\{N^+\circ\psi=1\}$ (resp. $\{N^+\circ\psi=2\}$) for the odomutant.

At the beginning of step n, for every $i\in\{0,\dots,\tilde{q}_n-1\}$ there are c_i subcolumns which have been defined with the same permutation[10] $\tau_i^{(n-1)}$ at step $n-1$, they actually play the role of c_i copies of the dynamics of a subcolumn that we would like to stack c_i times in each tower. When considering the partition $\tilde{\mathcal{P}}(n+1)$, we cannot distinguish between these "copies", as if it was the partition made up of the subcolumns that we would like to stack more than once in each tower.

The odomutants built with uniformly multiple permutations, equipped with the associated partitions $(\tilde{\mathcal{P}}(\ell))_{\ell\geqslant 1}$, better describe Boyle and Handelman's contructions [BH94] than odomutants equipped with $\mathcal{P}(\ell)_{\ell\geqslant 1}$. We refer the reader to Appendix II.B for more details, more precisely in Section II.B.d. The sequences $(c_n)_n$ and $(\tilde{q}_n)_n$ respectively correspond

[10]Note that the permutations $\tau_0^{(n-1)},\dots,\tau_{\tilde{q}_n-1}^{(n-1)}$ are not necessarily pairwise different.

to the sequences $(n_k)_k$ and $(m_k)_k$ introduced in their paper. Then, to prove Theorem I in the case $\alpha = +\infty$, we will partly reformulate the proof of their similar statement with our formalism. Our proof in the case $\alpha < +\infty$ will be different than theirs since we will build an odomutant with pairwise different permutations at each step.

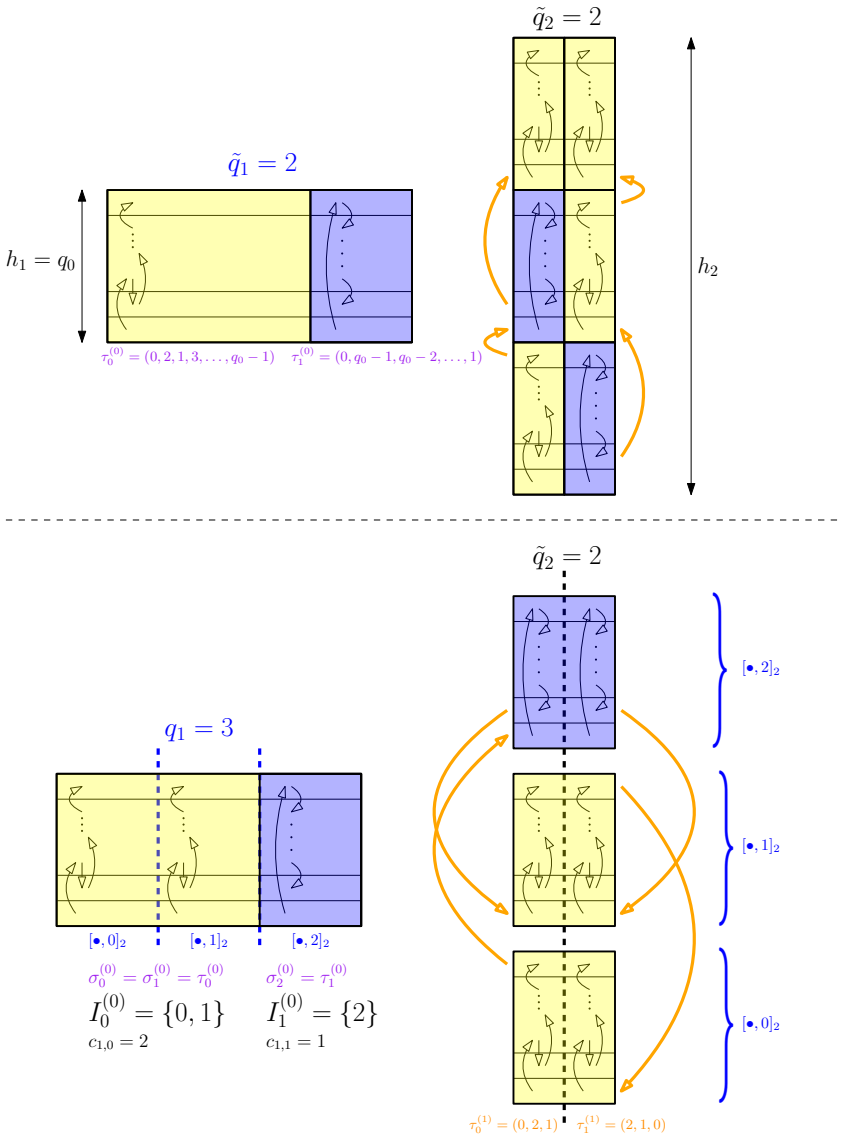


Figure II.3: At the top, the second step of a less restrictive cutting-and-stacking construction that we want to describe with an odomutant. At the bottom, the way we encode it with such a system. Here, the dynamics of the yellow tower appears twice in each new towers, so we divide it in two subtowers. Note that the partition $\tilde{\mathcal{P}}(2)$ is exactly the partition which gives the colour (yellow or blue) and the level in the h_0-tower to each points of the space, so that we cannot distinguish between points of the two yellow subtowers which are at the same level, contrary to the partition $\mathcal{P}(2)$. For the third step of the construction, the value of q_2 will depend on the number ($c_{2,0}$ and $c_{2,1}$) of copies for the dynamics of the two current towers in the next ones.

As mentioned in the introduction, our formalism of odomutants was inspired by Feldman's construction [Fel76] of a non-loosely Bernoulli system. As we will see in the proof of Theorem G, this system is an odomutant built with uniformly $\boldsymbol{c}$-multiple permutations where the integers c_n are powers of 2 and for a fixed n, the permutations $\tau_i^{(n)}$ are pairwise different at each step.

II.3.c Odomutants as p.m.p. bijections on a standard probability space

In this section, we study odomutants with a measure-theoretic viewpoint.

First properties

Proposition II.3.4. *T is a bijection from $\psi^{-1}(X_\infty^+)$ to $\psi^{-1}(X_\infty^-)$, its inverse is given by*

$$T^{-1}y = \psi_n^{-1}S^{-1}\psi_n(y)$$

for every $y \in \psi^{-1}(X_\infty^-)$ and any $n \geqslant N^-(\psi(y))$. Moreover T is an element of $\mathrm{Aut}(X,\mu)$ and ψ is a factor map from T to S.

Proof of Proposition II.3.4. The equality $\psi_n(Tx) = S\psi_n(x)$ implies $\psi(Tx) = S\psi(x)$ since ψ_n converges pointwise to ψ. Moreover, the map ψ preserves the measure μ and is onto (see Proposition II.3.1). Thus, assuming that T is in $\mathrm{Aut}(X,\mu)$, S is a factor of T via the factor map ψ.

Since X_∞^+ is the increasing union of the sets X_n^+, and for every $n \geqslant 0$, T and $\psi_n^{-1}S\psi_n$ coincide on X_n^+, the injectivity of T on $\psi^{-1}(X_\infty^+)$ follows from the injectivity of S and the maps ψ_n and ψ_n^{-1}.

For $x \in \psi^{-1}(X_\infty^+)$, we have $\psi(Tx) = S\psi(x)$ and $\psi(x) \neq x^+$, so $\psi(Tx)$ is not equal to x^-. Conversely, for $y \in \psi^{-1}(X_\infty^-)$, the element $x \coloneqq \psi_n^{-1}S^{-1}\psi_n(y)$ does not depend on the choice of an integer $n \geqslant N^-(\psi(y))$ (these are the same computations as before Definition II.3.2) and satisfies $Tx = y$.

By ψ-invariance, the sets $\psi^{-1}(X_\infty^+)$ and $\psi^{-1}(X_\infty^-)$ have full measure, so $T\colon X \to X$ is a bijection up to measure zero. It follows again from the properties of S and the maps ψ_n that T is bimeasurable and preserves the measure μ. □

The next result provides a criterion for ψ to be an isomorphism between T and S. We will not apply it in this paper but it enables us to understand that, in case permutations have common fixed points[11] (see Section II.3.5), we will need the sequence $(q_n)_{n\geqslant 0}$ to increase quickly enough, otherwise we get an odomutant T conjugate to S.

Lemma II.3.5. *For every $n \geqslant 0$, we set*

$$F_n \coloneqq \{x_n \in \{0,\dots,q_n-1\} \mid \forall x_{n+1} \in \{0,\dots,q_{n+1}-1\},\ \sigma_{x_{n+1}}^{(n)}(x_n) = x_n\}.$$

If the series $\sum \frac{|F_n|}{q_n}$ diverges, then ψ is an isomorphism between S and T.

Proof of Lemma II.3.5. By the Borel-Cantelli lemma, the set

$$X_0 \coloneqq \{(x_n)_{n\geqslant 0} \in X \mid x_n \in F_n \text{ for infinitely many integers } n\}$$

has full measure. It is also S-, T- and ψ-invariant and it is easy to check that $\psi\colon X_0 \to X_0$ is a bijection, using the fact that the equality $\sigma_{x_{n+1}}^{(n)}(x_n) = y_n$ implies $x_n = y_n$ when y_n is in F_n. □

Remark II.3.6. It is not hard to see, independently of Lemma II.3.5, that in order to prove Theorems G and H, one needs the sequence $(q_n)_{n\geqslant 0}$ to be unbounded. Otherwise, let K denote an upper bound of the sequence, then the underlying odomutant admits a cutting-and-stacking construction with at most K towers at each step. A system satisfying such property is said to have *rank K* (and more generally *finite rank*) and it is well-known that it is loosely Bernoulli and has zero entropy (see [Fer97]).

[11] For Theorem G (resp. Theorem I), we will require $\sigma_i^{(n)}(0) = 0$ (resp. $\sigma_i^{(n)}(0) = 0$ and $\sigma_i^{(n)}(q_n-1) = q_n-1$).

Question II.3.7. Is it possible to find a necessary and sufficient condition on the permutations $\sigma_i^{(n)}$ (for $n \geqslant 0$ and $0 \leqslant i < q_{n+1}$) for the factor map ψ to be an isomorphism? Since every odometer is coalescent (see Theorem II.2.14), this would enable us to know whether or not an odomutant is conjugate to its associated odometer.

The following two results will be useful for some computations in the proofs of Lemma II.3.11 and Proposition II.3.15. They deal with the well-definedness of powers (positive or negative) of an odomutant at some point of X.

Proposition II.3.8. *For $k \in \mathbb{N}$, the following assertion hold.*[12]

- *If $\psi(x)$ is in $\bigcap_{i=0}^{k-1} S^{-i}(X_\infty^+)$, then $Tx, T^2x, \dots, T^kx$ are well-defined and for every $i \in \{0, \dots, k\}$, we have*
$$T^ix = \psi_n^{-1} S^i \psi_n(x)$$
for any $n \geqslant \max_{0 \leqslant j \leqslant i-1} N^+(\psi(T^jx))$.
- *If $\psi(x)$ is in $\bigcap_{i=-(k-1)}^{0} S^{-i}(X_\infty^-)$, then $T^{-1}x, T^{-2}x, \dots, T^{-k}x$ are well-defined and for every $i \in \{-(k-1), \dots, 0\}$, we have*
$$T^{-i}x = \psi_n^{-1} S^{-i} \psi_n(x)$$
for any $n \geqslant \max_{-(i-1) \leqslant j \leqslant 0} N^-(\psi(T^jx))$.

Proof of Proposition II.3.8. For example, let us prove the first point by induction over $k \geqslant 1$. The proof of the second point is similar.

The result is clear for $k = 0$. Let $k \geqslant 1$. Let us assume that the result holds for $k-1$ and that
$$\psi(x) \in \bigcap_{i=0}^{k-1} S_n^{-i}(X_\infty^+).$$
This implies that $T^{k-1}x$ is well-defined and is equal to $\psi_n^{-1} S^{k-1} \psi_n(x)$ for any n greater than or equal to $\max_{0 \leqslant j \leqslant k-2} N^+(\psi(T^jx))$. Moreover $\psi(T^{k-1}x)$ is not equal to x^+. Indeed, the first $n+1$ coordinates of $\psi(T^{k-1}x)$ and $\psi_n(T^{k-1}x)$ are the same and we have
$$\psi_n(T^{k-1}x) = S^{k-1}\psi_n(x)$$
for any $n \geqslant \max_{0 \leqslant j \leqslant k-2} N^+(\psi(T^jx))$, so this follows from the fact that $S^{k-1}\psi(x)$ is not equal to x^+. This implies that T^kx is well-defined and equal to $\psi_n^{-1} S \psi_n(T^{k-1}x)$ for any $n \geqslant N^+(\psi(T^{k-1}x))$. Finally, for any $n \geqslant \max_{0 \leqslant j \leqslant k-1} N^+(\psi(T^jx))$, we get
$$T^kx = \psi_n^{-1} S \psi_n(T^{k-1}x) = \psi_n^{-1} S \psi_n(\psi_n^{-1} S^{k-1} \psi_n(x)) = \psi_n^{-1} S^k \psi_n(x),$$
hence the result for k. □

Corollary II.3.9. *Let $x, y \in X$ and $M \in \mathbb{N}^*$ such that $x_j = y_j$ for every $j \geqslant M$, and set*
$$K \coloneqq \sum_{j=0}^{M-1} h_j \left(\sigma_{y_{j+1}}^{(j)}(y_j) - \sigma_{x_{j+1}}^{(j)}(x_j) \right).$$
Assume that x and y are different. Then the following hold:

- *if $K > 0$, then $Tx, T^2x, \dots, T^Kx$ are well-defined;*
- *if $K < 0$, then $T^{-1}x, T^{-2}x, \dots, T^Kx$ are well-defined.*

[12] For instance, this holds for every $x \in X_\infty^+$ such that $\psi(x)$ is not in $\mathrm{Orb}_S(x^+)$ (which is also the S-orbit of x^-), so the hypothesis holds for a set of points x of full measure.

Moreover we have $T^K x = y$.

Remark II.3.10. The proof of the equality $T^K x = y$ is based on the well-understood case of an odometer, namely the permutations $\sigma_i^{(n)}$ are all identity maps and $T = S$. More precisely, given $w, z \in X$ satisfying $w_j = z_j$ for every j greater than or equal to some M, we know that $S^K w = z$ with

$$K = \sum_{j=0}^{M-1} h_j\,(z_j - w_j).$$

It remains to apply this well-known fact to $w = \psi_n(x)$ and $z = \psi_n(y)$ for a large enough integer n.

Proof of Corollary II.3.9. Let us consider the case $K > 0$ (the proof for the other case is similar). By the previous remark, it is clear that we have

$$y = \psi_n^{-1} S^K \psi_n(x)$$

for every $n \geqslant M$. Using Proposition II.3.8, it remains to prove that $S^i\psi(x)$ is not equal to x^+ for every $i \in \{0, \dots, K-1\}$. If there exists a positive integer i such that $S^i\psi(x) = x^+$, then we have

$$\sigma_{x_{j+1}}^{(j)}(x_j) = q_j - 1$$

for every sufficiently large integers j, and

$$\begin{aligned} i &= \sum_{j=0}^{+\infty} h_j \left(q_i - 1 - \sigma_{x_{j+1}}^{(j)}(x_j)\right) \\ &= \sum_{j=0}^{M-1} h_j \left(q_i - 1 - \sigma_{x_{j+1}}^{(j)}(x_j)\right) + \sum_{j=M}^{+\infty} h_j \left(q_i - 1 - \sigma_{x_{j+1}}^{(j)}(x_j)\right) \\ &\geqslant \sum_{j=0}^{M-1} h_j \left(\sigma_{y_{j+1}}^{(j)}(y_j) - \sigma_{x_{j+1}}^{(j)}(x_j)\right). \end{aligned}$$

Therefore i is greater than or equal to K and we are done. □

An odomutant and its associated odometer have the same point spectrum

Since every odomutant T factors onto its associated odometer S, we have the inclusion $\mathrm{Sp}(S) \subset \mathrm{Sp}(T)$ between the point spectrums. We actually show that this is an equality. The following lemma is inspired by Danilenko and Vieprik's methods to study the point spectrum of rank-one systems (see Proposition 3.7 in [DV23]).

Lemma II.3.11. *Let T be an odomutant built from the odometer S on $X = \prod_{n\geqslant 0} \{0, \dots, q_n - 1\}$ and the families of permutations $(\sigma_{x_{n+1}}^{(n)})_{0\leqslant x_{n+1} < q_{n+1}}$. If $\lambda \in \mathbb{T}$ is an eigenvalue of T, then for every $\varepsilon > 0$, there exists a positive integer n such that for every $m \geqslant n$, there exist $E_{n,m} \subset \prod_{j=n}^{m} \{0, \dots, q_j - 1\}$ and $x_{m+1} \in \{0, \dots q_{m+1} - 1\}$ satisfying the following:*

- $\dfrac{|E_{n,m}|}{q_n q_{n+1} \cdots q_m} > 1 - \varepsilon$;
- *for every $(y_n, \dots, y_m), (z_n, \dots, z_m) \in E_{n,m}$, we have*

$$\left|1 - \lambda^{\sum_{j=n}^{m} h_j \left(\sigma_{y_{j+1}}^{(j)}(y_j) - \sigma_{z_{j+1}}^{(j)}(z_j)\right)}\right| < \varepsilon,$$

with $y_{m+1} = z_{m+1} \coloneqq x_{m+1}$.

Proof of Lemma II.3.11. Let $\varepsilon > 0$, $\lambda \in \mathbb{T}$ an eigenvalue of T and g_λ an eigenfunction of T associated to λ. Without loss of generality, we assume that $\varepsilon \leqslant 1/2$. Moreover, the modulus of g_λ is almost everywhere constant (since it is T-invariant and T is ergodic), so we assume that g_λ has modulus 1. There exists $\alpha \in \mathbb{T}$ and a measurable subset $A \subset X$ of positive measure such that

$$\forall x \in A,\ |g_\lambda(x) - \alpha| < \varepsilon/2. \tag{II.2}$$

Since the partition given by the n-cylinders is increasing to the σ-algebra on X as $n \to +\infty$, we can find $n > 0$ and $(x_0, \dots, x_{n-1}) \in \prod_{j=0}^{n-1} \{0, \dots, q_j - 1\}$ such that

$$\mu(A \cap [x_0, \dots, x_{n-1}]_n) > (1 - \varepsilon^2)\mu([x_0, \dots, x_{n-1}]_n).$$

Let $m \geqslant n$. Then there exists $x_{m+1} \in \{0, \dots, q_{m+1} - 1\}$ such that

$$\mu(A \cap [x_0, \dots, x_{n-1}, \bullet, \dots, \bullet, x_{m+1}]_{m+2}) > (1 - \varepsilon^2)\mu([x_0, \dots, x_{n-1}, \bullet, \dots, \bullet, x_{m+1}]_{m+2}) \tag{II.3}$$

and we set

$$E_{n,m} \coloneqq \left\{ (y_n, \dots, y_m) \in \prod_{j=n}^{m} \{0, \dots, q_j - 1\} \;\middle|\; \begin{array}{c} \mu(A \cap [x_0, \dots, x_{n-1}, y_n, \dots, y_m, x_{m+1}]_{m+2}) > \\ (1 - \varepsilon)\mu([x_0, \dots, x_{n-1}, y_n, \dots, y_m, x_{m+1}]_{m+2}) \end{array} \right\}.$$

By Inequality (II.3), we get

$$\frac{|E_{n,m}|}{q_n \cdots q_m} > 1 - \varepsilon.$$

Let $(y_n, \dots, y_m), (z_n, \dots, z_m) \in E_{n,m}$. Let us set

$$B_y \coloneqq A \cap [x_0, \dots, x_{n-1}, y_n, \dots, y_m, x_{m+1}]_{m+2},$$

$$B_z \coloneqq A \cap [x_0, \dots, x_{n-1}, z_n, \dots, z_m, x_{m+1}]_{m+2}.$$

and

$$K \coloneqq \sum_{j=n}^{m} h_j \left(\sigma^{(j)}_{y_{j+1}}(y_j) - \sigma^{(j)}_{z_{j+1}}(z_j) \right)$$

(with $y_{m+1} = z_{m+1} \coloneqq x_{m+1}$). By Corollary II.3.9, the set $T^{-K}(B_y)$ is included in the cylinder

$$C \coloneqq [x_0, \dots, x_{n-1}, z_n, \dots, z_m, x_{m+1}]_{m+2},$$

which implies that $B \coloneqq T^{-K}(B_y) \cap B_z$ has positive measure. Indeed, if B were a null set, the cylinder C would contain two subsets $T^{-K}(B_y)$ and B_z of negligible intersection and we would get $\mu(C) > 2(1-\varepsilon)\mu(C)$ by definition of $E_{n,m}$, this is not possible since $\varepsilon \leqslant 1/2$.

Then we have $g_\lambda(T^K x) = \lambda^K g_\lambda(x)$ for almost every $x \in B$, and since every $x \in B$ is in A and satisfies $T^K x \in A$, we get $|1 - \lambda^K| < \varepsilon$ using (II.2). □

Lemma II.3.12. *Let $0 < \varepsilon < 2$ and $\theta = \theta(\varepsilon) > 0$ such that*

$$\{\nu \in \mathbb{T} \mid |1 - \nu| < \varepsilon\} = \{\exp(2i\pi\tau) \mid -\theta < \tau < \theta\}.$$

Let $\nu \in \mathbb{T}\backslash\{1\}$ satisfying $|1 - \nu| < \varepsilon$. We write it as $\nu = \exp(2i\pi\tau)$ with $-\theta < \tau < \theta$, $\tau \neq 0$. If ε is small enough so that $\theta < 1/4$, then for every interval[13] J of $\mathbb{Z}$, we have

$$\sum_{j \in J} \mathbb{1}_{|1-\nu^j|<\varepsilon} \leqslant \frac{3\theta}{1 - 2\theta}|J| + \frac{6\theta}{|\tau|}.$$

[13] By an interval of $\mathbb{Z}$, we mean a set of the form $\{k \in \mathbb{Z} \mid a \leqslant k \leqslant b\}$ for some integers a and b.

Proof of Lemma II.3.12. Without loss of generality, we assume that τ is positive. Let J be an interval of $\mathbb{Z}$. If we have

$$\sum_{j\in J} \mathbb{1}_{|1-\nu^j|} = 0,$$

then the result is clear. Now we assume that there exists $j \in J$ such that $|1-\nu^j| < \varepsilon$. Since ν is not equal to 1, this implies that we have $|1-\nu^k| < \varepsilon$ for infinitely many integers k. Since θ is less than $1/4$, we also have $|1-\nu^k| \geqslant \varepsilon$ for infinitely many integers k. Therefore we can find sequences $(n_\ell)_{\ell\in\mathbb{Z}}$ and $(m_\ell)_{\ell\in\mathbb{Z}}$ of integers such that $n_\ell < m_\ell < n_{\ell+1} < m_{\ell+1}$ for every $\ell \in \mathbb{Z}$, so that we can write

$$\mathbb{Z} = \ldots \sqcup C_{-2} \sqcup D_{-2} \sqcup C_{-1} \sqcup D_{-1} \sqcup C_0 \sqcup D_0 \sqcup C_1 \sqcup D_1 \sqcup \ldots$$

with intervals $C_\ell \coloneqq \{k \in \mathbb{Z} \mid n_\ell \leqslant k < m_\ell\}$ and $D_\ell \coloneqq \{k \in \mathbb{Z} \mid m_\ell \leqslant k < n_{\ell+1}\}$ such that

$$\forall k \in C_\ell,\ |1-\nu^k| < \varepsilon \text{ and } \forall k \in D_\ell,\ |1-\nu^k| \geqslant \varepsilon.$$

For every $\ell \in \mathbb{Z}$, we have

$$(m_\ell - n_\ell - 1)\tau < 2\theta < (m_\ell - n_\ell + 1)\tau$$

$$\text{and } (n_{\ell+1} - m_\ell + 1)\tau \geqslant 1 - 2\theta,$$

this implies

$$\frac{2\theta}{\tau} - 1 < |C_\ell| < \frac{2\theta}{\tau} + 1$$

$$\text{and } |D_\ell| \geqslant \frac{1-2\theta}{\tau} - 1.$$

Now we set $\ell_0 \coloneqq \max\{\ell \in \mathbb{Z} \mid n_\ell \leqslant \min J\}$ and $\ell_1 \coloneqq \max\{\ell \in \mathbb{Z} \mid n_\ell \leqslant \max J\}$. We then have the inclusion $\bigsqcup_{\ell_0+1\leqslant\ell\leqslant\ell_1-1} (C_\ell \sqcup D_\ell) \subset J$ which yields

$$|J| > (\ell_1 - \ell_0 - 1)\left(\frac{1}{\tau} - 2\right).$$

Finally, we have

$$\begin{aligned}
\sum_{j\in J} \mathbb{1}_{|1-\nu^j|<\varepsilon} &\leqslant \sum_{\ell=\ell_0}^{\ell_1} |C_\ell| \\
&\leqslant (\ell_1 - \ell_0 + 1)\left(\frac{2\theta}{\tau} + 1\right) \\
&\leqslant \left(\frac{\frac{2\theta}{\tau} + 1}{\frac{1}{\tau} - 2}\right)|J| + 2\left(\frac{2\theta}{\tau} + 1\right) \\
&\leqslant \left(\frac{\frac{2\theta}{\tau} + \frac{\theta}{\tau}}{\frac{1}{\tau} - 2\frac{\theta}{\tau}}\right)|J| + 2\left(\frac{2\theta}{\tau} + \frac{\theta}{\tau}\right) \text{ since } 1 \leqslant \frac{\theta}{\tau} \\
&= \frac{3\theta}{1-2\theta}|J| + \frac{6\theta}{\tau}
\end{aligned}$$

and we are done. □

Theorem II.3.13. *Let T be an odomutant built from the odometer S on $X = \prod_{n\geqslant 0}\{0,\ldots,q_n-1\}$. Then T and S have the same point spectrum.*

Using the Halmos-von Neumann theorem [HVN42], we get the following corollary.

Corollary II.3.14. *Let T be an odomutant built from the odometer S on $X = \prod_{n\geqslant 0}\{0,\dots,q_n-1\}$. If T is conjugate to an odometer, then T is conjugate to S.* □

Proof of Theorem II.3.13. Since T factors onto S, we already know that $\mathrm{Sp}(S)\subset\mathrm{Sp}(T)$. Let λ be an eigenvalue of T. Let us show that this is an eigenvalue of S. Let $\varepsilon>0$ small enough so that $\theta<1/4$ and $\frac{3\theta}{1-2\theta}\leqslant 1/4$, with $\theta=\theta(\varepsilon)$ introduced in Lemma II.3.12. We also assume that $\varepsilon\leqslant 1/2$. Let n be a positive integer given by Lemma II.3.11 for the eigenvalue λ, and $\nu\coloneqq\lambda^{h_n}$. If $\nu=1$, we are done.

Now assume $\nu\neq 1$. Let us choose a sufficiently large enough integer m so that $m\geqslant n$ and $\frac{6\theta}{\tau q_n\dots q_m}\leqslant\frac{1}{4}$. We consider a set $E_{n,m}\subset\prod_{j=n}^{m}\{0,\dots,q_j-1\}$ and an integer $x_{m+1}\in\{0,\dots,q_{m+1}-1\}$ satisfying

- $\dfrac{|E_{n,m}|}{q_nq_{n+1}\dots q_m}>1-\varepsilon$;
- for every $y=(y_n,\dots,y_m),z=(z_n,\dots,z_m)\in E_{n,m}$, we have

$$\left|1-\nu^{H(y)-H(z)}\right|<\varepsilon,$$

where $H\colon\prod_{j=n}^{m}\{0,\dots,q_j-1\}\to\{0,\dots,q_n\dots q_m-1\}$ is defined by

$$H\colon y=(y_n,\dots,y_m)\mapsto\sum_{j=n}^{m}\frac{h_j}{h_n}\sigma^{(j)}_{y_{j+1}}(y_j)\text{ with }y_{m+1}\coloneqq x_{m+1}.$$

The existence of $E_{n,m}$ and x_m is granted by Lemma II.3.11. Since $\varepsilon\leqslant 1/2$ and H is a bijection, there exists two different elements y and z in $E_{n,m}$ such that $H(y)-H(z)=1$. This implies

$$|1-\nu|<\varepsilon.$$

Let us fix $z\in E_{n,m}$ and set

$$J=\left\{H(y)-H(z)\mid y\in\prod_{j=n}^{m}\{0,\dots,q_j-1\}\right\}.$$

By Lemma II.3.12, we have

$$\sum_{j\in J}\mathbb{1}_{|1-\nu^j|<\varepsilon}\leqslant\frac{3\theta}{1-2\theta}|J|+\frac{6\theta}{|\tau|}\leqslant\frac{q_n\dots q_m}{2}$$

and we get a contradiction since we have

$$\sum_{j\in J}\mathbb{1}_{|1-\nu^j|<\varepsilon}\geqslant|E_{n,m}|>(1-\varepsilon)q_n\dots q_m$$

with $\varepsilon\leqslant 1/2$. Thus we have $\lambda^{h_n}=1$. □

II.3.d Orbit equivalence between odometers and odomutants

In this section, we prove that an odomutant and its associated odometer have the same orbits. Moreover, given a non-decreasing map $\varphi\colon\mathbb{R}_+\to\mathbb{R}_+$, we give sufficient conditions for the cocycles to be φ-integrable.

Proposition II.3.15. *For all $x \in \psi^{-1}(X_\infty^+)$, we have $Tx = S^{c_T(x)}x$ where the integer $c_T(x)$ is defined by*

$$c_T(x) = \sum_{i=0}^{N_1} h_i(y_i(x) - x_i) \tag{II.4}$$

with $N_1 \coloneqq N^+(\psi(x))$ and $y_0, \dots, y_{N_1}(x)$ inductively defined by

$$y_{N_1}(x) \coloneqq \left(\sigma^{(N_1)}_{x_{N_1+1}}\right)^{-1}(\sigma^{(N_1)}_{x_{N_1+1}}(x_{N_1}) + 1),$$
$$\forall\ 0 \leqslant i \leqslant N_1 - 1,\ y_i(x) \coloneqq \left(\sigma^{(i)}_{y_{i+1}(x)}\right)^{-1}(0).$$

For all $x \in X_\infty^+$, let us define the integer $c_S(x)$ by:

$$\begin{aligned} c_S(x) = {} & h_{N_2}\left(\sigma^{(N_2)}_{x_{N_2+1}}(1 + x_{N_2}) - \sigma^{(N_2)}_{x_{N_2+1}}(x_{N_2})\right) \\ & + h_{N_2-1}\left(\sigma^{(N_2-1)}_{1+x_{N_2}}(0) - \sigma^{(N_2-1)}_{x_{N_2}}(x_{N_2-1})\right) \\ & + \sum_{i=0}^{N_2-2} h_i\left(\sigma^{(i)}_0(0) - \sigma^{(i)}_{x_{i+1}}(x_i)\right) \end{aligned} \tag{II.5}$$

with $N_2 \coloneqq N^+(x)$. Then we have $Sx = T^{c_S(x)}x$ for every $x \in X_\infty^+$.

Proof of Proposition II.3.15. For $x \in \psi^{-1}(X_\infty^+)$, the value of $c_T(x)$ follows from the computations before Definition II.3.2. For $x \in X_\infty^+$ and $N_2 \coloneqq N^+(x)$, we have

$$x = (q_0 - 1, \dots, q_{N_2-1} - 1, \underbrace{x_{N_2}}_{\neq q_{N_2}-1}, x_{N_2+1}, x_{N_2+2}, \dots)$$

$$\text{and } Sx = (0, \dots, 0, 1 + x_{N_2}, x_{N_2+1}, x_{N_2+2}, \dots)$$

so the second result is clear by Corollary II.3.9. □

Theorem II.3.16. *The map $\Psi \coloneqq id_X$ is an orbit equivalence between T and S. Moreover, given a non-decreasing map $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$, this orbit equivalence is φ-integrable if one of the following two conditions is satisfied:*

(C1) the series $\sum \frac{\varphi(h_{n+1})}{h_n}$ converges;

(C2) the series

$$\sum_{n\geqslant 0} \frac{1}{h_{n+2}} \sum_{\substack{0\leqslant x_n < q_n,\\ 0\leqslant x_{n+1} < q_{n+1},\\ \sigma^{(n)}_{x_{n+1}}(x_n) \neq q_n - 1}} \varphi\left(h_n\left(1 + \left|\left(\sigma^{(n)}_{x_{n+1}}\right)^{-1}(\sigma^{(n)}_{x_{n+1}}(x_n) + 1) - x_n\right|\right)\right)$$

$$\text{and } \sum_{n\geqslant 0} \frac{1}{h_{n+2}} \sum_{\substack{0\leqslant x_n \leqslant q_n - 2,\\ 0\leqslant x_{n+1} < q_{n+1}}} \varphi\left(h_n\left(1 + \left|\sigma^{(n)}_{x_{n+1}}(1 + x_n) - \sigma^{(n)}_{x_{n+1}}(x_n)\right|\right)\right)$$

converge.

As we notice in the next proof, we need coarse bounds to get that Condition (C1) implies φ-integrably orbit equivalence, whereas Condition (C2) is a finer hypothesis. For Theorem J, Condition (C2) will enable us to exploit the sublinearity of the map φ, and Condition (C1) will be enough for Theorems G and I.

Proof of Theorem II.3.16. By Proposition II.3.15, the set of points $x \in X$ satisfying $Tx = S^{c_T(x)}x$ and $Sx = T^{c_S(x)}x$ for integers $c_T(x)$ and $c_S(x)$ defined by (II.4) and (II.5) have full measure, so the map id_X is an orbit equivalence between S and T.

The value of $c_T(x)$ gives the following bound:

$$|c_T(x)| \leqslant h_{N_1}\left|\left(\sigma^{(N_1)}_{x_{N_1+1}}\right)^{-1}(\sigma^{(N_1)}_{x_{N_1+1}}(x_{N_1})+1)-x_{N_1}\right| + \underbrace{\sum_{i=0}^{N_1-1} h_i\,|y_i(x)-x_i|}_{\leqslant h_{N_1}} \tag{II.6}$$

with $N_1 = N^+(\psi(x))$. Given $n \geqslant 0$, $z_n \in \{0,\dots,q_n-1\}$ and $z_{n+1} \in \{0,\dots,q_{n+1}-1\}$ such that $\sigma^{(n)}_{z_{n+1}}(z_n) \neq q_n-1$, we have

$$\mu(\{x \in X \mid N^+(\psi(x)) = n, x_n = z_n, x_{n+1} = z_{n+1}\}) = \frac{1}{h_{n+2}}.$$

We finally get

$$\begin{aligned}\int_X \varphi(|c_T(x)|)\mathrm{d}\mu(x) &= \sum_{n\geqslant 0}\sum_{\substack{0\leqslant z_n<q_n,\\ 0\leqslant z_{n+1}<q_{n+1},\\ \sigma^{(n)}_{z_{n+1}}(z_n)\neq q_n-1}} \int_{\substack{N^+(\psi(x))=n,\\ x_n=z_n,\\ x_{n+1}=z_{n+1}}} \varphi(|c_T(x)|)\mathrm{d}\mu(x)\\ &\leqslant \sum_{n\geqslant 0}\frac{1}{h_{n+2}}\sum_{\substack{0\leqslant z_n<q_n,\\ 0\leqslant z_{n+1}<q_{n+1},\\ \sigma^{(n)}_{z_{n+1}}(z_n)\neq q_n-1}} \varphi\left(h_n\left(1+\left|\left(\sigma^{(n)}_{z_{n+1}}\right)^{-1}(\sigma^{(n)}_{z_{n+1}}(z_n)+1)-z_n\right|\right)\right).\end{aligned}$$

From Inequality (II.6), we also get $|c_T(x)| \leqslant h_{N_1+1}$ and the following coarser bound:

$$\begin{aligned}\int_X \varphi(|c_T(x)|)\mathrm{d}\mu(x) &= \sum_{n\geqslant 0}\sum_{\substack{0\leqslant z_n<q_n,\\ 0\leqslant z_{n+1}<q_{n+1},\\ \sigma^{(n)}_{z_{n+1}}(z_n)\neq q_n-1}} \int_{\substack{N^+(\psi(x))=n,\\ x_n=z_n,\\ x_{n+1}=z_{n+1}}} \varphi(|c_T(x)|)\mathrm{d}\mu(x)\\ &\leqslant \sum_{n\geqslant 0}\frac{1}{h_{n+2}}\sum_{\substack{0\leqslant z_n<q_n,\\ 0\leqslant z_{n+1}<q_{n+1},\\ \sigma^{(n)}_{z_{n+1}}(z_n)\neq q_n-1}} \varphi\left(h_{n+1}\right)\\ &\leqslant \sum_{n\geqslant 0}\frac{1}{h_n}\varphi(h_{n+1}).\end{aligned}$$

For the other cocycle, we have

$$\begin{aligned}|c_S(x)| &\leqslant h_{N_2}\left|\sigma^{(N_2)}_{x_{N_2+1}}(1+x_{N_2})-\sigma^{(N_2)}_{x_{N_2+1}}(x_{N_2})\right|\\ &\quad+\underbrace{h_{N_2-1}\left|\sigma^{(N_2-1)}_{1+x_{N_2}}(0)-\sigma^{(N_2-1)}_{x_{N_2}}(x_{N_2-1})\right| + \sum_{i=0}^{N_2-2} h_i\left|\sigma^{(i)}_0(0)-\sigma^{(i)}_{x_{i+1}}(x_i)\right|}_{\leqslant h_{N_2}}.\end{aligned}$$

with $N_2 = N^+(x)$. Moreover it is easy to get

$$\mu(\{x \in X \mid N^+(x) = n, x_n = z_n, x_{n+1} = z_{n+1}\}) = \frac{1}{h_{n+2}}$$

for every $n \geqslant 0$, $z_n \in \{0,\dots,q_n-2\}$ and $z_{n+1} \in \{0,\dots,q_{n+1}-1\}$. Thus we find a bound on the φ-integral of c_S with the same method as c_T. □

II.3.e Extension to a homeomorphism on the Cantor set, strong orbit equivalence

We move on to a topological viewpoint. We give a sufficient condition for an odomutant to have an extension to a homeomorphism. It turns out that in this case the orbit equivalence that we obtained in the last section is a strong orbit equivalence.

Proposition II.3.17. *Assume that $\sigma_i^{(n)}(0) = 0$ and $\sigma_i^{(n)}(q_n - 1) = q_n - 1$ for every $n \geqslant 0$ and every $0 \leqslant i \leqslant q_n - 1$. Then the odomutant T admits a unique extension, also denoted by T, which is a homeomorphism on the whole compact set $X = \prod_{n\geqslant 0} \{0, 1, \dots, q_n - 1\}$. It is furthermore strongly orbit equivalent to the associated odometer S. In particular, it follows from Proposition II.2.20 that T is uniquely ergodic.*

Remark II.3.18. In this case, the equality $S \circ \psi(x) = \psi \circ T(x)$ holds for all $x \in X$.

Proof of Proposition II.3.17. Since, for every $n \geqslant 0$, the points 0 and $q_n - 1$ are fixed by the n-th permutations, x^- is the only point $x \in X$ satisfying $\psi(x) = x^-$ and x^+ is the only point $x \in X$ satisfying $\psi(x) = x^+$. This implies that we have

$$\psi^{-1}(X_\infty^-) = X_\infty^- = X\backslash\{x^-\} \text{ and } \psi^{-1}(X_\infty^+) = X_\infty^+ = X\backslash\{x^+\},$$

and T is a bijection from $X\backslash\{x^+\}$ to $X\backslash\{x^-\}$, so we set $Tx^+ \coloneqq x^-$. The map $T\colon X \to X$ is now a well-defined bijection.

The odometer S and the maps ψ_n are continuous on X so it is not difficult to see that T is continuous on each point of $X\backslash\{x^+\}$. It is easy to check the equality $T([q_0 - 1, \dots, q_n - 1]_{n+1}) = [0, \dots, 0]_{n+1}$, so the continuity at x^+ is clear. Therefore $T\colon X \to X$ is continuous and invertible, where X is a Haussdorf compact space, so T is a homeomorphism.

By Proposition II.3.15, we have $Tx = S^{c_T(x)}x$ and $Sx = T^{c_S(x)}$ for every $x \in X_\infty^+$, with $c_T(x)$ and $c_S(x)$ defined by (II.4) and (II.5). These relations are extended at x^+, with $c_T(x^+) = c_S(x^+) = 1$. Thus S and T have the same orbits and it is clear that the cocycles are continuous on X_∞^+ (x^+ is the only point of discontinuity if the cocycles are not continuous).[14] □

II.4 On non-preservation of loose Bernoullicity property under $\mathrm{L}^{<1/2}$ orbit equivalence

In this section, we prove that $\mathrm{L}^{<1/2}$ orbit equivalence (in particular Shannon orbit equivalence) does not imply even Kakutani equivalence.

Theorem II.4.1. *There exists an ergodic probability measure-preserving bijection T which is $\mathrm{L}^{<1/2}$ orbit equivalent (in particular Shannon orbit equivalent) to the dyadic odometer but not evenly Kakutani equivalent to it.*

Feldman [Fel76] has built a zero-entropy system which is not loosely Bernoulli. In his construction, for some partition that we will specify, the elements in $[0, \dots, 0]_n$ produce words, describing the future, which are not pairwise f-close for the f-metric introduced in Section II.2.d (therefore, the underlying system is not loosely Bernoulli). The goal is to describe his system as an odomutant built from the dyadic odometer and permutations that we are going to define. These permutations will fix 0, so that we will be able to read the words produced by the points at the bottom of the towers (using Lemmas II.A.1 and II.A.3), with respect to the partition that we will consider.

Let us set $\tilde{q}_n \coloneqq 2^{n+10}$, $q_n \coloneqq (\tilde{q}_n)^{2\tilde{q}_{n+1}+3}$ and $c_n \coloneqq \frac{q_n}{\tilde{q}_n} = (\tilde{q}_n)^{2\tilde{q}_{n+1}+2}$ for every $n \geqslant 0$, $h_0 \coloneqq 1$ and $h_{n+1} \coloneqq q_n h_n$. We inductively define words $a_i^{(n)}$ (we keep the notations of

[14] We can notice that we have $\{T^n x^- \mid n \in \mathbb{N}\} = \{S^n x^- \mid n \in \mathbb{N}\}$ and $\{T^{-n} x^+ \mid n \in \mathbb{N}\} = \{S^{-n} x^+ \mid n \in \mathbb{N}\}$.

Feldman in his paper) for every $n \geqslant 0$ and every $i \in \{0, \dots, \tilde{q}_n - 1\}$. Let us start with $\tilde{q}_0$ different letters $a_0^{(0)}, \dots, a_{\tilde{q}_0-1}^{(0)}$ seen as words of length $h_0 = 1$. For $n \geqslant 0$, if words $a_0^{(n)}, \dots, a_{\tilde{q}_n-1}^{(n)}$ of length h_n have been set, then we define new words $a_0^{(n+1)}, \dots, a_{\tilde{q}_{n+1}-1}^{(n+1)}$, of length h_{n+1}, by

$$a_j^{(n+1)} = \left\langle \langle a_0^{(n)} \rangle^{(\tilde{q}_n)^{2(j+1)}} \langle a_1^{(n)} \rangle^{(\tilde{q}_n)^{2(j+1)}} \dots \langle a_{\tilde{q}_n-1}^{(n)} \rangle^{(\tilde{q}_n)^{2(j+1)}} \right\rangle^{(\tilde{q}_n)^{2(\tilde{q}_{n+1}-j)}},$$

where $\langle w \rangle^k$ denotes the concatenation of k copies of a word w.

For $j \in \{0, 1, \dots, \tilde{q}_{n+1} - 1\}$, $\tau_j^{(n)}$ is a permutation of the set $\{0, 1, \dots, q_n - 1\}$ which permutes the entries of the finite sequence

$$\boldsymbol{u} \coloneqq (\underbrace{a_0^{(n)}, \dots, a_0^{(n)}}_{c_n \text{ times}}, \underbrace{a_1^{(n)}, \dots, a_1^{(n)}}_{c_n \text{ times}}, \dots, \underbrace{a_{\tilde{q}_n-1}^{(n)}, \dots, a_{\tilde{q}_n-1}^{(n)}}_{c_n \text{ times}})$$

so that the concatenation gives $a_j^{(n+1)}$, namely $\tau_j^{(n)}$ satisfies

$$a_j^{(n)} = u_{\tau_j^{(n)}(0)} \cdot u_{\tau_j^{(n)}(1)} \cdot \dots \cdot u_{\tau_j^{(n)}(q_n-1)}.$$

We now consider the odomutant T associated to the odometer S on the space $X = \prod_{n \geqslant 0} \{0, 1, \dots, q_n - 1\}$, and built with uniformly $\boldsymbol{c}$-multiple permutations $\tau_j^{(n)}$ where $\boldsymbol{c} \coloneqq (c_n, \tilde{q}_n)_{n \geqslant 0}$.

In view of the cutting-and-stacking construction behind the definition of this odomutant, we can be convinced that T is isomorphic to the non Bernoulli system built by Feldman. However we give more details on the fact that T is not loosely Bernoulli, based on the justifications of Feldman. Given $n \geqslant 0$, Lemmas II.A.1 and II.A.3 in Appendix II.A imply that the words $[\tilde{\mathcal{P}}(1)]_{h_n}(x)$ for $x \in [0, \dots, 0]_n$ (i.e. the points x at the bottom of the towers at step n) exactly correspond to the words $a_0^{(n)}, \dots, a_{\tilde{q}_n-1}^{(n)}$. As in [Fel76], the properties we are interested in can be deduced purely from this fact. Indeed, given any point x not necessarily at the bottom of the towers at step n, the word $[\tilde{\mathcal{P}}(1)]_{h_n}(x)$ is the concatenation of the tail of some $a_i^{(n)}$ and the beginning of some $a_j^{(n)}$, and this observation leads us to apply the same reasoning as in [Fel76, Step III and Step V in p. 36] to conclude that T has zero entropy and $(T, \tilde{\mathcal{P}}(1))$ is not loosely Bernoulli using the characterization provided by Theorem II.2.8. Therefore T is not loosely Bernoulli.

Proof of Theorem G. Let S be the odometer on $X = \prod_{n \geqslant 0} \{0, 1, \dots, q_n - 1\}$ (so S is loosely Bernoulli) and T the odomutant described above, which is not loosely Bernoulli, so that S and T are not evenly Kakutani equivalent by the theory of Ornstein, Rudolph and Weiss (see Theorem II.2.9). Note that S is the dyadic odometer since the integers q_n are powers of 2.

Let us prove that S and T are $\mathrm{L}^{<1/2}$ orbit equivalent, using Condition (C1) of Theorem II.3.16. Let us fix a real number p satisfying $0 < p < 1/2$. We have

$$h_n = h_{n-1} \tilde{q}_{n-1}^{2\tilde{q}_n+3} = h_{n-1} 2^{(n+9)(2^{n+11}+3)}$$

and this gives $h_n = 2^{S_n}$ with

$$S_n \coloneqq \sum_{i=1}^{n} (i+9)(2^{i+11}+3) = Cn2^n + D2^n + o(2^n)$$

for some positive constants C and D. For a fixed constant $C' \in [C, \frac{C}{2p}]$ and for a sufficiently large integer n, we have

$$Cn2^n < S_n < C'n2^n,$$

this gives

$$h_n < 2^{C'n2^n} = \left(2^{C(n-1)2^{n-1}}\right)^{2\frac{C'}{C}} 2^{C'2^n} < \left(2^{S_{n-1}}\right)^{2\frac{C'}{C}} 2^{C'2^n} = (h_{n-1})^{2\frac{C'}{C}} 2^{C'2^n}$$

and

$$\frac{{h_n}^p}{h_{n-1}} < (h_{n-1})^{2\frac{C'}{C}p-1} 2^{C'p2^n} < \left(2^{C'(n-1)2^{n-1}}\right)^{2\frac{C'}{C}p-1} 2^{C'p2^n}.$$

Since we have $2\frac{C'}{C}p < 1$, the series $\sum \frac{\varphi(h_n)}{h_{n-1}}$ converges for $\varphi(x) = x^p$, so we are done by Theorem II.3.16. □

II.5 On non-preservation of entropy under orbit equivalence with almost log-integrable cocycles

In this section, we prove that orbit equivalence with almost log-integrable cocycles does not preserve entropy. The statement is actually stronger, with a topological framework:

Theorem II.5.1. *Let α be either a positive real number or $+\infty$. Let S be an odometer whose associated supernatural number $\prod_{p\in\Pi} p^{k_p}$ satisfies the following property: there exists a prime number $p_\star$ such that $k_{p_\star} = +\infty$. Then there exists a Cantor minimal homeomorphism T such that*

1. $\mathrm{h_{top}}(T) = \alpha$;
2. *there exists a strong orbit equivalence between S and T, which is φ_m-integrable for all integers $m \geqslant 0$,*

where φ_m denotes the map $t \to \frac{\log t}{\log^{(\circ m)} t}$ and $\log^{(\circ m)}$ the composition $\log \circ \ldots \circ \log$ (m times).

We will crucially use the combinatorial lemmas stated in Appendix II.A. The cases $\alpha < +\infty$ and $\alpha = +\infty$ will be in fact separated, but in both proofs, we will apply the following lemma which will be useful for the quantification of the cocycles.

Lemma II.5.2. *Let $(q_n)_{n\geqslant 0}$ be a sequence of integers greater than or equal to 2, and let $\beta > 0$ such that*

$$\liminf_{n\to+\infty} \frac{\log q_n}{h_n} \geqslant \beta$$

where $h_n \coloneqq q_0 \ldots q_{n-1}$. Then, for every integer $m \geqslant 0$, we have

$$\frac{1}{\log^{\circ m}(q_{n+m})} \leqslant \exp(-\beta h_n)$$

for all sufficiently large integers n. In particular, the sequence $\left(\frac{1}{\log^{\circ m}(q_n)}\right)_{n\geqslant 0}$ is summable.

Proof of Lemma II.5.2. Let us consider an integer N such that

$$\forall n \geqslant N,\ \beta h_n \geqslant 1.$$

For every $n \geqslant N$, we have

$$\log q_{n+1} \geqslant \beta h_{n+1} = q_n \times \beta h_n \geqslant q_n.$$

By induction, we easily get for every $n \geqslant N$,

$$\log^{\circ m}(q_{n+m}) \geqslant q_n,$$

so we have $\log^{\circ m}(q_{n+m}) \geqslant \exp(\beta h_n)$. □

Before the proof of Theorem II.5.1 in the case $\alpha < +\infty$, we need two preliminary lemmas. The first one (Lemma II.5.3) provides permutations so that we can easily compute the entropy of the underlying odomutant. The second one (Lemma II.5.5) proves that the formula given by Lemma II.5.3 enables us to get all possible finite values of the entropy with a proper choice of parameters.

Lemma II.5.3. *Let $(q_n)_{n\geqslant 0}$ be a sequence of integers greater than or equal to 2 and satisfying $q_{n+1} \leqslant (q_n-2)!$. Then there exist permutations $\sigma_{x_{n+1}}^{(n)}$, for $n \geqslant 0$ and $x_{n+1} \in \{0,\dots,q_{n+1}-1\}$, satisfying the following properties:*

1. *for every $n \geqslant 0$, the maps $\sigma_0^{(n)}, \sigma_1^{(n)}, \dots, \sigma_{q_{n+1}-1}^{(n)}$ are pairwise different permutations of the set $\{0,\dots,q_n-1\}$, fixing 0 and q_n-1;*
2. *the topological entropy of the underlying odomutant is equal to*

$$\lim_{n\to+\infty} \frac{\log q_n}{h_n}.$$

Remark II.5.4. The entropy $\mathrm{h_{top}}(T)$ is well-defined by Lemma II.3.17, as well as the limit $\lim \frac{\log q_n}{h_n}$. Indeed, the sequence $\left(\frac{\log q_n}{h_n}\right)_{n\geqslant 0}$ is decreasing since we have $q_n < (q_{n-1})^{q_{n-1}}$.

Proof of Lemma II.5.3. By Lemma II.A.6 in Appendix II.A, we can choose permutations satisfying the first item and such that the following hold for every $n \geqslant \ell \geqslant 1$:

$$q_n \leqslant N(\mathcal{P}(\ell)_0^{h_n-1}) \leqslant h_{n-1} q_n q_{n-1}^2 2^{q_{n-1}}.$$

On the one hand, we have

$$\mathrm{h_{top}}(T) \geqslant \mathrm{h_{top}}(T,\mathcal{P}(1)) = \lim_{n\to+\infty} \frac{\log N(\mathcal{P}(1)_0^{h_n-1})}{h_n} \geqslant \lim_{n\to+\infty} \frac{\log q_n}{h_n}.$$

On the other hand, this gives

$$\begin{aligned}
\mathrm{h_{top}}(T,\mathcal{P}(\ell)) &= \lim_{n\to+\infty} \frac{\log N(\mathcal{P}(\ell)_0^{h_n-1})}{h_n} \\
&\leqslant \lim_{n\to+\infty} \frac{\log h_{n-1} + \log q_n + 2\log q_{n-1} + q_{n-1}\log 2}{h_n} \\
&= \lim_{n\to+\infty} \frac{\log q_n}{h_n}
\end{aligned}$$

and we finally get, using Theorem II.2.2,

$$\mathrm{h_{top}}(T) = \lim_{\ell\to+\infty} \mathrm{h_{top}}(T,\mathcal{P}(\ell)) \leqslant \lim_{n\to+\infty} \frac{\log q_n}{h_n}.$$

Hence the result. □

Lemma II.5.5. *Let α be a positive real number and $\prod_{p\in\Pi} p^{k_p}$ a supernatural number. We assume that there exists a prime number $p_\star$ such that $k_{p_\star} = +\infty$. Then there exists a sequence $(q_n)_{n\geqslant 0}$ of integers greater than or equal to 2, satisfying the following properties:*

1. *we have $q_{n+1} \leqslant (q_n-2)!$ for every $n \geqslant 0$;*
2. *the sequence $\left(\frac{\log q_n}{q_0\cdots q_{n-1}}\right)_{n\geqslant 0}$ tends to α;*
3. *we have $\sum_{n\geqslant 0} \nu_p(q_n) = k_p$ for every $p \in \Pi$.*

Proof of Lemma II.5.5. Let K be a large enough power of $p_\star$ so that the following property holds: for every integer q satisfying $q \geqslant K$, we have $(q-2)! \geqslant K$. Let

$$N \coloneqq \sum_{p \in \Pi \setminus \{p_\star\}} k_p \in \mathbb{N} \cup \{+\infty\}$$

and $(p_i)_{1 \leqslant i \leqslant N}$ be a sequence of prime numbers satisfying $\sum_{1 \leqslant i \leqslant N} \mathbb{1}_{p_i = p} = k_p$ for every $p \in \Pi \setminus \{p_\star\}$, and $\sum_{1 \leqslant i \leqslant N} \mathbb{1}_{p_i = p_\star} = 0$.[15] By induction, we build a sequence $(q_n)_{n \geqslant 0}$ of integers greater than or equal to 2, an increasing sequence $(i_n)_{n \geqslant 0}$ and a non-decreasing sequence $(j_n)_{n \geqslant 0}$ of non-negative integers, satisfying the following properties:

1. $q_0 > \frac{2}{\alpha} \log p_\star$ and $\log q_0 \geqslant \alpha + 5$;
2. $K \leqslant q_{n+1} \leqslant (q_n - 2)!$ for every $n \geqslant 0$;
3. for every $n \geqslant 1$, the following holds:
$$\alpha + \frac{5}{q_0 \dots q_{n-1}} \leqslant \frac{\log q_n}{q_0 \dots q_{n-1}} \leqslant \alpha + \frac{2}{q_0 \dots q_{n-2}},$$
where $q_0 \dots q_{n-2}$ is equal to 1 if $n = 1$;
4. $q_n = K^{i_n} \prod_{j=j_{n-1}+1}^{j_n} p_j$ for every $n \geqslant 0$, with $j_{-1} = j_0 = 0$ (so we have $q_0 = p_\star^{i_0}$);
5. $j_n \underset{n \to \infty}{\to} N$ if $N < +\infty$, $j_n \underset{n \to \infty}{\to} +\infty$ otherwise.

Such a sequence $(q_n)_{n \geqslant 0}$ satisfies the assumptions of the lemma.

We choose a large enough integer i_0 such that the hypotheses on $q_0 \coloneqq K^{i_0}$ are satisfied. Let $n \geqslant 0$. Assume that the integers $q_0, \dots, q_n, i_0, \dots, i_n, j_1, \dots, j_n$ have been defined and let us build $q_{n+1}, i_{n+1}, j_{n+1}$. In particular, the integers $q_0, \dots, q_n$ satisfy

$$\forall k \in \{0, \dots, n\}, \frac{\log q_k}{q_0 \dots q_{k-1}} \geqslant \alpha + \frac{5}{q_0 \dots q_{k-1}}.$$

Let j_{n+1} be the greatest integer k satisfying

- $k \geqslant j_n$ and, if $N < +\infty$, $k \leqslant N$;
- $K \prod_{j=j_n+1}^{k} p_j \leqslant (q_n - 2)!$;
- $\frac{\log\left(\prod_{j=j_n+1}^{k} p_j\right)}{q_0 \dots q_n} \leqslant \frac{\alpha}{2}$.

Let us consider the sequence $(\alpha_i)_{i \geqslant 1}$ defined by

$$\alpha_i \coloneqq \frac{\log\left(K^i \prod_{j=j_n+1}^{j_{n+1}} p_j\right)}{q_0 \dots q_n},$$

and let I be the greatest integer i such that $K^i \prod_{j=j_n+1}^{j_{n+1}} p_j \leqslant (q_n - 2)!$. The sequence $(\alpha_i)_{i \geqslant 1}$ is an arithmetic progression with common difference

$$\frac{\log K}{q_0 \dots q_n}.$$

[15] "$(p_i)_{i \geqslant 1}$" and "$\sum_{i \geqslant 1}$" in the case $N = +\infty$.

Moreover, we have

$$\alpha_1 \leqslant \frac{\log K}{q_0} + \frac{\log\left(\prod_{j=j_n+1}^{j_{n+1}} p_j\right)}{q_0 \dots q_n} \leqslant \alpha,$$

and, using the assumption on q_n and the inequalities $\log(k!) \geqslant k\log(k) - k$ and $\log k \leqslant k$,

$$\begin{aligned}
\alpha_{I+1} &= \frac{\log\left(K^{I+1}\prod_{j=j_n+1}^{j_{n+1}} p_j\right)}{q_0 \dots q_n} \\
&\geqslant \frac{\log((q_n-2)!)}{q_0 \dots q_n} \\
&= \frac{\log(q_n!)}{q_0 \dots q_n} - \frac{\log(q_n-1)}{q_0 \dots q_n} - \frac{\log(q_n)}{q_0 \dots q_n} \\
&\geqslant \frac{q_n\log(q_n) - q_n}{q_0 \dots q_n} - \frac{\log(q_n-1)}{q_0 \dots q_n} - \frac{\log(q_n)}{q_0 \dots q_n} \\
&\geqslant \frac{\log(q_n)}{q_0 \dots q_{n-1}} - \frac{3}{q_0 \dots q_{n-1}} \\
&\geqslant \alpha + \frac{2}{q_0 \dots q_{n-1}}.
\end{aligned}$$

Therefore, there exists $i' \in \{1, \dots, I\}$ such that

$$\alpha + \frac{2}{q_0 \dots q_{n-1}} - \frac{\log K}{q_0 \dots q_n} \leqslant \alpha_{i'} \leqslant \alpha + \frac{2}{q_0 \dots q_{n-1}}.$$

Since we have $2q_n \geqslant 2K \geqslant K + \log K \geqslant 5 + \log K$, we get

$$\alpha + \frac{5}{q_0 \dots q_n} \leqslant \alpha_{i'} \leqslant \alpha + \frac{2}{q_0 \dots q_{n-1}}.$$

It remains to set $i_{n+1} \coloneqq i'$ and $q_{n+1} \coloneqq K^{i_{n+1}}\prod_{j=j_n+1}^{j_{n+1}} p_j$.

Finally, we have to check that the increasing sequence $(j_n)_{n\geqslant 1}$ of integers diverges if $N = +\infty$, or converges to N if N is finite. If it was not the case, then there would exist a positive integer n such that the following hold for every $k \geqslant n$:

$$Kp_{j_n+1} > (q_k - 2)! \text{ or } \frac{\log p_{j_n+1}}{q_0 \dots q_k} > \frac{\alpha}{2}.$$

But the integers q_k are greater than or equal to 2, so it would mean that the sequence $(q_k)_{k\geqslant 0}$ is bounded, which is in contradiction with the inequality $\log q_k \geqslant \alpha q_0 \dots q_k$, so $(j_n)_{n\geqslant 1}$ satisfies the desired property. Hence the lemma. □

Proof of Theorem I in the case $\alpha < +\infty$. Let α be a positive real number and let S be an odometer whose associated supernatural number $\prod_{p\in\Pi} p^{k_p}$ satisfies the following property: there exists a prime number $p_\star$ such that $k_{p_\star} = +\infty$. Without loss of generality, S is the odometer on the Cantor set $X \coloneqq \prod_{n\geqslant 0}\{0, 1, \dots, q_n - 1\}$, where the sequence $(q_n)_{n\geqslant 0}$ satisfies $2 \leqslant q_n \leqslant (q_n - 2)!$ for every $n \geqslant 0$ and $\frac{\log q_n}{h_n} \to \alpha$. The existence of such a sequence is granted by Lemma II.5.5. By Lemma II.5.3 and Proposition II.3.17, we can find families of permutations such that the underlying odomutant T is a homeomorphism strongly orbit equivalent to S and its topological entropy is equal to α.

Finally, given an increasing map $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$, the orbit equivalence is φ-integrable if $(\varphi(h_{n+1})/h_n)_n$ is summable, by Theorem II.3.16 (see Condition (C1)). This holds for

$\varphi(x) = \frac{\log(x)}{\log^{\circ m}(x)}$. Indeed, we have

$$\begin{aligned}\frac{\varphi(h_{n+1})}{h_n} &= \frac{1}{\log^{\circ m}(h_{n+1})}\frac{\log(h_{n+1})}{h_n}\\ &\leqslant \frac{1}{\log^{\circ m}(q_n)}\left(\frac{\log(h_n)}{h_n}+\frac{\log q_n}{h_n}\right)\\ &\leqslant \frac{1}{\log^{\circ m}(q_n)}\left(1+\frac{\log q_n}{h_n}\right),\end{aligned}$$

so using the monotonicity of the sequence $\left(\frac{\log q_n}{h_n}\right)_{n\geqslant 0}$ (see Remark II.5.4), we get

$$\frac{\varphi(h_{n+1})}{h_n} \leqslant \frac{1}{\log^{\circ m}(q_n)}(1+\log q_0)$$

and we are done by Lemma II.5.2 with $\beta = \alpha$. □

In the case $\alpha = +\infty$, we prove Theorem I with the same methods as in [BH94], but with our formalism. We will consider an odomutant T on $\prod_{n\geqslant 0}\{0,1,\dots,q_n-1\}$, built with uniform $\boldsymbol{c}$-multiple permutations $\tau_j^{(n)}$, where $\boldsymbol{c} = (c_n,\tilde{q}_n)_{n\geqslant 0}$, and for every $n\geqslant 0$ and every $0\leqslant j<\tilde{q}_{n+1}$, $\tau_j^{(n)}$ is a permutation on $\{0,1,\dots,q_n-1\}$ fixing 0 and q_n-1. For every $n\geqslant 0$, we assume that the map

$$j\in\{0,1,\dots,\tilde{q}_{n+1}-1\}\mapsto(\tau_j^{(n)}(I_0^{(n)}),\dots,\tau_j^{(n)}(I_{\tilde{q}_n-1}^{(n)}))$$

is κ_{n+1}-to-one for some positive integer κ_{n+1} (as in the assumption of Lemma II.A.4). Finally, we write $\chi_n \coloneqq \frac{\tilde{q}_n}{\kappa_n}$ for every $n\geqslant 1$. Then we have $q_n = c_n\tilde{q}_n$ for every $n\geqslant 0$ and $\tilde{q}_n = \kappa_n\chi_n$ for every $n\geqslant 1$. The sequences (h_n), $(\tilde{q}_n)$, (c_n), (κ_n), (χ_n) respectively correspond to the sequences (l_k), (m_k), (n_k), (j_k), $(\overline{m}_k)$ in [BH94]. The integer χ_{n+1} is the number of sequences of the form $(\tau_j^{(n)}(I_0^{(n)}),\dots,\tau_j^{(n)}(I_{\tilde{q}_n-1}^{(n)}))$ for $j\in\{0,\dots,\tilde{q}_{n+1}-1\}$, so we have

$$1\leqslant\chi_{n+1}\leqslant\frac{(q_n-2)!}{c_n!^{\tilde{q}_n-2}(c_n-1)!^2},$$

thus motivating the following lemma.

Lemma II.5.6. *Let p, $\tilde{q}$ and c be positive integers and $q\coloneqq\tilde{q}c$. Assume that $p\geqslant 2$ and $q\geqslant 3$. Then the greatest power of p less than or equal to*

$$\frac{(q-2)!}{c!^{\tilde{q}-2}(c-1)!^2}$$

is greater than or equal to

$$\frac{1}{p}\frac{1}{\tilde{q}^2}\left(\frac{1}{ec}\right)^{\tilde{q}}\tilde{q}^q.$$

Proof of Lemma II.5.6. Using the inequalities

$$\left(\frac{k}{e}\right)^k\leqslant k!\leqslant e\left(\frac{k+1}{e}\right)^{k+1},$$

we get

$$\frac{(q-2)!}{c!^{\tilde{q}-2}(c-1)!^2} = \frac{c^2}{q(q-1)}\frac{q!}{(c-1)!^{\tilde{q}}}\frac{1}{c^{\tilde{q}}}\geqslant\left(\frac{c}{q}\right)^2\frac{\left(\frac{q}{e}\right)^q}{e^{\tilde{q}}\left(\frac{c}{e}\right)^{c\tilde{q}}}\frac{1}{c^{\tilde{q}}} = \frac{1}{\tilde{q}^2}\left(\frac{1}{ec}\right)^{\tilde{q}}\tilde{q}^q$$

and we are done. □

Proof of Theorem I in the case $\alpha = +\infty$. Let

$$N \coloneqq \sum_{p \in \Pi \setminus \{p_\star\}} k_p \in \mathbb{N} \cup \{+\infty\}$$

and $(p_i)_{1 \leqslant i \leqslant N}$ be a sequence of prime numbers satisfying $\sum_{1 \leqslant i \leqslant N} \mathbb{1}_{p_i = p} = k_p$ for every $p \in \Pi \setminus \{p_\star\}$, and $\sum_{1 \leqslant i \leqslant N} \mathbb{1}_{p_i = p_\star} = 0$.[16] Let us define $c_n = p_\star^n$ for every $n \geqslant 0$. By induction, we build sequences $(\kappa_n)_{n \geqslant 1}$ and $(\chi_n)_{n \geqslant 0}$ of integers, and a non-decreasing sequence $(j_n)_{n \geqslant 1}$ of non-negative integers, satisfying the following properties:

1. for every $n \geqslant 0$, χ_{n+1} is the greatest power of $p_\star$ less than or equal to $\frac{(q_n - 2)!}{c_n!^{\tilde{q}_n - 2}(c_n - 1)!^2}$, where $\tilde{q}_n = \kappa_n \chi_n$ (with $\kappa_0 \coloneqq 1$) and $q_n \coloneqq c_n \tilde{q}_n$;

2. $\kappa_n = p_\star^{h_n} \prod_{j = j_{n-1}+1}^{j_n} p_j$ for every $n \geqslant 1$, with $j_0 \coloneqq 0$;

3. $j_n \underset{n \to \infty}{\to} N$ if $N < +\infty$, $j_n \underset{n \to \infty}{\to} +\infty$ otherwise.

Let us define $\tilde{q}_0 \coloneqq p_\star$. Given $n \geqslant 0$, assume that $\chi_0, \dots, \chi_n, j_0, \dots, j_n, \kappa_1, \dots, \kappa_n$ have been set (if $n = 0$, then there is no integer κ_i). We define χ_{n+1} as the greatest power of $p_\star$ less than or equal to $\frac{(q_n - 2)!}{c_n!^{\tilde{q}_n}(c_n - 1)!^2}$, j_{n+1} as the greatest integer k satisfying

- $j_n \leqslant k$ and, if $N < +\infty$, $k \leqslant N$;
- $\prod_{j = j_n + 1}^{k} p_j \leqslant p_\star^{h_{n+1}}$,

and $\kappa_{n+1} \coloneqq p_\star^{h_{n+1}} \prod_{j = j_n + 1}^{j_{n+1}} p_j$. Let us define T as the odomutant built with uniform $\boldsymbol{c}$-multiple permutations $\tau_j^{(n)}$, with $\boldsymbol{c} \coloneqq (c_n, \tilde{q}_n)_{n \geqslant 0}$, and assume that the assumption of Lemma II.A.4 in Appendix II.A is satisfied: for every $n \geqslant 0$, the map

$$j \in \{0, 1, \dots, \tilde{q}_{n+1} - 1\} \mapsto (\tau_j^{(n)}(I_0^{(n)}), \dots, \tau_j^{(n)}(I_{\tilde{q}_n - 1}^{(n)}))$$

is κ_{n+1}-to-1. Note that the fact that χ_{n+1} is less than or equal to

$$\frac{(q_{n+1} - 2)!}{c_{n+1}!^{\tilde{q}_{n+1}}(c_{n+1} - 1)!^2}$$

enables us to find such families of permutations. It is straightforward to prove that $j_n \to +\infty$ if $N = \infty$, or $j_n \to N$ if $N < +\infty$, so T is an odomutant associated to S.

Lemma II.A.4 implies

$$N(\tilde{\mathcal{P}}(\ell)_0^{h_n - 1}) \geqslant \frac{\tilde{q}_n}{\prod_{k=\ell}^{n} \kappa_k^{h_n / h_k}}$$

for all $n \geqslant \ell \geqslant 1$. By Lemma II.5.6, we have for every $i \geqslant 1$,

$$\tilde{q}_i = \kappa_i \chi_i \geqslant \kappa_i \frac{1}{p_\star} \frac{1}{(\tilde{q}_{i-1})^2} \left(\frac{1}{e c_{i-1}} \right)^{\tilde{q}_{i-1}} (\tilde{q}_{i-1})^{q_{i-1}},$$

this gives

$$\tilde{q}_i^{1/h_i} \geqslant \kappa_i^{1/h_i} \left(\frac{1}{p_\star \tilde{q}_{i-1}^2} \right)^{1/h_i} \left(\frac{1}{e c_{i-1}} \right)^{1/(c_{i-1} h_{i-1})} \tilde{q}_{i-1}^{1/h_{i-1}}$$

[16] "$(p_i)_{i \geqslant 1}$" and "$\sum_{i \geqslant 1}$" in the case $N = +\infty$.

and we can apply this inequality many times to get

$$\begin{aligned}\tilde{q}_n^{1/h_n} &\geqslant \left(\prod_{i=1}^{n}\kappa_i^{1/h_i}\right)\left(\prod_{i=1}^{n}\left(\frac{1}{p_\star\tilde{q}_{i-1}^2}\right)^{1/h_i}\left(\frac{1}{ec_{i-1}}\right)^{1/(c_{i-1}h_{i-1})}\right)\tilde{q}_0\\ &\geqslant \left(\prod_{i=\ell}^{n}\kappa_i^{1/h_i}\right)\left(\prod_{i=1}^{n}\left(\frac{1}{p_\star\tilde{q}_{i-1}^2}\right)^{1/h_i}\left(\frac{1}{ec_{i-1}}\right)^{1/(c_{i-1}h_{i-1})}\right)p_\star^{\ell-1}\tilde{q}_0.\end{aligned}$$

Hence we have,

$$\frac{\log N(\tilde{\mathcal{P}}(\ell)_0^{h_n-1})}{h_n} \geqslant (\ell-1)\log p_\star + \log\tilde{q}_0 - \sum_{i=1}^{n}\left(\frac{\log\left(p_\star\tilde{q}_{i-1}^2\right)}{h_i} + \frac{\log\left(ec_{i-1}\right)}{c_{i-1}h_{i-1}}\right).$$

It is straightforward to check that the series $\sum_{i=1}^{+\infty}\left(\frac{\log\left(p_\star\tilde{q}_{i-1}^2\right)}{h_i} + \frac{\log\left(ec_{i-1}\right)}{c_{i-1}h_{i-1}}\right)$ converges and we denote by V its value. We are now able to get that T has infinite topological entropy:

$$\mathrm{h_{top}}(T) \geqslant \lim_{\ell\to+\infty}\mathrm{h_{top}}(T,\tilde{\mathcal{P}}(\ell)) \geqslant \lim_{\ell\to+\infty}\left((\ell-1)\log p_\star + \log\tilde{q}_0 - V\right) = +\infty.$$

Let us finally check Condition (C1) in Lemma II.3.16 to prove that there exists a strong orbit equivalence between T and S, which is φ_m-integrable for every $m \geqslant 0$, where $\varphi_m(x) = \frac{\log(x)}{\log^{(\circ m)}(x)}$. We first have $c_n \leqslant (p_\star)^{h_n}$, $\chi_n \leqslant (q_{n-1})^{q_{n-1}} \leqslant (h_n)^{q_{n-1}}$ and $\log\kappa_n \leqslant 2h_n\log p_\star$ by definition, so

$$\log h_{n+1} = \log h_n + \log c_n + \log\kappa_n + \log\chi_n \leqslant (1+3\log p_\star)h_n + q_{n-1}\log h_n,$$

this implies

$$\frac{\log h_{n+1}}{h_n} \leqslant (1+3\log p_\star) + \frac{\log h_n}{h_{n-1}}$$

and we get $\frac{\log h_{n+1}}{h_n} = O(n)$. Then, it remains to prove that the sequence $\left(\frac{n}{\log^{(\circ m)}(h_{n+1})}\right)_{n\geqslant 0}$ is summable. This is a consequence of Lemma II.5.2 with $\beta = \log p_\star$, since we have

$$\log q_n \geqslant \log\kappa_n \geqslant h_n\log p_\star$$

by definition of κ_n. So there exists a strong orbit equivalence between T and S, which is φ_m-integrable for every $m \geqslant 0$. □

II.6 Orbit equivalence with almost integrable cocycles

In this section, we prove that being orbit equivalent to an odometer, with almost integrable cocycles, does not imply being flip-conjugate to it.

Theorem II.6.1. *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear map and S an odometer. There exists a probability measure-preserving transformation T such that S and T are φ-integrably orbit equivalent but not flip-conjugate.*

For Theorems G and I, some invariants (loose Bernoullicity property, entropy) ensure that we build an odomutant T which is not flip-conjugate to the associated odometer S. For Theorem II.6.1, we use the fact that every odometer is coalescent (see Theorem II.2.14). Given a sublinear map $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$, the goal is to find families of permutations $\left(\sigma_{x_{n+1}}^{(n)}\right)_{0\leqslant x_{n+1}<q_{n+1}}$, for $n \geqslant 0$, such that the factor map

$$\psi\colon x \in X \to (\sigma_{x_1}^{(0)}(x_0), \sigma_{x_2}^{(1)}(x_1), \sigma_{x_3}^{(2)}(x_2), \ldots) \in X$$

from the associated odomutant T to S is not an isomorphism, with φ-integrable cocycles for the orbit equivalence between S and T.

Lemma II.6.2. *Let $(q_n)_{n\geqslant 0}$ be a sequence of integers greater or equal to 2. For every $n \geqslant 0$, let $\left(\sigma_{x_{n+1}}^{(n)}\right)_{0\leqslant x_{n+1}<q_{n+1}}$ be a family of permutations of the set $\{0,1,\dots,q_n-1\}$, defined by:*

$$\forall x_{n+1} \in \{0,\dots,q_{n+1}-1\},\ \forall i \in \{0,\dots,q_n-1\},\ \sigma_{x_{n+1}}^{(n)}(i) = i + x_{n+1} \bmod q_n.$$

Assume that the infinite product $\prod_{n\geqslant 0}\left(1-\frac{1}{q_n}\right)$ converges[17]*. Then $\psi\colon x \in X \to (\sigma_{x_{n+1}}^{(n)}(x_n))_{n\geqslant 0} \in X$ is not injective almost everywhere.*

Proof of Lemma II.6.2. Let $Y_1 \coloneqq \{x \in X \mid \forall n \geqslant 0, x_n \neq (q_n-1)\mathbb{1}_{n \text{ is even}}\}$ and $Y_2 \coloneqq \{x \in X \mid \forall n \geqslant 0, x_n \neq (q_n-1)\mathbb{1}_{n \text{ is odd}}\}$. It is straightforward to check that

$$\mu(Y_1) = \mu(Y_2) = \prod_{n\geqslant 0}\left(1-\frac{1}{q_n}\right) > 0.$$

Let $\theta\colon X \to X$ defined by:

$$\theta(x) \coloneqq (x_n + (-1)^n \bmod q_n)_{n\geqslant 0}.$$

The map θ is in $\mathrm{Aut}(X,\mu)$ since X can be seen as the compact group $\prod_{n\geqslant 0}\mathbb{Z}/q_n\mathbb{Z}$, with its Haar probability measure μ and θ as the translation by $((-1)^n)_{n\geqslant 0}$. Moreover, θ is a bijection from Y_1 to Y_2 and we have $\psi(\theta(x)) = \psi(x)$ for all $x \in Y_1$.

Let us prove by contradiction that ψ is not injective almost everywhere. Assume that ψ is injective on a measurable set X_0 of full measure. This hypothesis and the equality $\psi \circ \theta = \psi$ on Y_1 imply that the sets X_0 and $\theta(X_0 \cap Y_1)$ are disjoint. This finally gives

$$\mu((X_0)^c) \geqslant \mu\left(\theta(X_0 \cap Y_1)\right) = \mu(X_0 \cap Y_1) = \mu(Y_1) > 0$$

and we get a contradiction since $(X_0)^c$ has zero measure. □

Before the proof of Theorem J, we use a lemma stated in [CJLMT23] and which enables us to reduce to the case where the sublinear map φ is non-decreasing (actually the statement is stronger but we only need the monotonicity).

Lemma II.6.3 (Lemma 2.12 in [CJLMT23])**.** *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear function. Then there is a sublinear non-decreasing function $\tilde{\varphi}\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(t) \leqslant \tilde{\varphi}(t)$ for all t large enough.*

Proof of Theorem J. Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear map. If $\tilde{\varphi}$ is another sublinear map satisfying $\varphi(t) = O(\tilde{\varphi}(t))$, then $\tilde{\varphi}$-integrability implies φ-integrability. Therefore, by Lemma II.6.3, we assume without loss of generality that φ is non-decreasing.

Let $(q_n)_{n\geqslant 0}$ be a sequence of integers greater or equal to 2 and S the odometer on $X \coloneqq \prod_{n\geqslant 0}\{0,1,\dots,q_n-1\}$. The Halmos-von Neumann theorem implies that S is conjugate to the odometer on $\prod_{n\geqslant 0}\{0,1,\dots,q_{i_{n-1}}\dots q_{i_n-1}-1\}$ for any increasing sequence $(i_n)_{n\geqslant 0}$ satisfying $i_0 = 0$. Therefore, we can assume without loss of generality that the integers q_n are sufficiently large so that they satisfy the following properties:

1. $\prod_{n\geqslant 0}\left(1-\frac{1}{q_n}\right)$ converges[17];
2. the series $\sum \dfrac{\varphi(2h_n)}{h_n}$ converges.

[17] By definition, the infinite product $\prod_{n\geqslant 0}\left(1-\frac{1}{q_n}\right)$ converges if the sequence $\left(\prod_{k=0}^{n}\left(1-\frac{1}{q_k}\right)\right)_{n\geqslant 0}$ converges to a nonzero real number.

Let T be the odomutant built from S and the same families $\left(\sigma_{x_{n+1}}^{(n)}\right)_{0\leqslant x_{n+1}<q_{n+1}}$ as in Lemma II.6.2. By this lemma and Theorem II.2.14, S and T are not conjugate. Since S is conjugate to its inverse S^{-1} (by the Halmos-von Neumann theorem), S and T are not flip-conjugate.

It remains to quantify the cocycles, using Condition (C2) of Theorem II.3.16. Let $n \geqslant 0$ and $x_{n+1} \in \{0, \dots, q_{n+1}-1\}$, and $i \in \{0, \dots, q_n - 1\}$ such that $x_{n+1} = i \bmod q_n$. For every $x \in \{0, \dots, q_n - 2\} \backslash \{q_n - i - 1\}$, we have

$$\left(\sigma_{x_{n+1}}^{(n)}\right)^{-1} (\sigma_{x_{n+1}}^{(n)}(x_n) + 1) - x_n = \sigma_{x_{n+1}}^{(n)}(1 + x_n) - \sigma_{x_{n+1}}^{(n)}(x_n) = 1.$$

For $x_n = q_n - 1$, we consider the following bounds:

$$\left|\left(\sigma_{x_{n+1}}^{(n)}\right)^{-1} (\sigma_{x_{n+1}}^{(n)}(x_n) + 1) - x_n\right| \leqslant q_n$$

$$\text{and } \left|\sigma_{x_{n+1}}^{(n)}(1 + x_n) - \sigma_{x_{n+1}}^{(n)}(x_n)\right| \leqslant q_n.$$

We finally get

$$\begin{aligned}
&\sum_{n\geqslant 0} \frac{1}{h_{n+2}} \sum_{\substack{0\leqslant x_n<q_n,\\ 0\leqslant x_{n+1}<q_{n+1},\\ \sigma_{x_{n+1}}^{(n)}(x_n)\neq q_n-1}} \varphi\left(h_n\left(1 + \left|\left(\sigma_{x_{n+1}}^{(n)}\right)^{-1} (\sigma_{x_{n+1}}^{(n)}(x_n) + 1) - x_n\right|\right)\right) \\
&= \sum_{n\geqslant 0} \frac{1}{h_{n+2}} \sum_{\substack{0\leqslant x_n\leqslant q_n-2,\\ 0\leqslant x_{n+1}<q_{n+1}\\ x_n\neq q_n-i-1}} \varphi\left(h_n\left(1 + \left|\left(\sigma_{x_{n+1}}^{(n)}\right)^{-1} (\sigma_{x_{n+1}}^{(n)}(x_n) + 1) - x_n\right|\right)\right) \\
&\leqslant \sum_{n\geqslant 0} \frac{1}{h_{n+2}} \sum_{0\leqslant x_{n+1}<q_{n+1}} ((q_n - 2)\varphi(2h_n) + \varphi(h_n(1 + q_n))) \\
&\leqslant \sum_{n\geqslant 0} \frac{\varphi(2h_n)}{h_n} + \sum_{n\geqslant 0} \frac{\varphi(2h_{n+1})}{h_{n+1}} < +\infty
\end{aligned}$$

and similarly

$$\sum_{n\geqslant 0} \frac{1}{h_{n+2}} \sum_{\substack{0\leqslant x_n\leqslant q_n-2,\\ 0\leqslant x_{n+1}<q_{n+1}}} \varphi\left(h_n\left(1 + \left|\sigma_{x_{n+1}}^{(n)}(1 + x_n) - \sigma_{x_{n+1}}^{(n)}(x_n)\right|\right)\right) < \infty,$$

so S and T are φ-integrably orbit equivalent. □

Remark II.6.4. As Theorem I, the odomutants T in Theorem G and I can be built as homeomorphisms, with a strong orbit equivalence between them and the odometers S. This is clear for Theorem G since we may assume $\sigma^{(n,i)}(q_n - 1) = q_n - 1$ without loss of generality. For Theorem J, we have to slightly modify the settings in Lemma II.6.2 and its proof. For example, we can define $\sigma_{x_{n+1}}^{(n)}$ as the permutation mapping 0 to 0, $q_n - 1$ to $q_n - 1$ and $i \in \{1, \dots, q_n - 2\}$ to $1 + (i - 1 + x_{n+1} \bmod q_n - 2)$. The set Y_1 becomes the set of $x \in X$ such that $x_n \notin \{0, q_n - 2, q_n - 1\}$ if n is even, $x_n \notin \{0, 1, q_n - 1\}$ if n is odd, and vice versa for Y_2. Then the ideas remain the same.

II.A Some combinatorial properties

In this section, we fix an odomutant T built with uniformly $\boldsymbol{c}$-multiple permutations, with $\boldsymbol{c} = (c_n, \tilde{q}_n)_{n\geqslant 0}$ and $q_n \coloneqq c_n\tilde{q}_n$. We refer the reader to Definition II.3.3 for all the notations that we will use, although not defined in this section (for instance the partitions $\tilde{\mathcal{P}}(\ell)$, the segments $I_j^{(\ell)}$, etc).

In the proof of Theorem I, for combinatorial purposes appearing in the computation of topological entropy, we need to understand the dynamics of this odomutant with respect to the associated partition $\tilde{\mathcal{P}}(\ell)$ for some ℓ. Indeed, as explained in Example II.2.3, computing the topological entropy with respect to a clopen partition partly consists in counting words given by the associated coding map. Recall that, given $c_n = 1$ for every $n \geqslant 0$, and an odomutant built with $\boldsymbol{c}$-multiple permutations, $\tilde{\mathcal{P}}(\ell)$ is the partition $\mathcal{P}(\ell)$ in ℓ-cylinders of the space $X = \prod_{n\geqslant 0}\{0,\dots,q_n-1\}$, as introduced in Example II.2.3.

As we can notice in the proofs of the following results, it is more convenient for the computations that the permutations have common fixed points (here this is the point 0), as in Section II.3.e when one wants to extend an odomutant to a homeomorphism. With this assumption, at each step of the cutting-and-stacking construction, we can study the words produced by the points in the first level of the towers, and the recurrence relation describing such a word at step $n+1$ as a concatenation of words at step n (Lemmas II.A.1 and II.A.3). Counting only these words gives a lower bound of the number of all the words produced by the coding map, thus providing a lower bound of the topological entropy with respect to the clopen partition that we consider. If this lower bound of $\mathrm{h}_{\mathrm{top}}(T)$ diverges to $+\infty$, then we have built an odomutant of infinite entropy. This is the strategy that we will apply in the proof of Theorem I in the case $\alpha = +\infty$, using a lower bound on the number of words provided by Lemma II.A.4 when the odomutant satisfies some assumptions. Note that this lemma is a reformulation of the main ideas of Boyle and Handelman for the proof of their similar statement [BH94, Section 3]. In the case $\alpha < +\infty$, we will need an exact formula on the entropy. To this purpose, Lemma II.A.6 provides an upper bound of the number of all words produced by a coding map, and thus a finer upper bound of the entropy as we see in the proof of Theorem I.

Lemma II.A.1. *Let $\ell \geqslant 1$ and T be an odomutant built with uniformly $\boldsymbol{c}$-multiple permutations fixing 0.*

1. *For every $n \geqslant \ell-1$, for every $x_n \in \{0,1,\dots,q_n-1\}$, the set*
$$\{[\tilde{\mathcal{P}}(\ell)]_{h_n}(x) \mid x \in [0,\dots,0,x_n]_{n+1}\}$$
is a singleton, denoted by $\{W(\tilde{\mathcal{P}}(\ell))_{x_n}^{(n)}\}$.

2. *The following holds in the case $n = \ell-1$: the preimages of the map*
$$x_{\ell-1} \in \{0,1,\dots,q_{\ell-1}-1\} \mapsto W(\tilde{\mathcal{P}}(\ell))_{x_{\ell-1}}^{(\ell-1)}$$
are $I_0^{(\ell-1)},\dots,I_{\tilde{q}_{\ell-1}}^{(\ell-1)}$. Therefore this map is $c_{\ell-1}$-to-1.

3. *For every $n < \ell-1$, for every $(x_n,\dots,x_{\ell-1}) \in \prod_{n\leqslant i\leqslant \ell-1}\{0,1,\dots,q_i-1\}$, the set*
$$\{[\tilde{\mathcal{P}}(\ell)]_{h_n}(x) \mid x \in [0,\dots,0,x_n,\dots,x_{\ell-1}]_\ell\}$$
is a singleton, denoted by $\{W(\tilde{\mathcal{P}}(\ell))_{x_n,\dots,x_{\ell-1}}^{(n)}\}$.

4. *For every $n < \ell-1$ and every $(x_n,\dots,x_{\ell-2}) \in \prod_{n\leqslant i\leqslant \ell-2}\{0,1,\dots,q_i-1\}$, the preimages of the map*
$$x_{\ell-1} \in \{0,1,\dots,q_{\ell-1}-1\} \mapsto W(\tilde{\mathcal{P}}(\ell))_{x_n,\dots,x_{\ell-2},x_{\ell-1}}^{(n)}$$
are $I_0^{(\ell-1)},\dots,I_{\tilde{q}_{\ell-1}}^{(\ell-1)}$. Therefore this map is $c_{\ell-1}$-to-1.

Remark II.A.2.

- In the case of multiple permutations with $c_n = 1$ for every $n \geqslant 0$ (so $\tilde{q}_n = q_n$), we get $\tilde{\mathcal{P}}(\ell) = \mathcal{P}(\ell)$ and $I_j^{(\ell)} = \{j\}$ for every $\ell \geqslant 1$ and every $j \in \{0, \dots, q_\ell - 1\}$, so the map
$$x_{\ell-1} \in \{0, 1, \dots, q_{\ell-1} - 1\} \mapsto W(\mathcal{P}(\ell))_{x_{\ell-1}}^{(\ell-1)}$$
is injective.
- The first point of the above lemma remains true if we replace $\tilde{\mathcal{P}}(\ell)$ by any partition $\mathcal{P}$ refined by $\mathcal{P}(\ell)$. Indeed, the result is true for the partition $\mathcal{P}(\ell)$ (it suffices to consider T as an odomutant built with multiple permutations and $c_n = 1$). Moreover, the word $[\tilde{\mathcal{P}}]_{h_n}(x)$ is obtained from the word $[\tilde{\mathcal{P}}(\ell)]_{h_n}(x)$ by applying letters by letters the projection which maps $P \in \tilde{\mathcal{P}}(\ell)$ to the atom of $\mathcal{P}$ containing P.

Proof of Lemma II.A.1. Let $x \in [0, \dots, 0, x_n]_{n+1}$. We can write $x = (\underbrace{0, \dots, 0}_{n \text{ times}}, x_n, x_{n+1}, \dots)$. All the permutations fix 0, so for every $i \geqslant n-1$, we have
$$\psi_i(x) = (\underbrace{0, \dots, 0}_{n \text{ times}}, \sigma_{x_{n+1}}^{(n)}(x_n), \dots, \sigma_{x_{i+1}}^{(i)}(x_i), x_{i+1}, x_{i+2}, \dots).$$
For $k \in \{0, 1, \dots, h_n - 1\}$, let $(k_0, k_1, \dots, k_{n-1})$ be the n-tuple in $\prod_{0 \leqslant i \leqslant n-1} \{0, 1, \dots, q_i - 1\}$ satisfying
$$k = k_0 + h_1 k_1 + \dots + h_{n-1} k_{n-1}.$$
We then have
$$S^k \psi_i(x) = (k_0, k_1, \dots, k_{n-1}, \sigma_{x_{n+1}}^{(n)}(x_n), \dots, \sigma_{x_{i+1}}^{(i)}(x_i), x_{i+1}, x_{i+2}, \dots)$$
so $T^k x$ is equal to $(y_0^{(k)}, \dots, y_{n-1}^{(k)}, x_n, x_{n+1}, \dots)$ where $y_i^{(k)}$ defined by
$$y_n^{(k)} = x_n,$$
$$\forall\ 0 \leqslant i \leqslant n-1,\ y_i^{(k)} = \left(\sigma_{y_{i+1}^{(k)}}^{(i)}\right)^{-1}(k_i).$$
Denote by $j(k, \ell-1)$ the integer in $\{0, 1, \dots, \tilde{q}_{\ell-1}\}$ satisfying $y_{\ell-1}^{(k)} \in I_{j(k,\ell-1)}^{(\ell-1)}$. For every $k \in \{0, 1, \dots, h_n - 1\}$, $(y_0^{(k)}, \dots, y_n^{(k)})$ does not depend on $x_{n+1}, x_{n+2}, \dots$ and only depends on x_n, so does the h_n-tuple $([y_0^{(k)}, \dots, y_{\ell-2}^{(k)}, I_{j(k,\ell-1)}^{(\ell-1)}]_\ell)_{0 \leqslant k \leqslant h_n - 1}$ which is equal to $[\tilde{\mathcal{P}}(\ell)]_{h_n}(x)$.

In the case $n = \ell - 1$, we have $y_{\ell-1}^{(k)} = x_n$, so the value of the word $W(\tilde{\mathcal{P}}(\ell))_{x_n}^{(n)}$ only depends on the interval $I_j^{(n)}$ containing x_n.

We similarly prove the last two items. □

Lemma II.A.3. *Let $\ell \geqslant 1$ and T be an odomutant built with uniformly $\boldsymbol{c}$-multiple permutations fixing 0. Let us recall the words $W(\tilde{\mathcal{P}}(\ell))_{x_n}^{(n)}$ defined in Lemma II.A.1. Then, for every $n \geqslant \ell - 1$ and $x_n \in \{0, 1, \dots, q_n - 1\}$, we have*
$$W(\tilde{\mathcal{P}}(\ell))_{x_{n+1}}^{(n+1)} = W(\tilde{\mathcal{P}}(\ell))_0^{(n)} \cdot W(\tilde{\mathcal{P}}(\ell))_{\left(\sigma_{x_{n+1}}^{(n)}\right)^{-1}(1)}^{(n)} \cdot \dots \cdot W(\tilde{\mathcal{P}}(\ell))_{\left(\sigma_{x_{n+1}}^{(n)}\right)^{-1}(q_n-1)}^{(n)}.$$

Proof of Lemma II.A.3. Given $n \geqslant \ell - 1$, note that we have
$$\{0, 1, \dots, h_{n+1} - 1\} = \bigsqcup_{0 \leqslant i < q_n} \Big(\{0, 1, \dots, h_n - 1\} + h_n i\Big).$$

Moreover if i is in $\{0,1,\dots,q_n-1\}$, if x_{n+1} is in $\{0,1,\dots,q_{n+1}-1\}$, we have

$$T^{ih_n}([0,\dots,0,0,x_{n+1}]_{n+2}) = [0,\dots,0,\left(\sigma^{(n)}_{x_{n+1}}\right)^{-1}(i),x_{n+1}]_{n+2}.$$

This implies that, for a fixed $x \in [0,\dots,0,0,x_{n+1}]_{n+2}$, the element $y_i \coloneqq T^{ih_n}(x)$ is in $[0,\dots,0,\left(\sigma^{(n)}_{x_{n+1}}\right)^{-1}(i)]_{n+1}$ and we get

$$[\tilde{\mathcal{P}}(\ell)]_{ih_n,(i+1)h_n-1}(x) = [\tilde{\mathcal{P}}(\ell)]_{h_n}(T^{ih_n}(x)) = [\tilde{\mathcal{P}}(\ell)]_{h_n}(y_i) = W(\tilde{\mathcal{P}}(\ell))^{(n)}_{\left(\sigma^{(n)}_{x_{n+1}}\right)^{-1}(i)}$$

by Lemma II.A.1. Finally the h_{n+1}-word on x is the following concatenation:

$$\begin{aligned}
W(\tilde{\mathcal{P}}(\ell))^{(n)}_{x_{n+1}} &= [\tilde{\mathcal{P}}(\ell)]_{h_{n+1}}(x)\\
&= [\tilde{\mathcal{P}}(\ell)]_{0,h_{n+1}-1}(x)\\
&= [\tilde{\mathcal{P}}(\ell)]_{0,h_n-1}(x)\cdot[\tilde{\mathcal{P}}(\ell)]_{h_n,2h_n-1}(x)\cdot\ldots\cdot[\tilde{\mathcal{P}}(\ell)]_{h_n(q_{n+1}-1),h_{n+1}-1}(x)\\
&= W(\tilde{\mathcal{P}}(\ell))^{(n)}_{\left(\sigma^{(n)}_{x_{n+1}}\right)^{-1}(0)}\cdot W(\tilde{\mathcal{P}}(\ell))^{(n)}_{\left(\sigma^{(n)}_{x_{n+1}}\right)^{-1}(1)}\cdot\ldots\cdot W(\tilde{\mathcal{P}}(\ell))^{(n)}_{\left(\sigma^{(n)}_{x_{n+1}}\right)^{-1}(q_n-1)}\\
&= W(\tilde{\mathcal{P}}(\ell))^{(n)}_{0}\cdot W(\tilde{\mathcal{P}}(\ell))^{(n)}_{\left(\sigma^{(n)}_{x_{n+1}}\right)^{-1}(1)}\cdot\ldots\cdot W(\tilde{\mathcal{P}}(\ell))^{(n)}_{\left(\sigma^{(n)}_{x_{n+1}}\right)^{-1}(q_n-1)}
\end{aligned}$$

and we are done. □

Lemma II.A.4. *Let T be an odomutant built with uniformly $\boldsymbol{c}$-multiple permutations $\tau^{(n)}_j$ fixing 0. Let $(\kappa_n)_{n\geqslant 1}$ be a sequence of positive integers and assume that for every $n \geqslant 0$, the map*

$$j \in \{0,1,\dots,\tilde{q}_{n+1}-1\} \mapsto (\tau^{(n)}_j(I^{(n)}_0),\dots,\tau^{(n)}_j(I^{(n)}_{\tilde{q}_n-1}))$$

is κ_{n+1}-to-1 (in particular, κ_{n+1} divides $\tilde{q}_{n+1}$)[18]*. Then, for all $n \geqslant \ell \geqslant 1$, we have*

$$\left|\left\{W(\tilde{\mathcal{P}}(\ell))^{(n)}_{x_n} \mid 0 \leqslant x_n \leqslant q_n-1\right\}\right| \geqslant \frac{\tilde{q}_n}{\prod_{k=\ell}^{n}\kappa_k^{h_n/h_k}}.$$

Remark II.A.5. In the case of uniform permutations with pairwise different permutations, the lemma implies that

$$\left|\left\{W(\tilde{\mathcal{P}}(\ell))^{(n)}_{x_n} \mid 0 \leqslant x_n \leqslant q_n-1\right\}\right| = q_n$$

so $W(\tilde{\mathcal{P}}(\ell))^{(n)}_{x_n}$ is an injective function of x_n. This could also be deduced from Lemma II.A.3. Therefore, odomutants can have more words in their language than odometer, and then their entropy can be positive.

Proof of Lemma II.A.4. Let $(\tilde{\mathcal{P}}(\ell))_{\ell\geqslant 1}$ be the sequence of partitions associated to the construction of this odomutant with uniformly $\boldsymbol{c}$-multiple permutations. Given $n \geqslant \ell \geqslant 1$, we consider the projection $\pi_{n+1,\ell}\colon \tilde{\mathcal{P}}(n+1) \to \tilde{\mathcal{P}}(\ell)$ which maps $P \in \tilde{\mathcal{P}}(n+1)$ to the atom of $\tilde{\mathcal{P}}(\ell)$ containing P. This projection induces a map on the set of words with letters in $\tilde{\mathcal{P}}(n+1)$, it consists in projecting each entry on $\tilde{\mathcal{P}}(\ell)$.

[18] Let us go back to the intuition behind uniformly multiple permutations. Since we consider the partitions $\tilde{\mathcal{P}}(\ell)$ instead of $\mathcal{P}(\ell)$, we cannot distinguish between the copies of a subcolumn that we stack to form each tower. Therefore, given two permutations $\tau^{(n)}_j$ and $\tau^{(n)}_{j'}$, if $(\tau^{(n)}_j(I^{(n)}_0),\dots,\tau^{(n)}_j(I^{(n)}_{\tilde{q}_n-1})) = (\tau^{(n)}_{j'}(I^{(n)}_0),\dots,\tau^{(n)}_{j'}(I^{(n)}_{\tilde{q}_n-1}))$, then we cannot distinguish between the permutations that they encode, although these permutations are different.

Claim 1. *Let $x \in [0,\dots,0]_n$ and $k \in \{1,\dots,n\}$. For every $i \in \{0,1,\dots,\frac{h_n}{h_k}-1\}$ and every $j \in \{0,1,\dots,q_{k-1}-1\}$, the point $x^{(i,j)} \coloneqq T^{ih_k+jh_{k-1}}x$ is in $[0,\dots,0]_{k-1}$. Moreover, $(x^{(i,j)})_k$ does not depend on j and we have*

$$(x^{(i,j)})_k = \sigma^{(k)}_{(x^{(i,j)})_{k+1}}(i_k) \text{ and } (x^{(i,j)})_{k-1} = \sigma^{(k-1)}_{(x^{(i,j)})_k}(j)$$

where $i_k \coloneqq \left\lfloor \frac{i}{q_k} \right\rfloor$.

Proof of the claim. Let us write $ih_k = i_k h_k + i_{k+1}h_{k+1} + \dots + i_{n-1}h_{n-1}$ with $i_m \in \{0,\dots,q_m-1\}$ for every $m \in \{k,\dots,n-1\}$. Given $j \geqslant n$, we have

$$\psi_j(x) = (\underbrace{0,\dots,0}_{n \text{ times}}, \sigma^{(n)}_{x_{n+1}}(x_n), \dots)$$

and

$$S^{ih_k+jh_{k-1}}\psi_j(x) = (\underbrace{0,\dots,0}_{k-1 \text{ times}}, j, i_k, \dots, i_{n-1}, \sigma^{(n)}_{x_{n+1}}(x_n), \dots).$$

Hence we get $x^{(i,j)} = \psi_j^{-1} S^{ih_k+jh_{k-1}}\psi_j(x)$ for every $j \geqslant n$, which implies

$$\begin{aligned}
(x^{(i,j)})_n &= x_n,\\
(x^{(i,j)})_{n-1} &= \sigma^{(n-1)}_{x_n}(i_{n-1}),\\
(x^{(i,j)})_{n-2} &= \sigma^{(n-2)}_{(x^{(i,j)})_{n-1}}(i_{n-2}),\\
&\vdots\\
(x^{(i,j)})_k &= \sigma^{(k)}_{(x^{(i,j)})_{k+1}}(i_k),\\
(x^{(i,j)})_{k-1} &= \sigma^{(k-1)}_{(x^{(i,j)})_k}(j),
\end{aligned}$$

so we are done. □claim

Claim 2. *With the hypotheses of the lemma, for every $k \in \{\ell,\dots,n\}$, the map*

$$\pi_{k+1,k}\colon \left\{W(\tilde{\mathcal{P}}(k+1))^{(n)}_{x_n} \mid 0 \leqslant x_n \leqslant q_n-1\right\} \to \left\{W(\tilde{\mathcal{P}}(k))^{(n)}_{x_n} \mid 0 \leqslant x_n \leqslant q_n-1\right\}$$

is at most $\kappa_k^{h_n/h_k}$-to-1.

Proof of the claim. Let $x \in [0,\dots,0]_n$. We have $[\tilde{\mathcal{P}}(k+1)]_{h_n}(x) = W(\tilde{\mathcal{P}}(k+1))^{(n)}_{x_n}$ and $[\tilde{\mathcal{P}}(k)]_{h_n}(x) = W(\tilde{\mathcal{P}}(k))^{(n)}_{x_n}$. We first write these words as a concatenation of words of size h_{k-1}, namely the words $[\tilde{\mathcal{P}}(k+1)]_{h_{k-1}}(T^{mh_{k-1}}x)$ or $[\tilde{\mathcal{P}}(k)]_{h_{k-1}}(T^{mh_{k-1}}x)$ for $m \in \{0,1,\dots,\frac{h_n}{h_{k-1}}-1\}$. Given $m \in \{0,1,2,\dots,\frac{h_n}{h_{k-1}}-1\}$, the point $T^{mh_{k-1}}(x)$ is in $[0,\dots,0]_{k-1}$ by the last claim, so we have

$$[\tilde{\mathcal{P}}(k+1)]_{h_{k-1}}(T^{mh_{k-1}}(x)) = W(\tilde{\mathcal{P}}(k+1))^{(k-1)}_{(T^{mh_{k-1}}(x))_{k-1},(T^{mh_{k-1}}(x))_k}$$

and

$$[\tilde{\mathcal{P}}(k)]_{h_{k-1}}(T^{mh_{k-1}}(x)) = W(\tilde{\mathcal{P}}(k))^{(k-1)}_{(T^{mh_{k-1}}(x))_{k-1}}.$$

Secondly, we gather these words of length h_{k-1} in groups $M(k+1)_{x_n,i}$ or $M(k)_{x_n,i}$ of q_{k-1} words:

$$\begin{aligned}
M(k+1)_{x_n,i} \coloneqq\ & W(\tilde{\mathcal{P}}(k+1))^{(k-1)}_{(x^{(i,0)})_{k-1},(x^{(i,0)})_k} \cdot W(\tilde{\mathcal{P}}(k+1))^{(k-1)}_{(x^{(i,1)})_{k-1},(x^{(i,1)})_k}\\
&\cdot \dots \cdot W(\tilde{\mathcal{P}}(k+1))^{(k-1)}_{(x^{(i,q_{k-1}-1)})_{k-1},(x^{(i,q_{k-1}-1)})_k}
\end{aligned}$$

and

$$M(k)_{x_n,i} \coloneqq W(\tilde{\mathcal{P}}(k))^{(k-1)}_{(x^{(i,0)})_{k-1}} \cdot W(\tilde{\mathcal{P}}(k))^{(k-1)}_{(x^{(i,1)})_{k-1}} \cdot \ldots \cdot W(\tilde{\mathcal{P}}(k))^{(k-1)}_{(x^{(i,q_{k-1}-1)})_{k-1}}$$

for all $i \in \{0,1,\ldots,\frac{h_n}{h_k}-1\}$ in such a way that we have

$$W(\tilde{\mathcal{P}}(k+1))^{(n)}_{x_n} = M(k+1)_{x_n,0} \cdot M(k+1)_{x_n,1} \cdot \ldots \cdot M(k+1)_{x_n,\frac{h_n}{h_k}-1}$$

and

$$W(\tilde{\mathcal{P}}(k))^{(n)}_{x_n} = M(k)_{x_n,0} \cdot M(k)_{x_n,1} \cdot \ldots \cdot M(k)_{x_n,\frac{h_n}{h_k}-1}.$$

To prove the lemma, it now remains to prove that, for every $i \in \{0,1,\ldots,\frac{h_n}{h_k}-1\}$, the map

$$\pi_{k+1,k}\colon \{M(k+1)_{x_n,i} \mid 0 \leqslant x_n \leqslant q_n-1\} \to \{M(k)_{x_n,i} \mid 0 \leqslant x_n \leqslant q_n-1\}$$

is at most κ_k-to-1. Let us fix a word $M(k)_{x_n,i}$ with $i \in \{0,1,\ldots,\frac{h_n}{h_k}-1\}$ and $x_n \in \{0,1,\ldots,q_n-1\}$. We write $i_k = \lfloor i/q_k \rfloor$. By the last claim, the quantities $(x^{(i,0)})_k,\ldots,(x^{(i,q_{k-1}-1)})_k$ are equal and their common value is denoted by X_k, and we have

$$(x^{(iq_{k-1},j)})_{k-1} = \left(\sigma^{(k-1)}_{X_k}\right)^{-1}(j) \tag{II.7}$$

for every $j \in \{0,1,\ldots,q_{k-1}\}$. This first implies that

$$(x^{(i,0)})_{k-1}, (x^{(i,1)})_{k-1}, \ldots, (x^{(i,q_{k-1}-1)})_{k-1}$$

are q_{k-1} pairwise different elements of $\{0,1,\ldots,q_{k-1}-1\}$. Since we know each subword $W(\tilde{\mathcal{P}}(k))^{(k-1)}_{(x^{(i,j)})_{k-1}}$ of $M(k)_{x_n,i}$, the third item of Lemma II.A.1 implies that we completely know the sets $I^{(k-1)}_0,\ldots,I^{(k-1)}_{\tilde{q}_{k-1}-1}$, so

$$\left(\sigma^{(k-1)}_{X_k}(I^{(k-1)}_0),\ldots,\sigma^{(k-1)}_{X_k}(I^{(k-1)}_{\tilde{q}_{k-1}-1})\right)$$

is also completely determined. By assumptions, X_k is in the disjoint union of κ_k sets of the form $I^{(k)}_j$.

To conclude, we have proved that, if we have $\pi_{k+1,k}(M(k+1)_{y_n,i}) = M(k)_{x_n,i}$ for some $y_n \in \{0,1,\ldots,q_n-1\}$, then $M(k+1)_{y_n,i}$ is of the form

$$W(\tilde{\mathcal{P}}(k+1))^{(k-1)}_{(x^{(i,0)})_{k-1},X_k} \cdot W(\tilde{\mathcal{P}}(k+1))^{(k-1)}_{(x^{(i,1)})_{k-1},X_k} \cdot \ldots \cdot W(\tilde{\mathcal{P}}(k+1))^{(k-1)}_{(x^{(i,q_{k-1}-1)})_{k-1},X_k}$$

with X_k in the disjoint union of κ_k sets of the form $I^{(k)}_j$, and which completely determines $(x^{(i,0)})_{k-1},(x^{(i,1)})_{k-1},\ldots,(x^{(i,q_{k-1}-1)})_{k-1}$ by Equality (II.7). But since two elements X_k and X'_k in the same $I^{(k)}_j$ provide the same word $M(k+1)_{y_n,i}$ (by the last item of Lemma II.A.1), we get that there are at most κ_k possible values for the word $M(k+1)_{y_n,i}$. □claim

By the last claim, the map

$$\pi_{n+1,\ell}\colon \left\{W(\tilde{\mathcal{P}}(n+1))^{(n)}_{x_n} \mid 0 \leqslant x_n \leqslant q_n-1\right\} \to \left\{W(\tilde{\mathcal{P}}(\ell))^{(n)}_{x_n} \mid 0 \leqslant x_n \leqslant q_n-1\right\}$$

is at most $\left(\prod_{k=\ell}^{n} \kappa_k^{h_n/h_k}\right)$-to-1, so we have

$$\left|\left\{W(\tilde{\mathcal{P}}(\ell))^{(n)}_{x_n} \mid 0 \leqslant x_n \leqslant q_n-1\right\}\right| \geqslant \frac{\left|\left\{W(\tilde{\mathcal{P}}(n+1))^{(n)}_{x_n} \mid 0 \leqslant x_n \leqslant q_n-1\right\}\right|}{\prod_{k=\ell}^{n} \kappa_k^{h_n/h_k}}$$

and the result follows from the second item of Lemma II.A.1. □

Lemma II.A.6. *Let $(q_n)_{n\geqslant 0}$ be a sequence of integers greater than or equal to 2 and satisfying $q_{n+1} \leqslant (q_n-2)!$. Then there exist permutations $\sigma_{x_{n+1}}^{(n)}$, for $n \geqslant 0$ and $x_{n+1} \in \{0,\dots,q_{n+1}-1\}$, satisfying the following properties:*

1. *for every $n \geqslant 0$, the maps $\sigma_0^{(n)}, \sigma_1^{(n)}, \dots, \sigma_{q_{n+1}-1}^{(n)}$ are pairwise different permutations of the set $\{0,\dots,q_n-1\}$, fixing 0 and q_n-1;*
2. *the following bounds hold for every $n \geqslant \ell \geqslant 1$:*

$$q_n \leqslant N(\mathcal{P}(\ell)_0^{h_n-1}) \leqslant h_{n-1}q_n q_{n-1}^2 2^{q_{n-1}}$$

Proof of Lemma II.A.6. Let us recall that $N(\mathcal{P}(\ell)_0^{h_n-1})$ is equal to the cardinality of $\{[\mathcal{P}(\ell)]_{h_n}(x) \mid x \in X\}$. If the permutations $\sigma_0^{(n)}, \sigma_1^{(n)}, \dots, \sigma_{q_{n+1}-1}^{(n)}$ are pairwise different for every $n \geqslant 0$, then we get $N(\mathcal{P}(\ell)_0^{h_n-1}) \geqslant q_n$ for the underlying odomutant (see Remark II.A.5 following Lemma II.A.4).

Given $n \geqslant 0$, let $i_n \in \{2,\dots,q_n-2\}$ be such that $(i_n-1)! < q_{n+1} \leqslant i_n!$ and let us choose any family $\left(\sigma_{x_{n+1}}^{(n)}\right)_{0\leqslant x_{n+1}<q_{n+1}}$ of pairwise different permutations of the set $\{0,\dots,q_n-1\}$ fixing $0, i_n+1, i_n+2, \dots, q_n-1$. Given an integer $\ell \geqslant 1$, let us find an upper bound of $N(\mathcal{P}(\ell)_0^{h_n-1})$ for every $n \geqslant \ell$. Let $n \geqslant \ell$ and $x \in X$. There exists $i \in \{0,1,\dots,h_n-1\}$ such that $y \coloneqq T^{-i}x$ is in $[0,\dots,0,x_n]_{n+1}$. Let us write $z \coloneqq T^{h_n}x$. Thus $[\mathcal{P}(\ell)]_{h_n}(x)$ is the concatenation of a final subword of $W(\mathcal{P}(\ell))_{x_n}^{(n)}$ and an initial subword of $W(\mathcal{P}(\ell))_{z_n}^{(n)}$. Writing $i = jh_{n-1}+r$ with integers $j \in \{0,\dots,q_{n-1}-1\}$ and $r \in \{0,\dots,h_{n-1}-1\}$, and using Lemma II.A.3, we have

$$\begin{aligned}[\mathcal{P}(\ell)]_{h_n}(x) = w \cdot W(\mathcal{P}(\ell))_{\left(\sigma_{x_n}^{(n-1)}\right)^{-1}(j+1)}^{(n-1)} \cdot \ldots \cdot W(\mathcal{P}(\ell))_{\left(\sigma_{x_n}^{(n-1)}\right)^{-1}(q_{n-1}-2)}^{(n-1)} \cdot W(\mathcal{P}(\ell))_{q_{n-1}-1}^{(n-1)} \\ \cdot W(\mathcal{P}(\ell))_0^{(n-1)} \cdot W(\mathcal{P}(\ell))_{\left(\sigma_{z_n}^{(n-1)}\right)^{-1}(1)}^{(n-1)} \cdot \ldots \cdot W(\mathcal{P}(\ell))_{\left(\sigma_{z_n}^{(n-1)}\right)^{-1}(j-1)}^{(n-1)} \cdot w'\end{aligned}$$

where w is a final subword of $W(\mathcal{P}(\ell))_{\left(\sigma_{x_n}^{(n-1)}\right)^{-1}(j)}^{(n-1)}$ of length $h_{n-1}-r$, and w' an initial subword of $W(\mathcal{P}(\ell))_{\left(\sigma_{z_n}^{(n-1)}\right)^{-1}(j)}^{(n-1)}$ of length r. Therefore, to every word of the form $[\mathcal{P}(\ell)]_{h_n}(x)$ for the points $x \in X$ sharing the same integers x_n, z_n, j and r, we can associate the family

$$\left(\left(\sigma_{x_n}^{(n-1)}\right)^{-1}(j), \dots, \left(\sigma_{x_n}^{(n-1)}\right)^{-1}(q_{n-1}-2), \left(\sigma_{z_n}^{(n-1)}\right)^{-1}(1), \dots, \left(\sigma_{z_n}^{(n-1)}\right)^{-1}(j)\right).$$

In the particular cases $j=0$ and $j=q_{n-1}-1$, this family is respectively equal to

$$\left(\left(\sigma_{x_n}^{(n-1)}\right)^{-1}(1), \dots, \left(\sigma_{x_n}^{(n-1)}\right)^{-1}(q_{n-1}-2)\right)$$

and

$$\left(\left(\sigma_{z_n}^{(n-1)}\right)^{-1}(1), \dots, \left(\sigma_{z_n}^{(n-1)}\right)^{-1}(q_{n-1}-2)\right).$$

Moreover, this association is injective, as a consequence of Remark II.A.2 following Lemma II.A.1. Thus we have

$$\begin{aligned}N(\mathcal{P}(\ell)_0^{h_n-1}) &\leqslant \sum_{r=0}^{h_{n-1}-1}\left(a_1^{(n)} + b_{q_{n-1}-2}^{(n)} + \sum_{j=1}^{q_{n-1}-2} a_j^{(n)} \times b_j^{(n)}\right) \\ &\leqslant h_{n-1}\left(a_1^{(n)} + b_{q_{n-1}-2}^{(n)} + \sum_{j=1}^{q_{n-1}-2} a_j^{(n)} \times b_j^{(n)}\right)\end{aligned}$$

where $a_j^{(n)}$ and $b_j^{(n)}$ are respectively the cardinality of

$$\left\{ \left(\left(\sigma_{x_n}^{(n-1)}\right)^{-1}(j), \ldots, \left(\sigma_{x_n}^{(n-1)}\right)^{-1}(q_{n-1}-2) \right) \mid x_n \in \{0,1,\ldots,q_n-1\} \right\}$$

and

$$\left\{ \left(\left(\sigma_{x_n}^{(n-1)}\right)^{-1}(1), \ldots, \left(\sigma_{x_n}^{(n-1)}\right)^{-1}(j) \right) \mid x_n \in \{0,1,\ldots,q_n-1\} \right\}.$$

We now find upper bounds of the quantities $a_j^{(n)}$ and $b_j^{(n)}$, using the properties of the permutations that we have chosen at the beginning of this proof. We have $a_1^{(n)} = q_n$, $a_j^{(n)} \leqslant i_{n-1} \times \ldots \times j$ if $1 \leqslant j \leqslant i_{n-1}$ and $a_j^{(n)} = 1$ if $i_{n-1}+1 \leqslant j \leqslant q_{n-1}-2$. We also have $b_j^{(n)} \leqslant i_{n-1} \times \ldots \times (i_{n-1}-j+1)$ if $1 \leqslant j \leqslant i_{n-1}-1$ and $b_j^{(n)} = q_n$ if $i_{n-1} \leqslant j \leqslant q_{n-1}-2$. We then get

$$\begin{aligned}
&a_1^{(n)} + b_{q_{n-1}-2}^{(n)} + \sum_{j=1}^{q_{n-1}-2} a_j^{(n)} \times b_j^{(n)} \\
&\leqslant 2q_n + \left(\sum_{j=1}^{i_{n-1}-1} \frac{i_{n-1}!}{(j-1)!} \frac{i_{n-1}!}{(i_{n-1}-j)!} \right) + q_n i_{n-1} + \left(\sum_{j=i_{n-1}+1}^{q_{n-1}-2} q_n \right) \\
&\leqslant q_n q_{n-1} + i_{n-1}^2 (i_{n-1}-1)! \sum_{j=1}^{i_{n-1}-1} \binom{i_{n-1}-1}{j-1} \\
&\leqslant q_n q_{n-1} + q_{n-1}^2 q_n 2^{i_{n-1}-1} \\
&\leqslant q_n q_{n-1}^2 2^{i_{n-1}} \\
&\leqslant q_n q_{n-1}^2 2^{q_{n-1}},
\end{aligned}$$

and we are done □

II.B Further comments on odomutants: Bratteli diagrams, strong orbit equivalence

II.B.a Bratteli diagrams, strong orbit equivalence

We introduce the most important definitions and results in the context of strong orbit equivalence. For more details, we refer the reader to [HPS92] and [GPS95].

Bratteli diagrams

A **Bratteli diagram** is a graph $B = (V, E)$ with the set of vertices

$$V = \bigsqcup_{k \geqslant 0} V_k$$

and the set of edges

$$E = \bigsqcup_{k \geqslant 0} E_k,$$

where V_k and E_k are finite, $V_0 = \{v^{(0)}\}$ and the edges in E_k connect vertices in V_k to vertices in V_{k+1} (multiple edges between two vertices are allowed). If $e_k \in E_k$ connects $v_k \in V_k$ to $v_{k+1} \in V_{k+1}$, we write $s(e_k) = v_k$ and $r(e_k) = v_{k+1}$, this provides maps $s\colon E \to V$ (**source**

map) satisfying $s(E_k) \subset V_k$ and $r\colon E \to V$ (**range map**) satisfying $r(E_k) \subset V_{k+1}$. We assume that

$$\forall v \in V,\ s^{-1}(v) \neq \varnothing$$

and

$$\forall v \in V\backslash V_0,\ r^{-1}(v) \neq \varnothing.$$

For $k < \ell$, a **path** from $v_k \in V_k$ to $v_\ell \in V_\ell$ is a tuple $(e_k, e_{k+1}, \dots, e_{\ell-1})$ satisfying $s(e_k) = v_k$, $r(e_i) = s(e_{i+1})$ for every $i \in \{k, \dots, \ell-2\}$ and $r(e_{\ell-1}) = v_\ell$.

An **ordered** Bratteli diagram is a Bratteli diagram together with a linear order in $r^{-1}(v)$ for every $v \in V\backslash V_0$, namely we consider a bijection

$$r^{-1}(v) \to \{0, 1, \dots, |r^{-1}(v)| - 1\}$$

for every $v \in V\backslash V_0$. Then we consider E_k as a subset of $V_k \times V_{k+1} \times \mathbb{N}$: an edge $e_k \in E_k$ is written as (v_k, v_{k+1}, ρ_k) where $v_k = s(e_k)$, $v_{k+1} = r(e_k)$ and $\rho_k \in \{0, \dots, |r^{-1}(e_k)| - 1\}$ is the **rank** of e_k for the linear order in $r^{-1}(v_{k+1})$, we write $\rho_k = \mathrm{rk}(e_k)$.

Let us set

$$X_B \coloneqq \left\{ (e_k)_{k\geqslant 0} \in \prod_{k\geqslant 0} E_k \mid \forall k \geqslant 0,\ r(e_k) = s(e_{k+1}) \right\},$$

$$X_{B,\min} \coloneqq \{(e_k)_{k\geqslant 0} \in X_B \mid \forall k \geqslant 0,\ \mathrm{rk}(e_k) = 0\}$$

$$\text{and } X_{B,\max} \coloneqq \left\{(e_k)_{k\geqslant 0} \in X \mid \forall k \geqslant 0,\ \mathrm{rk}(e_k) = |r^{-1}(r(e_k))| - 1\right\}.$$

As a subset of $\prod_{k\geqslant 0} E_k$, X_B is endowed with the induced product topology. X_B is a compact and totally disconnected metric space. By definition, the cylinders[19] of X_B are clopen sets and form a basis of the topology.

A Bratteli diagram is **simple** if there exists a subsequence (k_n) such that for every pair of vertices in $V_{k_n} \times V_{k_{n+1}}$, there exits a path between them. If an ordered Bratteli diagram is simple, then X_B has no isolated points, so it is a Cantor set.

Given a Bratteli diagram $B = (V, E)$, we can enumerate the vertices of each V_n:

$$V_n = \left\{ v_0^{(n)}, \dots, v_{|V_n|-1}^{(n)} \right\};$$

and define the **incidence matrices**

$$M_n \coloneqq \left(m_{i,j}^{(n)} \right)_{\substack{0\leqslant i\leqslant |V_{n+1}|-1 \\ 0\leqslant j\leqslant |V_n|-1}}$$

where $m_{i,j}^{(n)}$ is the number of edges of E_n connecting $v_j^{(n)}$ to $v_i^{(n+1)}$.

Bratteli-Vershik systems

Given an ordered Bratteli diagram B, we define a map $T_B\colon X_B\backslash X_{B,\max} \to X_B\backslash X_{B,\min}$ in the following way.

Let $x = (e_k)_{k\geqslant 0} \in X_B\backslash X_{B,\max}$ and

$$N \coloneqq \min\{i \geqslant 0 \mid \mathrm{rk}(e_i) < |r^{-1}(r(e_i))| - 1\}.$$

Let f_N be the edge in $r^{-1}(r(e_N))$ satisfying $\mathrm{rk}(f_N) = \mathrm{rk}(e_N) + 1$ and $(f_0, \dots, f_{N-1})$ the minimal path from $v^{(0)}$ to $s(f_N)$, namely this is the unique path satisfying $\mathrm{rk}(f_i) = 0$ for every $i \in \{0, \dots, N-1\}$.[20] Then we define

$$T_B x \coloneqq (f_1, \dots, f_N, e_{N+1}, e_{N+2}, \dots).$$

[19] defined as in Section II.2.e

[20] We find this path in an inductive way: f_{N-1} is the unique edge satisfying $r(f_{N-1}) = s(f_N)$ and $\mathrm{rk}(f_{N-1}) = 0$, f_{N-2} is the unique edge satisfying $r(f_{N-2}) = s(f_{N-1})$ and $\mathrm{rk}(f_{N-2}) = 0$, and so on.

The map T_B is called the **Bratteli-Vershik system** associated to the ordered Bratteli diagram B.

An ordered and simple Bratteli diagram is **properly ordered** if $X_{B,\min}$ and $X_{B,\max}$ are singletons. Given a properly ordered Bratteli diagram, we extend T to the whole set X_B by setting

$$T_B(x_{\max}) \coloneqq x_{\min}$$

where $X_{B,\max} \coloneqq \{x_{\max}\}$ and $X_{B,\min} = \{x_{\min}\}$. In this case, we can check that T_B is a Cantor minimal homeomorphism.

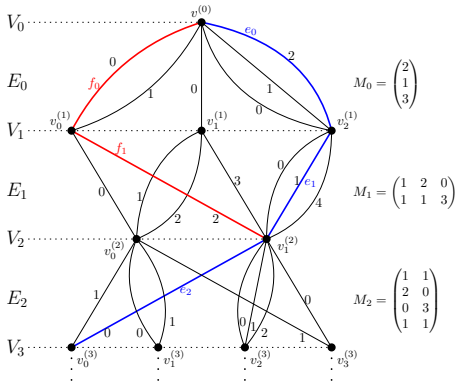


Figure II.4: Example of ordered Bratteli diagram B. The image of $(e_0, e_1, e_2, \ldots)$ by T_B is $(f_0, f_1, e_2, \ldots)$.

For example, the Bratteli-Vershik system of the diagram in Figure II.5 is topologically conjugate to the odometer on $X \coloneqq \prod_{n\geqslant 0} \{0, 1, \ldots, q_n - 1\}$, the following map

$$\Psi\colon (x_n)_{n\geqslant 0} \in X \mapsto \left(\left(v^{(0)}, v^{(1)}, x_0\right), \left(v^{(1)}, v^{(2)}, x_1\right), \left(v^{(2)}, v^{(3)}, x_2\right), \ldots\right) \in X_B$$

is a conjugation between them. As we explain in the next part, every Cantor minimal homeomorphism can be described by a Bratteli diagram.

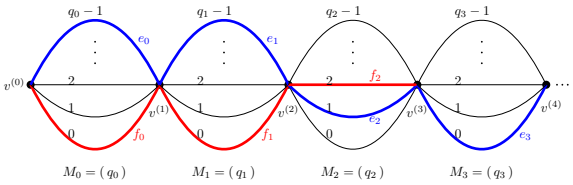


Figure II.5: An ordered Bratteli diagram describing the odometer on $\prod_{n\geqslant 0} \{0, \ldots, q_n - 1\}$. The image of $(e_0, e_1, e_2, e_3, \ldots)$ by T_B is $(f_0, f_1, f_2, e_3, \ldots)$.

Cantor minimal homeomorphisms

The Bratteli-Vershik systems of properly ordered Bratteli diagrams describe all the Cantor minimal homeomorphisms.

Theorem (Herman, Putnam, Skau [HPS92]). *If T is a Cantor minimal homeomorphism, then there exists a properly ordered Bratteli diagram B such that the associated Bratteli-Vershik system T_B is topologically conjugate to T.*

We briefly describe how a Cantor minimal homeomorphism $T\colon X \to X$ is encoded by a properly ordered Bratteli diagram. All we have to find is an increasing sequence $(\mathcal{P}_n)_{n\geqslant 0}$ of partitions generating the topology, of the form

$$\mathcal{P}_n \coloneqq \{T^j(B_{n,i}) \mid 0 \leqslant i \leqslant k_n - 1,\ 0 \leqslant j \leqslant h_i^{(n)} - 1\}$$

where $k_n, h_1^{(n)}, \dots, h_{k_n}^{(n)}$ are positive integers, and the sequence $(B_n)_{n\geqslant 0}$ defined by

$$B_n \coloneqq \bigsqcup_{0\leqslant i\leqslant k_n-1} B_{n,i},$$

is decreasing to a singleton $\{y\}$.

By "increasing sequence of partitions", we mean that $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$, namely the atoms of $\mathcal{P}_n$ are unions of atoms of $\mathcal{P}_{n+1}$. The partition $\mathcal{P}_n$ is composed of k_n towers and given $i \in \{0, \dots, k_n - 1\}$, the tower

$$\mathcal{T}_{n,i} \coloneqq \{B_{n,i}, T(B_{n,i}), \dots, T^{h_i^{(n)}-1}(B_{n,i})\}$$

has height $h_i^{(n)}$.

Without the assumption that the partitions have to generate the topology, the construction only consists in considering in an inductive way a clopen subset B_{n+1} of B_n, that we partition in $B_{n,0}, \dots, B_{n,k_n-1}$ according to the value of the first return time. The underlying sequence of partitions does not necessarily generate the topology. For a generating sequence, we refer the reader to Lemma 4.1 of in [HPS92] and Lemma 3.1 in [Put89] for more details.

The properly ordered Bratteli diagram $B = (V, E)$ is defined as follows. Assume that $\mathcal{P}_0 = X$ (so $k_0 = h_1 = 1$) and define

$$V_n \coloneqq \{\mathcal{T}_{n,i} \mid 0 \leqslant i \leqslant k_n - 1\}.$$

Given $n \geqslant 1$ and $i \in \{0, \dots, k_n - 1\}$, the tower $\mathcal{T}_{n,i}$ visits successively the towers

$$\mathcal{T}_{n-1,\ell_1^{(n,i)}}, \mathcal{T}_{n-1,\ell_2^{(n,i)}}, \dots, \mathcal{T}_{n-1,\ell_{r_{n,i}}^{(n,i)}}$$

with integers $\ell_j^{(n,i)} \in \{0, \dots, k_{n-1} - 1\}$ and $r_{n,i} \geqslant 1$. Then E is defined so that $r^{-1}(\mathcal{T}_{n,i})$ has cardinality $r_{n,i}$ and

$$r^{-1}(\mathcal{T}_{n,i}) \coloneqq \left\{ \left(\mathcal{T}_{n-1,\ell_{j+1}^{(n,i)}}, \mathcal{T}_{n,i}, j \right) \mid 0 \leqslant j \leqslant r_{n,i} - 1 \right\}.$$

The underlying Bratteli diagram is properly ordered and the associated Bratteli-Vershik system $T_B \colon X_B \to X_B$ is topologically conjugate to $T \colon X \to X$. Note that $x_{\min} \in X_B$ corresponds to the point $y \in X$.

To sum up, a Bratteli diagram encodes a cutting-and-stacking process defining a system (see Figure II.6).

Classification up to strong orbit equivalence

Here we present a complete invariant of strong orbit equivalence, due to Giordano, Putnam and Skau.

Recall the incidence matrices

$$M_n \coloneqq \left(m_{i,j}^{(n)} \right)_{\substack{0\leqslant i\leqslant |V_{n+1}|-1 \\ 0\leqslant j\leqslant |V_n|-1}}$$

given an enumeration of the vertices of each V_n. Then let us define the group $G(B)$ as the following inductive limit

$$G(B) \coloneqq \lim \mathbb{Z}^{|V_0|} \xrightarrow{M_0} \mathbb{Z}^{|V_1|} \xrightarrow{M_1} \mathbb{Z}^{|V_2|} \xrightarrow{M_2} \dots.$$

With the usual ordering on each $\mathbb{Z}^{|V_n|}$, $G(B)$ has a structure of ordered group, the unit order is chosen as the image of 1 in $\mathbb{Z} = \mathbb{Z}^{|V_0|}$. The ordered group $G(B)$ is called the **dimension group** of B. We refer the reader to [GPS95] for more details.

For instance, for the dyadic odometer, the incidence matrices are all (1×1)-matrices equal to (2), and the dimension group is $\mathbb{Z}[1/2]$.

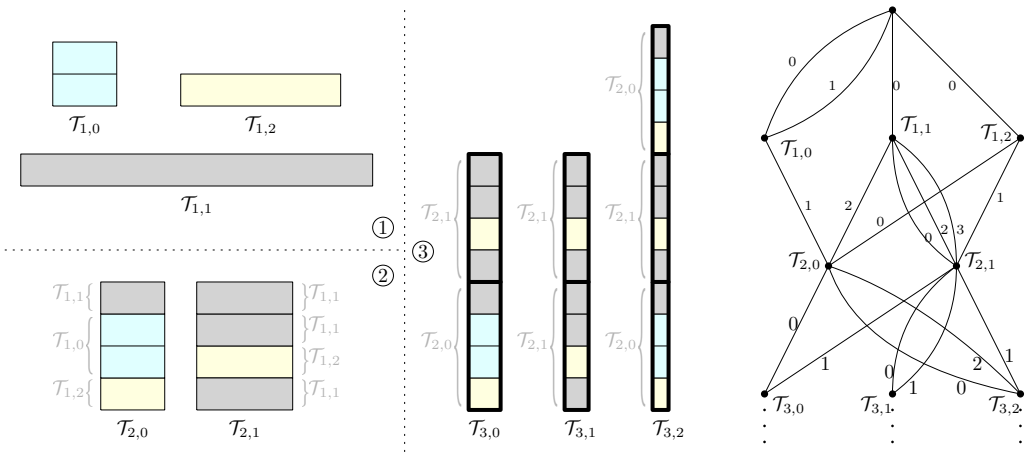


Figure II.6: Example of towers $\mathcal{T}_{n,i}$, and the associated Bratteli diagram.

Theorem II.B.1 (Giordano, Putnam, Skau [GPS95]). *Let S and T be two Cantor minimal homeomorphisms. The following assertions are equivalent:*

1. *S and T are strongly orbit equivalent;*
2. *If B (resp. B') denotes a Bratteli diagram associated to S (resp. T), then the dimension groups $G(B)$ and $G(B')$ with distinguished order unit are order isomorphic.*

II.B.b Bratteli diagrams of odomutants

Let $X \coloneqq \prod_{n\geqslant 0}\{0,\dots,q_n-1\}$. Denoting by $\mathcal{P}(n)$ the partition whose atoms are the n-cylinders, with $\mathcal{P}(0)=(X)$, note that the sequence $(\mathcal{P}(n))_{n\geqslant 0}$ generates the infinite product topology on X and $\mathcal{P}(n+1)$ is composed of q_n towers of height h_n, denoted by

$$\mathcal{T}_{n+1,i} \coloneqq \left\{B_{n+1,i}, T(B_{n+1,i}),\dots,T^{h_n-1}(B_{n+1,i})\right\}$$

where $B_{n+1,i} \coloneqq [0,\dots,0,i]_{n+1}$, for every $i\in\{0,\dots,q_n-1\}$ (see Figure II.2). The atoms of $\mathcal{T}_{n+1,i}$ are the cylinders of the form $[x_0,\dots,x_{n-1},i]_{n+1}$ with $x_k\in\{0,\dots,q_k-1\}$ for every $k\in\{0,\dots,n-1\}$.

Given $n\geqslant 1$ and $i\in\{0,\dots,q_n-1\}$, the tower $\mathcal{T}_{n+1,i}$ visits the n-th towers with the following order:

$$\mathcal{T}_{n,\left(\sigma_i^{(n-1)}\right)^{-1}(0)}, \mathcal{T}_{n,\left(\sigma_i^{(n-1)}\right)^{-1}(1)},\dots,\mathcal{T}_{n,\left(\sigma_i^{(n-1)}\right)^{-1}(q_n-1)}.$$

According to Section II.B.a, we get the Bratteli diagram B of T illustrated in Figure II.7.

The following map

$$\Psi\colon (x_n)_{n\geqslant 0}\in X \mapsto \left(\left(v^{(0)},v^{(1)}_{x_0},0\right),\left(v^{(1)}_{x_0},v^{(2)}_{x_1},\sigma^{(0)}_{x_1}(x_0)\right),\left(v^{(2)}_{x_1},v^{(3)}_{x_2},\sigma^{(1)}_{x_2}(x_1)\right),\dots\right)\in X_B$$

is a conjugation between T and the Bratteli-Vershik system T_B, it satisfies

$$\Psi(\psi^{-1}(X^+_\infty)) = X_B\backslash X_{B,\max}$$
$$\text{and } \Psi(\psi^{-1}(X^-_\infty)) = X_B\backslash X_{B,\min}.$$

In the case the permutations satisfy $\sigma_i^{(n)}(0)=0$, $\sigma_i^{(n)}(q_n-1)=q_n-1$ for every $n\geqslant 0$, the Bratteli diagram is properly ordered and we have

$$X_{B,\max} = \left\{\Psi(x^+)\right\}$$
$$\text{and } X_{B,\min} = \left\{\Psi(x^-)\right\}.$$

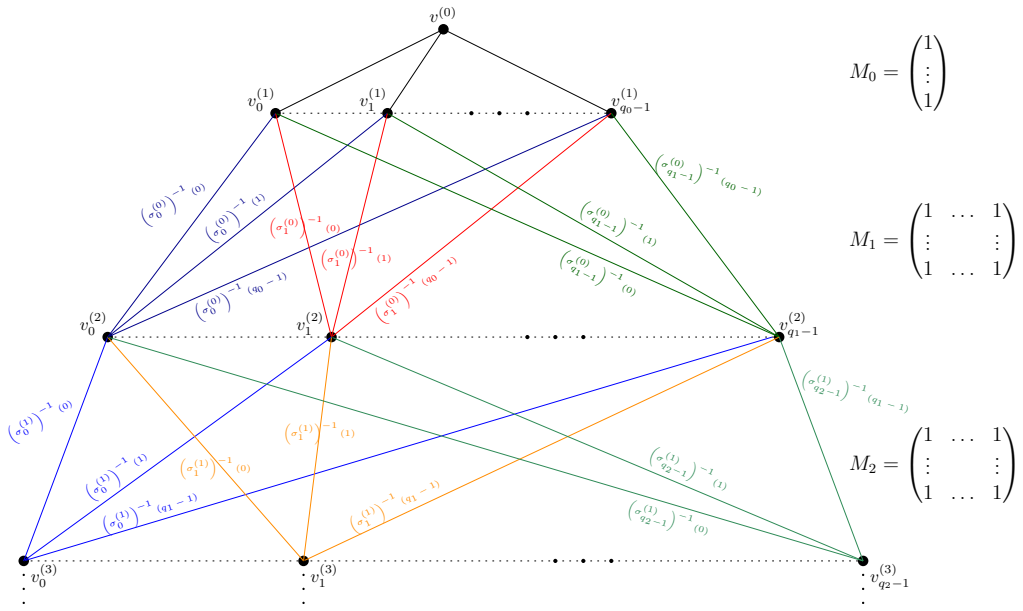


Figure II.7: An ordered Bratteli diagram describing the odomutant built from the odometer on $\prod_{n\geqslant 0}\{0,\dots,q_n-1\}$ and families of permutations $\left(\sigma_i^{(n)}\right)_{0\leqslant i<q_{n+1}}$ for $n\geqslant 0$.

II.B.c Comparisons between Boyle and Handelman's system and our odomutants.

As mentioned in the introduction, Boyle and Handelman have shown the following result.

Theorem (Boyle, Handelman [BH94]). *Let S be the dyadic odometer. Let α be either a positive real number or $+\infty$. Then there exists a Cantor minimal homeomorphism T such that:*

1. *S and T are strongly orbit equivalent;*
2. $\mathrm{h_{top}}(T)=\alpha$.

In their proof, they build a Bratteli diagram B_{BH} (see Figure II.8) similar to the diagram in Figure II.7, the only difference is that for every $k\geqslant 1$, for every $v_i^{(k)}\in V_k$, $v_j^{(k+1)}\in V_{k+1}$, with $0\leqslant i<q_{k-1}$ and $0\leqslant j\leqslant q_k-1$, there are n_k edges connecting these vertices. Then the ideas remain almost the same. Every vertex $v_j^{(k+1)}\in V_{k+1}$ provides a permutation $\sigma_j^{(k-2)}$ on the n_kq_{k-1} edges of range $v_j^{(k+1)}$, satisfying

$$\sigma_j^{(k-2)}(0)=0$$

$$\text{and } \sigma_j^{(k-2)}(n_kq_{k-1}-1)=n_kq_{k-1}-1,$$

so that the diagram is properly ordered and the associated Bratteli-Vershik system $T\coloneqq T_{B_{\mathrm{BH}}}$ can be extended to a homeomorphism on the Cantor set. The permutations are chosen in order to get $\mathrm{h_{top}}(T)=\alpha$ (we refer the reader to their proof for more details, note that their proof in the case $\alpha=+\infty$ has been here entirely reformulated in our formalism).

It turns out that their Bratteli diagram is a diagram for an odomutant. Let us recall the following facts.

- In a Bratteli diagram, for some fixed vertex $v_j^{(k+1)}\in V_{k+1}$, with $0\leqslant j\leqslant q_k-1$, the set of edges $r^{-1}(v_j^{(k+1)})$, with its linear ordering, encodes the stacking of subtowers of $\mathcal{T}_{k,0},\dots,\mathcal{T}_{k,q_{k-1}-1}$ to build the tower $\mathcal{T}_{k+1,j}$ (see Figure II.6).

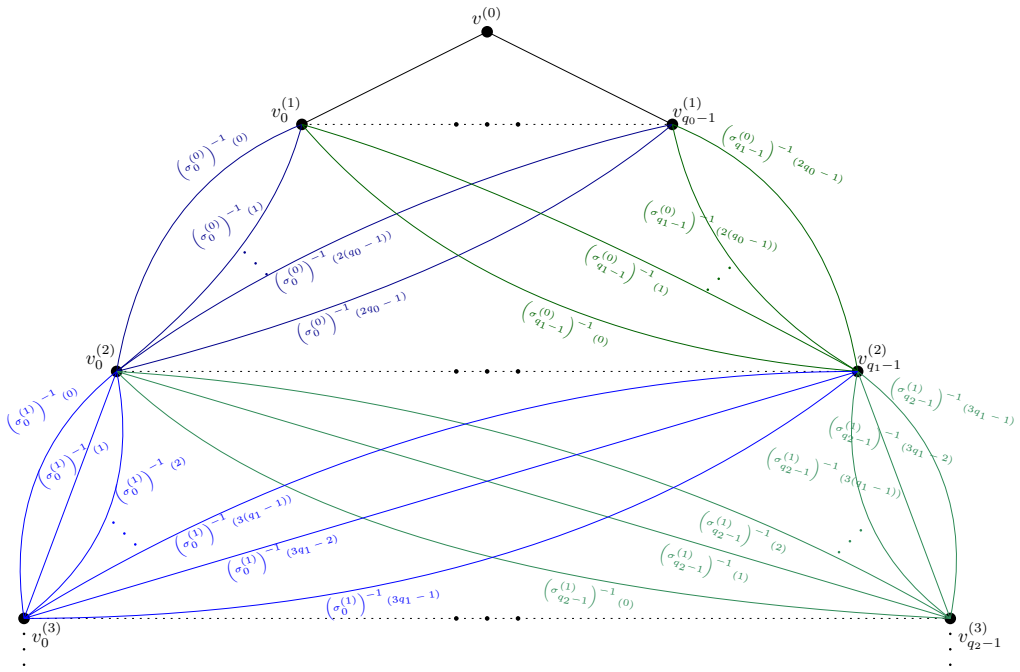


Figure II.8: Bratteli diagram built by Boyle and Handelman in the proof of their Theorem 2.8 [BH94], with $n_1 = 2$, $n_2 = 3$.

- In the cutting-and-stacking construction of an odomutant described in Figure II.2, every tower $\mathcal{T}_{k,i}$, with $0 \leqslant i < q_{k-1}$, is uniformly cut in q_{k+1} subtowers $(\mathcal{T}_{k,i}(\ell))_{0\leqslant\ell<q_{k+1}}$ and we build every tower $\mathcal{T}_{k+1,j}$, with $0 \leqslant j < q_k$, by choosing only one subtower in each $\mathcal{T}_{k,0}, \dots, \mathcal{T}_{k,q_{k-1}-1}$ and stacking them.

The Bratteli diagram B_{BH} of Boyle and Handelman describes the following cutting-and-stacking construction: every tower $\mathcal{T}_{k,i}$ is uniformly cut in $n_k q_{k+1}$ subtowers $(\mathcal{T}_{k,i}(\ell))_{0\leqslant\ell\leqslant n_k q_{k+1}-1}$ and we build every tower $\mathcal{T}_{k+1,j}$ by choosing exactly n_k subtowers in each $\mathcal{T}_{k,0}, \dots, \mathcal{T}_{k,q_{k-1}-1}$ and stacking them.

As explained in Section II.3.b, to understand why this is equivalent to the construction of an odomutant, it suffices to cut each k-th tower $\mathcal{T}_{k,i}$ in n_k (sub)towers $\mathcal{T}_{k,(i,0)}, \dots, \mathcal{T}_{k,(i,n_k-1)}$, in such a manner that for every $(k+1)$-th tower $\mathcal{T}_{k+1,j}$, each tower $\mathcal{T}_{k,(i,m)}$ contains only one of the n_k subtowers $\mathcal{T}_{k,i}(\ell)$ which form $\mathcal{T}_{k+1,j}$. We then replace the former k-th towers $\mathcal{T}_{k,i}$, with $0 \leqslant i < q_{k-1}$, by the new ones $\mathcal{T}_{k,(i,m)}$, with $0 \leqslant i < q_{k-1}$ and $0 \leqslant m \leqslant n_k - 1$, and we recover the cutting-and-stacking process of an odomutant described above (with new integers n_k equal to 1). In other words, each vertex in V_k is split in n_k copies, and we get the Bratteli diagram B'_{BH} illustrated in Figure II.9.

An infinite path of $X_{B_{\mathrm{BH}}}$ can be uniquely written as

$$\left((v_{i_k}^{(k)}, v_{i_{k+1}}^{(k+1)}, \left(\sigma_{i_{k+1}}^{(k-1)}\right)^{-1} (n_k i_k + m_k)) \right)_{k\geqslant 1}$$

with $0 \leqslant i_k < q_{k-1}$ and $0 \leqslant m_k \leqslant n_k - 1$ (we omit the first edge $(v^{(0)}, v_{i_1}^{(1)}, 0)$). With the notations of Figure II.9, the map

$$\begin{aligned} &\left((v_{i_k}^{(k)}, v_{i_{k+1}}^{(k+1)}, \left(\sigma_{i_{k+1}}^{(k-1)}\right)^{-1} (n_k i_k + m_k)) \right)_{k\geqslant 1} \in X_{B_{\mathrm{BH}}} \\ &\mapsto \left((v_{(i_k,m_k)}^{(k)}, v_{(i_{k+1},m_{k+1})}^{(k+1)}, \left(\sigma_{i_{k+1}}^{(k-1)}\right)^{-1} (n_k i_k + m_k)) \right)_{k\geqslant 1} \in X_{B'_{\mathrm{BH}}} \end{aligned}$$

is a conjugation between the Bratteli-Vershik systems $T_{B_{\mathrm{BH}}}$ and $T_{B'_{\mathrm{BH}}}$.

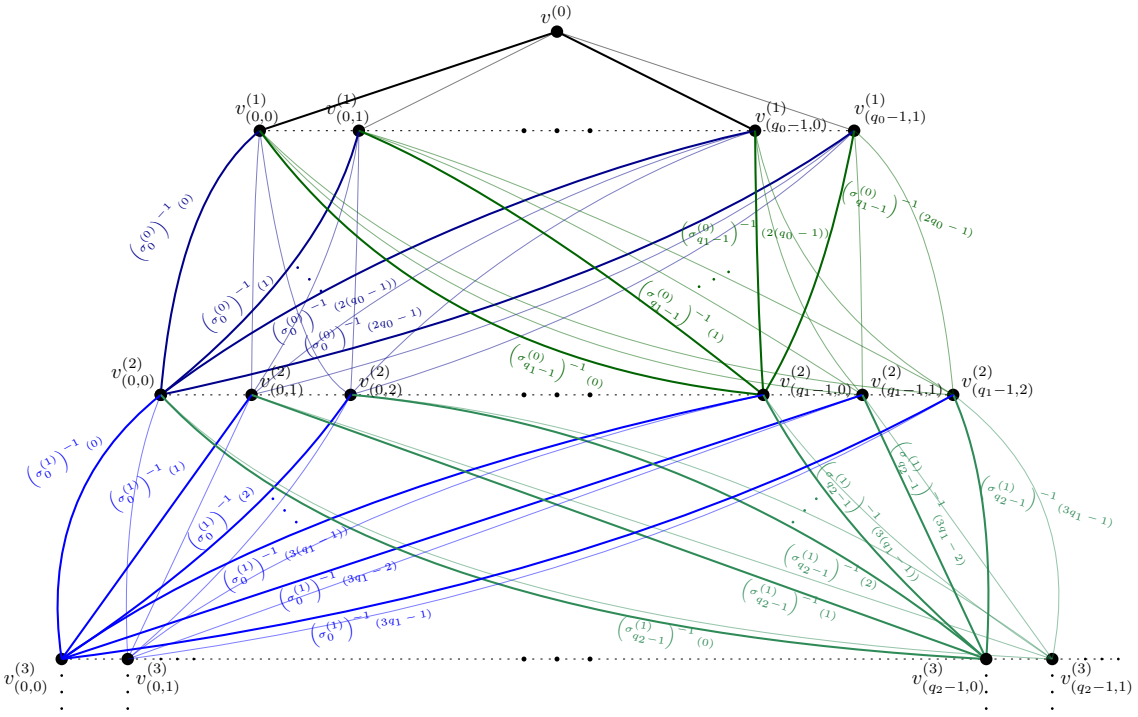


Figure II.9: To get $B'_{\mathrm{BH}} = (V', E')$ from the Bratteli diagram B_{BH} of Boyle and Handelman (see Figure II.8), we successively duplicate the vertices and the edges.
In B_{BH}, the integer n_1 is equal to 2 (from each vertex in V_1 to each one in V_2, there are two edges), so each vertex $v_i^{(1)}$ (with $0 \leqslant i < q_0$) is split in two new vertices $v_{(i,0)}^{(1)}$ and $v_{(i,1)}^{(1)}$, each one being associated to one of the two edges in $r^{-1}(v_j^{(2)})$ (for every $0 \leqslant j < q_1$). For every $0 \leqslant m_1 \leqslant n_1 - 1$, there is only one edge between $v^{(0)}$ to $v_{i,m_1}^{(1)}$ (this edge can be considered as a copy of the edge from $v^{(0)}$ to the former vertex $v_i^{(1)}$).
The integer n_2 is equal to 3 (from each vertex in V_2 to each one in V_3, there are three edges), so each vertex $v_j^{(2)}$ (with $0 \leqslant j < q_1$) is split in three new vertices $v_{(j,0)}^{(2)}$, $v_{(j,1)}^{(2)}$ and $v_{(j,2)}^{(2)}$, each one being associated to one of the three edges in $r^{-1}(v_k^{(3)})$ (for every $0 \leqslant k < q_2$). For every $0 \leqslant m_2 \leqslant n_2 - 1$, we define the edges of range $v_{j,m_2}^{(2)}$ as copies of the edges of range the former vertex $v_j^{(2)}$. A thicker edge corresponds to one copy. We do not indicate the rank of the other edges (the thinner ones) for clarity.
Then we apply the same algorithm to define the new vertices and edges in $E'_2, V'_3, E'_3, V'_4, \ldots$.

II.B.d Comparisons between Boyle and Handelman's proof and our techniques.

Unlike Boyle and Handelman, we prove the case $\alpha < +\infty$ of Theorem I with a cutting-and-stacking process where all new towers contain only one copy of each former tower. This is naturally the construction encoded by an odomutant endowed with the sequence $(\mathcal{P}(\ell))_{\ell \geqslant 1}$ of partitions in ℓ-cylinders (see Figure II.2). In order to get the case $\alpha = +\infty$, the main trick is to understand that a less restrictive cutting-and-stacking process, namely where every former tower may appear many times in the new ones, is encoded by an odomutant equipped with another sequence of partitions. Here the partitions are the ones associated to a description of this odomutant by multiple permutations, namely the partitions $\tilde{\mathcal{P}}(\ell)$ (see Definition II.3.3). A first way to understand why this is relevant is to notice that with these partitions, we cannot distinguish between towers of the same step, as if they were the copies of the same former tower which appear in a new one (see Figure II.3). Another remark is that Boyle and Handelman use the partitions in cylinders in the Cantor space $X_{B_{\mathrm{BH}}}$ on which their system $T_{B_{\mathrm{BH}}}$ is defined. Therefore, if we want to reformulate their proof in our formalism and with the odomutant conjugate to $T_{B_{\mathrm{BH}}}$, we have to consider the image of these partitions by the conjugation that we explicit above. It turns out that

we get the partitions $\tilde{\mathcal{P}}(\ell)$.

To prove that the system $T_{B_{\mathrm{BH}}}$ is strongly orbit equivalent to the dyadic odometer S, Boyle and Handelman use the Giordano-Putnam-Skau theorem and the fact that the dimension group of $T_{B_{\mathrm{BH}}}$ is $\mathbb{Z}[1/2]$. In our proof of Theorem I, the orbit equivalence is explicit and this enables us to directly show that the cocycles have at most one point of discontinuity. This also enables us to quantify the integrability of this orbit equivalence.

II.C Equivalence between definitions of loose Bernoullicity in the zero-entropy case

To our knowledge, justifications for the equivalence between two definitions of loose Bernoullicity in the zero-entropy case (see Theorem II.2.8) is missing is the literature. Here we provide a proof. Let us first recall these definitions, that we already wrote in Section II.2.d.

Definition II.C.1. Let $T \in \mathrm{Aut}(X,\mu)$ and $\mathcal{P}$ be a partition of X.

- $(T,\mathcal{P})$ is loosely Bernoulli, and we write T is **LB**, if for every $\varepsilon > 0$, for every sufficiently large integer N and for each $M > 0$, there exists a collection $\mathcal{G}$ of "good" atoms in $\mathcal{P}^0_{-M}$ whose union has measure greater than or equal to $1-\varepsilon$, and so that for each pair A, B of atoms in $\mathcal{G}$, the following holds: there is a probability measure $n_{A,B}$ on $\mathcal{P}^N \times \mathcal{P}^N$ satisfying

  (I) $n_{A,B}(\{w\} \times \mathcal{P}^N) = \mu_A(\{[\mathcal{P}]_{1,N}(.) = w\})$ for every $w \in \mathcal{P}^N$;

  (II) $n_{A,B}(\mathcal{P}^N \times \{w'\}) = \mu_B(\{[\mathcal{P}]_{1,N}(.) = w'\})$ for every $w' \in \mathcal{P}^N$;

  (III) $n_{A,B}(\{(w,w') \in \mathcal{P}^N \times \mathcal{P}^N \mid f_N(w,w') > \varepsilon\}) < \varepsilon$.

- We say that $(T,\mathcal{P})$ is $\mathbf{LB_0}$ if for every $\varepsilon > 0$ and for every sufficiently large integer N, there exists a collection $\mathcal{H}$ of "good" atoms in $\mathcal{P}_1^N$ whose union has measure greater than or equal to $1-\varepsilon$ and so that we have $f_N(w,w') \leqslant \varepsilon$ for every $w, w' \in [\mathcal{P}]_{1,N}(\mathcal{H})$.

Theorem II.C.2. *Let $T \in \mathrm{Aut}(X,\mu)$ and $\mathcal{P}$ be a partition of X. If $\mathrm{h}_\mu(T,\mathcal{P}) = 0$, then $(T,\mathcal{P})$ is LB if and only if it is LB_0.*

This theorem relies on the following key lemma which crucially uses the assumption on the entropy. Note that, when considering a set $\mathcal{Q}$ of subsets of X, for instance a set of atoms of a partition, $\mu(\mathcal{Q})$ will abusively denote the measure of $\bigcup_{A\in\mathcal{Q}} A$.

Lemma II.C.3. *Let $T \in \mathrm{Aut}(X,\mu)$ and $\mathcal{P}$ be a partition of X, such that $\mathrm{h}_\mu(T,\mathcal{P}) = 0$. Let $\alpha > 0$ and N be a positive integer. Then there exists an integer $M_0 \geqslant 0$ such that the following holds for every $M \geqslant M_0$: there exists a collection $(\mathcal{Q}_C)_{C\in\mathcal{P}_1^N}$ of disjoints subsets of $\mathcal{P}^0_{-M}$ such that*

- *for every $C \in \mathcal{P}_1^N$, for every $A \in \mathcal{Q}_C$, we have $\mu_A(C) \geqslant 1-\sqrt{\alpha}$;*
- *for every $C \in \mathcal{P}_1^N$, we have $\mu\left(\mathcal{Q}_C\right) \geqslant (1-2\sqrt{\alpha})\mu(C)$.*

Note that the second item implies $\mu\left(\bigcup_{C\in\mathcal{P}_1^N} \mathcal{Q}_C\right) \geqslant 1-2\sqrt{\alpha}$.

Proof of Lemma II.C.3. By [Dow11, Fact 2.3.12], the assumption $\mathrm{h}_\mu(T,\mathcal{P}) = 0$ implies that $\mathcal{P}_1^N$ is $\mathcal{P}^0_{-\infty}$-measurable, where

$$\mathcal{P}^0_{-\infty} \coloneqq \sigma\left(\mathcal{P}^0_{-M}, M \geqslant 0\right),$$

namely $\mathcal{P}^0_{-\infty}$ is the σ-algebra generated by the increasing sequence of algebras $\big(\sigma(\mathcal{P}^0_{-M})\big)_{M\geqslant 0}$. Then the following holds for every $C \in \mathcal{P}_1^N$: for every $\eta > 0$, there exists $B_C \in \bigcup_{M\geqslant 0}\sigma(\mathcal{P}^0_{-M})$ such that $\mu(C\Delta B_C) \leqslant \eta$. Applied to $\eta = \alpha\mu(C)$, this fact provides an integer $M_0 \geqslant 0$ such that every atom $C \in \mathcal{P}_1^N$ is closed to some $B_C \in \sigma(\mathcal{P}^0_{-M_0})$, namely $\mu(C\Delta B_C) \leqslant \alpha\mu(C)$. Let us fix an integer $M \geqslant M_0$, and notice that B_C is also in $\sigma(\mathcal{P}^0_{-M})$.

For every $C \in \mathcal{P}_1^N$, let us set

$$\mathcal{Q}_C \coloneqq \left\{A \in \mathcal{P}^0_{-M} \mid A \subset B_C,\ \mu_A(C) > \min\left(1-\sqrt{\alpha}, 1/2\right)\right\}.$$

Given two distinct atoms $C, C' \in \mathcal{P}_1^N$, the sets $\mathcal{Q}_C$ and $\mathcal{Q}_{C'}$ are disjoint, otherwise we would have an atom $A \in \mathcal{P}^0_{-M}$ lying in $\mathcal{Q}_C$ and $\mathcal{Q}_{C'}$ and such that the following occurs:

$$\mu(A) \geqslant \mu(A\cap C) + \mu(A\cap C') = \big(\mu_A(C) + \mu_A(C')\big)\,\mu(A) > \mu(A),$$

a contradiction.

Given $C \in \mathcal{P}_1^N$, it remains to prove $\mu(\mathcal{Q}_C) \geqslant (1-2\sqrt{\alpha})\mu(C)$. Let us write

$$\mathcal{Q}_C^- \coloneqq \{A \in \mathcal{P}^0_{-M} \mid A \subset B_C\}\backslash\mathcal{Q}_C.$$

On the one hand, we have

$$\begin{aligned}
\mu(B_C\cap C) &= \sum_{A\in\mathcal{Q}_C}\mu(A\cap C) + \sum_{A\in\mathcal{Q}_C^-}\mu(A\cap C)\\
&\leqslant \mu(\mathcal{Q}_C) + (1-\sqrt{\alpha})\mu(\mathcal{Q}_C^-)\\
&= \mu(\mathcal{Q}_C) + (1-\sqrt{\alpha})(\mu(B_C) - \mu(\mathcal{Q}_C))\\
&= (1-\sqrt{\alpha})\mu(B_C) + \sqrt{\alpha}\mu(\mathcal{Q}_C)\\
&\leqslant (1-\sqrt{\alpha})(1+\alpha)\mu(C) + \sqrt{\alpha}\mu(\mathcal{Q}_C).
\end{aligned}$$

where the last inequality comes from

$$\mu(B_C) \leqslant \mu(B_C\Delta C) + \mu(C) \leqslant (1+\alpha)\mu(C)$$

On the other hand, we have

$$\mu(B_C\cap C) \geqslant \mu(C) - \mu(B_C\Delta C) \geqslant (1-\alpha)\mu(C).$$

Combining all these inequalities, we get

$$\mu(\mathcal{Q}_C) \geqslant \frac{1}{\sqrt{\alpha}}\left(1-\alpha-(1-\sqrt{\alpha})(1+\alpha)\right)\mu(C) = (1-\sqrt{\alpha})^2\mu(C) \geqslant (1-2\sqrt{\alpha})\mu(C),$$

as wanted. □

Proof of Theorem II.C.2. Assume that $(T,\mathcal{P})$ is LB. Let us fix $\varepsilon \in]0,1[$ and a sufficiently large integer N as in the definition of LB. With $\alpha > 0$ small enough so that

$$(1-\sqrt{\alpha})(1-\sqrt{\alpha}-\varepsilon) \geqslant 1-2\varepsilon$$

$$\text{and } 1-2\sqrt{\alpha} \geqslant \varepsilon,$$

we apply Lemma II.C.3 to get M and $(\mathcal{Q}_C)_{C\in\mathcal{P}_1^N}$ as described in the statement. By definition of LB associated to the quantities ε, N and M, we get $\mathcal{G} \subset \mathcal{P}^0_{-M}$ covering at least $1-\varepsilon$ of the space, and a family $(n_{A,B})_{A,B\in\mathcal{G}}$ of probabilities on $\mathcal{P}^N \times \mathcal{P}^N$ satisfying items (I), (II) and (III). Let us define

$$\mathcal{H} \coloneqq \left\{C \in \mathcal{P}_1^N \mid \mathcal{G}\cap\mathcal{Q}_C \neq \varnothing\right\}.$$

We first have

$$\begin{aligned}\mu(\mathcal{H}) = \sum_{C\in\mathcal{H}} \mu(C) &\geqslant \sum_{C\in\mathcal{H}} \sum_{A\in\mathcal{Q}_C\cap\mathcal{G}} \mu(C\cap A) \geqslant (1-\sqrt{\alpha}) \sum_{C\in\mathcal{H}} \sum_{A\in\mathcal{Q}_C\cap\mathcal{G}} \mu(A)\\ &= (1-\sqrt{\alpha})\mu\left(\mathcal{G}\cap \bigcup_{C\in\mathcal{P}_1^N} \mathcal{Q}_C\right)\\ &\geqslant (1-\sqrt{\alpha})(1-\sqrt{\alpha}-\varepsilon)\\ &\geqslant 1-2\varepsilon.\end{aligned}$$

Secondly, let us consider $C, C' \in \mathcal{H}$ and let us prove that $w \coloneqq [\mathcal{P}]_{1,N}(C)$ and $w' \coloneqq [\mathcal{P}]_{1,N}(C')$ are f_N-close. By definition, we can pick $A \in \mathcal{G}\cap\mathcal{Q}_C$ and $B \in \mathcal{G}\cap\mathcal{Q}_{C'}$, and using items (I) and (II) we have

$$n_{A,B}(\{w\}\times\mathcal{P}^N) \geqslant \mu_A(C) \geqslant 1-\sqrt{\alpha}$$

$$\text{and } n_{A,B}(\mathcal{P}^N\times\{w'\}) \geqslant \mu_A(C') \geqslant 1-\sqrt{\alpha}.$$

This implies

$$n_{A,B}(\{(w,w')\}) \geqslant 1-2\sqrt{\alpha} \geqslant \varepsilon,$$

so $f_N(w,w') \leqslant \varepsilon$ by item (III). We have proved that $(T,\mathcal{P})$ satisfies LB_0 for 2ε.

Let us now assume that $(T,\mathcal{P})$ is LB_0, we fix $\varepsilon > 0$, a sufficiently large integer $N > 0$ and an associated $\mathcal{H} \subset \mathcal{P}_1^N$ as in the definition of LB_0. With $\alpha > 0$ small enough so that

$$(1-\sqrt{\alpha})^2 \geqslant 1-\varepsilon$$

$$\text{and } (1-2\sqrt{\alpha})(1-\varepsilon) \geqslant 1-2\varepsilon,$$

we apply Lemma II.C.3 to get M_0 and for every $M \geqslant M_0$, an associated collection $(\mathcal{Q}_C)_{C\in\mathcal{P}_1^N}$ as described in the statement. Let us fix $M \geqslant M_0$ and let us consider

$$\mathcal{G} \coloneqq \bigcup_{C\in\mathcal{H}} \mathcal{Q}_C$$

and for every $A, B \in \mathcal{G}$, the probability $n_{A,B}$ on $\mathcal{P}^N\times\mathcal{P}^N$ defined by

$$n_{A,B}(\{(w,w')\}) = \mu_A(\{[\mathcal{P}]_{1,N}(.) = w\})\mu_B(\{[\mathcal{P}]_{1,N}(.) = w'\}),$$

they automatically satisfy items (I) and (II). Given $C, C' \in \mathcal{H}$, $A \in \mathcal{Q}_C$ and $B \in \mathcal{Q}_{C'}$, and $w \coloneqq [\mathcal{P}]_{1,N}(C)$ and $w' \coloneqq [\mathcal{P}]_{1,N}(C')$, we have

$$n_{A,B}(\{(w,w')\}) \geqslant \mu_A(C)\mu_B(C') \geqslant (1-\sqrt{\alpha})^2 \geqslant 1-\varepsilon,$$

and since $f_N(w,w') \leqslant \varepsilon$, we get item (III). Finally, we have

$$\mu(\mathcal{G}) = \sum_{C\in\mathcal{H}} \mu(\mathcal{Q}_C) \geqslant (1-2\sqrt{\alpha}) \sum_{C\in\mathcal{H}} \mu(C) = (1-2\sqrt{\alpha})\mu(\mathcal{H}) \geqslant (1-2\sqrt{\alpha})(1-\varepsilon) \geqslant 1-2\varepsilon.$$

We have proved that $(T,\mathcal{P})$ satisfies LB for 2ε, N large enough and $M \geqslant M_0$. By [Fel76, Corollary 2], we can replace "for each $M > 0$" by "for every sufficiently large $M > 0$" in the definition of LB, so we are done. □

Chapter III

On the absence of quantitatively critical measure equivalence couplings

This chapter corresponds to the article [Cor25c].

Abstract

Given a measure equivalence coupling between two finitely generated groups, Delabie, Koivisto, Le Maître and Tessera have found explicit upper bounds on how integrable the associated cocycles can be. These bounds are optimal in many cases but the integrability of the cocycles with respect to these critical thresholds remained unclear. For instance, a cocycle from $\mathbb{Z}^{k+\ell}$ to $\mathbb{Z}^k$ can be L^p for all $p < \frac{k}{k+\ell}$ but not for $p > \frac{k}{k+\ell}$, and the case $p = \frac{k}{k+\ell}$ was an open question which we answer by the negative. Our main result actually yields much more examples where the integrability threshold given by Delabie-Koivisto-Le Maître-Tessera Theorems cannot be reached.

Contents

III.1 Introduction

Measure equivalence is an equivalence relation on countable groups introduced by Gromov as a measured analogue of quasi-isometry. A first example of measure equivalent groups is given by two lattices in the same locally compact group.

Another source of examples is provided by orbit equivalence. Two groups Γ and Λ are *orbit equivalent* if there exist two free probability measure-preserving Γ- and Λ-actions α_Γ and α_Λ on a standard probability space (X, μ), having the same orbits. This yields measurable functions $c_{\Gamma,\Lambda} \colon \Gamma \times X \to \Lambda$ and $c_{\Lambda,\Gamma} \colon \Lambda \times X \to \Gamma$ describing the distortions on the orbits, called the *cocycles* and defined almost everywhere by the equations

$$\alpha_\Gamma(\gamma)x = \alpha_\Lambda(c_{\Gamma,\Lambda}(\gamma, x))x \text{ and } \alpha_\Lambda(\lambda)x = \alpha_\Gamma(c_{\Lambda,\Gamma}(\lambda, x))x.$$

More generally, the notion of measure equivalence also yields cocycles $c_{\Gamma,\Lambda} \colon \Gamma \times X_\Lambda \to \Lambda$ and $c_{\Lambda,\Gamma} \colon \Lambda \times X_\Gamma \to \Gamma$, where $(X_\Gamma, \mu_{X_\Gamma})$ and $(X_\Lambda, \mu_{X_\Lambda})$ are probability spaces arising from the measure equivalence coupling between the groups (see Section III.2).

When the two groups are finitely generated, a stronger notion called L^1 *measure equivalence* can be defined. It requires that the measurable functions $|c_{\Gamma,\Lambda}(\gamma, \cdot)|_{S_\Lambda}$ and $|c_{\Lambda,\Gamma}(\lambda, \cdot)|_{S_\Gamma}$ are integrable for every $\gamma \in \Gamma$ and $\lambda \in \Lambda$, where $|.|_{S_\Gamma}$ and $|.|_{S_\Lambda}$ respectively denote the *word-length metrics* with respect to some finite generating sets S_Γ and S_Λ of the groups. This definition does not depend on the choice of S_Γ and S_Λ and we simply say that $c_{\Gamma,\Lambda}$ and $c_{\Lambda,\Gamma}$ are integrable. Many rigidity results have been uncovered in this context (see e.g. [BFS13] and [Aus16b]). Most of the time, these results tell us that L^1 measure equivalence captures the geometry of the groups, in contrast to Ornstein-Weiss Theorem [OW80] which states that all infinite countable amenable groups are measure equivalent.

To get finer rigidity results among finitely generated groups, Delabie, Koivisto, Le Maître and Tessera [DKLMT22] introduced more general quantitative restrictions on the cocycles. Given positive real numbers p and q, we say that two finitely generated groups Γ and Λ are $(\mathrm{L}^p, \mathrm{L}^q)$ *measure equivalent* (resp. $(\mathrm{L}^p, \mathrm{L}^q)$ *orbit equivalent*) if there exists a measure equivalence (resp. an orbit equivalence) between them and the associated cocycles $c_{\Gamma,\Lambda}$ and $c_{\Lambda,\Gamma}$ are respectively L^p and L^q, i.e. the real-valued measurable functions $|c_{\Gamma,\Lambda}(\gamma, \cdot)|_{S_\Lambda}$ and $|c_{\Lambda,\Gamma}(\lambda, \cdot)|_{S_\Gamma}$ are respectively L^p and L^q for every $\gamma \in \Gamma$ and $\lambda \in \Lambda$. We also replace L^p or L^q by L^0 when no requirement is made on the corresponding cocycle.

We can also define (φ, ψ)-integrability measure equivalence (resp. orbit equivalence) for non-decreasing maps $\varphi, \psi \colon \mathbb{R}_+ \to \mathbb{R}_+$ (see Definition III.2.4). In particular, L^p means that we consider the map $x \mapsto x^p$.

In the case of the groups $\mathbb{Z}^d$, for $d \geqslant 1$, Delabie, Koivisto, Le Maître and Tessera prove that there is no $(\mathrm{L}^p, \mathrm{L}^0)$ measure equivalence coupling from $\mathbb{Z}^{k+\ell}$ to $\mathbb{Z}^k$ for $p > \frac{k}{k+\ell}$ ([DKLMT22, Corollary 3.4]). On the other hand, they explicitly build a measure equivalence from $\mathbb{Z}^{k+\ell}$ to $\mathbb{Z}^k$ which is $(\mathrm{L}^p, \mathrm{L}^0)$ for every $p < \frac{k}{k+\ell}$ ([DKLMT22, Theorem 1.9]).

The existence of an $(\mathrm{L}^{\frac{k}{k+\ell}}, \mathrm{L}^0)$ measure equivalence coupling from $\mathbb{Z}^{k+\ell}$ to $\mathbb{Z}^k$ remained unclear (see also [DKLMT22, Question 1.10]). Our contribution provides a negative answer to this question (see Corollary III.4.3), thus yielding the following complete description:

Theorem K (see Theorem III.4.4). *If k and ℓ are positive integers, then there exists an $(\mathrm{L}^p, \mathrm{L}^0)$ measure equivalence coupling from $\mathbb{Z}^{k+\ell}$ to $\mathbb{Z}^k$ if and only if $p < \frac{k}{k+\ell}$.*

The absence of measure equivalence coupling from $\mathbb{Z}^{k+\ell}$ to $\mathbb{Z}^k$ with the critical integrability $(\mathrm{L}^{\frac{k}{k+\ell}}, \mathrm{L}^0)$ was the initial goal of the paper. As we will see later in this introduction, this is actually a particular case of more general statements (see Theorems L and N).

Our result relies on the following key lemma (Lemma III.3.2 in the easier case $\Gamma = \mathbb{Z}^{k+\ell}$ and $\Lambda = \mathbb{Z}^k$): given a measure equivalence coupling from $\mathbb{Z}^{k+\ell}$ and $\mathbb{Z}^k$, if a cocycle is φ-integrable, then it is ψ-integrable for another non-decreasing map $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\psi(x) = O(\varphi(x))$ does *not* hold as x goes to $+\infty$. If now we assume that the cocycle $c_{\mathbb{Z}^k,\mathbb{Z}^{k+\ell}}$ is φ-integrable where $\varphi(x) = x^{\frac{k}{k+\ell}}$, we can combine this with a more precise version of [DKLMT22, Corollary 3.4]: by their Theorem 3.1, we must have $\psi(x) = O(\varphi(x))$, a contradiction, thus proving our result. It is interesting to note that while the statement does not mention φ-integrability, its proof crucially uses it.

This key lemma is a natural adaptation of the following elementary, yet fundamental fact.

Fact. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers which is summable. Then there exists a sequence $(v_n)_{n\in\mathbb{N}}$ of non-negative real numbers which is summable and such that $u_n = o(v_n)$.*

Proof of the fact. We can find an increasing sequence $(N_k)_{k\geqslant 1}$ of positive integers satisfying $N_1 = 0$ and $\sum_{n\geqslant N_k}^{+\infty} u_n \leqslant \frac{1}{k^3}$ for every $k \geqslant 2$. Then for every integer $n \geqslant 0$, we define $v_n \coloneqq k u_n$ if $N_k \leqslant n < N_{k+1}$. We have

$$\sum_{n=0}^{+\infty} v_n = \sum_{k=1}^{+\infty} \sum_{n=N_k}^{N_{k+1}-1} k u_n \leqslant \sum_{k=1}^{+\infty} \frac{1}{k^2} < +\infty$$

and $u_n = o(v_n)$. □

Let us now present generalizations to other groups, using the isoperimetric profile (Theorem L) and then the growth (Theorem N). First, recall that given non-decreasing real-valued functions f and g defined on a neighborhood of $+\infty$, we say that f is *asymptotically less* than g, denoted by $f \preccurlyeq g$, if there exists a constant $C > 0$ such that $f(x) = O\left(g(Cx)\right)$ as $x \to +\infty$. We say that f is *asymptotically equivalent* to g, denoted by $f \approx g$, if $f \preccurlyeq g$ and $f \succcurlyeq g$. The *asymptotic behavior* of f is its equivalence class modulo $\approx$.

Given a finitely generated group Γ, its *isoperimetric profile* is a real-valued function $j_{1,\Gamma}$ defined on the set of positive integers and given modulo $\approx$ by the formula

$$j_{1,\Gamma}(x) \approx \sup_{A\subset\Gamma, |A|\leqslant x} \frac{|A|}{|\partial A|},$$

where $\partial A \coloneqq S_\Gamma A \,\Delta\, A$ and S_Γ is a finite generating subset of Γ. It has been computed for many groups, for instance $j_{1,\mathbb{Z}^d}(x) \approx x^{1/d}$ [Cou00], $j_{1,(\mathbb{Z}/2\mathbb{Z})\wr\mathbb{Z}}(x) \approx \log x$ [Ers03], where $(\mathbb{Z}/2\mathbb{Z})\wr\mathbb{Z}$ is a lamplighter group (the definition is recalled in Section III.4.c). Note that it is an unbounded function if and only if the group is amenable. It can thus be interpreted as a measurement of amenability: the faster it goes to infinity, the "more amenable" the group is. We refer the reader to [DKLMT22] for more details on the isoperimetric profile and more generally the ℓ^p-isoperimetric profile.

Now we state the theorem of Delabie, Koivisto, Le Maître and Tessera on the behaviour of the isoperimetric profile under quantitative measure equivalence.

Theorem III.1.1 ([DKLMT22, Theorem 1.1])**.** *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that φ and $t \mapsto t/\varphi(t)$ are non-decreasing, let Γ and Λ be finitely generated groups. Assume that there exists a (φ, L^0)-integrable measure equivalence coupling from Γ to Λ. Then their isoperimetric profiles satisfy the following asymptotic inequality:*

$$\varphi \circ j_{1,\Lambda} \preccurlyeq j_{1,\Gamma}.$$

If $j_{1,\Lambda}$ is injective, then $\varphi \circ j_{1,\Lambda} \preccurlyeq j_{1,\Gamma}$ means that there exists a constant $C > 0$ such that the following holds as x goes to $+\infty$:

$$\varphi(x) = O(j_{1,\Gamma}(Cj_{1,\Lambda}^{-1}(x))), \tag{III.1}$$

so Theorem III.1.1 provides upper bounds $x \mapsto j_{1,\Gamma}(Cj_{1,\Lambda}^{-1}(x))$ for $C > 0$. In order to generalize our first contribution ("there is no $(\mathrm{L}^{\frac{k}{k+\ell}}, \mathrm{L}^0)$ measure equivalence coupling from $\mathbb{Z}^{k+\ell}$ from $\mathbb{Z}^k$") to other groups, we must pay attention to a few obstacles which do not appear in the case $\Gamma = \mathbb{Z}^{k+\ell}$ and $\Lambda = \mathbb{Z}^k$.

- The isoperimetric profile of a finitely generated group Λ is not necessarily injective, so (III.1) is not well-defined in full generality. But when studying this function, we only take into account its asymptotic behaviour. Moreover, we will check that it suffices to consider an injective function h_Λ with the same asymptotic behavior (the existence of such a map is granted by Remark III.1.2).

- Given two different positive constants C and C', we do not know if the functions $j_{1,\Gamma}(Cj_{1,\Lambda}^{-1}(.))$ and $j_{1,\Gamma}(C'j_{1,\Lambda}^{-1}(.))$ have the same asymptotic behavior, so Theorem III.1.1 does not provide a precise upper bound of φ *a priori*. This is the reason why we will assume that the isoperimetric profile of Γ satisfies $j_{1,\Gamma}(Cx) = O(j_{1,\Gamma}(x))$ for every $C > 0$. For other technical reasons arising from the existence of a constant in the definition of "φ-integrability" (see Definition III.2.4), we will also require this hypothesis on $j_{1,\Gamma} \circ j_{1,\Lambda}^{-1}$. These requirements motivate Assumptions (III.3) and (III.4) in Theorem L below.

- In Lemma III.3.2, where we build a new map ψ from the original one $\varphi := j_{1,\Gamma} \circ j_{1,\Lambda}^{-1}$ (for the case $\Gamma = \mathbb{Z}^{k+\ell}$ and $\Lambda = \mathbb{Z}^k$, see the paragraph after the proof of the elementary fact), we need φ to be sublinear[1], hence Assumption (III.2) in Theorem L.

Hence, a first generalization is the following.

Theorem L. *Let Γ and Λ be finitely generated groups. Assume that there exist a non-decreasing function h_Γ and an increasing function h_Λ satisfying $h_\Gamma \approx j_{1,\Gamma}$, $h_\Lambda \approx j_{1,\Lambda}$ and the following assumptions as $x \to +\infty$:*

$$h_\Gamma(x) = o\left(h_\Lambda(x)\right), \tag{III.2}$$

$$\forall C > 0,\ h_\Gamma(Cx) = O\left(h_\Gamma(x)\right), \tag{III.3}$$

$$\forall C > 0,\ h_\Gamma \circ h_\Lambda^{-1}(Cx) = O\left(h_\Gamma \circ h_\Lambda^{-1}(x)\right). \tag{III.4}$$

Then there is no $(h_\Gamma \circ h_\Lambda^{-1}, \mathrm{L}^0)$-integrable measure equivalence coupling from Γ to Λ.

Remark III.1.2. The isoperimetric profile of a finitely generated group Γ is always asymptotically equivalent to an increasing function h_Γ. For instance, if $j_{1,\Gamma}$ satisfies

$$0 < j_{1,\Gamma}(n-1) < j_{1,\Gamma}(n) = \ldots = j_{1,\Gamma}(n+k-1) < j_{1,\Gamma}(n+k)$$

for some positive integers n and k, then we can set

$$h_\Gamma(n+i) := \frac{k-i}{k} j_{1,\Gamma}(n) + \frac{i}{k} \min\left(j_{1,\Gamma}(n+k), 2j_{1,\Gamma}(n)\right)$$

for every $i \in \{0, \ldots, k-1\}$. We do not provide the details.

[1] This is necessary to assume that $j_{1,\Gamma} \circ j_{1,\Lambda}^{-1}$ is sublinear. Indeed, we cannot apply the same strategy in the case $\Gamma = \mathbb{Z}$ and $\Lambda = \mathbb{Z}^2$, since Escalier and Joseph have built a measure equivalence coupling from $\mathbb{Z}$ to $\mathbb{Z}^2$ which is $(\mathrm{L}^\infty, \mathrm{L}^p)$ for every $p < \frac{1}{2}$ (not yet published work).

It is straightforward to check that the equivalence relation $\approx$ preserves Assumption (III.3) for a non necessarily injective function. Moreover satisfying Assumptions (III.2) and (III.3) is also preserved under this equivalence relation, as well as satisfying Assumptions (III.2), (III.3) and (III.4) when the inverse of one of the functions is well-defined.

Question III.1.3. Does the isoperimetric profile of a finitely generated group always satisfy Assumption (III.3)? In the case $j_{1,\Gamma}(x) = o(j_{1,\Lambda}(x))$, does there always exist a pair (h_Γ, h_Λ) of functions satisfying the assumptions of Theorem L?

The following corollary allows us to answer a question of Delabie, Koivisto, Le Maître and Tessera (see [DKLMT22, Question 1.2]) by the negative for many of finitely generated group Γ.

Corollary M (see Corollary III.4.1). *Let Γ be a finitely generated group which is not virtually cyclic. Assume that its isoperimetric profile $j_{1,\Gamma}$ satisfies*

$$\forall C > 0,\ j_{1,\Gamma}(Cx) = O\left(j_{1,\Gamma}(x)\right) \text{ as } x \to +\infty. \tag{III.5}$$

Then there is no $(j_{1,\Gamma}, \mathrm{L}^0)$-integrable measure equivalence coupling from Γ to $\mathbb{Z}$.

Given an increasing function satisfying a mild regularity condition, Brieussel and Zheng [BZ21] build a group whose isoperimetric profile is asymptotically equivalent to this function. It turns out that this regularity condition implies our condition (III.5) (see Section III.4.a). Moreover, if Γ is such a group[2], it follows from the work of Escalier [Esc24] that there exists an orbit equivalence from Γ to $\mathbb{Z}$ which is almost $(j_{1,\Gamma}, \mathrm{L}^0)$-integrable, thus providing a complete description similar to Theorem K (see Theorem III.4.2).

Explicit constructions of orbit equivalences in [DKLMT22] show that the upper bound given in Theorem III.1.1 is sharp for other groups than $\mathbb{Z}^d$, such as lamplighter groups or iterated wreath products. The existence of a measure equivalence coupling with this critical threshold remained unclear and our Theorem L enables us to answer by the negative. We refer the reader to Theorems III.4.6, III.4.8, III.4.10 and III.4.11 for precise statements.

Another rigidity result in [DKLMT22] deals with the notion of volume growth. Given a finitely generated group Γ and finite generating set S_Γ of Γ, we define

$$V_\Gamma(n) \coloneqq \left|\left\{\gamma_1 \dots \gamma_n \mid \gamma_1, \dots, \gamma_n \in S_\Gamma \cup (S_\Gamma)^{-1} \cup \{e_\Gamma\}\right\}\right|$$

for every positive integer n, where e_Γ denotes the identity element of Γ. As for the isoperimetric profile, we extend V_Γ to a continuous and non-decreasing function. The *volume growth* of Γ is the asymptotic behavior of V_Γ, it does not depend on the choice of S_Γ, nor does its extension to $\mathbb{R}_+$. We say that Γ has *polynomial growth of degree d* if $V_\Gamma(x) \approx x^d$. Finally, note that the volume growth is increasing but the isoperimetric profile may fail to be injective.

Theorem III.1.4 ([DKLMT22, Theorem 3.1]). *Let φ be an increasing, subadditive function such that $\varphi(0) = 0$, let Γ and Λ be finitely generated groups. Assume that there exists a (φ, L^0)-integrable measure equivalence coupling from Γ to Λ. Then*

$$V_\Gamma \preccurlyeq V_\Lambda \circ \varphi^{-1},$$

where φ^{-1} denotes the inverse function of φ.

With the same strategy as Theorem L, we get the following statement.

[2] We call it a Brieussel-Zheng group, although their construction is more general.

Theorem N. *Let Γ and Λ be finitely generated groups. Assume that there exist two increasing functions h_Γ and h_Λ satisfying $h_\Gamma \approx V_\Gamma$, $h_\Lambda \approx V_\Lambda$ and the following properties as $x \to +\infty$:*

$$h_\Gamma^{-1}(x) = o\left(h_\Lambda^{-1}(x)\right), \tag{III.6}$$

$$\forall C > 0,\ h_\Gamma^{-1}(Cx) = O\left(h_\Gamma^{-1}(x)\right), \tag{III.7}$$

$$\forall C > 0,\ h_\Gamma^{-1} \circ h_\Lambda(Cx) = O\left(h_\Gamma^{-1} \circ h_\Lambda(x)\right). \tag{III.8}$$

Then there is no $(h_\Gamma^{-1} \circ h_\Lambda, \mathrm{L}^0)$-integrable measure equivalence coupling from Γ to Λ.

We will prove Theorems L and N in Section III.3 and give the main applications in Section III.4.

More general statements of Delabie, Koivisto, Le Maître and Tessera deal with asymmetric weakenings of measure equivalence coupling: measure subgroup, quotient and subquotient couplings. We can still apply our ideas to these generalizations.

Theorems L and N still hold in the context of quantitative orbit equivalence, since the existence of a (φ, ψ)-integrable orbit equivalence from Γ to Λ is equivalent to the existence of a (φ, ψ)-integrable measure equivalence coupling with equal fundamental domains.

III.2 Quantitative measure equivalence

The groups Γ and Λ are always assumed to be finitely generated. By a *smooth* action of a countable group Γ, we mean a measure-preserving Γ-action on a standard measured space (Ω, μ) which admits a fundamental domain, namely a Borel subset X_Γ of Ω that intersects every Γ-orbit exactly once.

Definition III.2.1. A **measure equivalence coupling** between Γ and Λ is a quadruple $(\Omega, X_\Gamma, X_\Lambda, \mu)$ where (Ω, μ) is a standard Borel measure space equipped with commuting measure-preserving smooth Γ- and Λ-actions such that

1. both the Γ- and Λ-actions are free;
2. X_Γ (resp. X_Λ) is a fixed fundamental domain for the Γ-action (resp. for the Λ-action);
3. X_Γ and X_Λ have finite measures.

We will always use the notations $\gamma * x$ and $\lambda * x$ (with $\gamma \in \Gamma$, $\lambda \in \Lambda$, $x \in \Omega$) for these smooth actions on Ω. The notations $\gamma \cdot x$ and $\lambda \cdot x$ refers to the induced actions that we now define, as well as the cocycles.

Definition III.2.2. A measure equivalence coupling $(\Omega, X_\Gamma, X_\Lambda, \mu)$ between Γ and Λ induces a finite measure-preserving Γ-action on $(X_\Lambda, \mu_{X_\Lambda})$ in the following way: for every $\gamma \in \Gamma$ and every $x \in X_\Lambda$, $\gamma \cdot x \in X_\Lambda$ is defined by the identity

$$(\Lambda * \gamma * x) \cap X_\Lambda = \{\gamma \cdot x\},$$

it is unique since X_Λ is a fundamental domain for the smooth Λ-action.

This also yields a **cocycle** $c_{\Gamma,\Lambda} \colon \Gamma \times X_\Lambda \to \Lambda$ uniquely (by freeness) defined by

$$c_{\Gamma,\Lambda}(\gamma, x) * \gamma * x = \gamma \cdot x,$$

or equivalently $c_{\Gamma,\Lambda}(\gamma, x) * \gamma * x \in X_\Lambda$, for almost every $x \in X_\Lambda$ and every $\gamma \in \Gamma$. We similarly define a finite measure-preserving Λ-action on $(X_\Gamma, \mu_{X_\Gamma})$ and the associated cocycle $c_{\Lambda,\Gamma} \colon \Lambda \times X_\Gamma \to \Gamma$.

Remark III.2.3. The cocycle $c_{\Gamma,\Lambda}\colon \Gamma \times X_\Lambda \to \Lambda$ satisfies the cocycle identity

$$\forall \gamma_1, \gamma_2 \in \Gamma,\ \forall x \in X_\Lambda,\ c_{\Gamma,\Lambda}(\gamma_1\gamma_2, x) = c_{\Gamma,\Lambda}(\gamma_1, \gamma_2 \cdot x) c_{\Gamma,\Lambda}(\gamma_2, x).$$

Definition III.2.4 (Delabie, Koivisto, Le Maître and Tessera [DKLMT22])**.** Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing map. Given a measure equivalence coupling between Γ and Λ, we say that the cocycle $c_{\Gamma,\Lambda}\colon \Gamma \times X_\Lambda \to \Lambda$ is **φ-integrable** if for every $\gamma \in \Gamma$, there exists $c_\gamma > 0$ such that

$$\int_{X_\Lambda} \varphi\left(\frac{|c_{\Gamma,\Lambda}(\gamma, x)|_{S_\Lambda}}{c_\gamma}\right) \mathrm{d}\mu_{X_\Lambda}(x) < +\infty$$

where S_Λ is a finite generating set of Λ and for every λ, $|\lambda|_{S_\Lambda}$ denotes its **word-length metric** with respect to S_Λ, defined by

$$|\lambda|_{S_\Lambda} \coloneqq \min\{n \geqslant 0 \mid \exists \lambda_1, \dots, \lambda_n \in S_\Lambda \cup (S_\Lambda)^{-1} \cup \{e_\Lambda\}, \lambda = \lambda_1 \dots \lambda_n\}.$$

We define φ-integrability for $c_{\Lambda,\Gamma}$ in a similar way.

Remark III.2.5. Defining φ-integrability for the cocycle $c_{\Gamma,\Lambda}$ with the use of constants c_γ is necessary because we need the following properties:

- this notion of φ-integrability does not depend on the choice of the finite generating set of Λ, since for any finitely generated sets S_Λ, S'_Λ, there exists a constant $C > 0$ such that
$$\frac{1}{C}|\lambda|_{S'_\Lambda} \leqslant |\lambda|_{S_\Lambda} \leqslant C|\lambda|_{S'_\Lambda}$$
for every $\lambda \in \Lambda$;

- if $\varphi \approx \psi$, then φ-integrability and ψ-integrability are equivalent notions;

- to prove that the cocycle $c_{\Gamma,\Lambda}\colon \Gamma \times X_\Lambda \to \Lambda$ is φ-integrable, it suffices to check the finiteness of
$$\int_{X_\Lambda} \varphi\left(\frac{|c_{\Gamma,\Lambda}(\gamma, x)|_{S_\Lambda}}{c_\gamma}\right) \mathrm{d}\mu_{X_\Lambda}(x)$$
for every element γ in a finite generating set of Γ. This follows from [DKLMT22, Proposition 2.22].

Definition III.2.6 (Delabie, Koivisto, Le Maître and Tessera [DKLMT22])**.** A measure equivalence coupling $(\Omega, X_\Gamma, X_\Lambda, \mu)$ between the groups Γ and Λ is a **(φ, ψ)-integrable measure equivalence coupling** from Γ to Λ if $c_{\Gamma,\Lambda}\colon \Gamma \times X_\Lambda \to \Lambda$ is φ-integrable and $c_{\Lambda,\Gamma}\colon \Lambda \times X_\Gamma \to \Gamma$ is ψ-integrable.

For $p > 0$, we write L^p instead of φ or ψ if we consider the map $t \mapsto t^p$, and we write L^0 when no requirement is made on the cocycle. For example, the measure equivalence coupling is (φ, L^p)-integrable if $c_{\Gamma,\Lambda}$ is φ-integrable and $c_{\Lambda,\Gamma}$ is in $\mathrm{L}^p(X_\Lambda, \mu_{X_\Lambda})$; it is $(\mathrm{L}^p, \mathrm{L}^0)$-integrable if $c_{\Gamma,\Lambda}$ is $\mathrm{L}^p(X, \mu)$. Finally, a measure equivalence coupling is **φ-integrable** if it is (φ, φ)-integrable.

Note that a (φ, ψ)-integrable measure equivalence coupling from Γ to Λ is a (ψ, φ)-integrable measure equivalence coupling from Λ to Γ.

III.3 Proof of the main results

We now prove Theorems L and N. The key result is Lemma III.3.2, which uses Lemma III.3.1.

Lemma III.3.1. *Let $x \in \mathbb{R}$ and $\theta\colon [x,+\infty) \to \mathbb{R}$ be a continuous sublinear function. If y is a real number satisfying $y < \theta(t)$ for every $t \in [x,+\infty)$, then the set*

$$E(x,y,\theta) \coloneqq \left\{ t > x \mid \forall s \in [x,t], \theta(s) \geqslant \frac{\theta(t)-y}{t-x}(s-x)+y \right\}$$

is not bounded above.

Proof of Lemma III.3.1. Let us consider the continuous maps $a : t \in (x,+\infty) \mapsto \mathbb{R}$ and $m : t \in (x,+\infty) \mapsto \mathbb{R}$ defined by

$$a(t) = \frac{\theta(t)-y}{t-x} \text{ and } m(t) = \min_{s\in(x,t]} a(s).$$

Note that the set $E(x,y,\theta)$ is equal to $\{t > x \mid m(t) = a(t)\}$. Let us also define the set

$$E' \coloneqq \{t \in (x,+\infty) \mid \forall s \in (x,t),\ m(s) > m(t)\}.$$

By the assumptions, the non-increasing map m satisfies the following properties:

- $m(t) > 0$ for every $t \in (x,+\infty)$;
- $m(t) \underset{t\to+\infty}{\longrightarrow} 0$;
- if t is in E', then we have $m(t) = a(t)$.

Therefore the set E' is not bounded above and is included in $E(x,y,\theta)$. □

Lemma III.3.2. *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous, sublinear and increasing function. Given an integer $\ell \geqslant 1$ and a probability space (X,μ), let $f_1,\dots,f_\ell\colon X \to \mathbb{N}$ be measurable maps satisfying*

$$\int_X \varphi(f_i(x))\mathrm{d}\mu(x) < +\infty$$

for every $i \in \{1,\dots,\ell\}$. Then there exists a subadditive map $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\psi(0) = 0$, ψ and $t \mapsto t/\psi(t)$ are non-decreasing, and

1. *$\varphi(x_k) = o(\psi(x_k))$ for some increasing sequence $(x_k)_{k\geqslant 0}$ of non-negative real numbers tending to $+\infty$;*
2. *for every $i \in \{1,\dots,\ell\}$,*
$$\int_X \psi(f_i(x))\mathrm{d}\mu(x) < +\infty.$$

Proof of Lemma III.3.2. For every $n \geqslant 0$ and every $i \in \{1,\dots,\ell\}$, let us define the non-negative real number $u_n^{(i)} \coloneqq \varphi(n)\mu(\{f_i = n\})$. For every $i \in \{1,\dots,\ell\}$, the sequence $(u_n^{(i)})_{n\geqslant 0}$ is summable since

$$\sum_{n=0}^{+\infty} u_n^{(i)} = \sum_{n=0}^{+\infty} \varphi(n)\mu(\{f_i = n\}) = \int_X \varphi(f_i(x))\mathrm{d}\mu(x) < \infty.$$

Let $(N_k)_{k\geqslant 1}$ be an increasing sequence of positive integers satisfying $N_1 = 0$ and

$$\forall k \geqslant 2,\ \forall i \in \{1,\dots,\ell\},\ \sum_{n=N_k}^{+\infty} u_n^{(i)} \leqslant \frac{1}{k^3}.$$

Then for every integer $n \geqslant 1$, we define $K_n \coloneqq k$ if $N_k \leqslant n < N_{k+1}$. The sequence $(K_n)_{n\geqslant 1}$ tends to $+\infty$ and the sequences $(K_n u_n^{(i)})_{n\geqslant 1}$ are summable (see the proof of the fact in the introduction).

We inductively build an increasing sequence $(x_k)_{k\geqslant 0}$ of integers satisfying $x_0 = 0$ and $x_k \geqslant N_{k+1}$ for every $k \geqslant 1$, a decreasing sequence $(a_k)_{k\geqslant 0}$ of positive real numbers, a sequence $(b_k)_{k\geqslant 0}$ of non-negative real numbers satisfying $b_0 = 0$, and a continuous piecewise linear map $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following properties:

- for every $k \geqslant 0$, for every $t \in [x_k, x_{k+1}]$, $\psi(t) = b_k + a_k t$ and $\psi(t) \leqslant (k+1)\varphi(t)$;
- for every $k \geqslant 0$, $\psi(x_k) = k\varphi(x_k)$.

Let us set $x_0 \coloneqq 0$, $x_1 \coloneqq N_2$, $a_0 = \varphi(N_2)/N_2$, $b_0 = 0$ and for every $t \in [0, N_2]$,

$$\psi(t) \coloneqq \frac{\varphi(N_2)}{N_2} t.$$

Given an integer $k \geqslant 2$, assume that we have already defined $0 = x_0 < x_1 < \ldots < x_{k-1}$, $a_0 > \ldots > a_{k-2}$, $b_0, \dots, b_{k-2}$ and the map ψ on $[0, x_{k-1}]$. By the assumptions on φ and since

$$\psi(x_{k-1}) = (k-1)\varphi(x_{k-1}) < k\varphi(x_{k-1}),$$

we can apply Lemma III.3.1 to $x \coloneqq x_{k-1}$, $y \coloneqq \psi(x_{k-1})$, $\theta \coloneqq k \times \varphi$. We choose $x_k \in E(x_{k-1}, \psi(x_{k-1}), k \times \varphi)$ sufficiently large so that

- $x_k \geqslant N_{k+1}$;
- $a_{k-1} \coloneqq \dfrac{k\varphi(x_k) - \psi(x_{k-1})}{x_k - x_{k-1}}$ is less than a_{k-2},

the last condition being possible since φ is sublinear. Let us define

$$b_{k-1} \coloneqq \psi(x_{k-1}) - \frac{k\varphi(x_k) - \psi(x_{k-1})}{x_k - x_{k-1}} x_{k-1}.$$

We then extend ψ on $[x_{k-1}, x_k]$ by setting

$$\psi(t) \coloneqq b_{k-1} + a_{k-1} t = \frac{k\varphi(x_k) - \psi(x_{k-1})}{x_k - x_{k-1}} (t - x_{k-1}) + \psi(x_{k-1}),$$

so that ψ satisfies $\psi(x_k) = k\varphi(x_k)$ and $\psi(t) \leqslant k\varphi(t)$ for every $t \in [x_{k-1}, x_k]$ (by definition of the set $E(x_{k-1}, \psi(x_{k-1}), k \times \varphi)$). The real number b_{k-1} is necessarily non-negative since we have $b_{k-2} + a_{k-2}x_{k-1} = b_{k-1} + a_{k-1}x_{k-1}$ with $a_{k-1} < a_{k-2}$ and $b_{k-2} \geqslant 0$.

Let us prove that ψ satisfies the desired conditions. The map ψ is increasing since the real numbers a_i are positive. It is easy to prove that $\varphi(x_k) = o(\psi(x_k))$. Since the map $t \in (0, +\infty) \mapsto \frac{t}{at+b}$ is non-decreasing if $a > 0$ and $b \geqslant 0$, we get that the map $t \mapsto t/\psi(t)$ is non-decreasing. We build ψ as a concave and increasing map satisfying $\psi(0) = 0$, so ψ

is subadditive. Finally, given an integer $i \in \{1, \dots, \ell\}$, we have

$$\begin{aligned}\sum_{n=x_1}^{+\infty} \psi(n)\mu(\{f_i = n\}) &= \sum_{k=1}^{+\infty} \sum_{n=x_k}^{x_{k+1}-1} \psi(n)\mu(\{f_i = n\}) \\ &\leqslant \sum_{k=1}^{+\infty} \sum_{n=x_k}^{x_{k+1}-1} (k+1)\varphi(n)\mu(\{f_i = n\}) \\ &\leqslant \sum_{k=1}^{+\infty} \sum_{n=x_k}^{x_{k+1}-1} K_n\varphi(n)\mu(\{f_i = n\}) \\ &= \sum_{n=1}^{+\infty} K_n u_n^{(i)} < \infty,\end{aligned}$$

where the second inequality follows from the inequalities $k+1 \leqslant K_n$ for every integers n and k satisfying $n \geqslant x_k$ (since we have $x_k \geqslant N_{k+1}$). The equality

$$\int_X \psi(f_i(x))\mathrm{d}\mu(x) = \sum_{n=0}^{x_1-1} \psi(n)\mu(\{f_i = n\}) + \sum_{n=x_1}^{+\infty} \psi(n)\mu(\{f_i = n\})$$

implies that the integral is finite. □

Proof of Theorem L. Suppose that there exist a non-decreasing function h_Γ and an increasing function h_Λ satisfying $h_\Gamma \approx j_{1,\Gamma}$, $h_\Lambda \approx j_{1,\Lambda}$ and the following assumptions as $x \to +\infty$:

$$h_\Gamma(x) = o\left(h_\Lambda(x)\right), \tag{III.9}$$

$$\forall C > 0,\ h_\Gamma(Cx) = O\left(h_\Gamma(x)\right), \tag{III.10}$$

$$\forall C > 0,\ h_\Gamma \circ h_\Lambda^{-1}(Cx) = O\left(h_\Gamma \circ h_\Lambda^{-1}(x)\right). \tag{III.11}$$

Let us assume by contradiction that there exists a $(h_\Gamma \circ h_\Lambda^{-1}, \mathrm{L}^0)$-integrable measure equivalence coupling $(\Omega, X_\Gamma, X_\Lambda, \mu)$ from Γ to Λ. Let us fix finite generating sets S_Γ of Γ and S_Λ of Λ. We write $S_\Gamma = \{\gamma_1, \dots, \gamma_\ell\}$. For every $i \in \{1, \dots, \ell\}$, there is a constant $c_{\gamma_i} > 0$ such that

$$\int_{X_\Lambda} h_\Gamma \circ h_\Lambda^{-1}\left(\frac{|c_{\Gamma,\Lambda}(\gamma_i, x)|_{S_\Lambda}}{c_{\gamma_i}}\right) \mathrm{d}\mu_{X_\Lambda}(x) < +\infty.$$

Using Assumption (III.11) for $C = c_{\gamma_i}$, we may and do assume that $c_{\gamma_i} = 1$ for every $i \in \{1, \dots, \ell\}$. We now apply Lemma III.3.2 to $\varphi = h_\Gamma \circ h_\Lambda^{-1}$ (φ is sublinear by Assumption (III.9)), $(X, \mu) = (X_\Lambda, \mu_{X_\Lambda})$ and $f_i \colon x \mapsto |c_{\Gamma,\Lambda}(\gamma_i, x)|_{S_\Lambda}$. We thus get that $(\Omega, X_\Gamma, X_\Lambda, \mu)$ is a (ψ, L^0)-integrable measure equivalence coupling from Γ to Λ, for some map $\psi \colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following properties:

(A) $h_\Gamma \circ h_\Lambda^{-1}(x_k) = o(\psi(x_k))$ for some sequence $(x_k)_{k\geqslant 0}$ of non-negative real numbers tending to $+\infty$;

(B) ψ and $t \mapsto \frac{t}{\psi(t)}$ are non-decreasing;

(C) ψ is subadditive;

If we have

$$h_\Gamma \geqslant \psi \circ h_\Lambda, \tag{III.12}$$

namely $\psi(x) = O\left(h_\Gamma(Ch_\Lambda^{-1}(x))\right)$ for some constant $C > 0$, then we get a contradiction with Assumption (III.10) and Property (A). Now it remains to prove Inequality (III.12).

First, Property (B) and Theorem III.1.1 imply that

$$j_{1,\Gamma} \succcurlyeq \psi \circ j_{1,\Lambda},$$

which means that there exist constants $C, D > 0$ such that $\psi(j_{1,\Lambda}(x)) \leqslant D j_{1,\Gamma}(Cx)$ for every $x \geqslant 0$. Secondly there also exist constants $C_1, C_2, D_1, D_2 > 0$ such that $h_\Lambda(x) \leqslant D_1 j_{1,\Lambda}(C_1 x)$ and $j_{1,\Gamma}(x) \leqslant D_2 h_\Gamma(C_2 x)$ for every $x \geqslant 0$. Moreover, by Property (C) and the monotonicity of ψ, we have $\psi(cx) \leqslant \lceil c \rceil \psi(x)$ for every $c > 0$. Finally, this gives

$$\begin{aligned} \psi(h_\Lambda(x)) &\leqslant \psi(D_1 j_{1,\Lambda}(C_1 x)) \\ &\leqslant \lceil D_1 \rceil \psi(j_{1,\Lambda}(C_1 x)) \\ &\leqslant \lceil D_1 \rceil D j_{1,\Gamma}(C C_1 x) \\ &\leqslant \lceil D_1 \rceil D D_2 h_\Gamma(C C_1 C_2 x) \end{aligned}$$

and we get Inequality (III.12). □

Proof of Theorem N. This is the same proof as Theorem L, except that we get a contradiction with Theorem III.1.4, using the fact that Lemma III.3.2 yields a map ψ which can be increasing and subadditive and satisfy $\psi(0) = 0$. Moreover we similarly prove that $V_\Gamma \succcurlyeq V_\Lambda \circ \psi^{-1}$ implies $h_\Gamma \succcurlyeq h_\Lambda \circ \psi^{-1}$. □

III.4 Applications

III.4.a Coupling from a finitely generated group to $\mathbb{Z}$

Corollary III.4.1. *Let Γ be a finitely generated group which is not virtually cyclic. Assume that its isoperimetric profile $j_{1,\Gamma}$ satisfies*

$$\forall C > 0,\ j_{1,\Gamma}(Cx) = O\left(j_{1,\Gamma}(x)\right) \text{ as } x \to +\infty. \tag{III.13}$$

Then there is no $(j_{1,\Gamma}, \mathrm{L}^0)$-integrable measure equivalence coupling from Γ to $\mathbb{Z}$.

Proof of Corollary III.4.1. A group Γ is not virtually cyclic if and only if $j_{1,\Gamma}(x) = o(x)$. This is a consequence of the Coulhon Saloff-Coste isoperimetric inequality [CS93, Theorem 1] and the fact that the volume growth of such a group is at least quadratic if it is not virtually cyclic (see e.g. [Man11, Corollary 3.5]). We then apply Theorem L and Remark III.1.2 to get Corollary III.4.1. □

In [BZ21, Theorem 1.1] Brieussel and Zheng prove that for any non-decreasing function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $x \mapsto x/f(x)$ is non-decreasing, there exists a group Γ such that $j_{1,\Gamma} \approx \frac{\log}{f \circ \log}$, we call it a Brieussel-Zheng group (although their construction is more general).

Defining the map $F \coloneqq \frac{\log}{f \circ \log}$, the monotonicity of f (resp. of $x \mapsto x/f(x)$) implies that $F/\log$ is non-increasing (resp. F is non-decreasing) and the converse is true. Therefore, any non-decreasing function $F\colon [1, \infty) \to [1, \infty)$ such that $F/\log$ is non-increasing is the isoperimetric profile of a group. This equivalent statement was already noticed in [DKLMT22, Theorem 4.26].

From this we deduce that the isoperimetric profiles provided by Brieussel and Zheng satisfy Assumption (III.13). Indeed, let F be a non-decreasing function such that $F/\log$ is non-increasing, and let C be a positive constant. If $C < 1$, then the monotonicity of F directly implies the inequality $F(Ct) \leqslant F(t)$. If $C \geqslant 1$, we get

$$\frac{F(Cx)}{\log(Cx)} \leqslant \frac{F(x)}{\log(x)}$$

by monotonicity of $F/\log$, so we have $F(Cx) \leqslant F(x)\frac{\log(Cx)}{\log(x)}$, where the right-hand side is less than $2F(x)$ when x is large enough.

As mentioned in the introduction, Escalier [Esc24, Theorem 1.7] proves that for every[3] Brieussel-Zheng group Γ mentioned above, there exists an orbit equivalence coupling from Γ to $\mathbb{Z}$ which is $(\varphi_\varepsilon, \mathrm{L}^0)$-integrable for all $\varepsilon > 0$, where $\varphi_\varepsilon(x) = \frac{j_{1,\Gamma}(x)}{(\log j_{1,\Gamma}(x))^{1+\varepsilon}}$. Hence, we deduce the following.

Theorem III.4.2. *Let Γ be a Brieussel-Zheng group and $p > 0$. Then there exists a $((j_{1,\Gamma})^p, \mathrm{L}^0)$-integrable measure equivalence from Γ to $\mathbb{Z}$ if and only if $p < 1$.*

III.4.b Coupling between groups of polynomial growth

Corollary III.4.3. *Assume that Γ and Λ have polynomial growth of degree b and a respectively, with $b > a$. Then there is no $(\mathrm{L}^{a/b}, \mathrm{L}^0)$ measure equivalence coupling from Γ to Λ.*

Proof of Corollary III.4.3. The isoperimetric profiles satisfy $j_{1,\Gamma}(x) \approx x^{1/b}$ and $j_{1,\Lambda}(x) \approx x^{1/a}$ (see [CS93, Theorem 1]), so the corollary follows from Theorem L. □

As mentioned in the introduction, Delabie, Koivisto, Le Maître and Tessera [DKLMT22] explicitly build an orbit equivalence in the special case of the groups $\mathbb{Z}^d$ for $d \geqslant 1$, and then show that there exists a measure equivalence coupling from $\mathbb{Z}^b$ to $\mathbb{Z}^a$ (with $b > a$) which is $(\mathrm{L}^p, \mathrm{L}^0)$-integrable for every $p < a/b$. But the existence of an $(\mathrm{L}^{a/b}, \mathrm{L}^0)$-integrable coupling remained unclear. Our Corollary III.4.3 then gives the following complete description.

Theorem III.4.4. *Given positive integers $b > a$, there exists an $(\mathrm{L}^p, \mathrm{L}^0)$ measure equivalence coupling from $\mathbb{Z}^b$ to $\mathbb{Z}^a$ if and only if $p < a/b$.*

III.4.c Lamplighter groups

Let G and F be two countable groups and $\bigoplus_{g\in G} F$ be the subgroup of F^G consisting of all functions with finite support[4]. We define the action of G on $\bigoplus_{g\in G} F$ as follows. For every $g \in G$ and every $f \in \bigoplus_{g\in G} F$, the function $g \cdot f \in \bigoplus_{g\in G} F$ is defined by:

$$\forall g' \in G,\ (g \cdot f)(g') = f(g^{-1}g').$$

Then the *wreath product* $F \wr G$ is the semi-direct product

$$F \wr G \coloneqq \left(\bigoplus_{g\in G} F\right) \rtimes G.$$

When F is a non-trivial finite group, $F \wr G$ is also called a *lamplighter group*.

Corollary III.4.5. *Assume that G and H have polynomial growth of degree b and a respectively, with $b > a$, and let F and K be non-trivial finite groups. Then there is no $(\mathrm{L}^{a/b}, \mathrm{L}^0)$ measure equivalence coupling from $F \wr G$ to $K \wr H$.*

[3] Actually, the statement of Theorem 1.7 in [Esc24] is the following : given a non-decreasing function F such that $F/\log$ is non-decreasing, there exists a group Γ such that $j_{1,\Gamma} \approx F$ and there exists an orbit equivalence coupling from Γ to $\mathbb{Z}$ which is $(\varphi_\varepsilon, \exp\circ F\circ\exp)$ for every $\varepsilon > 0$, where $\varphi_\varepsilon(x) = F(x)/(\log F(x))^{1+\varepsilon}$. The group Γ is in fact a Brieussel-Zheng group and the proof of the theorem shows that the existence of such an orbit equivalence holds for every such groups.

[4] The support of a function $f\colon G \to F$ is the set $\{g \in G \mid f(g) \neq e_F\}$ where e_F is the identity element of F.

Proof of Corollary III.4.5. The isoperimetric profiles satisfy $j_{1,F\wr G}(x) \approx (\log x)^{1/b}$ and $j_{1,K\wr H}(x) \approx (\log x)^{1/a}$ (see [Ers03, Theorem 1]), so the corollary follows from Theorem L. □

In the case $F = K$, $G = \mathbb{Z}^b$ and $H = \mathbb{Z}^a$, using the notion of wreath product for measure-preserving equivalence relations, Corollary 7.4 in [DKLMT22] implies that there exists an $(\mathrm{L}^p, \mathrm{L}^0)$ measure equivalence coupling from $F \wr \mathbb{Z}^b$ to $F \wr \mathbb{Z}^a$ for every $p < a/b$. Combined with Corollary III.4.5, this yields the following theorem.

Theorem III.4.6. *Given positive integers $b > a$, there exists an $(\mathrm{L}^p, \mathrm{L}^0)$ measure equivalence coupling from $F \wr \mathbb{Z}^b$ to $F \wr \mathbb{Z}^a$ if and only if $p < a/b$.*

Corollary III.4.7. *Assume that G and Λ have polynomial growth of degree b and a respectively, with $b > a$, and let F be a non-trivial finite group. Then there is no $(\log^{1/b}, \mathrm{L}^0)$-integrable measure equivalence coupling from $F \wr G$ to Λ.*

Proof of Corollary III.4.7. The isoperimetric profiles satisfy $j_{1,F\wr G}(x) \approx (\log x)^{1/b}$ and $j_{1,\Lambda}(x) \approx x^{1/a}$ (see [Ers03, Theorem 1] and [CS93, Theorem 1]), so we are done by Theorem L. □

In the case $G = \mathbb{Z}$ and $\Lambda = \mathbb{Z}$, it is shown in [DKLMT22, Proposition 6.20] that there exists a $(\log^p, \mathrm{L}^0)$-integrable measure equivalence coupling from $F \wr \mathbb{Z}$ to $\mathbb{Z}$ for every $p < 1$ (this statement deals with $F = \mathbb{Z}/m\mathbb{Z}$ but remains true for any finite group), and Corollary III.4.7 completes this result.

Theorem III.4.8. *Given a finite group F, there exists a $(\log^p, \mathrm{L}^0)$-integrable measure equivalence coupling from $F \wr \mathbb{Z}$ to $\mathbb{Z}$ if and only if $p < 1$.*

III.4.d Iterated wreath products

Given an integer $k \geqslant 1$ and a finite group F, we define groups $H_n(k)$ inductively as follows: $H_0(k) = \mathbb{Z}^k$ and $H_{n+1}(k) = F \wr H_n(k)$. Given a positive integer n, the map $\log^{\circ n}$ denotes the composition $\log \circ \ldots \circ \log$ (n times).

Corollary III.4.9.

- *If $b > a$, then there is no $(\mathrm{L}^{a/b}, \mathrm{L}^0)$ measure equivalence coupling from $H_n(b)$ to $H_n(a)$.*
- *Given integers $d, k \geqslant 1$, there is no $((\log^{\circ n})^{1/k}, \mathrm{L}^0)$-integrable measure equivalence coupling from $H_n(k)$ to $\mathbb{Z}^d$.*

Proof of Corollary III.4.9. The isoperimetric profiles satisfy $j_{1,H_n(k)}(x) \approx (\log^{\circ n} x)^{1/k}$ (see [Ers03, Theorem 1]), and $j_{1,\mathbb{Z}^d}(x) \approx x^{1/d}$. Then the corollary follows from Theorem L. □

Using the notion of wreath products of measure-preserving equivalence relations, it is proven in [DKLMT22, Corollary 7.5] that there exists an $(\mathrm{L}^p, \mathrm{L}^0)$ measure equivalence coupling from $H_n(b)$ to $H_n(a)$ for every $p < a/b$. Moreover the composition of couplings yields a $((\log^{\circ n})^p, \mathrm{L}^0)$ measure equivalence coupling from $H_n(1)$ to $\mathbb{Z}$ for every $p < 1$ (see [DKLMT22, Corollary 7.6]). Our results allow us to complete these observations.

Theorem III.4.10. *Given positive integers $b > a$, there exists an $(\mathrm{L}^p, \mathrm{L}^0)$ measure equivalence coupling from $H_n(b)$ to $H_n(a)$ if and only if $p < a/b$.*

Theorem III.4.11. *Given integers $d, k \geqslant 1$, there exists a $((\log^{\circ n})^p, \mathrm{L}^0)$-integrable measure equivalence coupling from $H_n(1)$ to $\mathbb{Z}$ if and only if $p < 1$.*

Remark III.4.12. All the measure equivalence couplings provided in [Esc24] and [DKLMT22] and that we have mentioned in Section III.4 actually come from a construction of orbit equivalences between the groups, with the same integrability for the cocycles. Then Theorems III.4.4, III.4.6, III.4.8, III.4.10 and III.4.11 remain valid in the context of quantitative orbit equivalence.

Conclusion

These works naturally lead to several open questions. The most important one is probably the question of a quantitative Dye theorem.

Question 40. Does there exist an unbounded map $\varphi\colon \mathbb{R} \to \mathbb{R}$ such that any two ergodic transformations $S, T \in \mathrm{Aut}(X, \mu)$ are φ-integrably orbit equivalent?

Such a map φ would be less than log since log-integrable orbit equivalence preserves entropy. To answer this question by the negative, a strategy would be to find finer invariants.

Question 41. Given any map $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$, what could be an invariant of φ-integrable orbit equivalence?

Fixing two ergodic transformations $S, T \in \mathrm{Aut}(X, \mu)$, we would also like to know what is the "smallest" map $\varphi_0\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that S and T are not φ_0-integrably orbit equivalent, namely a map φ_0, such that S and T are not φ-integrably orbit equivalent if and only if $\varphi \succcurlyeq \varphi_0$. If we now fix an orbit equivalence, coming with fixed cocycles, we want to know if the "minimal" integrability can be easily read on the cocycles. It follows from [CJLMT23, Theorem 4.5] that if the cocycle $c_T\colon X \to \mathbb{Z}$ is φ-integrable, then

$$\frac{\varphi(|c_T(n,x)|)}{n} \underset{n\to\pm\infty}{\to} 0,$$

for almost every $x \in X$, where $c_T(n,x)$ stands for the cocycle $\mathbb{Z} \times X \to \mathbb{Z}$ considered more generally for group actions and that we usually restrict to the generator $+1$ in the particular case of the group $\mathbb{Z}$.

Question 42. What is the asymptotics of $\max_{0\leqslant i\leqslant n} |c_T(n,x)|$ as n goes to $+\infty$? Is it related to the "smallest" φ_0 such that c_T is not φ_0-integrable ?

Note that the proof of Belinskaya's theorem tells us that if $c_T(n,x)$ diverges to $\pm\infty$, then S and T are flip-conjugate, and S is conjugate to T^ε for $\varepsilon \in \{-1,+1\}$ such that $c_T(n,x) \underset{n\to\pm\infty}{\sim} \varepsilon n$. If we can answer the previous question, the asymptotic information we can derive from it could enable us to find a direct proof of the fact that log-integrable orbit equivalence preserves the entropy, without using Shannon orbit equivalence, and maybe using a more combinatorial definition of entropy. With such a proof, we could more deeply understand the reason why the logarithm plays a particular role.

Conversely, we wonder whether some classical relations between transformations imply quantitative forms of orbit equivalence.

Question 43. Given any two transformations $T, S \in \mathrm{Aut}(X, \mu)$ with the same entropy, are they Shannon orbit equivalent or φ-integrably orbit equivalent for some unbounded $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$? Does even Kakutani equivalence imply Shannon orbit equivalence or φ-integrable orbit equivalence for some unbounded $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$? Does Kakutani equivalence imply φ-integrable orbit equivalence for some unbounded $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ dominated by log?

Examples of evenly Kakutani equivalent transformations are rank-one systems.

Question 44. Are any two rank-one systems Shannon orbit equivalent? φ-integrably orbit equivalent for some unbounded $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$?

We may also wonder whether any two rank-one systems are $\mathrm{L}^{<1}$ orbit equivalent, since a recent result of Naryshkin and Petrakos states that it holds true for odometers [NP25].

Appendix A

Some basics of ergodic theory and amenable groups

For the readers who are not acquainted with the basic notions considered in this thesis, this appendix gives some recalls. We start with an introduction to ergodic theory, with the main tools. Then we present all the terminologies on finitely generated groups and amenable groups. Finally we briefly explain how ergodic theory is generalized to actions of amenable groups.

Contents

A.1 Basics of ergodic theory

Let us recall some basic notions of ergodic theory used in this thesis.

In the sequel, we focus on a probability space (X, μ) which is standard and atomless, and elements of $\mathrm{Aut}(X, \mu)$, namely bimeasurable bijections $X \to X$ which preserve the measure μ, but the results we will state also hold for non invertible dynamical systems. In fact, invertible systems yield actions of the group $\mathbb{Z}$ and we will explain in Section A.3 how the theory generalises for actions of amenable groups. Finally we will sometimes deal with dynamical systems in a more topological setting.

The reader may refer to [Gla03], [Dow11] and [VO16] (which contains a lot of examples) for more details.

A.1.a Motivations behind dynamical systems

Given a set X and a map $T\colon X \to X$, we want to study the sequence $(T^n x)_{n\geqslant 0}$ for every $x \in X$.

1. Does it converge? diverge?
2. Does it visit every "region" of the set X?
3. For which x can we say something about $(T^n x)_{n\geqslant 0}$?

These questions make sense if X is equipped with some structure. For instance, for the questions 1 and 2, we need a topology on X. For question 3, the answer could be: $(T^n x)_{n\geqslant 0}$ is periodic for every x in a dense subset (X equipped with a topology), or for almost every $x \in X$ (X is a measured space).

Example A.1.1.

- Given $\lambda > 1$, let us consider $X = \mathbb{R}$ and $T\colon x \in \mathbb{R} \to \lambda x \in \mathbb{R}$. The dynamics is very easy to study since we have the following closed formula:

$$\forall x \in \mathbb{R},\ \forall n \geqslant 0,\ T^n x = \lambda^n x,$$

  so the sequence $(T^n x)_{n\geqslant 0}$ diverges if and only if $x \neq 0$.

- Let us now consider $X = [-1, 1]$ and $T\colon x \in [-1, 1] \to 2x^2 - 1 \in [-1, 1]$. We can still find a closed formula, up to some change of variable, since we have

$$T^n(\cos(\pi t)) = \cos(2^n \pi t)$$

  for every $n \geqslant 0$ and every $t \in \mathbb{R}$. Using dyadic real numbers, we can therefore prove that the sequence $(T^n x)_{n\geqslant 0}$ is periodic for every x in a dense subset. For points x outside this subset, the sequence is more difficult to study.

- Given $r \in]0, 4]$, for $X = [0, 1]$ and $T\colon x \in [0, 1] \to rx(1 - x)$ (the logistic map), the situation is more complicated since we do not have any closed formula for $T^n x$. Then the goal is to find qualitative properties (existence of a dense orbit, existence of a fixed point, an attractor, etc).

Measure-theoretic setting

Some dynamical systems are difficult to study and we can only have a partial description (see for instance the last example). Moreover, the behaviour of the sequence $(T^n x)_{n\geqslant 0}$ depends on x and, if X is a metric space, $(T^n x)_{n\geqslant 0}$ and $(T^n x')_{n\geqslant 0}$ can have completely

different behaviours even if x and x' are very close. For instance, the sequence can be periodic for x, but can diverge for a point x' arbitrarily close to x, this is chaos!

Imagine that the evolution of some physical quantity is driven by $(T^n x)_{n\geqslant 0}$, and imagine that, in an experiment, some noise prevents us from choosing exactly the same initial condition x twice (due to measurement errors for instance). Then the successive simulations could give different behaviours due to chaos, as if it was random. This motivates the following definition of dynamical systems in a measure-theoretic context, where the initial condition can be seen as a random variable.

Definition A.1.2. Given a probability space (X,μ), a probability measure-preserving transformation is a measurable map $T\colon X\to X$ which preserves the measure: $T_\star\mu \coloneqq \mu(T^{-1}(.))$ is equal to μ.

The assumption $T_\star\mu=\mu$ is important: if the initial condition x is random with respect to μ, then so is Tx, so Tx can be considered as a new initial condition. In fact, it follows from the ergodic theorem (see Section A.1.f for the statement) that if T is ergodic (see Section A.1.c for the definition), then the repartition of the points $\{x,Tx,\dots,T^n x\}$ approximates the measure μ, as n goes to $+\infty$ and for μ-almost every x. So we can somehow recover μ from T (with the good points x, namely the one "supported" by μ).

We will in fact work with invertible transformations on "nice" probability space.

Definition A.1.3. A measured space X is *standard* if X is a Polish topological space, endowed with the associated Borel σ-algebra. A probability space (X,μ) is *atomless* if for every $x\in X$, $\mu(\{x\})=0$.

Given a standard and atomless probability space (X,μ), $\mathrm{Aut}(X,\mu)$ is the set of probability-measure preserving bimeasurable bijections $X\to X$, two such maps being identified if they coincide on a subset of full measure. Dynamical systems have been intensively studied in the more general case of non-invertible transformations, but we only introduce the topic in the invertible case.

It is well-known that any standard and atomless probability space is isomorphic to $([0,1],\mathrm{Leb})$, namely there exists a bimeasurable map $\Psi\colon X\to[0,1]$ such that $\Psi_\star\mu=\mathrm{Leb}$ (roughly speaking, Ψ transfers the structure of X to $[0,1]$, and vice versa).

Here are some examples.

Example A.1.4.

- The *Bernoulli shift* on $X=\{0,1\}^{\mathbb{Z}}$ is the transformation $T\in\mathrm{Aut}(X,\mu)$ defined by $T(x_n)_{n\in\mathbb{Z}}\coloneqq(x_{n+1})_{n\in\mathbb{Z}}$, where X is endowed with the probability measure $\mu=\nu^{\mathbb{Z}}$ for any probability measure ν on $\{0,1\}$. We can more generally look at the Bernoulli shift on $\Sigma^{\mathbb{Z}}$ for any finite set Σ.

- The *irrational rotation* of angle $\theta\in\mathbb{R}\backslash\mathbb{Q}$ is an example of $\mathbb{Z}$-action. This is given by the transformation $R_\theta\in\mathrm{Aut}(X,\mu)$, where X is the unit circle $\mathbb{T}=\{z\in\mathbb{C}\mid |z|=1\}$ endowed with its Lebesgue measure, defined by $R_\theta z\coloneqq e^{2i\pi\theta}z$.

- The most famous $\mathbb{Z}$-actions we work with in this thesis are *odometers*. Given integers $q_n\geqslant 2$ for every $n\geqslant 0$, let $X=\prod_{n\geqslant 0}\{0,1,\dots,q_n-1\}$ be endowed with the product of uniform distributions on each $\{0,1,\dots,q_n-1\}$. The associated odometer is the transformation $S\in\mathrm{Aut}(X,\mu)$ defined by $S(x_n)_{n\in\mathbb{Z}}\coloneqq(x_n)_{n\in\mathbb{Z}}+(1,0,0,\dots)$ with carry over to the right, namely

$$Sx=\begin{cases}(0,\dots,0,x_i+1,x_{i+1},\dots) & \text{if } i\coloneqq\min\{j\geqslant 0\mid x_j\neq q_j-1\} \text{ is finite}\\ (0,0,0,\dots) & \text{if } x=(q_0-1,q_1-1,q_2-1,\dots)\end{cases}.$$

Topological setting

We can also study dynamical systems with a topological viewpoint, when the set we act on is a topological space.

Definition A.1.5. Given a topological space X, a topological dynamical system is a continuous map $T\colon X \to X$.

We will focus on invertible transformations, namely homeomorphisms. Topological dynamics is often studied in the case of a compact set X, or even the Cantor set.

The dynamical systems presented Example A.1.4 are also topological dynamical systems (consider the product topology on $\Sigma^{\mathbb{Z}}$ for the Bernoulli shifts, and on $\prod_{n\geqslant 0}\{0,1,\dots,q_n-1\}$ for odometers, and the usual topology on the unit circle for the irrational rotations).

How to compare dynamical systems: conjugacy, factor

The main problem in ergodic theory is to classify transformations up to conjugacy, namely we want to know whether two transformations are the "same".

For instance, we all know that multiplying a number by 10 consists in shifting the digits of its decimal representation. The crucial word of this sentence is "consists". It means that we have to translate numbers in sequences of digits (their decimal representations) and to explain what happens in the world of such sequences when something happens in the world of numbers, and vice versa. The translation is a change of viewpoint, this is exactly the map Ψ of the following definition.

Definition A.1.6. Two systems $T \in \mathrm{Aut}(X,\mu)$ and $S \in \mathrm{Aut}(Y,\nu)$ are *conjugate*, or *isomorphic*, if there exists a bimeasurable bijection $\Psi\colon X \to Y$ such that $\Psi_\star\mu = \nu$ and the equality $\Psi T = S\Psi$ holds almost everywhere.

In our intuitive example above, we only focus on the equality "$\Psi T = S\Psi$" (which can also be written as $T\Psi^{-1} = \Psi^{-1}S$, conjugacy is a symmetric notion), where T is the multiplication by 10 in the world X of numbers, S is the shift of sequences of Y and Ψ tells us how to encode a number in a sequence. This can be called *setwise conjugacy*, since it does not use any structure on the sets.

The other ingredients of the definition ("bimeasurable" and "$\Psi_\star\mu = \nu$") are useful when the sets X and Y have a structure and we want Ψ to transfer the structure of X in Y and vice versa. In the measure-theoretic context, X (resp. Y) is equipped with a σ-algebra and a probability measure μ (resp. ν) and the ingredients of the definition aims at finding a correspondence which respects these "equipments".

In the topological setting, if X and Y are both endowed with a topology, we want Ψ to provide a correspondence between open sets of X and open sets of Y, namely Ψ is a homeomorphism. This motivates the definition of conjugacy in the topological context.

Definition A.1.7. Two homeomorphisms T and S of two topological spaces X and Y respectively, are *(topologically) conjugate* if there exists a homeomorphism $\Psi\colon X \to Y$ such that the equality $\Psi T = S\Psi$ holds on X.

Let us now introduce the notion of factor. We keep the above ideas for conjugacy, but we remove all the "vice versa": if something happens in X, there exists some translator Ψ which tells us what it means in Y, but we cannot necessarily go the other way because we loose a lot of information when passing from X to Y. For instance, when we add 1 to an integer (X is the set of integers and T is the addition by 1), we change its parity ($Y = \{\text{even}, \text{odd}\}$ and S is the map which swaps the two elements of Y). But conversely if we know that we change the parity of an integer by adding 1, then we do not know which

integer it is, since there are infinitely many odd/even integers, and we do not know if T is the addition by 1, for instance it could be the addition by -1. Here is the definition of factor/extension in the measure-theoretic setting.

Definition A.1.8. Let $S, T \in \mathrm{Aut}(Y, \nu)$. We say that S is a *factor* of T, or T is an *extension* of S, if there exists a measurable map $\Psi\colon X \to Y$ which is onto and such that $\Psi_\star \mu = \nu$ and the equality $\Psi T = S\Psi$ holds almost everywhere.

Here we want Ψ to be onto (almost everywhere), otherwise we would only have part of the information on Y. In the topological context, factor is defined in the same way, replacing "homeomorphism" by "continuous map" for Ψ in the definition of topological conjugacy, as well as we replaced "bimeasurable" by "measurable" in the last definition.

Definition A.1.9. Given two topological space X and Y, and homeomorphisms T and S on X and Y respectively, we say that S is a *(topological) factor* of T, or T is a *(topological) extension* of S, if there exists a continuous map $\Psi\colon X \to Y$ which is onto and such that $\Psi_\star \mu = \nu$ and the equality $\Psi T = S\Psi$ holds on X.

If we want systems to be the "same", they have to share similar properties, otherwise we did not find the good definition of conjugacy. The goal is thus to find invariants (ergodicity, mixing properties, point spectrum, entropy, ...), namely characteristics of systems preserved under conjugacy. If systems do not share such a property, then we know that they are not conjugate.

Every time we find an invariant, we hope that it is a complete invariant, this occurs when restricting to some subclasses, here are the two examples to keep in mind. For instance, we can classify Bernoulli shifts up to conjugacy using entropy, by Ornstein's theory [Orn70], as well as systems of discrete spectrum using the point spectrum [HVN42].

However it is *a priori* impossible to find a complete invariant of conjugacy which classifies systems in full generality. Conjugacy is a hard problem.

A.1.b Poincaré recurrence theorem, return time, induced transformation

Let $T \in \mathrm{Aut}(X, \mu)$ and A be a measurable subset of positive measure. The return time $r_{T,A}\colon A \to \mathbb{N}^* \cup \{\infty\}$ is defined by :

$$\forall x \in A,\ r_{T,A}(x) := \inf\{k \geqslant 1 \mid T^k x \in A\},$$

also written r_A if the context is clear. Given $x \in A$, we would like to know if the orbit of x for the dynamics of T will return in A in the future. In fact, there will be infinitely many visits, as stated in the following theorem.

Theorem A.1.10 (Poincaré recurrence theorem)**.** *If $\mu(A) > 0$, then for almost every $x \in A$, the set $\{k \in \mathbb{N}^* \mid T^k x \in A\}$ is infinite.*

By "almost every $x \in A$", we mean with respect to the probability measure $\mu_A := \mu(. \cap A)/\mu(A)$ which is well defined if $\mu(A) > 0$. As we will see in Section A.1.c, this statement can be improved when the transformation T is ergodic, since it will hold for almost every $x \in X$ (not only $x \in A$). The proof of Theorem A.1.10 is elementary, it does not require a lot of assumptions since it only uses the fact that the measure is preserved by the transformation.

Proof of Theorem A.1.10. For every $n \in \mathbb{N} \cup \{\infty\}$, we define

$$E_n := \{x \in A \mid \sup\{k \geqslant 0 \mid T^k x \in A\} = n\}.$$

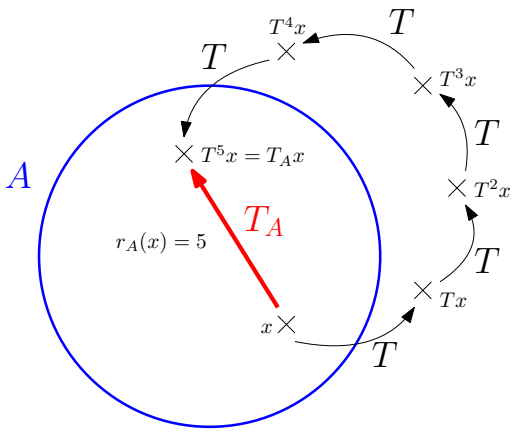
For every finite n, we have $T^n(E_n) = E_0$ so E_n has the same measure as E_0, but this measure is necessarily zero since A is the disjoint union of the E_n for $n \in \mathbb{N} \cup \{\infty\}$ and μ is finite. Thus A is equal to E_∞ up to a null set. □

We can then define a transformation T_A on the set of x given by Poincaré recurrence theorem, namely on A up to a null set, called the *induced tranformation* on A :

$$T_A x := T^{r_A(x)} x.$$

The set A is endowed with the σ-algebra of measurable sets of X included in A, and (A, μ_A) is a standard and atomless probability space.

The map T_A is an element of $\mathrm{Aut}(A, \mu_A)$. Indeed, every measurable subset B of A is the disjoint union of the sets $B_n := B \cap \{r_A = n\}$ for $n \geqslant 1$, and $T_A(B_n) = T^n(B_n)$ by definition, then the equality $\mu_A(T_A(B)) = \mu_A(B)$ follows from the T-invariance of μ.



The next theorem states that r_A is μ_A-integrable, but it is incomplete since we can say more about the value of its integral in the ergodic case; see Theorem A.1.15 once ergodicity has been defined.

Theorem A.1.11 (Kac's theorem (incomplete version))**.** *Given $T \in \mathrm{Aut}(X, \mu)$ and A a measurable set of positive measure, we have*

$$\int_A r_A \mathrm{d}\mu_A \leqslant \frac{1}{\mu(A)}.$$

As for Poincaré recurrence theorem, the goal is to cut the space into interesting pieces and to use the T-invariance of the measure.

Proof of Theorem A.1.11. If a T-orbit visits A infinitely many times, then we can decompose it in countably many "intervals" $x_i, Tx_i, \ldots, T^{r_A(x_i)-1}x_i$ for $i \in \mathbb{Z}$, where

- the intersection of this orbit with A is exactly $\{x_i \mid i \in \mathbb{Z}\}$;
- $x_{i+1} = T^{r_A(x_i)}x_i$.

Then for every point x lying in such an orbit, there is a unique such interval which contains it and x is thus completely determined by the data of a point $x_i \in A$ and positive integers k, n such that $0 \leqslant n < k$, where $x = T^n x_i$ and $r_A(x_i) = k$. In other words, there are unique integers $0 \leqslant n < k$ such that $x \in T^n(\{r_A = k\})$.

We finally have proved that X contains the disjoint union $\bigcup_{k>0} \bigcup_{0 \leqslant n < k} T^n(\{r_A = k\})$. Considering the measure of both sets, we get the desired formula. □

A.1.c Ergodicity

Definition A.1.12. Let $T \in \mathrm{Aut}(X,\mu)$. We say that a measurable subset A of X is *T-invariant* if $\mu(A\Delta T(A)) = 0$. We say that T is *ergodic* if every T-invariant measurable subset is null or conull.

This is the definition of T-invariance in the simpler case of invertible transformations (when T is not invertible, A is T-invariant if $\mu(A\Delta T^{-1}(A)) = 0$).

Example A.1.13. Let us consider integers $q_n \geqslant 2$ for every $n \geqslant 0$, and T the odometer on $X \coloneqq \prod_{n\geqslant 0}\{0,\dots,q_n-1\}$ introduced in Example A.1.4. Then T is ergodic. Indeed, X is endowed with the product σ-algebra, namely the σ-algebra generated by the cylinders

$$[y_0,\dots,y_{n-1}]_n \coloneqq \{(x_i)_{i\geqslant 0} \mid x_0 = y_0,\dots,x_{n-1} = y_{n-1}\},$$

(n is called the length of the cylinder) so given a T-invariant subset A of positive measure, we can approximate it with these sets. More precisely, given $\varepsilon > 0$, there is a cylinder C (let us say of length n) such that $\mu(A \cap C) \geqslant (1-\varepsilon)\mu(C)$. T-invariance implies that we have $\mu(A \cap T^k(C)) \geqslant (1-\varepsilon)\mu(C)$, moreover T acts cyclically on the set of cylinders of length n which forms a partition of the space with elements of the same measure, so summing these inequalities, we get $\mu(A) \geqslant 1-\varepsilon$ where ε is arbitrary, so $\mu(A) = 1$.

It is not hard to adapt this proof to rank-one systems that we introduce in Chapter I.

Irrational rotations and Bernoulli shifts are also ergodic, but we will need additional tools to prove it, using a functional approach explained in Section A.1.e. For irrational rotations, we can also use the fact that they are rank-one systems (by Del Junco [Jun76]), which are ergodic as mentioned at the end of the previous example.

If A is T-invariant, then so is A^c, and T can be decomposed in two subsystems $A \to A$ and $A^c \to A^c$. Ergodicity thus means that we cannot decompose a system in a non trivial way: one of the two sets A or A^c must have full measure and the other must be negligible.

Moreover, if a set A satisfies $A = T(A)$, then for all points $x \in A$, its T-orbit is contained in A. This means that A is a union of orbits. In the case A is T-invariant, then A is a union of orbits up to zero measure. Therefore, ergodicity means that a measurable property only concerning the orbits is null or conull. For instance, given some measurable subset B, the property "there exist infinitely many positive integers n such that $T^n x$ lies in B" is satisfied for x if and only if it is satisfied for $T^k x$ for every $k \in \mathbb{Z}$, so it is a property on the orbits. Since Poincaré recurrence theorem implies that B satisfies this property, ergodicity enables us to improve this theorem.

Corollary A.1.14. *If B is a measurable subset of X of positive measure and if $T \in \mathrm{Aut}(X,\mu)$ is ergodic, then for almost every $x \in X$, there exists infinitely many positive integers n such that $T^n x \in B$.*

In Section A.1.f, the ergodic theorem will enable us to compute the asymptotic frequency of visits in B. Let us notice that Kac's theorem is also improved when the system is ergodic.

Theorem A.1.15 (Kac's theorem (ergodic case))**.** *Let $T \in \mathrm{Aut}(X,\mu)$ be an ergodic transformation and A a measurable set of positive measure. Then we have*

$$\int_A r_A \mathrm{d}\mu_A = \frac{1}{\mu(A)}.$$

Proof of Theorem A.1.15. In the proof of the incomplete version (Theorem A.1.11), we considered points lying in the T-orbits visiting A infinitely many times, a property which occurs almost surely if T is ergodic. The inclusion

$$\bigcup_{k>0}\bigcup_{0\leqslant n<k} T^n(\{r_A = k\}) \subset X$$

is thus an equality up to a null set and, considering the measure of both sets, we get the result. □

In Section A.1.b, we also defined induced transformations. The following proposition states that ergodicity is stable under this operation.

Proposition A.1.16. *Let $T \in \mathrm{Aut}(X,\mu)$ and A be a measurable subset of positive measure. If T is μ-ergodic, then the induced transformation T_A is μ_A-ergodic.*

Idea of proof. If B is a T_A-invariant subset B of A, then $B' \coloneqq \bigcup_{n\in\mathbb{N}} T^n(B)$ is T-invariant and we can prove that $B' \cap A$ is equal to B, so the result follows from the ergodicity of T. □

A fact that we often use in this thesis is that ergodicity implies aperiodicity.

Proposition A.1.17. *If $T \in \mathrm{Aut}(X,\mu)$ is ergodic, then it is aperiodic: for almost every $x \in X$, the T-orbit of x is infinite (in other words, the associated $\mathbb{Z}$-action is free).*

Proof. In the space X, we can find countably and infinitely many measurable subsets $X_1, X_2, X_3, \dots$ of positive measure and which are pairwise disjoint. Indeed, since (X,μ) is standard and atomless, we can assume without loss of generality that it is $([0,1],\mathrm{Leb})$, and we set

$$X_i \coloneqq \left[\sum_{j=1}^{i-1} \frac{1}{2^j}, \sum_{j=1}^{i} \frac{1}{2^j}\right]$$

for every $i \geqslant 1$. Then Corollary A.1.14 and the fact that a countable intersection of conull sets is a conull set imply that for almost every $x \in X$, the T-orbit of x visits every X_i, so it is infinite since the subsets X_i are pairwise disjoint. □

Ergodicity for $T \in \mathrm{Aut}(X,\mu)$ does not imply ergodicity of its powers T^k for $k \in \mathbb{Z}\backslash\{0\}$ (but it implies ergodicity of T^{-1}). For a counter-example, let us look at the dyadic odometer, namely the odometer on $\prod_{n\geqslant 0}\{0,\dots,q_n-1\}$ with $q_n = 2$ for every $n \geqslant 0$, and let us notice that its square preserves the set of elements $(x_n)_{n\geqslant 0} \in \{0,1\}^{\mathbb{N}}$ satisfying $x_0 = 0$ (resp. $x_0 = 1$), which has measure $1/2$.

In fact, if T is the odometer on $X \coloneqq \prod_{n\geqslant 0}\{0,\dots,q_n-1\}$, then none of its non trivial powers (namely T^k for every $k \in \mathbb{Z}\backslash\{-1,0,1\}$) is ergodic if and only if the integers q_n satisfy the following condition: for every prime number p, there exists $n \geqslant 0$ such that p divides q_n.

A.1.d Mixing properties

A stronger property than ergodicity is the *weak mixing property*.

Definition A.1.18. $T \in \mathrm{Aut}(X,\mu)$ is *weakly mixing* if for all measurable subsets A and B, the following holds:

$$\frac{1}{n}\sum_{i=0}^{n} \left|\mu(T^{-i}(A)\cap B) - \mu(A)\mu(B)\right| \underset{n\to+\infty}{\longrightarrow} 0.$$

Weak mixing property implies ergodicity (consider a T-invariant subset A and apply weak mixing property to $A = B$). Weak mixing property is in fact a "Cesaro" version of *strong mixing property* defined above (from this we deduce that **strong** mixing property is... **stronger**).

Definition A.1.19. T is *strongly mixing* if for all measurable subsets A and B, the following holds:

$$\left|\mu(T^{-n}(A)\cap B) - \mu(A)\mu(B)\right| \underset{n\to+\infty}{\longrightarrow} 0. \tag{A.1}$$

It is easy to see that if T is weakly (resp. strongly) mixing, then so is T^k for every $k \in \mathbb{Z}\backslash\{0\}$. From this we deduce that there exist ergodic systems that are not weakly mixing (for instance odometers, see Example A.1.13). There also exist weakly mixing systems that are not strongly mixing, for instance the Chacon map which is an example of rank-one system, see Section I.3.a in Chapter I.

In the next section, we will see equivalent definitions of ergodicity and weak mixing property in terms of eigenvalues, easier to apply than the original definitions, even for very simple dynamical systems like irrational rotations.

To check strong mixing property, we can restrict its definition to a smaller class of measurable subsets A and B for which (A.1) is easier to prove. Here is an example for Bernoulli shifts.

Example A.1.20. Let $X = \{0,1\}^{\mathbb{Z}}$ and let us consider the Bernoulli shift $T\colon (x_n)_{n\in\mathbb{Z}} \to (x_{n+1})_{n\in\mathbb{Z}}$. T is an element of $\mathrm{Aut}(X,\mu)$, with $\mu = \nu^{\otimes\mathbb{Z}}$ and where ν is the uniform measure on $\{0,1\}$. As for odometers (but in a two-sided manner), let us consider the cylinders:

$$[y_k,\dots,y_\ell]_{k,\ell} \coloneqq \{(x_i)_{i\in\mathbb{Z}} \mid x_k = y_k, x_{k+1} = y_{k+1}, \dots, x_\ell = y_\ell\},$$

with integers $k < \ell$. The space X is endowed with the product σ-algebra, namely the one generated by the cylinders. Also the class of cylinders is stable under intersections, this is what we usually call a π-system. Using the Dynkin $\pi-\lambda$ theorem, we can thus check that we can reduce the proof of strong mixing property to subsets A, B being in this class of cylinders.

Why is it easier with cylinders? Let us consider two cylinders of the form

$$A = [y_k,\dots,y_\ell]_{k,\ell} \text{ and } B = [y'_{k'},\dots,y'_{\ell'}]_{k',\ell'}.$$

Then we can notice that

$$T^{-n}(A) = [y_{k+n},\dots,y_{\ell+n}]_{k+n,\ell+n},$$

so if n is sufficiently large, we have $k+n > \ell'$, and $T^{-n}(A)$ and B get independent by definition of μ, namely $\mu(T^{-n}(A)\cap B) = \mu(T^{-n}(A))\mu(B) = \mu(A)\mu(B)$. So T is strongly mixing.

A.1.e Functional approach, introduction to spectral properties

Given $T \in \mathrm{Aut}(X,\mu)$, we say that a measurable map $f\colon X \to \mathbb{R}$ is *T-invariant* if $f(Tx) = f(x)$ for almost every $x \in X$. Given a measurable subset A and $T \in \mathrm{Aut}(X,\mu)$, we have $\mathbb{1}_A \circ T = \mathbb{1}_{T^{-1}(A)}$, so a (trivial) equivalent definition of ergodicity is that every T-invariant characteristic function is either $\mathbb{1}_X$ or $\mathbb{1}_\varnothing$.

Moreover, for every measurable subset C, we have $\mu(C) = \int_X \mathbb{1}_C \mathrm{d}\mu$, so the quantity $\mu(T^{-i}(A)\cap B) - \mu(A)\mu(B)$ appearing in the definitions of weakly and strongly mixing can be written as

$$\int_X \left(\mathbb{1}_A \circ T^i\right)\mathbb{1}_B \mathrm{d}\mu - \int_X \mathbb{1}_A\mu \int_X \mathbb{1}_B\mu.$$

All these remarks lead us to a functional viewpoint of ergodic theory. Let us start with the reformulations of the properties defined in the previous sections.

Proposition A.1.21. *Let $T \in \mathrm{Aut}(X,\mu)$.*

- *T is ergodic if and only if every T-invariant measurable map $X \to \mathbb{R}$ is constant almost everywhere.*

- *T is weakly mixing if and only if for every $f, g \in \mathrm{L}^2(X, \mu)$, we have*

$$\frac{1}{n}\sum_{i=0}^{n}\left|\int_X \left(f \circ T^i\right) g \mathrm{d}\mu - \int_X f\mathrm{d}\mu \int_X g\mathrm{d}\mu\right| \underset{n\to+\infty}{\longrightarrow} 0.$$

- *T is strongly mixing if and only if for every $f, g \in \mathrm{L}^2(X, \mu)$, we have*

$$\int_X (f \circ T^n)\, g\mathrm{d}\mu \underset{n\to+\infty}{\longrightarrow} \int_X f\mathrm{d}\mu \int_X g\mathrm{d}\mu.$$

Ideas of proof. By the above discussion, the "if" part of each point is immediate, considering characteristic functions.

Assume that T is ergodic and let $f\colon X \to \mathbb{R}$ be a T-invariant function. For every $s \in \mathbb{R}$, the set $\{f > s\}$ is T-invariant. By ergodicity, $\mu(\{f > s\})$ is equal to 0 or 1, so the real number

$$s_0 \coloneqq \sup\{s \in \mathbb{R} \mid \mu(\{f > s\}) = 1\} = \inf\{s \in \mathbb{R} \mid \mu(\{f > s\}) = 0\}$$

is well defined and it is a routine to prove that $f = s_0$ almost everywhere.

If T is weakly mixing or strongly mixing, then the characterization with square-integrable functions f, g comes from the approximation (with the L^2-norm) by linear combination of characteristic functions (for which the result immediately holds). □

Then it is natural to study the unitary operator

$$U_T\colon f \in \mathrm{L}^2(X, \mu) \to f \circ T \in \mathrm{L}^2(X, \mu),$$

called the *Koopman operator* of T (it is unitary since μ is T-invariant). For instance, ergodicity means that the only fixed points of this operator are the constant functions, so we can more generally look at its point spectrum and its eigenspaces.

Definition A.1.22. We say that $\lambda \in \mathbb{C}$ is an *eigenvalue* of T if there exists $f \in \mathrm{L}^2(X, \mu)\backslash\{0\}$ such that $f \circ T = \lambda f$. The function f is called an eigenfunction of T. The *point spectrum* is the set $\mathrm{Sp}(T)$ of all its eigenvalues. Since U_T is unitary, $\mathrm{Sp}(T)$ is a subset of the unit circle $\mathbb{T}$.

Example A.1.23. The constants functions are always eigenfunctions associated to the eigenvalue 1.

The two following propositions show that the spectral properties of the Koopman operator provide reformulations of some dynamical properties.

Proposition A.1.24. *Given $T \in \mathrm{Aut}(X, \mu)$, the following hold.*

1. *$\mathrm{Sp}(T)$ is a subgroup of $\mathbb{T}$.*
2. *Eigenfunctions associated to eigenvalues different than 1 have zero integral.*
3. *T is ergodic if and only if every eigenspace is a line.*
4. *If T is ergodic, then every eigenfunction has constant modulus.*

It is immediate to prove that if λ is an eigenvalue of T, then λ^{-1} is an eigenvalue of T^{-1}. Using the first item of this proposition, we thus get that T and T^{-1} have the same point spectrum.

Proof. Considering constants functions $X \to \mathbb{C}$, 1 is always an eigenvalue. If λ and μ are eigenvalues, with eigenfunctions f and g respectively, then fg and and $\overline{f}$ are eigenfunctions associated to the eigenvalues $\lambda\mu$ and λ^{-1} respectively. So $\mathrm{Sp}(T)$ is a subgroup of $\mathbb{T}$.

If λ is an eigenvalue and f is an associated eigenfunction, then the preservation of the measure implies

$$\lambda \int_X f \mathrm{d}\mu = \int_X f \circ T \mathrm{d}\mu = \int_X f \mathrm{d}\mu$$

so the integral of f is zero if $\lambda \neq 1$.

Let us now assume that T is ergodic. If f is an eigenfunction, then $|f|$ is eigenfunction with eigenvalue 1, so it is constant almost everywhere. This implies in particular that eigenfunctions are almost surely nonzero, so if f and g are eigenfunctions associated to the same eigenvalue, we can look at f/g which is an eigenfunction associated to 1, so it is constant almost everywhere, namely f and g are colinear. Conversely, if eigenspaces are lines, then the eigenspace with eigenvalue 1 only contains constant functions, which is a reformulation of ergodicity by Proposition A.1.21. □

Proposition A.1.25. *If $T \in \mathrm{Aut}(X, \mu)$ is ergodic, then it is weakly mixing if and only if 1 is the only eigenvalue.*

Proof. The "if" part is more technical and requires additional tools, that we do not introduce here (see Sections 1 and 2 in [Gla03, Chapter 3]). Let us only prove the "only if" part. Assume that T is weakly mixing and ergodic, and let f be an eigenfunction associated to an eigenvalue λ. We have

$$\int_X (f \circ T^i)\overline{f} \mathrm{d}\mu = \lambda^i \int_X |f|^2 \mathrm{d}\mu$$

for every $i \geqslant 0$. If λ were not equal to 1, then f would have zero integral and the weak mixing property would imply

$$\frac{1}{n} \sum_{i=0}^{n} \left| \int_X (f \circ T^i)\overline{f} \mathrm{d}\mu \right| \underset{n \to +\infty}{\to} 0,$$

namely $\|f\|_2 = 0$ since $|\lambda| = 1$, a contradiction. So λ is equal to 1, this is the only eigenvalue of T. □

Let us now exploit the useful fact that $\mathrm{L}^2(X, \mu)$ is a Hilbert space to study the point spectrum of some transformations. Let us first point out the fact that any two eigenfunctions associated to distinct eigenvalues are orthogonal, this directly follows from the fact that the Koopman operator is unitary.

It is easy to compute the point spectrum of systems of *discrete spectrum*, thanks to the structure of separable Hilbert space of $\mathrm{L}^2(X, \mu)$ (see the examples below), but let us first define this class of transformations.

Definition A.1.26. A transformation $T \in \mathrm{Aut}(X, \mu)$ has *discrete spectrum* if the span of all its eigenfunctions is dense in $\mathrm{L}^2(X, \mathcal{A}, \mu)$.

Example A.1.27.

1. Let θ be an irrational number, and $R_\theta \colon z \in \mathbb{T} \mapsto z \exp(2i\pi\theta) \in \mathbb{T}$ be the irrational rotation of angle θ. For every $n \in \mathbb{Z}$, the map $f_n \colon z \in \mathbb{T} \to z^n \in \mathbb{T}$ is an eigenfunction of R_θ associated to the eigenvalue $\exp(2in\pi\theta)$. By Fourier analysis, the span of all the functions f_n is dense in $\mathrm{L}^2(\mathbb{T})$, so this system has discrete spectrum.

These functions f_n in fact describe (up to multiplicative constants) all the eigenfunctions of R_θ and provide all the eigenvalues. Indeed, if $f\colon \mathbb{T} \to \mathbb{R}$ is an eigenfunction associated to some eigenvalue $\lambda \in \mathbb{T}$, then we can decompose it in the Fourier basis:

$$f = \sum_{n\in\mathbb{Z}} a_n f_n,$$

and apply the Koopman operator to get:

$$f = \sum_{n\in\mathbb{Z}} a_n \frac{e^{2in\pi\theta}}{\lambda} f_n.$$

This gives $a_n = a_n \frac{e^{2in\pi\theta}}{\lambda}$ for every $n \in \mathbb{Z}$, so there exists some $n_0 \in \mathbb{Z}$ such that $\lambda = e^{2in_0\pi\theta}$ and $a_n = 0$ for every $n \neq n_0$, so $f = a_{n_0} f_{n_0}$. We thus have proved that the point spectrum of the R_θ is exactly $\{\exp(2in\pi\theta) \mid n \in \mathbb{Z}\}$.

2. Given a sequence $(q_n)_{n\geqslant 0}$ of integers greater than or equal to 2, and S the odometer on $X = \prod_{n\geqslant 0} \{0, 1, \dots, q_n - 1\}$, its point spectrum is exactly

$$\mathrm{Sp}(S) = \left\{ \exp\left(\frac{2i\pi k}{h_n}\right) \mid n \geqslant 1, 0 \leqslant k \leqslant h_n - 1 \right\}.$$

where $h_n \coloneqq q_0 \dots q_{n-1}$. Indeed, it is straightforward to check that $f_\lambda\colon x \in X \mapsto \sum_{j=0}^{h_n-1} \lambda^j \mathbb{1}_{S^j([0,\dots,0]_n)}(x)$ is an eigenfunction associated to $\lambda = \exp\left(\frac{2i\pi k}{h_n}\right)$. Moreover, the span of all the eigenfunctions is dense in $\mathrm{L}^2(X)$, since a linear combination $\sum_{\ell=0}^{h_n-1} a_\ell f_{\lambda^\ell}$ is of the form $\sum_{j=0}^{h_n-1} P(\lambda^j)\mathbb{1}_{S^j([0,\dots,0]_n)}$ with the polynomial $P = a_0 + a_1 Y + \dots + a_{h_n-1} Y^{h_n-1}$, and good choices of P yield the characteristic functions of the n-cylinders. We conclude as for irrational rotations.

Using Propositions A.1.24 and A.1.25, we get the following.

Corollary A.1.28. *Odometers and irrational rotations are ergodic but not weakly mixing.*

Moreover, there is no isomorphism between an odometer and an irrational rotation, since they do not share the same point spectrum. Indeed, the point spectrum is preserved under isomorphism, and irrational rotations have irrational eigenvalues whereas odometers only have rational eigenvalues.

The following result, due to Halmos and von Neumann, provides a classification of ergodic discrete-spectrum systems up to conjugacy.

Theorem A.1.29 (Halmos, von Neumann [HVN42])**.** *Two ergodic systems of discrete spectrum are conjugate if and only if they have the same point spectrum.*

This is also a classification up to flip-conjugacy since the point spectrum is symmetric. This theorem enables us to classify odometers and irrational rotations up to conjugacy.

Example A.1.30.

- Let R_θ and $R_{\theta'}$ be two irrational rotations. Their point spectrum are respectively equal to

$$\{e^{2in\pi\theta} \mid n \in \mathbb{Z}\} \text{ and } \{e^{2in\pi\theta'} \mid n \in \mathbb{Z}\}$$

and their are the same if and only if $\theta' = \pm\theta$. By Halmos-von Neumann theorem, R_θ and $R_{\theta'}$ are isomorphic if and only if $\theta' = \pm\theta$.

- Let S and S' be two odometers, respectively on $\prod_{n\geqslant 0}\{0,1,\dots,q_n-1\}$ and $\prod_{n\geqslant 0}\{0,1,\dots,q'_n-1\}$. Their point spectrum are respectively equal to

$$\left\{\exp\left(\frac{2i\pi k}{h_n}\right) \mid n\geqslant 1, 0\leqslant k\leqslant h_n-1\right\} \text{ and } \left\{\exp\left(\frac{2i\pi k}{h'_n}\right) \mid n\geqslant 1, 0\leqslant k\leqslant h'_n-1\right\}$$

with $h_n = q_0\dots q_{n-1}$ and $h'_n = q'_0\dots q'_{n-1}$. For every prime number p, let us set

$$k_p = \sum_{n\geqslant 0}\nu_p(q_n)\in\mathbb{N}\cup\{+\infty\}$$

and similarly k'_p, where ν_p is the p-adic valuation. Then it is not hard to prove that S and S' have the same point spectrum if and only if $k_p = k'_p$ for every prime number p. So Halmos-von Neumann theorem implies that the family $(k_p)_{\text{prime number } p}$ is a total invariant of conjugacy among odometers. We will usually refer to the formal product $\prod_p p^{k_p}$ as the *supernatural number* associated to (the conjugacy class of) an odometer. We say that an odometer is *dyadic* if $k_2 = +\infty$ and $k_p = 0$ for the other prime numbers p, *universal* if $k_p = +\infty$ for every p.

A.1.f Ergodic theorems

Given $T\in \mathrm{Aut}(X,\mu)$ and a measurable subset A of X, what is the asymptotic density of $\{n\geqslant 0 \mid T^n x\in A\}$ in $\mathbb{N}$, in other words the frequency of visits in A? First, the answer seems to depend on x. Indeed, if T is not ergodic and admits a T-invariant subset B disjoint from A and of positive measure, then for every $x\in B$, there is no integer $n\geqslant 0$ such that $T^n x$ visits A. It is thus natural to ask this question when T is ergodic. Secondly, when T is ergodic, the frequency of visits in A depends on the measure of A, since the set $\{n\geqslant 0 \mid T^n x\in A\}$ is almost surely equal to $\mathbb{N}$ if $\mu(A)=1$ and to $\varnothing$ if $\mu(A)=0$.

We will answer these questions using the Birkhoff ergodic theorem. Before that, note that the expected value of $|\{0\leqslant i\leqslant n \mid T^i x\in A\}|$ is $(n+1)\mu(A)$:

$$\begin{aligned}\int_X |\{0\leqslant i\leqslant n \mid T^i x\in A\}|\mathrm{d}\mu(x) = \int_X\left(\sum_{i=0}^n \mathbb{1}_{T^{-i}(A)}(x)\right)\mathrm{d}\mu(x) &= \sum_{i=0}^n\int_X \mathbb{1}_{T^{-i}(A)}(x)\mathrm{d}\mu(x)\\ &= \sum_{i=0}^n \mu(T^{-i}(A))\\ &= (n+1)\mu(A).\end{aligned}$$

This asymptotic behaviour is actually pointwise.

Theorem A.1.31 (Birkhoff ergodic theorem)**.** *Let $T\in\mathrm{Aut}(X,\mu)$ be an ergodic system and $f\in \mathrm{L}^1(X,\mu)$. Then for μ-almost every $x\in X$, the following holds:*

$$\frac{1}{n}\sum_{i=0}^n f(T^i(x)) \underset{n\to+\infty}{\to} \int_X f\mathrm{d}\mu,$$

and the convergence also holds in $\mathrm{L}^1(X,\mu)$.

We refer to Section 8 in [Gla03, Chapter 3] for a proof.

Applied to $f=\mathbb{1}_A$, Birkhoff ergodic theorem gives

$$\frac{1}{n}|\{n\geqslant 0 \mid T^n x\in A\}| \underset{n\to+\infty}{\to} \mu(A)$$

for μ-almost every $x\in X$, if T is ergodic.

In the non-ergodic case, the limit still exists but is not constant almost everywhere:

$$\frac{1}{n}\sum_{i=0}^{n} f(T^i(x)) \underset{n\to+\infty}{\to} \mathbb{E}[f \mid \mathcal{I}]$$

for μ-almost every $x \in X$, where $\mathcal{I}$ is the σ-algebra of T-invariant subsets and $\mathbb{E}[f \mid \mathcal{I}]$ is the expectation of f conditionally to $\mathcal{I}$. This is closely related to the ergodic decomposition (see Sections 4 and 8 in [Gla03, Chapter 3] for more details). Furthermore, the convergence holds also in $\mathrm{L}^1(X,\mu)$.

Note that in the L^2-case, the expectation conditionally to a σ-algebra $\mathcal{C}$ is in fact the orthogonal projection onto the subspace of $\mathcal{C}$-measurable functions. When $\mathcal{C}$ is the σ-algebra $\mathcal{I}$ of T-invariant subsets, $\mathcal{I}$-measurable functions are exactly the T-invariant functions. This is thus natural to prove a L^2-version of Birkhoff ergodic theorem using the structure of Hilbert space.

Theorem A.1.32 (Von Neumann ergodic theorem)**.** *Let $T \in \mathrm{Aut}(X,\mu)$ and $f \in \mathrm{L}^2(X,\mu)$. Then the following convergence holds in $\mathrm{L}^2(X,\mu)$:*

$$\frac{1}{n}\sum_{i=0}^{n} f \circ T^i \underset{n\to+\infty}{\to} Pf,$$

where P is the orthogonal projection onto the subspace of T-invariant functions.

Idea of proof. Since the Koopman operator U_T is unitary, we have the following orthogonal decomposition:

$$\mathrm{L}^2(X,\mu) = \mathrm{Ker}(U_T - Id) \oplus \overline{\mathrm{Im}(U_T - Id)},$$

where $\mathrm{Ker}(U_T - Id)$ is the subspace of T-invariant functions, so it suffices to prove the statement for f lying in either $\mathrm{Ker}(U_T - Id)$ or $\mathrm{Im}(U_T - Id)$. □

In the ergodic case, the projection P is the orthogonal projection onto the constant functions, and $\left(\frac{1}{n}\sum_{i=0}^{n} f \circ T^i\right)_{n\geqslant 1}$ converges to $\int_X f \mathrm{d}\mu$ with the L^2-norm.

A.1.g About T-invariant probability measures

Let us deal with the set of T-invariant probability measures, with a brief presentation of its topological nature.

Given a bimeasurable bijection (or simply a measurable map) $T\colon X \to X$, does there always exist a T-invariant probability measure μ? The answer is **YES!**, using functional analysis. More precisely, the set of T-invariant probability measure is non-empty, convex and compact in the set of probability measures, for the weak-$\star$ topology. We refer the reader to [VO16, Chapter 2] for a deep study on the topic.

Let us mention this crucial fact on T-ergodic probability measures on X, namely T-invariant probability measures μ such that T, seen as an element of $\mathrm{Aut}(X,\mu)$, is ergodic.

Proposition A.1.33 (see [Gla03, Theorem 4.2])**.** *Given a standard space X, let $T\colon X \to X$ be a measurable map. Then the extremal points of the set of T-invariant probability measures are exactly the T-ergodic ones.*

Therefore, by Krein-Milman Theorem, there exists at least one T-ergodic probability measure, and moreover the set of T-invariant probability measures is equal to the closed convex hull of its T-ergodic probability measures.

Finally, let us point out the following property among T-ergodic probability measures.

Proposition A.1.34 (see [Gla03, Theorem 4.2])**.** *If μ and ν are distinct T-ergodic probability measures, then they are mutually singular: there exists a measurable subset Y such that $\mu(Y) = 1$ and $\nu(Y) = 0$.*

After these descriptions, let us focus on the particular case of systems having only one T-invariant probability measure.

Definition A.1.35. We say that a bimeasurable bijection $T\colon X \to X$ is *uniquely ergodic* if it admits a unique T-invariant probability measure.

It follows from Proposition A.1.33 that if T is uniquely ergodic, then its only T-invariant probability measure is T-ergodic.

Example A.1.36.

- The irrational rotation of angle θ on the unit circle is uniquely ergodic. Indeed, if μ is an R_θ-invariant probability measure, then for every $n \in \mathbb{Z}\backslash\{0\}$, we have

$$\int_{\mathbb{T}} z^n \mathrm{d}\mu = \int_{\mathbb{T}} z^n \mathrm{d}(\mu \circ R_\theta^{-1}) = \int_{\mathbb{T}} (ze^{2in\theta\pi})^n \mathrm{d}\mu = e^{2in\theta\pi} \int_{\mathbb{T}} z^n \mathrm{d}\mu$$

  which forces $\int_{\mathbb{T}} z^n \mathrm{d}\mu = 0$ for every $n \neq 0$ since θ is irrational. By Fourier analysis, we conclude that μ is the Lebesgue measure on the unit circle.

- Odometers are also uniquely ergodic. Indeed, the odometer on $\prod_{n\geqslant 0} \{0, \dots, q_n - 1\}$ cyclically acts on the cylinders of same length, so for every $n \geqslant 1$, cylinders of length n must have measure $1/(q_0 \dots, q_{n-1})$. Since the σ-algebra of the space is generated by the cylinders, the Dynkin π-λ theorem implies that an invariant probability measure must be the product of the uniform distributions on the finite sets $\{0, \dots, q_n - 1\}$.

A.1.h Entropy

We now introduce numerical invariants which quantify how much a transformation complexifies the space: measure-theoretic entropy (in the measured context) and topological entropy (for topological dynamical systems). We refer the reader to [Gla03, Part 2] and [Dow11].

Measurable partitions

A set $\mathcal{P}$ of measurable subsets of X is a *measurable partition* of X if:

- for every $P_1, P_2 \in \mathcal{P}$, we have $\mu(P_1 \cap P_2) = 0$;
- the union $\bigcup_{P\in\mathcal{P}} P$ has full measure.

The elements of $\mathcal{P}$ are called the *atoms*. If $\mathcal{P}$ and $\mathcal{Q}$ are measurable partitions of (X, μ), we say that $\mathcal{P}$ *refines* (or is a refinement of, or is finer than) $\mathcal{Q}$, denoted by $\mathcal{P} \geqslant \mathcal{Q}$, if every atom of $\mathcal{Q}$ is a union of atoms of $\mathcal{P}$ (up to a null set). More generally, their *joint partition* is

$$\mathcal{P} \vee \mathcal{Q} \coloneqq \{P \cap Q \mid P \in \mathcal{P}, Q \in \mathcal{Q}\},$$

namely the least fine partition which refines $\mathcal{P}$ and $\mathcal{Q}$. This operation $\vee$ is associative.

A measurable partition $\mathcal{P}$ defines almost everywhere a map $\mathcal{P}(.)\colon X \to \mathcal{P}$ where $\mathcal{P}(x)$ is the atom of $\mathcal{P}$ which contains x. Given a measurable map $T\colon X \to X$, $\mathcal{P}$ provides *coding maps*

$$[\mathcal{P}]_{i,n}\colon x \in X \mapsto (\mathcal{P}(T^j x))_{i\leqslant j\leqslant n} \in \mathcal{P}^{\{i,\dots,n\}}.$$

In particular, $[\mathcal{P}]_n(x) \coloneqq [\mathcal{P}]_{0,n-1}(x)$ is the *n-word* of x.

Given atoms $P_i, P_{i+1}, \dots, P_n$ of $\mathcal{P}$, the equality $[\mathcal{P}]_{i,n}(x) = (P_i, \dots, P_n)$ exactly means that x is an element of $T^{-i}(P_i) \cap T^{-(i+1)}(P_{i+1}) \cap \dots \cap T^{-n}(P_n)$. Therefore the partition which gives the values of $[\mathcal{P}]_{i,n}$ is the following joint partition

$$\mathcal{P}_i^n \coloneqq \bigvee_{j=i}^{n} T^{-j}(\mathcal{P})$$

with $T^{-j}(\mathcal{P}) \coloneqq \{T^{-j}(P) \mid P \in \mathcal{P}\}$, this is a division of the space given by the dynamic of T, over the timeline $\{i, \dots, n\}$ and with respect to $\mathcal{P}$.

Entropy of a partition

If $\mathcal{P} = \{P_1, P_2\}$ is a partition of X with two atoms of equal measure, and $\mathcal{Q} = \{Q_1, Q_2\}$ is another partition with two atoms, such that $\mu(Q_1) = 0,9999$, then $\mathcal{P}$ brings more information than $\mathcal{Q}$. Given a random variable x with law μ, the answer to the question "In which atom is x?" is of more interest for $\mathcal{P}$ since it is uncertain. We want a function (called *entropy*) from the set of measurable partitions to $\mathbb{R}_+$ which quantifies the uncertainty of the answer, or equivalently which quantifies how much a partition divides the space.

First, we define the *information function* I from the measurable subsets to $\mathbb{R}_+$, such that, given a subset $P \subset X$ and a random variable x with law μ, $I(P)$ quantifies how much it is surprising to find out that x lies in P. We heuristically get the following axioms:

- $I(X) = 0$;
- $I(\varnothing) = +\infty$;
- $I(P) = f(\mu(P))$ for a decreasing map f;
- if A and B are independant, then $I(A \cap B) = I(A) + I(B)$.

The map f is necessarily $-\log$ (up to a multiplicative constant), so we define

$$I(P) \coloneqq -\log \mu(P).$$

Then we define the *information function* of a partition $\mathcal{P}$ as

$$I_{\mathcal{P}} \coloneqq \sum_{P \in \mathcal{P}} I(P)\mathbb{1}_P \colon X \to \mathbb{R}_+$$

and the *entropy* is the mean of this function:

$$H_\mu(\mathcal{P}) \coloneqq \int_X I_{\mathcal{P}}(x)\mathrm{d}\mu(x) = -\sum_{P \in \mathcal{P}} \mu(P)\log\mu(P) \in \mathbb{R}_+.$$

Let us now point out all the properties satisfied by entropy and that we were looking for when motivating this notion at the beginning of the paragraph.

- If $\mathcal{P}$ is a measurable partition of X of cardinality $n \geqslant 2$. Then we have

  $$\mathrm{H}_\mu(\mathcal{P}) \leqslant \log n$$

  with equality if and only if $\mathcal{P}$ is uniform. This follows from the fact that $x \in [0,1] \mapsto -x\log x$ is concave. As desired, it says that uniform partitions are the partitions for which the uncertainty is maximal.

- If $\mathcal{P}$ and $\mathcal{Q}$ are measurable partitions of X such that $\mathcal{P} \geqslant \mathcal{Q}$, then $\mathrm{H}_\mu(\mathcal{P}) \geqslant \mathrm{H}_\mu(\mathcal{Q})$. To prove it, we only have to notice that $I_{\mathcal{P}}(x) \geqslant I_{\mathcal{Q}}(x)$ almost everywhere and to integrate this inequality. It simply tells us that the entropy of a partition is larger and larger as the partition divides more and more the space

- Using the concavity of log, we prove that the entropy is subadditive: if $\mathcal{P}$ and $\mathcal{Q}$ are measurable partitions of X, then we have

$$\mathrm{H}_\mu(\mathcal{P} \vee \mathcal{Q}) \leqslant \mathrm{H}_\mu(\mathcal{P}) + \mathrm{H}_\mu(\mathcal{Q}).$$

  This crucial property enables us to use the subadditive lemma in order to define the entropy of a transformation.

- Finally, the inequality $\mathrm{H}_\mu(\mathcal{P} \vee \mathcal{Q}) \leqslant \mathrm{H}_\mu(\mathcal{P}) + \mathrm{H}_\mu(\mathcal{Q})$ is an equality if and only if $\mathcal{P}$ and $\mathcal{Q}$ are independent, meaning that for every $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$, we have $\mu(P \cap Q) = \mu(P)\mu(Q)$.

  The interpretation of subadditivity and its equality case is the following: the information brought by $\mathcal{P}$ *and* $\mathcal{Q}$ is less than the information brought by both partition separately, and it is not an equality if and only if $\mathcal{P}$ brings information that $\mathcal{Q}$ has already brought, or the converse (this is the interpretation behind "non-independance").

  Notice that the quantity $\mathrm{H}_\mu(\mathcal{P} \vee \mathcal{Q}) - \mathrm{H}_\mu(\mathcal{P})$ is a measurement of what $\mathcal{Q}$ brings more if we already know the information given by $\mathcal{P}$. We therefore define the conditional entropy by

$$\mathrm{H}_\mu(\mathcal{Q} \mid \mathcal{P}) = \mathrm{H}_\mu(\mathcal{Q} \vee \mathcal{P}) - \mathrm{H}_\mu(\mathcal{P}),$$

  and we get the following intuitive properties:

$$\mathrm{H}_\mu(\mathcal{Q} \mid \mathcal{P}) = \mathrm{H}_\mu(\mathcal{Q}) \iff \mathcal{P} \text{ and } \mathcal{Q} \text{ are independent},$$

$$\mathrm{H}_\mu(\mathcal{Q} \mid \mathcal{P}) = 0 \iff \mathcal{P} \geqslant \mathcal{Q}.$$

Measure-theoretic entropy of a transformation

Entropy, or measure-theoretic entropy, or metric entropy, of a measurable transformation is an invariant of conjugacy which quantifies how much a transformation complexifies the partitions.

Let $T \in \mathrm{Aut}(X, \mu)$ and $\mathcal{P}$ be a finite measurable partition of X. By the properties of H_μ, the map $n \in \mathbb{N} \mapsto \mathrm{H}_\mu(\mathcal{P}_0^n)$ is subadditive, so we deduce from the subadditive lemma that the following quantity

$$\mathrm{h}_\mu(T, \mathcal{P}) \coloneqq \lim_{n \to +\infty} \frac{\mathrm{H}_\mu(\mathcal{P}_0^n)}{n}$$

is well-defined, this is the *entropy of* T with respect to $\mathcal{P}$, and it tells us how quickly the dynamic of T is dividing the space X with the partition $\mathcal{P}$. Finally, let us define the *entropy of* T by

$$\mathrm{h}_\mu(T) \coloneqq \sup_{\mathcal{P}} \mathrm{h}_\mu(T, \mathcal{P}),$$

where the supremum is over all the finite measurable partitions $\mathcal{P}$ of X. This quantity is non-negative and can be infinite.

It is usually complicated to compute $\mathrm{h}_\mu(T, \mathcal{P})$ for every finite measurable partitions $\mathcal{P}$. Fortunately, the following result, due to Kolmogorov and Sinaï, states that we can restrict to some partitions. We say that a finite partition $\mathcal{P}$ is *T-generating* if the σ-algebra of X is generated by $(\mathcal{P}_0^n)_{n \geqslant 0}$ up to null sets. If T is invertible, it is equivalent to saying that the σ-algebra of X is generated by $\left(\mathcal{P}_{-n}^n\right)_{n \geqslant 0}$ up to null sets.

Theorem A.1.37 (see [Dow11, Theorem 4.2.2]). *If $\mathcal{P}$ is T-generating, then we have*

$$\mathrm{h}_\mu(T, \mathcal{P}) = \mathrm{h}_\mu(T).$$

Example A.1.38.

- Let Σ be a finite set equipped with a probability measure ν, and let T be the Bernoulli shift on $(X,\mu) \coloneqq (\Sigma^{\mathbb{Z}}, \nu^{\otimes\mathbb{Z}})$, where X is endowed with the σ-algebra generated by the cylinders.

  If $\mathcal{P}$ is the partition in cylinders of the form $[y_0]_{0,0}$, for $y_0 \in \Sigma$, then $\mathcal{P}_{-n}^n$ is the partition in cylinders of the form $[y_{-n},\dots,y_n]_{-n,n}$ (see Example A.1.20 for the definition of two-sided cylinders), so $\mathcal{P}$ is T-generating.

  By definition of the measure, the partitions $T^{-i}(\mathcal{P})$ for $i \geqslant 0$ are pairwise independent, so $\mathrm{H}_\mu(\mathcal{P}_0^n) = (n+1)\mathrm{H}_\mu(\mathcal{P})$ and

$$\mathrm{h}_\mu(T) = \mathrm{h}_\mu(T,\mathcal{P}) = \mathrm{H}_\mu(\mathcal{P}) = \sum_{x\in\Sigma} \nu(x)\log\nu(x).$$

- Let θ be an irrational real number and $T \coloneqq R_\theta$ the irrational rotation of angle θ on the unit circle $\mathbb{U}$. Without loss of generality, we can assume $0 < \theta < 1$. Let $\mathcal{P} = \{[0,\theta],[\theta,1]\}$, with the notation $[x,y] \coloneqq \{e^{2i\pi\tau} \mid x \leqslant \tau \leqslant y\}$. For every $n \in \mathbb{Z}$, we have $T^{-i}(\mathcal{P}) = \{[-i\theta,(-i+1)\theta],[(-i+1)\theta,1-i\theta]\}$, and we deduce from the fact that $\{e^{2in\pi\theta} \mid n \in \mathbb{Z}\}$ is dense in $\mathbb{U}$ that $\mathcal{P}$ is T-generating.

  For the computation of $\mathrm{H}_\mu(\mathcal{P})$, we have to notice that $\mathcal{P}_0^n$ is the partition in intervals with endpoints $-i\theta$ for $i \in \{0,\dots,n\}$, so it has cardinality $n+1$. We thus get

$$\mathrm{h}_\mu(T) = \mathrm{h}_\mu(T,\mathcal{P}) = \lim_{n\to+\infty} \frac{\mathrm{H}_\mu(\mathcal{P}_0^n)}{n} \leqslant \lim_{n\to+\infty} \frac{\log(n+1)}{n} = 0.$$

Let us finally mention Abramov's formula which gives the entropy of an induced map.

Theorem A.1.39 (Abramov's formula, see [Dow11, Theorem 4.3.3])**.** *Let $T \in \mathrm{Aut}(X,\mu)$ be an ergodic transformation and A a measurable subset of X. Then*

$$\mu(A)h(T_A) = h(T).$$

Topological entropy

In the topological setting, topological entropy is an invariant of topological conjugacy and is defined with similar ideas. As a definition, let us present the analogies with the measure-theoretic context.

Settings for measure-theoretic entropy	Settings for topological entropy
Probability space (X,μ)	Compact space X
Measurable map $T\colon X \to X$ which preserves the measure	Continuous map $T\colon X \to X$
Measurable partition $\mathcal{P}$	Open cover $\mathcal{U}$
Joint of partitions $\mathcal{P}\vee\mathcal{Q} = \{P\cap Q \mid P\in\mathcal{P}, Q\in\mathcal{Q}\}$ $\mathcal{P}_0^n = \bigvee_{i=0}^n T^{-i}(\mathcal{P})$	Joint of open covers $\mathcal{U}\vee\mathcal{V} \coloneqq \{U\cap V \mid U\in\mathcal{U}, V\in\mathcal{V}\}$ $\mathcal{U}_0^n = \bigvee_{i=0}^n T^{-i}(\mathcal{U})$
Entropy of a partition $\mathrm{H}_\mu(\mathcal{P}) = -\sum_{P\in\mathcal{P}} \mu(P)\log\mu(P)$	Entropy of an open cover $\log N(\mathcal{U})$ where $N(\mathcal{U}) \coloneqq$ $\min\{\lvert\mathcal{U}'\rvert \mid \mathcal{U}' \text{ is an open subcover of } \mathcal{U}\}$

Subadditivity: $\mathrm{H}_\mu(\mathcal{P} \vee \mathcal{Q}) \leqslant \mathrm{H}_\mu(\mathcal{P}) + \mathrm{H}_\mu(\mathcal{Q})$	Submultiplicativity: $N(\mathcal{U} \vee \mathcal{V}) \leqslant N(\mathcal{U})N(\mathcal{V})$, hence subadditivity: $\log N(\mathcal{U} \vee \mathcal{V}) \leqslant \log N(\mathcal{U}) + \log N(\mathcal{V})$
Entropy of T with respect to $\mathcal{P}$: $\mathrm{h}_\mu(T,\mathcal{P}) = \lim_{n\to+\infty} \frac{\mathrm{H}_\mu(\mathcal{P}_0^n)}{n}$	Topological entropy of T with respect to $\mathcal{U}$: $\mathrm{h}_{\mathrm{top}}(T,\mathcal{U}) = \lim_{n\to+\infty} \frac{\log N(\mathcal{U}_0^n)}{n}$
Entropy of T: $\mathrm{h}_\mu(T) = \sup_{\mathcal{P}} \mathrm{h}_\mu(T,\mathcal{P})$	Topological entropy of T: $\mathrm{h}_{\mathrm{top}}(T) = \sup_{\mathcal{U}} \mathrm{h}_{\mathrm{top}}(T,\mathcal{U})$

Analogously to Kolmogorov-Sinaï theorem for measure-theoretic entropy, we can reduce the computation of topological entropy to particular open covers. Assuming that the compact space X is metrizable, we say that an open cover $\mathcal{U}$ is T-generating for the topology on X if for every $\varepsilon > 0$, there exists $N \geqslant 0$ such that for every $n \geqslant N$, the open sets of $\mathcal{U}_0^n$ have a diameter less than ε.

Theorem A.1.40 (see [Dow11, Remark 6.1.7]). *Let T be a topological system on X and $\mathcal{U}$ a T-generating open cover. Then we have*

$$\mathrm{h}_{\mathrm{top}}(T,\mathcal{U}) = \mathrm{h}_{\mathrm{top}}(T).$$

Example A.1.41. Let us go back to the dynamical systems considered in Example A.1.38: Bernoulli shifts and irrational rotations which are also topological systems. We keep the same notations.

- The compact space $\Sigma^{\mathbb{Z}}$ can be endowed with the metric

$$d((x_i)_{i\in\mathbb{Z}},(y_i)_{i\in\mathbb{Z}}) \coloneqq \sum_{n\in\mathbb{Z}} \frac{1}{2^{|n|}} \mathbb{1}_{x_n \neq y_n}$$

  (it is easy to prove that it generates the product topology). The partition $\mathcal{P}$ of $\Sigma^{\mathbb{Z}}$ considered in Example A.1.38 is an open cover and is also T-generating in the topological sense. Since $\mathcal{P}$ is a partition, so is $\mathcal{P}_0^n$ and we simply have $N(\mathcal{P}_0^n) = |\mathcal{P}_0^n| = |\Sigma|^{n+1}$ (topological entropy is particularly interesting in Cantor sets since we can find clopen sets, so open covers which are partitions). We thus have

$$\mathrm{h}_{\mathrm{top}}(T) = \log|\Sigma|.$$

- For the irrational rotation of angle θ, recall that we considered the partition $\mathcal{P} = \{[0,\theta],[\theta,1]\}$ to compute the measure-theoretic entropy. To get open sets covering the unit circle $\mathbb{U}$, we start with the open sets $]0,\theta[$ and $]\theta,1[$, but since they do not cover 0 and θ, we add the open sets $]\theta/2,3\theta/2[$ and $]3\theta/2,1+\theta/2[$. It is not difficult to prove that the open cover $\mathcal{U}$ composed of these four open sets is topologically T-generating and to conclude that $\mathrm{h}_{\mathrm{top}}(T) = \mathrm{h}_{\mathrm{top}}(T,\mathcal{U}) = 0$.

The variational principle

The variational principle enables us to connect topological and measure-theoretic entropies.

Theorem A.1.42 (Variational principle, see [Dow11, Theorem 6.8.1]). *Let $T\colon X \to X$ be a topological system on a metric compact set X. Then we have*

$$\mathrm{h}_{\mathrm{top}}(T) = \sup_{\mu} \mathrm{h}_\mu(T)$$

where the supremum is over all T-invariant Borel probability measures μ on X.

As a consequence, if T is uniquely ergodic, then we have

$$\mathrm{h}_{\mathrm{top}}(T) = \mathrm{h}_{\mu}(T),$$

where μ denotes the only T-invariant Borel probability measure.

Example A.1.43. Let us consider the dynamical systems of Examples A.1.38 and A.1.41.

- We proved that the Bernoulli shift on $\Sigma^{\mathbb{Z}}$ has topological entropy $\mathrm{h}_{\mathrm{top}}(T) = \log|\Sigma|$ and its measure-theoretic entropy $\mathrm{h}_{\mu}(T) = \sum_{x\in\Sigma} \nu(x)\log\nu(x)$ when the space is endowed with $\mu = \nu^{\otimes\mathbb{Z}}$. When ν is the uniform distribution on Σ, we get $\mathrm{h}_{\mu}(T) = \mathrm{h}_{\mathrm{top}}(T)$.
- Irrational rotations are uniquely ergodic (see Example A.1.36), so we could prove that they have zero topological entropy from the fact that they have zero measure-theoretic entropy, and vice-versa.

A.1.i Kakutani equivalence

Let us finally introduce *Kakutani equivalence* and *even Kakutani equivalence*. These are equivalence relations weaker than conjugacy but the theory is as rich as the conjugacy problem, see [ORW82].

Definition A.1.44. Let $T, S \in \mathrm{Aut}(X,\mu)$ be two ergodic transformations.

1. T and S are said to be *Kakutani equivalent*, if there exist measurable subsets A and B of positive measure such that the induced transformations T_A and S_B are conjugate.
2. T and S are *evenly Kakutani equivalent* if moreover $\mu(A) = \mu(B)$.

Proposition A.1.45. *Kakutani equivalence and even Kakutani equivalence are equivalence relations.*

The proof of this proposition crucially uses ergodicity.

Proof of Proposition A.1.45. Kakutani equivalence is obviously reflexive (take $A = B$) and symmetric. For transitivity, let $S, T, U \in \mathrm{Aut}(X,\mu)$ be three ergodic transformations and A, B_1, B_2, C be measurable subsets of positive measure such that S_A is isomorphic to T_{B_1} and T_{B_2} is isomorphic to U_C. By ergodicity, there exists an integer i such that $B \coloneqq T^i(B_2) \cap B_1$ has positive measure. It is not difficult to prove the equality

$$T_{B_2} = T^{-i} T_{T^i(B_2)} T^i$$

so U_C is conjugate to $T_{T^i(B_2)}$. Moreover, if $\varphi_1 \colon (B_1, \mu_{B_1}) \to (A, \mu_A)$ (resp. $\varphi_2 \colon (T^i(B_2), \mu_{T^i(B_2)}) \to (C, \mu_C)$) is a conjugation between T_{B_1} and S_A (resp. $T_{T^i(B_2)}$ and U_C), then it induces a conjugation between T_B and $S_{\varphi_1(B)}$ (resp. T_B and $U_{\varphi_2(B)}$), so $S_{\varphi_1(B)}$ and $U_{\varphi_2(B)}$ are isomorphic.

Reflexivity and symmetry are also obvious for even Kakutani equivalence. For transitivity, we make the same proof as for Kakutani equivalence, using the fact that if A, B_1, B_2, C have the same measure, then $\varphi_1(B)$ and $\varphi_2(B)$ also have the same measure. □

An immediate consequence of Abramov's formula (Theorem A.1.39) is the following.

Corollary A.1.46. *Entropy is an invariant of even Kakutani equivalence.*

As mentioned in Section A.1.a, the conjugacy problem admits a complete invariant among Bernoulli shifts: this has been proved by Ornstein [Orn70]. He first classified up to conjugacy a class of systems called *very weak Bernoully*, which turned out to be exactly the class of Bernoulli shift. The definition of very weak Bernoulli systems uses the Hamming distance between words.

Similarly to Ornstein's theory [Orn70] for the conjugacy problem, Ornstein, Rudolph and Weiss [ORW82] found a class of systems, called loosely Bernoulli system, where Kakutani and even Kakutani equivalences are well understood. These systems were first introduced by par Feldman [Fel76], their definition is similar to that of very weak Bernoulli systems, replacing the Hamming word-metric by a more flexible one. We give more details in Chapter II, see Section II.2.d. For instance, Bernoulli shifts, odometers and irrational rotations are loosely Bernoulli.

The classification theorem is the following.

Theorem A.1.47 ([ORW82, Theorems 5.1 and 5.2])**.** *Let $S, T \in \mathrm{Aut}(X, \mu)$ be two ergodic transformations.*

1. *If S is loosely Bernoulli and is Kakutani equivalent to T, then T is also loosely Bernoulli.*
2. *If S and T are loosely Bernoulli, then they are evenly Kakutani equivalent if and only if they have the same entropy.*

A.2 Some basics of finitely generated groups and amenability

A.2.a Finitely generated groups

If S_Γ is a generating subset of Γ, then we can define the Cayley graph of Γ with respect to S_Γ as the graph whose vertices are the elements of Γ and edges are all the pairs $(\gamma, \gamma s)$ with $\gamma \in \Gamma$ and $s \in S_\Gamma \cup S_\Gamma^{-1}$.

This graph is connected and not directed, and enables us to endow Γ with the path metric: the distance between γ and $\gamma' \in \Gamma$ is

$$d_{S_\Gamma}(\gamma, \gamma') \coloneqq \min\{n \geqslant 0 \mid \exists s_1, \dots, s_n \in S_\Gamma \cup S_\Gamma^{-1}, \gamma = \gamma' s_1 \dots s_n\}.$$

This metric is also called the *word-length metric*. We then define the norm of $\gamma \in \Gamma$ as

$$|\gamma|_{S_\Gamma} \coloneqq d_{S_\Gamma}(\gamma, 1_G) = \min\{n \geqslant 0 \mid \exists s_1, \dots, s_n \in S_\Gamma \cup S_\Gamma^{-1}, \gamma = s_1 \dots s_n\}.$$

Note that we have $d_{S_\Gamma}(\gamma, \gamma') = |\gamma'^{-1}\gamma|_{S_\Gamma}$. We denote by $V_{S_\Gamma}(n)$ the cardinality of the closed ball of radius n centered at 1_Γ, namely the cardinality of $\{\gamma \in \Gamma \mid |\gamma|_{S_\Gamma} \leqslant n\}$. The map $V_{S_\Gamma} \colon \mathbb{N} \to \mathbb{R}_+$ is called the *volume growth* of Γ with respect to S_Γ.

Example. Let $d \geqslant 1$ be an integer and let us endow $\mathbb{Z}^d$ with the finite generating subset $S_{\mathbb{Z}^d}$ consisting in its canonical basis. Then the associated length function is defined by

$$|\boldsymbol{k}|_{S_{\mathbb{Z}^d}} = |k_1| + \ldots + |k_d|$$

for every $\boldsymbol{k} = (k_1, \dots, k_d) \in \mathbb{Z}^d$. In particular, the length function $|.|_{\{+1\}}$ for $\mathbb{Z}$ is the absolute value.

In the context of quantitative orbit equivalence, we only focus on finitely generated groups. For such groups, the two metrics obtained from two different finite generating sets S_Γ and S'_Γ are bi-Lipschitz equivalent: there exists a constant $C > 0$ such that the following holds:

$$\forall \gamma \in \Gamma, \frac{1}{C}|\gamma|_{S'_\Gamma} \leqslant |\gamma|_{S_\Gamma} \leqslant C|\gamma|_{S'_\Gamma}.$$

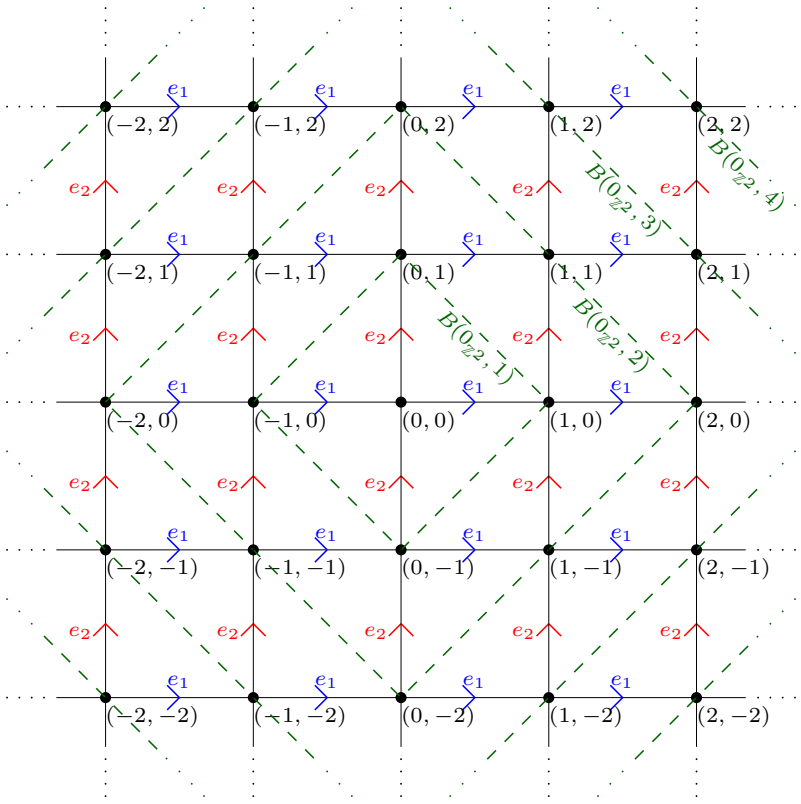


Figure A.1: The Cayley graph of $\mathbb{Z}^2$ with the finite generating set $\{e_1, e_2\}$ where $e_1 = (1,0)$ and $e_2 = (0,1)$.

The proof is very simple: if we can write γ as $s_1 \dots s_n$, with $s_i \in S_\Gamma$ and $n = |\gamma|_{S_\Gamma}$, then writting each s_i as a product of generators in S'_Γ gives the following bound on the number of generators in S'_Γ that we need to write γ:

$$|\gamma|_{S'_\Gamma} \leqslant \left(\max_{1 \leqslant i \leqslant n} |s_i|_{S'_\Gamma} \right) n = \left(\max_{1 \leqslant i \leqslant n} |s_i|_{S'_\Gamma} \right) |\gamma|_{S_\Gamma}.$$

We similarly prove the reverse inequality.

From this important fact, we deduce that the information on the geometry of the group is asymptotically the same whatever the finite generating set we consider. Let us introduce some terminologies. Given two increasing maps $f, g \colon \mathbb{R}_+ \to \mathbb{R}_+$, we say that g dominates f, written as $g \succcurlyeq f$, if there exists a constant $C > 0$ such that $f(x) \leqslant Cg(Cx)$ for sufficiently large real numbers x. If $g \succcurlyeq f$ and $f \succcurlyeq g$, then we say that f and g are *asymptotically equivalent*, and we write $f \approx g$.

Therefore, given two different finite generating sets S_Γ and S'_Γ, the above fact implies that the volume growth V_{S_Γ} and $V_{S'_\Gamma}$ are asymptotically equivalent. The asymptotic behaviour, namely the equivalence class modulo $\approx$, is denoted by V_Γ. We say that a group has polynomial growth of degree $d \geqslant 1$ if $V_\Gamma(x) \approx x^d$, and has exponential growth if $V_\Gamma(x) \approx e^x$ (note that we have $e^x \approx e^{Cx}$ for every $C > 0$).

Example A.2.1.

- For every integer $d \geqslant 1$, $\mathbb{Z}^d$ has polynomial growth of degree d.
- Let us consider the set $\mathbb{Z}^3$ endowed with the group operation:

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + xy'),$$

coming from the identification with the group of matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

This group, called the *Heisenberg group*, is generated by $\{(1,0,0),(0,1,0)\}$ and has polynomial growth of degree 4.

- Given groups Γ and Λ, we define the *wreath product*

$$\Lambda \wr \Gamma \coloneqq \left(\bigoplus_{\Gamma} \Lambda \right) \rtimes \Gamma$$

where Γ acts on the direct sum by shifting the coordinate. If Γ and Λ are finitely generated, with finite generating subsets S_Γ and S_Λ, then so is $\Lambda \wr \Gamma$. Indeed, $\Lambda \wr \Gamma$ is generated by:

$$\{(\iota(s), 1_\Gamma) \mid s \in S_\Lambda\} \cup \{(\iota(1_\Lambda), s) \mid s \in S_\Gamma\},$$

where $\iota(s)\colon \Gamma \to \Lambda$ is defined by $\iota(s)(1_\Gamma) = s$ and $\iota(s)(\gamma) = 1_\Lambda$ for every $\gamma \in \Gamma \backslash \{1_\Gamma\}$. If Γ is infinite and Λ is not trivial, then $\Lambda \wr \Gamma$ has exponential growth.

- Given integers $n, m \geqslant 1$, the *Baumslag-Solitar group*

$$\mathrm{BS}(m,n) = \langle s, t \mid ts^n t^{-1} = s^m \rangle$$

is finitely generated (by definition). If $(m,n) = (1,1)$, this is the group $\mathbb{Z}^2$ (polynomial growth of degree 2), otherwise it has exponential growth.

- Given an integer $n \geqslant 2$, the free group with n generators has exponential growth. Moreover the Cayley graph with respect to the canonical generators is a $2n$-tree.

These metric properties give algebraic information on the group. For instance, Gromov [Gro81] proved that finitely generated groups have polynomial growth if and only if they are virtually nilpotent (i.e. there exists a finite index subgroup which is nilpotent).

A.2.b Amenable groups

Quantitative orbit equivalence suits with finitely generated groups that are *amenable*, since it has rigidity results using the *isoperimetric profiles*. Let us introduce these notions. We refer to [BO08, Section 2.6] for more details on amenability and its "$10^{10^{10}}$ different characterisations".

Let Γ be a countable group. A sequence $(F_n)_{n \geqslant 0}$ of finite subsets of Γ is a *(left) Følner sequence* of Γ if the following holds:

$$\forall \gamma \in \Gamma, \ \frac{|\gamma F_n \ \Delta \ F_n|}{|F_n|} \underset{n \to +\infty}{\longrightarrow} 0,$$

where $A \ \Delta \ B \coloneqq (A \backslash B) \cup (B \backslash A)$ denotes the symmetric difference between two sets A and B. We could also define *right Følner sequences*, writting $F_n \gamma$ instead of γF_n. Note that (F_n) is a *left Følner sequence* if and only if (F_n^{-1}) is a *right Følner sequence*.

We now say that a countable group Γ is *amenable* if it admits a Følner sequence.

Notice that $|\cdot \ \Delta \ \cdot|$ defines a distance in the set of the finite subsets of Γ. Indeed, it suffices to write $|A \ \Delta \ B| = \sum_{\gamma \in \Gamma} |\mathbb{1}_A(\gamma) - \mathbb{1}_B(\gamma)|$ and to use the properties of the absolute value. In particular, $|\cdot \ \Delta \ \cdot|$ satisfies the triangle inequality which enables us to give the

following equivalent definition when Γ is generated by a subset S_Γ: a sequence $(F_n)_{n\geqslant 0}$ of finite subsets of Γ is a *Følner sequence* of Γ if the following holds:

$$\forall \gamma \in S_\Gamma,\ \frac{|\gamma F_n \,\Delta\, F_n|}{|F_n|} \underset{n\to+\infty}{\longrightarrow} 0.$$

When Γ is finitely generated, the problem thus reduces to finitely many properties to check.

Amenability is closed under many operations between groups: extensions, increasing unions, passing to quotients, to subgroups, etc. This provides many examples.

Example A.2.2.

- Any finite group Γ is amenable (take $F_n = \Gamma$).
- Given an integer $d \geqslant 1$, $\mathbb{Z}^d$ is amenable (take $F_n = \{0, \dots, n\}^d$).
- The Heisenberg group is amenable.
- A wreath product $\Lambda \wr \Gamma$ is amenable if and only if Λ and Γ are amenable.
- However the free group with $n \geqslant 2$ generators is not amenable.
- Given integers $n, m \geqslant 1$, the Baumslag-Solitar group $\mathrm{BS}(m,n)$ is amenable if and only if m or n is equal to 1.

Let us mention the following equivalent definition of amenability (the definition we give above is its sequential characterisation somehow). Given a finite subset L of Γ and $\delta > 0$, we say that a finite subset F of Γ is (L,δ)-invariant if the following hold:

$$\forall \gamma \in L,\ \frac{|\gamma F \,\Delta\, F|}{|F|} \leqslant \delta.$$

Then Γ is amenable if and only if for every such pair (L,δ), there exists an (L,δ)-invariant finite subset. Let us also mention that a sequence $(F_n)_{n\geqslant 0}$ of finite subsets is Følner if and only if for every such pair (L,δ), there exists $n_0 \geqslant 0$ such that F_n is (L,δ)-invariant for every $n \geqslant n_0$. Intuitively, an amenable group is a group admitting finite subsets which give an idea of its structure.

Amenability has many other equivalent definitions, let us explain the *Reiter definition*, easily understandable from the Følner definition above. Indeed, we noticed for instance that triangle inequality for the metric $|\cdot\ \Delta\ \cdot|$ is easier to prove when we replace sets A by their characteristic functions $\mathbb{1}_A$. For amenability we can also find functional characterizations: the ℓ^p-Reiter conditions, for $p \geqslant 1$. *ℓ^p-Reiter condition* means that there exists a sequence $(f_n)_{n\geqslant 0}$ of functions in $\ell^p(\Gamma)$ (the set of function $f\colon \Gamma \to \mathbb{R}$ with $\|f\|_p^p \coloneqq \sum_{\gamma\in\Gamma} |f(\gamma)|^p < \infty$, such that for every $\gamma \in \Gamma$,

$$\frac{\|f_n(\gamma^{-1}\cdot) - f_n(\cdot)\|_p}{\|f_n\|_p} \underset{n\to+\infty}{\to} 0.$$

Notice that, if $(F_n)_{n\geqslant 0}$ is a Følner sequence, then $f_n \coloneqq \mathbb{1}_{F_n}$ satisfies ℓ^p-Reiter condition, since $\|f_n(\gamma^{-1}\cdot) - f_n(\cdot)\|_p^p = |\gamma F_n \,\Delta\, F_n|$ and $\|f_n\|_p^p = |F_n|$.

Let us finally introduce the isoperimetric profiles which provide a more quantitative description of amenability among finitely generated groups. Given a finitely generated group Γ, and $p \geqslant 1$, the *ℓ^p-isoperimetric profile* of Γ is the map $j_{p,\Gamma}\colon \mathbb{N} \to \mathbb{R}_+$ defined by

$$\forall n \in \mathbb{N},\ j_{p,\Gamma}(n) \coloneqq \sup_{\substack{f\colon \Gamma\to\mathbb{R} \\ |\mathrm{supp}\ f|\leqslant n}} \frac{\|f\|_p}{\|\nabla_{S_\Gamma} f\|_p},$$

where the *support* of f is $\operatorname{supp} f \coloneqq \{\gamma \in \Gamma \mid f(\gamma) \neq 0\}$ and the ℓ^p-norm of its *gradient* is given by $\|\nabla_{S_\Gamma} f\|_p^p \coloneqq \sum_{s\in S_\Gamma} \|f(s^{-1}\cdot) - f(\cdot)\|_p^p = \sum_{s\in S_\Gamma}\sum_{\gamma\in\Gamma} |f(s^{-1}\gamma) - f(\gamma)|^p$. For $p = 1$, the ℓ^1-isoperimetric profile, simply called the isoperimetric profile, has a simpler definition modulo $\approx$:

$$j_{1,\Gamma}(n) \approx \sup_{\substack{A\subset\Gamma \\ |A|\leqslant n}} \frac{|A|}{|\partial_{S_\Gamma} A|},$$

where $\partial_{S_\Gamma} A \coloneqq S_\Gamma A \,\Delta\, A$ is the boundary of A. Reiter conditions imply that a finitely generated group is amenable if and only if its isoperimetric profiles are not bounded. Intuitively, the faster isoperimetric profiles go to infinity, the "more amenable" the group is.

Coulhon Saloff-Coste isoperimetric inequality [CS93, Theorem 1] gives a connection between the ℓ^1-isoperimetric profile and the volume growth of a finitely generated group:

$$j_{1,\Gamma}(n) \preccurlyeq V_\Gamma^{-1}(n).$$

Finally, ℓ^2-isoperimetric profile is particularly studied for its relation with return probabilities of random walks on groups, see for instance [SZ15; SZ16; SZ18; BZ21].

Example A.2.3.

- Given an integer $d \geqslant 1$, we have $j_{p,\mathbb{Z}^d}(n) \approx n^{1/d}$. The same holds more generally for $j_{p,\Gamma}$ when Γ has polynomial growth of degree d.
- Given a non trivial finite group Λ and an integer $d \geqslant 1$, we have $j_{p,\Lambda\wr\mathbb{Z}^d}(n) \approx (\log n)^{1/d}$ [Ers03].
- $j_{p,\mathbb{Z}\wr\mathbb{Z}}(n) \approx \frac{\log n}{\log\log n}$ [Ers03].
- Given an integer $k \geqslant 2$, we have $j_{p,\mathrm{BS}(1,k)}(n) \approx \log n$.

A.3 Ergodic theory for actions of amenable groups

Let us now move on to ergodic theory in the more general context of group actions (we refer to [KL16]). Given an amenable group Γ, we consider a pmp Γ-action on a standard and atomless probability space (X, μ), namely a morphism $\Gamma \to \mathrm{Aut}(X, \mu)$. The only idea to keep in mind is the following.

"A group Γ is amenable if we can do ergodic theory as if Γ were the group $\mathbb{Z}$."

So let us generalize the content of Section A.1 to these groups.

First, ergodicity does not need amenability for a definition. Given a group Γ, a pmp action $\Gamma \curvearrowright (X, \mu)$ is ergodic if for every measurable set A, the following holds: if $\mu(\gamma A \,\Delta\, A) = 0$ for every $\gamma \in \Gamma$, then $\mu(A) \in \{0, 1\}$.

Let us now focus on amenable groups. In Birkhoff and Von Neumann ergodic theorems and in the definition of entropy for $T \in \mathrm{Aut}(X, \mu)$, we always focus on the action of T in the timeline $\{0, 1, \dots, n\}$. Notice that this set provides a Følner sequence of $\mathbb{Z}$ as n grows, so for an action of an amenable group Γ, let us replace it by F_n, where $(F_n)_{n\geqslant 0}$ is a Følner sequence of Γ.

- Von Neumann ergodic theorem holds for a pmp Γ-action on (X, μ), replacing $\frac{1}{n}\sum_{i=0}^{n} f \circ T^i$ by $\frac{1}{|F_n|}\sum_{\gamma\in F_n} f(\gamma.)$ (see [KL16, Theorem 4.22]). In particular, it does not depend on the Følner sequence $(F_n)_{n\geqslant 0}$ we consider.

- For entropy, we replace $\frac{1}{n}\mathrm{H}_\mu(\bigvee_{i=0}^{n} T^{-i}\mathcal{P})$ by $\frac{1}{|F_n|}\mathrm{H}_\mu(\bigvee_{\gamma\in F_n}\gamma^{-1}\mathcal{P})$, and we get an analogous theory (see [KL16, Section 9.3]). It is worth noticing that the quantity that we get when passing to the limit, namely the entropy, does not depend on the Følner sequence $(F_n)_{n\geqslant 0}$ we consider. We can similarly generalize the notion of topological entropy (see [KL16, Section 9.9]) and extend the variational principle (see [KL16, Section 9.10]) to amenable groups.

- For Birkhoff ergodic theorem (see [KL16, Theorem 4.28]), it is not enough to proceed as for Von Neumann theorem. We have to assume that $(F_n)_{n\geqslant 0}$ is *tempered*, this means that there exists $b > 0$ such that

$$\left|\bigcup_{i=0}^{n-1} F_k^{-1}F_n\right| \leqslant b|F_n|$$

  for every $n \geqslant 1$. This is the definition for *left* Følner sequences, we can analogously define temperedness for right Følner sequences. This property is not so restrictive since every Følner sequence admits a subsequence which is tempered. Indeed, being a left Følner sequence means that, as n grows, F_n can "absorb" more and more points when multiplying to the left, so if k is large enough, F_{n+k} can "absorb" $\bigcup_{i=0}^{n-1} F_k^{-1}$.

Finally, Ornstein's theory for Bernoulli shifts has been generalized to amenable groups (the Bernoulli shift $\Gamma \curvearrowright \Sigma^\Gamma$ is defined by $\gamma\cdot(x_{\gamma'})_{\gamma'\in\Gamma} \coloneqq (x_{\gamma^{-1}\gamma'})_{\gamma'\in\Gamma}$). Indeed, given an amenable group Γ, Ornstein and Weiss [OW87] proved that two Bernoulli shifts of Γ are isomorphic if and only if they have the same entropy.

Appendix B

Background on quantitative orbit equivalence and related notions

This appendix provides a state of the art on orbit equivalence (between actions or between groups), with more details than the introduction, and emphasizes on the connections between this topic and many areas such as ergodic theory and geometric group theory that we presented in Appendix A.

Contents

B.1 Orbit equivalence and related notions

B.1.a Framework

In this thesis, groups act on a standard and atomless probability space (X, μ), where "standard" means that X is a Polish space, endowed with its Borel σ-algebra. Such spaces are all isomorphic to $([0,1], \mathrm{Leb})$, meaning that there exists a bimeasurable bijection $\Psi\colon X \to [0,1]$ such that $\Psi_\star\mu = \mathrm{Leb}$, where $\Psi_\star\mu$ denotes the probability measure on $[0,1]$ defined by $\Psi_\star\mu(A) = \mu(\Psi^{-1}(A))$ for every measurable subset A. We will denote by $\mathrm{Aut}(X,\mu)$ the set of bimeasurable bijections which preserve the measure, two such maps being identified if they coincide on a subset of full measure.

Given a group Γ, a Γ-action on (X,μ) is *pmp* (for *probability measure-preserving*) if for every $\gamma \in \Gamma$, the map $x \in X \mapsto \gamma \cdot x \in X$ is a bimeasurable bijection which preserves the probability measure : $\mu(\gamma A) = \mu(A)$ for every measurable subset $A \subset X$. A pmp Γ-action on (X,μ) is in fact a group morphism from Γ to $\mathrm{Aut}(X,\mu)$.

We say that the group action is *(essentially) free* if for μ-almost every $x \in X$, for every $\gamma \in \Gamma\backslash\{1_\Gamma\}$, $\gamma \cdot x \neq x$. Given a countable group Γ, there always exists a free pmp Γ-action on a standard and atomless probability space, consider for instance the Bernoulli shift $\Gamma \curvearrowright \{0,1\}^\Gamma$ defined by $\gamma \cdot (x_{\gamma'})_{\gamma' \in \Gamma} \coloneqq (x_{\gamma^{-1}\gamma'})_{\gamma' \in \Gamma}$. Freeness offers to every orbit a graph structure inherited from the Cayley graph of the group (see Figure B.1).

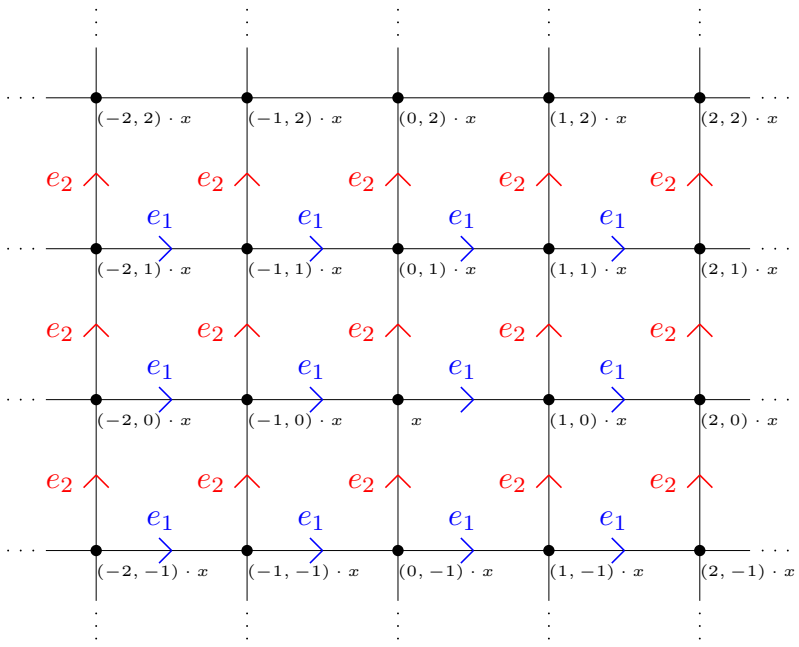


Figure B.1: Given a free action of $\mathbb{Z}^2$ on X, and given $x \in X$, the $\mathbb{Z}^2$-orbit of x has the structure inherited from the Cayley graph of $\mathbb{Z}$ with respect to the finite generating set $\{e_1, e_2\}$ with $e_1 = (1,0)$ and $e_2 = (0,1)$, where x is identified with $(0,0)$.

In the case of the group $\mathbb{Z}$, a $\mathbb{Z}$-action can be summarized by the data of the action of $+1$, since it generates the group. Therefore, instead of considering a $\mathbb{Z}$-action, we will consider an element of $\mathrm{Aut}(X,\mu)$, that we will call a *(dynamical) system*, a *transformation*, etc. A transformation $T \in \mathrm{Aut}(X,\mu)$ is *aperiodic* if the associated $\mathbb{Z}$-action is free. In this case of the group $\mathbb{Z}$, this amounts to saying that for μ-almost every $x \in X$, the orbit of x is infinite.

Given a Γ-action, the orbit of some point $x \in X$ will be denoted by $\Gamma \cdot x$. In the case of the group $\mathbb{Z}$, we will write $\mathrm{Orb}_T(x)$, where $T \in \mathrm{Aut}(X,\mu)$ is the action of the generator $+1$.

B.1.b Orbit equivalence

Let us now move on to the definition of orbit equivalence. First, let us define this notion for group actions.

Definition B.1.1. Let Γ and Λ be groups. Two pmp actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ on standard and atomless probability spaces are *orbit equivalent* if there exists a measured isomorphism $\Psi\colon X \to Y$ such that for almost every $x \in X$, the following equality holds:

$$\Psi(\Gamma \cdot x) = \Lambda \cdot \Psi(x).$$

If furthermore the actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ are free, we can then define measurable maps $c_{\Gamma,\Lambda}\colon \Gamma \times X \to \Lambda$ and $c_{\Lambda,\Gamma}\colon \Lambda \times Y \to \Gamma$, called the *cocycles* associated to this orbit equivalence, and defined by the following equations:

$$\Psi(\gamma \cdot x) = c_{\Gamma,\Lambda}(g, x) \cdot \Psi(x) \text{ and } \Psi^{-1}(\lambda \cdot y) = c_{\Lambda,\Gamma}(\lambda, y) \cdot \Psi^{-1}(y)$$

for every $\gamma \in \Gamma$, $\lambda \in \Lambda$ and for almost every $x \in X$ and $y \in Y$ (these functions are well-defined by freeness).

These cocycles satisfy the cocycle identity:

$$c_{\Gamma,\Lambda}(\gamma\gamma', x) = c(\gamma, \gamma' \cdot x)c(\gamma', x)$$

for every $\gamma, \gamma' \in \Gamma$ and for almost every $x \in X$, and similarly for $c_{\Lambda,\Gamma}$. Moreover it is not hard to prove that for almost every $x \in X$, $c_{\Gamma,\Lambda}(., x)\colon \Gamma \to \Lambda$ is a bijection mapping 1_Γ to 1_Λ.

In the case of finitely generated groups, the equality between the orbits is easier to check. Indeed, if S_Γ is a finite generating subset of Γ, then the inclusion $S_\Gamma \cdot x \subset \Lambda \cdot x$ automatically implies $\Gamma \cdot x \subset \Lambda \cdot x$. Thus it is interesting to look at the particular case of $\mathbb{Z}$-actions.

Example B.1.2. Since the group $\mathbb{Z}$ is generated by $+1$, a pmp $\mathbb{Z}$-action is completely determined by an element of $\mathrm{Aut}(X, \mu)$. Furthermore, if aperiodic transformations $S, T \in \mathrm{Aut}(X, \mu)$ are orbit equivalent, with an orbit equivalence Ψ such that S and $\Psi^{-1}T\Psi$ have the same orbits, then we consider the associated cocycles only on the generator $+1$, namely we consider maps $c_S\colon X \to \mathbb{Z}$ and $c_T\colon X \to \mathbb{Z}$, still called *cocycles* and defined by:

$$\Psi(Sx) = T^{c_S(x)}\Psi(x) \text{ and } \Psi^{-1}(Tx) = S^{c_T(x)}\Psi^{-1}(x)$$

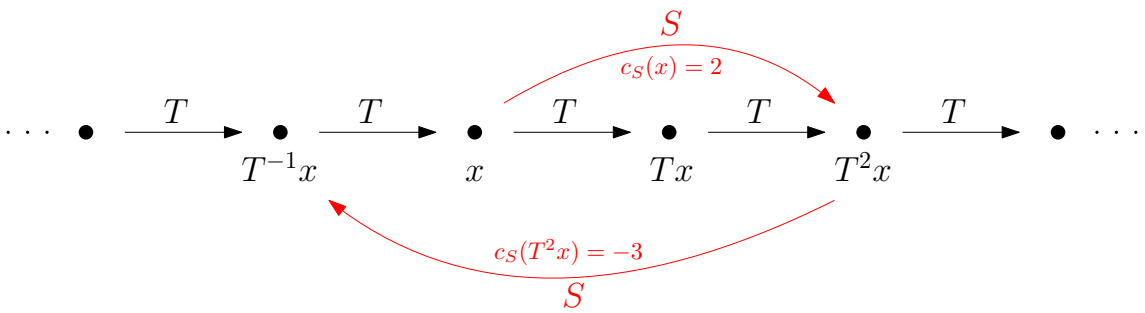


Figure B.2: Orbit equivalence between aperiodic transformations in $\mathrm{Aut}(X, \mu)$, when $\Psi = id_X$.

The definition of orbit equivalence between groups, and not between prescribed actions, is the following.

Definition B.1.3. Let Γ and Λ be groups. We say that they are orbit equivalent if there exist two free pmp Γ- and Λ-actions on a standard and atomless probability space (X, μ) such that for almost every $x \in X$, $\Gamma \cdot x = \Lambda \cdot x$. The space (X, μ) is called an *orbit equivalence coupling*.

In other words, Γ and Λ are orbit equivalent if there exist two orbit equivalent free pmp Γ- and Λ-actions. By Definition B.1.1, an orbit equivalence coupling (X, μ) between two groups Γ and Λ provides cocycles $c_{\Gamma,\Lambda} \colon \Gamma \times X \to \Lambda$ and $c_{\Lambda,\Gamma} \colon \Lambda \times Y \to \Gamma$.

Freeness is useful to define cocycles, which allow us to pass from a group to the other, but the goal is also to make sure that the orbits keep track of the structure of the group, what will in fact go wrong for amenable groups.

Theorem B.1.4 (Dye [Dye59] for $\mathbb{Z}$-actions; Ornstein, Weiss [OW80] for the general case)**.** *Any two ergodic pmp free actions of infinite amenable groups are orbit equivalent. In particular, any two infinite amenable groups are orbit equivalent.*

Note that ergodicity is necessary in this theorem since it is preserved by orbit equivalence (both notions are properties on the orbits). Moreover, orbit equivalence between groups preserves the cardinality of the groups (by freeness and orbit equalities) so $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ cannot be orbit equivalent. It is not hard to see that any two finite groups Γ and Λ with same cardinality are orbit equivalent. Indeed, denoting $n \coloneqq |\Gamma| = |\Lambda|$, any bijection between Γ and Λ yields a measured isomorphism Ψ between the probability spaces $[0, 1/n] \times \Gamma$ and $[0, 1/n] \times \Lambda$ (both endowed with the product of the Lebesgue measure and the counting measure on the group) and the left translation of the groups on themselves yields a free pmp action on the corresponding space we just defined, and it is not difficult to prove that these actions are orbit equivalent via Ψ.

Let us also introduce orbit equivalence in a topological setting. We first draw an analogy with the measure-theoretic context.

Measure-theoretic setting	Topological setting
Standard and atomless probability space (X, μ)	Cantor set X
Any such spaces (X, μ) and (Y, ν) are isomorphic (there exists a bimeasurable bijection $\Psi \colon X \to Y$ such that $\Psi_\star \mu = \mu$)	Any such spaces X and Y are isomorphic (there exists a homeomorphism $\Psi \colon X \to Y$)
Example: $([0, 1], \mathrm{Leb})$	Example: $\{0, 1\}^{\mathbb{N}}$
Dynamical systems: Bimeasurable bijections $T \colon X \to X$ which preserve the measure (i.e. $T \in \mathrm{Aut}(X, \mu)$)	Dynamical systems: Homeomorphisms $T \colon X \to X$
Orbit equivalence between $S, T \in \mathrm{Aut}(X, \mu)$: there exists a bimeasurable bijection $\Psi \colon X \to X$ such that $\Psi_\star \mu = \mu$ and such that $\Psi(\mathrm{Orb}_S(x)) = \mathrm{Orb}_T(\Psi(x))$ for **almost every** $x \in X$	Orbit equivalence between homeomorphisms S, T: there exists a homeomorphism $\Psi \colon X \to X$ such that $\Psi(\mathrm{Orb}_S(x)) = \mathrm{Orb}_T(\Psi(x))$ for **every** $x \in X$

Moreover, while we often look at **ergodic** transformations in a measure-theoretic con-

text, the dynamical systems we consider in the topological setting are most of the time **minimal**, namely every orbit is dense, and it is not hard to prove that ergodicity (resp. minimality) is invariant under measure-theoretic (resp. topological) orbit equivalence. Ergodicity and minimality thus play somehow the same role.

A (topological) orbit equivalence Ψ between two homeomorphisms $S, T\colon X \to X$ induces the bijection $\mu \mapsto \Psi_\star \mu$ from the S-invariant probability measures to the T-invariant ones (using the same ideas as in the proof of Proposition II.2.20 in Chapter II). Consequently, S is uniquely ergodic if and only if T is.

Let us now introduce a particular instance of orbit equivalence in the topological setting, this is a crucial notion in Chapter II.

Definition B.1.5. Let X be a Cantor set. We say that two minimal homeomorphisms $S, T\colon X \to X$ are *strongly orbit equivalent* if there exists a homeomorphism $\Psi\colon X \to X$ such that $\Psi(\mathrm{Orb}_S(x)) = \mathrm{Orb}_T(\Psi(x))$ for every $x \in X$, and such that the associated cocycles $c_S, c_T\colon X \to \mathbb{Z}$ each have at most one point of discontinuity.

This equivalence relation has been introduced by Giordano, Putnam and Skau [GPS95] which moreover found a complete invariant among minimal homeomorphisms, the *dimension group*. This invariant is obtained from a combinatorial description of these dynamical systems: the *Bratteli diagrams*. We refer the reader to Appendix II.B in Chapter II for a brief overview.

B.1.c Measure equivalence, stable orbit equivalence

Let us introduce the notion of *measure equivalence*, a measured analogue of quasi-isometry introduced by Gromov, and which is closely related to orbit equivalence, since it coincides with the notion of *stable orbit equivalence*. For the group $\mathbb{Z}$, Kakutani equivalence is an example of the latter, thus highlighting the interactions between orbit equivalence and classical problems in ergodic theory.

By a *smooth* action of a countable group Γ, we mean a measure-preserving Γ-action on a standard measured space (Ω, μ) which admits a fundamental domain, namely a Borel subset X_Γ of Ω that intersects every Γ-orbit exactly once.

Definition B.1.6. Two groups Γ and Λ are *measure equivalent* if there exists a standard Borel measure space (Ω, μ) equipped with commuting measure-preserving smooth Γ- and Λ-actions such that

1. both the Γ- and Λ-actions are free;
2. the Γ-action (resp. the Λ-action) admits a fixed fundamental domain X_Γ (resp. X_Λ) of finite measure.

The quadruple $(\Omega, X_\Gamma, X_\Lambda, \mu)$ is called a *measure equivalence coupling*. We will always use the notations $\gamma * x$ and $\lambda * x$ (with $\gamma \in \Gamma$, $\lambda \in \Lambda$, $x \in \Omega$) for these smooth actions on Ω. The notations $\gamma \cdot x$ and $\lambda \cdot x$ refers to the induced actions that we now define, as well as the cocycles.

Definition B.1.7. A measure equivalence coupling $(\Omega, X_\Gamma, X_\Lambda, \mu)$ between Γ and Λ induces a finite measure-preserving Γ-action on $(X_\Lambda, \mu_{X_\Lambda})$ in the following way: for every $\gamma \in \Gamma$ and every $x \in X_\Lambda$, $\gamma \cdot x \in X_\Lambda$ is defined by the identity

$$(\Lambda * \gamma * x) \cap X_\Lambda = \{\gamma \cdot x\},$$

it is unique since X_Λ is a fundamental domain for the smooth Λ-action.

This also yields a *cocycle* $c_{\Gamma,\Lambda}\colon \Gamma \times X_\Lambda \to \Lambda$ uniquely (by freeness) defined by

$$c_{\Gamma,\Lambda}(\gamma, x) * \gamma * x = \gamma \cdot x,$$

or equivalently $c_{\Gamma,\Lambda}(\gamma, x) * \gamma * x \in X_\Lambda$, for almost every $x \in X_\Lambda$ and every $\gamma \in \Gamma$. We similarly define a finite measure-preserving Λ-action on $(X_\Gamma, \mu_{X_\Gamma})$ and the associated cocycle $c_{\Lambda,\Gamma}\colon \Lambda \times X_\Gamma \to \Gamma$. As for orbit equivalence, these cocycles satisfy the cocycle identity. However, when fixing the coordinate in X_Λ or X_Γ, they do not provide bijections between the groups.

Example B.1.8. Two lattices Γ and Λ of a locally compact group G are measure equivalent. Indeed, $(G, X_\Gamma, X_\Lambda, \mu)$ is a measure equivalence coupling, where μ is a Haar measure of G, Γ acts on Γ by left multiplication, Λ acts on G by right multiplication, and X_Γ and X_Λ are the finite-measure fundamental domains provided by the definition of a lattice.

Let us now deal with the relations between measure and orbit equivalences.

Proposition B.1.9 ([Fur11])**.** *Two countable groups Γ and Λ are orbit equivalent if and only if they are measure equivalent and admit a measure equivalence coupling with equal fundamental domains.*

Note that given a measure equivalence coupling $(\Omega, X_\Gamma, X_\Lambda, \mu)$ with equal fundamental domains $X \coloneqq X_\Gamma = X_\Lambda$, the induced actions on X share the same orbits. However these actions are not necessarily free, so the idea is to combine the commuting actions on Ω with free pmp Γ- and Λ-actions of these groups on probability spaces Y_Γ and Y_Λ, namely we consider the product $\Omega \times Y_\Gamma \times Y_\Lambda$ where Γ (resp. Λ) diagonally acts on $\Omega \times Y_\Gamma$ (resp. $\Omega \times Y_\Lambda$). We easily check that both actions commute. Then the induced actions for the new measure equivalence coupling (with common fundamental domain $X \times Y_\Gamma \times Y_\Lambda$) share the same orbits and are free.

From the cocycles for the measure equivalence, defined on X, we thus get cocycles for the orbit equivalence, defined on $X \times Y_\Gamma \times Y_\Lambda$. In fact they are in some way the same, namely when we fix the coordinate in X in the product $X \times Y_\Gamma \times Y_\Lambda$, the cocycles for the orbit equivalence are constant in the corresponding fiber and coincide with the cocycles for the measure equivalence.

Conversely, the classical construction of a measure equivalence coupling from an orbit equivalence keeps the same cocycles.

When the fundamental domains are not equal, we can still give a reformulation of measure equivalence in terms of orbit equivalence. In fact, we can assume without loss of generality that the fundamental domains intersect non trivially. Indeed $\bigcup_{\lambda\in\Lambda} \lambda X_\Lambda$ is equal to Ω (up to a null set), so it non trivially intersects X_Γ, so there exists some $\lambda_0 \in \Lambda$ such that $\mu(X_\Gamma \cap \lambda_0 X_\Lambda) > 0$, we thus replace X_Λ by its translate $\lambda_0 X_\Lambda$. Then the induced actions $\Gamma \curvearrowright (X_\Lambda, \mu_\Lambda)$ and $\Lambda \curvearrowright (X_\Gamma, \mu_\Gamma)$ have this non-trivial intersection in common, and it is not hard to see that the portions of orbits visiting this intersection coincide. With this discussion, it is not hard to understand that the following weaker notion of orbit equivalence is a reformulation of measure equivalence.

Definition B.1.10. Two free pmp actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ of countable groups on standard and atomless probability spaces are *stably orbit equivalence* if there exist subset $A \subset X$ and $B \subset Y$ of positive measure and a measured isomorphism $\Psi\colon (A, \mu_A) \to (B, \nu_B)$ such that the following equality holds:

$$\Psi((\Gamma \cdot x) \cap A) = (\Lambda \cdot \Psi(x)) \cap B$$

for almost every $x \in A$.

Theorem B.1.11 ([Fur11]). *Two groups Γ and Λ are measure equivalent if and only if they are stably orbit equivalent. More precisely, we have the followings:*

- *if Γ and Λ admit stably orbit equivalent actions $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$, with subsets $A \subset X$ and $B \subset Y$ and a map $\Psi\colon A \to B$ as in the definition, then they admit a measure equivalence coupling $(\Omega, X_\Gamma, X_\Lambda, \rho)$ with $X_\Gamma = Y$, $X_\Lambda = X$, $\rho_{X_\Gamma} = \nu$ and $\rho_{X_\Lambda} = \mu$, where A is identified to B using Ψ, and we have the equality $\frac{\mu(A)}{\nu(B)} = \frac{\rho(X_\Gamma)}{\rho(X_\Lambda)}$;*
- *conversely, if $(\Omega, X_\Gamma, X_\Lambda, \rho)$ is a measure equivalence coupling between Γ and Λ, where $A = X_\Gamma \cap X_\Lambda$ has positive μ-measure, then the induced actions $\Gamma \curvearrowright (X_\Lambda, \mu)$ and $\Lambda \curvearrowright (X_\Gamma, \nu)$, with $\mu = \rho_{X_\Lambda}$ and $\nu = \rho_{X_\Gamma}$, are stably orbit equivalent, since the orbits intersected with A coincide. The stable orbit equivalence is the identity map $(A, (\mu_{X_\Lambda})_A) \to (A, (\mu_{X_\Gamma})_A)$ and we have $\frac{\mu(A)}{\nu(A)} = \frac{\rho(X_\Gamma)}{\rho(X_\Lambda)}$.*

In fact, the ratio $\mu(A)/\nu(B)$ is a rescalling, called the *compression* of the stable orbit equivalence and related to the ratio between the measures of the fundamental domains of an associated measure equivalence coupling. Considering a subset A' of A and $B' = \Psi(A') \subset B$, the restricted map $\Psi\colon (A', \mu_{A'}) \to (B', \mu_{B'})$ still defines a stable orbit equivalence between the actions, with the same compression.

When the stably orbit equivalent actions are ergodic, the following happens.

Theorem B.1.12 ([Fur99b]). *Let $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$ be free pmp actions of countable groups on standard and atomless probability spaces. Assume that these actions are ergodic and stably orbit equivalent, with subsets $A \subset X$ and $B \subset Y$ as in the definition. Then the following holds: if $\mu(A) \geqslant \nu(B)$ (resp. $\mu(A) \leqslant \nu(B)$) then there exists a stable orbit equivalence with subsets $A' \subset X$ and $B' \subset Y$ such that $A' = X$, $B \subset B'$ and $\nu(B') = \nu(B)/\mu(A)$ (resp. $B' = Y$, $A \subset A'$ and $\mu(A') = \mu(A)/\nu(B)$), so that the compression remains the same. In particular, if $\mu(A) = \nu(B)$, then the actions are in fact orbit equivalent.*

For the group $\mathbb{Z}$, Kakutani equivalence is a particular instance of stable orbit equivalence (see Section A.1.i in Appendix A for the definition). Indeed, given $T \in \mathrm{Aut}(X,\mu)$ and $S \in \mathrm{Aut}(Y,\nu)$, if there exist subsets $A \subset X$ and $B \subset Y$ of positive measure such that T_A and S_B are isomorphic, with an isomorphism $\Psi\colon (A,\mu_A) \to (B,\nu_B)$, then in particular T_A and S_B share the same orbits (up to Ψ), which are the orbits of T and S but restricted to A and B. So Ψ is a stable orbit equivalence between T and S.

Finally, if T and S are evenly Kakutani equivalent (i.e. T_A and S_B are isomorphic with $\mu(A) = \nu(B)$), then the last theorem implies that they are orbit equivalent.

B.2 Quantitative orbit/measure equivalence

By Ornstein-Weiss theorem, orbit equivalence and measure equivalence are not interesting theories among ergodic actions of infinite amenable groups. Let us strengthen their definitions, using the cocycles provided by these notions.

B.2.a Quantitative orbit/measure equivalence between finitely generated groups

Let Γ and Λ be two finitely generated groups, with finite generating subset S_Γ and S_Λ. We refer the reader to Section A.2.a in Appendix A for all the terminologies about finitely generated groups and their metric structure.

Assume that $\Gamma \curvearrowright (X,\mu)$ is a free pmp action on a standard and atomless probability space and let $c\colon \Gamma \times X \to \Lambda$ be a measurable cocycle, namely it satisfies the cocycle identity:

$$c(\gamma\gamma', x) = c(\gamma, \gamma' \cdot x) c(\gamma', x)$$

for every $\gamma, \gamma' \in \Gamma$ and for almost every $x \in X$.

Definition B.2.1 ([DKLMT22]). Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing map. We say that the cocycle $c\colon \Gamma \times X \to \Lambda$ defined above is *φ-integrable* if for every $\gamma \in \Gamma$, there exists $c_\gamma > 0$ such that

$$\int_X \varphi\left(\frac{|c(\gamma,x)|_{S_\Lambda}}{c_\gamma}\right) \mathrm{d}\mu(x) < +\infty.$$

We say that the cocycle is L^p for some $p \geqslant 0$ if it is φ-integrable for $\varphi(x) = x^p$, and it is L^∞ if for every $\gamma \in \Gamma$, the map $|c(\gamma,\cdot)|_{S_\Lambda}\colon X \to \mathbb{N}$ is essentially bounded.

For instance, every cocycle is L^0, so this assumption will simply mean that we have no requirement on the cocycle.

Remark B.2.2. The constants c_γ appearing in the definition of φ-integrability of the cocycle c are necessary because we need the following properties:

- these notions of φ-integrability and L^∞ does not depend on the choice of the finite generating set of Λ, since for any two finite generating sets S_Λ, S'_Λ of Λ, there exists a constant $C > 0$ such that

  $$\frac{1}{C}|\lambda|_{S'_\Lambda} \leqslant |\lambda|_{S_\Lambda} \leqslant C|\lambda|_{S'_\Lambda}$$

  for every $\lambda \in \Lambda$;
- if $\varphi \approx \psi$, then φ-integrability and ψ-integrability are equivalent notions;
- to prove that the cocycle $c\colon \Gamma \times X \longrightarrow \Lambda$ is φ-integrable, it suffices to check the finiteness of

  $$\int_X \varphi\left(\frac{|c(\gamma,x)|_{S_\Lambda}}{c_\gamma}\right) \mathrm{d}\mu(x)$$

  for every γ in a finite generating set of Γ. This follows from [DKLMT22, Proposition 2.22]. The same remark holds for L^∞.

We can now introduce the quantitative versions of orbit/measure equivalence. Recall that an orbit equivalence coupling (X,μ) (resp. a measure equivalence coupling $(\Omega, X_\Gamma, X_\Lambda, \mu)$) between Γ and Λ gives rise to cocycles $c_{\Gamma,\Lambda}\colon \Gamma \times X \to \Lambda$ and $c_{\Lambda,\Gamma}\colon \Lambda \times X \to \Gamma$ (resp. $c_{\Gamma,\Lambda}\colon \Gamma \times X_\Lambda \to \Lambda$ and $c_{\Lambda,\Gamma}\colon \Lambda \times X_\Gamma \to \Gamma$).

Definition B.2.3 ([DKLMT22]). Let Γ and Λ be finitely generated groups and $\varphi, \psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be non-decreasing maps. We say that there exists a *(φ,ψ)-integrable orbit equivalence coupling* (resp. *(φ,ψ)-integrable measure equivalence coupling*) from Γ to Λ if the groups admit an orbit equivalence coupling (resp. measure equivalence coupling) whose associated cocycles satisfy the following : $c_{\Gamma,\Lambda}$ is φ-integrable and $c_{\Lambda,\Gamma}$ is ψ-integrable.

A *φ-integrable orbit/measure equivalence coupling* refers to a (φ,φ)-integrable orbit/measure equivalence coupling.

We also deal with (L^∞,ψ)-integrable or $(\varphi,\mathrm{L}^\infty)$-integrable or even L^∞ orbit/measure equivalent, if we want the corresponding cocycles to be L^∞. Historically, asking for integrable cocycles was the first and most natural quantitative version of orbit/measure equivalence. Given $p \geqslant 1$, the notion of L^p orbit/measure equivalence has been introduced in [BFS13], and more generally (φ,ψ)-integrable orbit/measure equivalence was first defined in [DKLMT22] to study the weaker notion of L^p orbit equivalence for $p < 1$.

Remark B.2.4. It follows from the comments after Proposition B.1.9 that two groups are (φ,ψ)-integrably orbit equivalent if and only if they admit a (φ,ψ)-integrable measure equivalence coupling with equal fundamental domains.

B.2.b Quantitative orbit equivalence between actions

When focusing on prescribed group actions, the definitions of quantitative forms of orbit equivalence are the following.

Definition B.2.5. Let $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$ be free pmp actions of finitely generated groups on standard and atomless probability spaces. Let $\varphi,\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be non-decreasing maps. We say that $\Gamma \curvearrowright (X,\mu)$ is (φ,ψ)-*integrable orbit equivalent* to $\Lambda \curvearrowright (Y,\nu)$ if these actions admit an orbit equivalence whose cocycles satisfy the following: $c_{\Gamma,\Lambda}$ is φ-integrable and $c_{\Lambda,\Gamma}$ is ψ-integrable.

We say that these actions are φ-integrably orbit equivalent if furthermore both cocycles are φ-integrable.

Kerr and Li introduce another quantitative form of orbit equivalence, asking for the following restriction on the cocycles.

Definition B.2.6 ([KL21; KL24])**.** Given a free pmp action $\Gamma \curvearrowright (X,\mu)$ on a standard and atomless probability space, a cocycle $c\colon \Gamma \times X \to \Lambda$ is *Shannon* if for every $\gamma \in \Gamma$, the partition associated to $c(\gamma,.)\colon X \to \Lambda$ has finite entropy, namely

$$\mathrm{H}_\mu\Big(\Big\{\{c(\gamma,.)=\lambda\} \mid \lambda \in \Lambda\Big\}\Big) = -\sum_{\lambda\in\Lambda} \mu(\{c(\gamma,.)=\lambda\}) \log \mu(\{c(\gamma,.)=\lambda\}) < +\infty.$$

As for φ-integrability, we can prove that the above definition of Shannon property can be reduced to the elements γ in a finite generating subset.

Definition B.2.7 ([KL21; KL24])**.** Let $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$ be free pmp actions of finitely generated groups on standard and atomless probability spaces. We say that $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$ are *Shannon orbit equivalent* if there exists an orbit equivalence whose associated cocycles $c_{\Gamma,\Lambda}$ and $c_{\Lambda,\Gamma}$ are Shannon.

Remark B.2.8. Shannon property and φ-integrability do not exactly bring the same information. When φ is non-decreasing, φ-integrability of a cocycle at some element of the group, let us say $c_{\Gamma,\Lambda}(\gamma,.)\colon X \to \Lambda$, gives information on the tail of $|c_{\Gamma,\Lambda}(\gamma,.)|_{S_\Lambda}\colon X \to \mathbb{N}$, using Markov's inequality:

$$\mu\left(\{|c_{\Gamma,\Lambda}(\gamma,.)|_{S_\Lambda} > n\}\right) = \mu\left(\left\{\varphi\left(\frac{|c_{\Gamma,\Lambda}(\gamma,.)|_{S_\Lambda}}{c_\gamma}\right) > \varphi(n)\right\}\right) \leqslant \frac{\int_X \varphi\left(\frac{|c_{\Gamma,\Lambda}(\gamma,x)|_{S_\Lambda}}{c_\gamma}\right) \mathrm{d}\mu(x)}{\varphi(n)}.$$

On the contrary to the statistical information provided by φ-integrability, Shannon property quantifies the uncertainty of the value of $c_{\Gamma,\Lambda}(\gamma,x)$, if x is random with respect to μ (see Section A.1.h in Appendix A for the motivations behind entropy).

Finally, φ-integrability is a more geometric restriction. For every $\gamma \in S_\Gamma$, it statistically quantifies how far is $c_{\Gamma,\Lambda}(\gamma,x)$ from lying in S_Λ. Roughly speaking, if Γ is a "bigger" group than Λ (for instance $\Gamma = \mathbb{Z}^2$ and $\Lambda = \mathbb{Z}$), there is not enough room in Λ for the generators of Γ to be mapped to generators of Λ, this is the reason why quantitative forms of measure/orbit equivalence will capture the geometry of the groups.

Remark B.2.9. Shannon orbit equivalence and φ-integrably orbit equivalence between prescribed actions are not equivalence relation a priori (except when the groups are $\mathbb{Z}$ and φ is at least linear, by Belinskaya's theorem, see Section B.3).

B.2.c The particular case of $\mathbb{Z}$

In the previous section, the quantitative forms of orbit equivalence between actions we introduced do not depend on the finite generating subsets we consider, and are easy to check since we only have to prove the finiteness of finitely many integrals (the integrals associated to the generators, see Remark B.2.2). For the group $\mathbb{Z}$, we only care about the finite generating set $\{+1\}$ and, given an orbit equivalence between T and $S \in \mathrm{Aut}(X,\mu)$ (which describe $\mathbb{Z}$-actions via the action of $+1$), we only consider the cocycles on the generator $+1$. This is the reason why Carderi, Joseph, Le Maître and Tessera forget the constant c_γ in Definition B.2.1 and give the following definition for (φ,ψ)-integrable orbit equivalence.

Definition B.2.10 ([DKLMT22]). Let $S,T \in \mathrm{Aut}(X,\mu)$ and $\varphi,\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be non-decreasing maps. We say that there exists a *(φ,ψ)-integrable orbit equivalence coupling* from T to S if these transformations admit an orbit equivalence whose associated cocycles c_S and c_T satisfy

$$\int_X \varphi(|c_S(x)|)\mathrm{d}\mu(x) < \infty \text{ and } \int_X \psi(|c_T(x)|)\mathrm{d}\mu(x) < \infty,$$

where we recall that they are defined by the equations $Sx = T^{c_S(x)}x$ and $Tx = S^{c_T(x)}x$.

As well as the analogous definitions of the previous sections, L^p refers to $\varphi(x) = x^p$ and a φ-integrable orbit equivalence coupling refers to a (φ,φ)-integrable orbit equivalence coupling.

This definition of φ-integrability for cocycles is stronger than Definition B.2.1 in the case of the group $\mathbb{Z}$, so we consider the latter in every statements, except in Chapters I and II where we use Definition B.2.10.

B.3 Main results on quantitative orbit equivalence between amenable groups

Behaviour of dynamical properties of group actions under quantitative orbit equivalence

Ergodicity is preserved under orbit equivalence between free pmp actions of infinite amenable groups, and we know by Dye [Dye59] (for $\mathbb{Z}$) and Ornstein and Weiss [OW80] (for the general case) that any two such actions are orbit equivalent if they are ergodic. We would like to capture other dynamical properties, using quantitative versions of orbit equivalence.

In the case of $\mathbb{Z}$-actions, orbit equivalence is a weakening of the hard problem of conjugacy but is trivial, and we want its strengthenings to be more interesting, but still weaker than conjugacy. By Belinskaya's theorem, it is not a relevant choice to require integrable cocycles.

Theorem B.3.1 (Belinskaya [Bel69]). *Let $S,T \in \mathrm{Aut}(X,\mu)$. If there exists an orbit equivalence between S and T such that one of the cocycles is integrable, then S and T are flip-conjugate, meaning that S is conjugate to T or to T^{-1}.*

In particular, integrable orbit equivalence boils down to flip-conjugacy. This theorem fails in the case of $\mathbb{Z}^d$ for $d \geqslant 2$. Indeed, L^∞ orbit equivalence does not preserve strong mixing property.

Theorem B.3.2 (Fieldsteel, Friedman [FF86]). *Let $d \geqslant 2$. There exists a non strongly mixing free pmp $\mathbb{Z}^d$-action which is L^∞ orbit equivalent to a strongly mixing one.*

Which dynamical properties are preserved under L^1 orbit equivalence between actions of groups bigger than $\mathbb{Z}$? Austin launched the study of the preservation of measure-theoretic entropy and proved the following result.

Theorem B.3.3 (Austin [Aus16a]). *Let Γ and Λ be amenable groups and $\Gamma, \Lambda \curvearrowright (X, \mu)$ be free pmp actions on a standard and atomless probability space. If these actions are integrably orbit equivalent, then they have the same entropy.*

His result is in fact more general since it deals with the notion of stable orbit equivalence which also admits quantitative forms, we do not give more details on it.

Then Kerr and Li noticed than Austin's result also holds more generally in the context of Shannon orbit equivalence in many cases depending on algebraic properties of the groups. The typical statement is that if two pmp free actions of amenable groups Γ and Λ (with some assumptions) admit an orbit equivalence whose associated cocycle $c_{\Gamma,\Lambda} \colon \Gamma \times X \to \Lambda$ is Shannon, then $\mathrm{h}_\mu(\Gamma \curvearrowright (X, \mu)) \leqslant \mathrm{h}_\mu(\Lambda \curvearrowright (X, \mu))$. The first article [KL21] treats the case of non locally finite amenable groups Γ and Λ, where Γ is not virtually cyclic, and the second one [KL24] the case of a virtually cyclic group Γ and a virtually abelian group Λ. In fact, the content of [KL21] went beyond the amenable case using sofic entropy. In the particular case of the group $\mathbb{Z}$, Kerr and Li thus proved that entropy is invariant under Shannon orbit equivalence.

Theorem B.3.4 (Kerr, Li [KL24]). *Let $S, T \in \mathrm{Aut}(X, \mu)$. If they are Shannon orbit equivalent, then $\mathrm{h}_\mu(S) = \mathrm{h}_\mu(T)$.*

About entropy, let us mention this result of Boyle and Handelman in the topological setting. They proved that strong orbit equivalence does not preserve topological entropy.

Theorem B.3.5 (Boyle, Handelman [BH94]). *Let α be either a positive real number or $+\infty$. Let S be the dyadic odometer. Then there exists a Cantor minimal homeomorphism T such that*

1. *$\mathrm{h}_{\mathrm{top}}(T) = \alpha$;*
2. *S and T are strongly orbit equivalent.*

The dyadic odometer is any odometer on $\prod_{n \geqslant 0} \{0, \dots, q_n - 1\}$ where the integers q_n are powers of two (they are all isomorphic, see Example A.1.30 in Appendix A).

Carderi, Joseph, Le Maître and Tessera then found connections between Shannon orbit equivalence and φ-integrable orbit equivalence, and proved that the latter does not boil down to flip-conjugacy when φ is sublinear. Since integrable orbit equivalence is exactly φ-integrable orbit equivalence for any nonzero linear map φ, this implies that Belinskaya's theorem is optimal.

Theorem B.3.6 (Carderi, Joseph, Le Maître, Tessera [CJLMT23, Theorem 3.16]). *If $\varphi \colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $\log(t) = O(\varphi(t))$ as t goes to ∞, then φ-integrable orbit equivalence implies Shannon orbit equivalence.*

They in fact prove that if a measurable map $f \colon X \to \mathbb{Z}$ satisfies

$$\int_X \log |f(x)| \mathrm{d}\mu(x) < +\infty,$$

then it is Shannon:

$$-\sum_{n \in \mathbb{Z}} \mu(\{f = n\}) \log \mu(\{f = n\}) < +\infty.$$

We apply this fact to $f = c_T$ and c_S to deduce the theorem. In particular, when $\log(t) = O(\varphi(t))$, φ-integrable orbit equivalence preserves the entropy, so this is not a trivial relation. It is neither a flip-conjugacy problem, according to the following results.

The first one is asymmetric.

Theorem B.3.7 (Carderi, Joseph, Le Maître, Tessera [CJLMT23, Theorem 1.2]). *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear map and let $S \in \mathrm{Aut}(X,\mu)$ be ergodic. There exists $T \in \mathrm{Aut}(X,\mu)$ and an orbit equivalence between S and T such that the cocycle c_T is φ-integrable but T and S are not flip-conjugate.*

What about adding restrictions on both cocycles?

Theorem B.3.8 (Carderi, Joseph, Le Maître, Tessera [CJLMT23, Theorem 1.3]). *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear map and let $S \in \mathrm{Aut}(X,\mu)$ be ergodic. Assume that there exists an integer $n \geqslant 2$ such that S^n is ergodic. Then there exists $T \in \mathrm{Aut}(X,\mu)$ such that S and T are φ-integrably orbit equivalent but not flip-conjugate.*

Many comments are in order.

1. The assumption on S in Theorem B.3.8 holds for many dynamical systems, like irrational rotations, weakly mixing systems, etc; and non-examples are some odometers as explained at the end of Section A.1.c in Appendix A.

2. If S satisfies this assumption, then an explicit construction provides the counter-example T in Theorem B.3.8, it is built so that T^n is not ergodic.

3. Moreover, if S satisfies this assumption, then Theorem B.3.7 with respect to S is an immediate corollary of Theorem B.3.8. If not, then S is not weakly mixing and a "G_δ argument" provides the counter-example T in Theorem B.3.7, which is weakly mixing.

By the second comment, we know that ergodicity of non-trivial powers is not preserved under almost integrable orbit equivalence. Furthermore, we know for every prime number p that S^p is ergodic if and only if $\exp(2ip\pi)$ is not an eigenvalue of S; so this also shows that almost integrable orbit equivalence does not preserve the point spectrum. Moreover, it does not preserve weakly mixing since, starting from a weakly mixing transformation S, we can find a non weakly mixing counter-example T. By the third comment, this last remark remains true when we start from a non weakly mixing, but under the weaker relation of (φ, L^0)-integrable orbit equivalence (if φ is sublinear) and not under φ-integrable orbit equivalence.

Let us end this section with an explicit construction of orbit equivalence with explicit transformations.

Theorem B.3.9 (Kerr, Li [KL24]). *Every odometer is Shannon orbit equivalent to the universal odometer.*

We refer the reader to Example A.1.30 in Appendix A for the definition of a universal odometer. This statement provides explicit Shannon orbit equivalent transformations which do not share the same point spectrum.

Geometric properties of groups captured by quantitative orbit/measure equivalence

As we already mentioned, measure equivalence is a measured analogue of quasi-isometry. There is in fact a link between these notions. We say that a measure equivalence coupling $(\Omega, X_\Gamma, X_\Lambda, \mu)$ is *mutually cobounded* if there exist finite subsets $F_\Lambda \subset \Lambda$ and $F_\Gamma \subset \Gamma$ such that $X_\Gamma \subset F_\Lambda \cdot X_\Lambda$ and $X_\Lambda \subset F_\Gamma \cdot X_\Gamma$.

Theorem B.3.10 ([Sha04]). *Two amenable groups are quasi-isometric if and only if there exists a mutually cobounded L^∞ measure equivalence between them.*

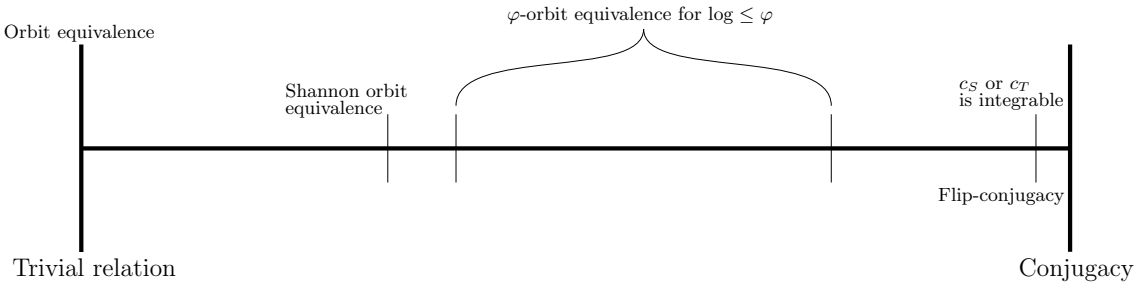


Figure B.3: Here is a schematic view of the interplay between the relations on ergodic bijections we have seen so far.

We can also relate bi-Lipschitz equivalence and orbit equivalence.

Theorem B.3.11 ([Sha04]). *Two amenable groups are bi-Lipschitz equivalent if and only if they are* L^∞ *orbit equivalent.*

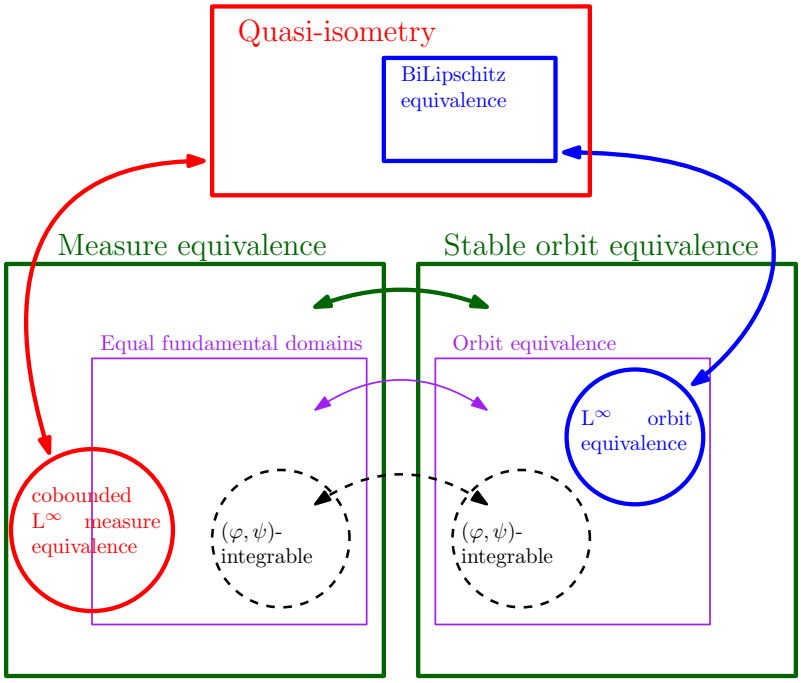


Figure B.4: Here is a schematic view of the interplay between relations on amenable groups we have seen so far.

For quantitative versions weaker than L^∞ measure/orbit equivalence, which properties of amenable groups are preserved? Many rigidity results have been uncovered in the context of L^1 measure equivalence, showing that this notion captures the geometry of groups. For non-amenable groups, we refer the reader to the work of Bader, Furman and Sauer on lattices of $\mathrm{SO}(n,1)$ [BFS13]. Focusing on amenable groups, Austin shows that integrably orbit equivalent groups of polynomial growth have bi-Lipschitz equivalent asymptotic cones [Aus16b]. Moreover, in the appendix of the aforementioned paper, Bowen proves the invariance of the growth function under L^1 measure equivalence.

Theorem B.3.12 ([Aus16b, Theorem B.2]). *Let* Γ *and* Λ *be finitely generated groups. If* Γ *and* Λ *are* L^1 *measure equivalent, then* $V_\Gamma(n) \approx V_\Lambda(n)$.

As an application, $\mathbb{Z}$ and $\mathbb{Z}^2$ are not integrably measure equivalent. It is therefore natural to wonder whether these rigidity results still hold for the more general notions

of (φ, ψ)-integrability which encompass for instance L^p for $p < 1$. In this wider setup, Delabie, Koivisto, Le Maître and Tessera refined Bowen's result as follows.

Theorem B.3.13 ([DKLMT22, Theorem 3.1]). *Let $\varphi\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be an increasing and subadditive map. If there is a (φ, L^0)-integrable measure equivalence coupling from Γ to Λ, then*

$$V_\Gamma(n) \preccurlyeq V_\Lambda(\varphi^{-1}(n))$$

where φ^{-1} is the inverse function of φ.

This inequality provides explicit upper bounds on how integrable the cocycles of an orbit equivalence coupling can be, thus implying that quantitative orbit equivalence is more complex than the trivial relation of orbit equivalence (among finitely generated amenable groups). For instance, for every integers $k > d \geqslant 1$, there is no L^p measure equivalence between $\mathbb{Z}^k$ and $\mathbb{Z}^d$ if $p > \frac{d}{k}$.

Going further, Delabie, Koivisto, Le Maître and Tessera also proved in [DKLMT22] an inequality that involves rather the isoperimetric profile, which is more suitable to distinguish amenable groups of exponential growth for instance. For recalls about the isoperimetric profile, we refer the reader to Section A.2.b in Appendix A.

Theorem B.3.14 ([DKLMT22, Theorem 1.1]). *Let $\varphi\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a non-decreasing function such that $t \longmapsto \frac{t}{\varphi(t)}$ is non-decreasing. Let Γ and Λ be finitely generated groups. Assume that there exists a (φ, L^0)-integrable measure equivalence coupling from Γ to Λ. Then their isoperimetric profiles satisfy the asymptotic inequality*

$$\varphi \circ j_{1,\Lambda}(n) \preccurlyeq j_{1,\Gamma}(n).$$

Explicit constructions show that the bound given by this theorem is almost sharp. As an example:

Theorem B.3.15 ([DKLMT22, Theorem 6.12]). *Let $k > d$ be two positive integers. Then there exists an orbit equivalence coupling from $\mathbb{Z}^k$ to $\mathbb{Z}^d$ which is $(\varphi_\varepsilon, \psi_\varepsilon)$-integrable for every $\varepsilon > 0$, where*

$$\varphi_\varepsilon(x) = \frac{x^{\frac{d}{k}}}{\ln(x)^{1+\varepsilon}} \text{ and } \psi_\varepsilon(x) = \frac{x^{\frac{k}{d}}}{\ln(x)^{1+\varepsilon}}.$$

In particular, they are L^p-orbit equivalent for every $p < \frac{d}{k}$.

They also provide results about the ℓ^p-isoperimetric profile for $p \geqslant 1$: if there exists an $(\mathrm{L}^p, \mathrm{L}^0)$ measure equivalence from Γ to Λ, then $j_{p,\Lambda}(n) \preccurlyeq j_{p,\Gamma}(n)$. It is worth noticing that they find a connection (in the same vein as Theorems B.3.10 and B.3.11) between the existence of regular maps and quantitative forms of asymmetric version of measure equivalence, which enables them to prove the following: if there exists a regular map from Γ to Λ, then $j_{p,\Lambda}(n) \preccurlyeq j_{p,\Gamma}(n)$ for every $p \geqslant 1$.

Bibliography

[Aus16a] Tim Austin. "Behaviour of Entropy Under Bounded and Integrable Orbit Equivalence". In: *Geometric and Functional Analysis* 26.6 (2016), pp. 1483–1525. ISSN: 1016-443X, 1420-8970. DOI: 10.1007/s00039-016-0392-5.

[Aus16b] Tim Austin. "Integrable Measure Equivalence for Groups of Polynomial Growth". In: *Groups, Geometry, and Dynamics* 10.1 (2016), pp. 117–154. ISSN: 1661-7207. DOI: 10.4171/GGD/345.

[BFS13] Uri Bader, Alex Furman, and Roman Sauer. "Integrable Measure Equivalence and Rigidity of Hyperbolic Lattices". In: *Inventiones mathematicae* 194.2 (2013), pp. 313–379. DOI: 10.1007/s00222-012-0445-9.

[Bel69] R. M. Belinskaya. "Partitions of Lebesgue Space in Trajectories Defined by Ergodic Automorphisms". In: *Functional Analysis and Its Applications* 2.3 (1969), pp. 190–199. ISSN: 0016-2663, 1573-8485. DOI: 10.1007/BF01076120.

[Boy83] M. Boyle. "Topological Orbit Equivalence and Factor Maps in Symbolic Dynamics". PhD thesis. University of Wahington, 1983.

[BH94] McBlaine Boyle and David Handelman. "Entropy versus Orbit Equivalence for Minimal Homeomorphisms". In: *Pacific Journal of Mathematics* 164.1 (1994), pp. 1–13. ISSN: 0030-8730, 0030-8730. DOI: 10.2140/pjm.1994.164.1.

[BZ21] Jérémie Brieussel and Tianyi Zheng. "Speed of Random Walks, Isoperimetry and Compression of Finitely Generated Groups". In: *Annals of Mathematics* 193.1 (2021), pp. 1–105. ISSN: 0003-486X, 1939-8980. DOI: 10.4007/annals.2021.193.1.1.

[BO08] N. P. Brown and N. Ozawa. *C*-Algebras and Finite-Dimensional Approximations*. Vol. 88. Graduate Studies in Mathematics. American Mathematical Society, 2008. ISBN: 978-0-8218-4381-9.

[CJLMT23] Alessandro Carderi, Matthieu Joseph, François Le Maître, and Romain Tessera. "Belinskaya's Theorem Is Optimal". In: *Fundamenta Mathematicae* 263.1 (2023), pp. 51–90. ISSN: 0016-2736, 1730-6329. DOI: 10.4064/fm266-4-2023.

[Cha69] R. V. Chacon. "Weakly Mixing Transformations Which Are Not Strongly Mixing". In: *Proceedings of the American Mathematical Society* 22.3 (1969), pp. 559–562. ISSN: 0002-9939, 1088-6826. DOI: 10.1090/S0002-9939-1969-0247028-5.

[Cor25b] Corentin Correia. *Odomutants and Flexibility Results for Quantitative Orbit Equivalence*. 2025. DOI: 10.48550/ARXIV.2504.02021.

[Cor25c] Corentin Correia. "On the Absence of Quantitatively Critical Measure Equivalence Couplings". In: *Proceedings of the American Mathematical Society* (2025). DOI: 10.1090/proc/17291.

[Cor25a] Corentin Correia. "Rank-One Systems, Flexible Classes and Shannon Orbit Equivalence". In: *Ergodic Theory and Dynamical Systems* 45.7 (2025), pp. 2132–2182. DOI: 10.1017/etds.2024.118.

[Cou00] Thierry Coulhon. "Random Walks and Geometry on Infinite Graphs". In: *Lecture Notes on Analysis in Metric Spaces* (2000), pp. 5–30.

[CS93] Thierry Coulhon and Laurent Saloff-Coste. "Isopérimétrie pour les groupes et les variétés." In: *Revista Matemática Iberoamericana* 9.2 (1993), pp. 293–314. ISSN: 0213-2230.

[DV23] Alexandre I. Danilenko and Mykyta I. Vieprik. "Explicit Rank-1 Constructions for Irrational Rotations". In: *Studia Mathematica* 270.2 (2023), pp. 121–144. ISSN: 0039-3223, 1730-6337. DOI: 10.4064/sm220214-13-10.

[DKLMT22] T. Delabie, J. Koivisto, F. Le Maître, and R. Tessera. "Quantitative Measure Equivalence between Amenable Groups". In: *Annales Henri Lebesgue* 5 (2022), pp. 1417–1487. ISSN: 2644-9463. DOI: 10.5802/ahl.155.

[Dow11] T. Downarowicz. *Entropy in Dynamical Systems*. New Mathematical Monographs 18. Cambridge ; New York: Cambridge University Press, 2011. ISBN: 978-0-521-88885-1.

[DEJLMS23] Hindy Drillick, Alonso Espinosa-Dominguez, Jennifer N. Jones-Baro, James Leng, Yelena Mandelshtam, and Cesar E. Silva. "Non-Rigid Rank-One Infinite Measures on the Circle". In: *Dynamical Systems* 38.2 (2023), pp. 275–300. ISSN: 1468-9367. DOI: 10.1080/14689367.2023.2174412.

[Dye59] H. A. Dye. "On Groups of Measure Preserving Transformations. I". In: *American Journal of Mathematics* 81.1 (1959), pp. 119–159. ISSN: 0002-9327. DOI: 10.2307/2372852.

[EW11] Manfred Einsiedler and Thomas Ward. *Ergodic Theory: With a View towards Number Theory*. London: Springer London, 2011. ISBN: 978-0-85729-020-5 978-0-85729-021-2. DOI: 10.1007/978-0-85729-021-2.

[Ers03] Anna Erschler. "On Isoperimetric Profiles of Finitely Generated Groups". In: *Geometriae Dedicata* 100.1 (2003), pp. 157–171. ISSN: 1572-9168. DOI: 10.1023/A:1025849602376.

[Esc24] Amandine Escalier. "Building Prescribed Quantitative Orbit Equivalence with the Integers". In: *Groups, Geometry, and Dynamics* (2024). ISSN: 1661-7207, 1661-7215. DOI: 10.4171/ggd/766.

[Fel76] J. Feldman. "New K-automorphisms and a Problem of Kakutani". In: *Israel Journal of Mathematics* 24.1 (1976), pp. 16–38. ISSN: 0021-2172, 1565-8511. DOI: 10.1007/BF02761426.

[Fer97] Sébastien Ferenczi. "Systems of Finite Rank". In: *Colloquium Mathematicum* 73.1 (1997), pp. 35–65. ISSN: 0010-1354, 1730-6302. DOI: 10.4064/cm-73-1-35-65.

[FF86] Adam Fieldsteel and N. A. Friedman. "Restricted Orbit Changes of Ergodic $\mathbb{Z}^d$-Actions to Achieve Mixing and Completely Positive Entropy". In: *Ergodic Theory and Dynamical Systems* 6.4 (1986), pp. 505–528. DOI: 10.1017/S0143385700003667.

[FRW11] Matthew Foreman, Daniel Rudolph, and Benjamin Weiss. "The Conjugacy Problem in Ergodic Theory". In: *Annals of Mathematics* 173.3 (2011), pp. 1529–1586. ISSN: 0003-486X. DOI: 10.4007/annals.2011.173.3.7.

[Fur11] A. Furman. "A Survey of Measured Group Theory". In: *Geometry, Rigidity, and Group Actions*. Chicago and London: The University of Chicago Press, 2011, pp. 296–374. ISBN: 978-0-226-23788-6.

[Fur99a] Alex Furman. "Gromov's Measure Equivalence and Rigidity of Higher Rank Lattices". In: *The Annals of Mathematics* 150.3 (1999), p. 1059. DOI: 10.2307/121062. JSTOR: 121062.

[Fur99b] Alex Furman. "Orbit Equivalence Rigidity". In: *The Annals of Mathematics* 150.3 (1999), p. 1083. DOI: 10.2307/121063. JSTOR: 121063.

[Gab00] Damien Gaboriau. "Coût des relations d'équivalence et des groupes". In: *Inventiones mathematicae* 139.1 (2000), pp. 41–98. DOI: 10.1007/s002229900019.

[GZ19] Su Gao and Caleb Ziegler. "Topological Mixing Properties of Rank-One Subshifts". In: *Transactions of the London Mathematical Society* 6.1 (2019), pp. 1–21. ISSN: 2052-4986. DOI: 10.1112/tlm3.12016.

[GPS95] T. Giordano, I. F. Putnam, and V. & Ch F. Skau. "Topological orbit equivalence and C*-crossed products." In: *Journal für die reine und angewandte Mathematik* 469 (1995), pp. 51–112. ISSN: 0075-4102; 1435-5345/e.

[Gla03] E. Glasner. *Ergodic Theory via Joinings*. Vol. 101. Mathematical Surveys and Monographs. Providence, Rhode Island: American Mathematical Society, 2003. ISBN: 978-0-8218-3372-8 978-1-4704-1328-6. DOI: 10.1090/surv/101.

[Gro81] Michael Gromov. "Groups of polynomial growth and expanding maps (with an appendix by Jacques Tits)". In: *Publications Mathématiques de l'IHÉS* 53 (1981), pp. 53–78.

[HP68] Frank Hahn and William Parry. "Some Characteristic Properties of Dynamical Systems with Quasi-Discrete Spectra". In: *Mathematical systems theory* 2.2 (1968), pp. 179–190. ISSN: 1433-0490. DOI: 10.1007/BF01692514.

[HVN42] Paul R. Halmos and John Von Neumann. "Operator Methods in Classical Mechanics, II". In: *The Annals of Mathematics* 43.2 (1942), pp. 332–350. ISSN: 0003486X. DOI: 10.2307/1968872.

[Hal56] Paul Richard Halmos. *Lectures on Ergodic Theory*. American Mathematical Soc., 1956. ISBN: 978-0-8218-4125-9.

[HPS92] R. Herman, I. Putnam, and Christian F. Skau. "Ordered Bratteli Diagrams, Dimension Groups and Topological Dynamics". In: *International Journal of Mathematics* (1992).

[IT16] Adrian Ioana and Robin Tucker-Drob. "Weak Containment Rigidity for Distal Actions". In: *Advances in Mathematics* 302 (2016), pp. 309–322. ISSN: 00018708. DOI: 10.1016/j.aim.2016.07.024.

[Jun76] Andrés Del Junco. "Transformations With Discrete Spectrum Are Stacking Transformations". In: *Canadian Journal of Mathematics* 28.4 (1976), pp. 836–839. ISSN: 0008-414X, 1496-4279. DOI: 10.4153/CJM-1976-080-3.

[KL16] David Kerr and Hanfeng Li. *Ergodic Theory: Independence and Dichotomies*. Springer Monographs in Mathematics. Cham: Springer International Publishing, 2016. ISBN: 978-3-319-49845-4 978-3-319-49847-8. DOI: 10.1007/978-3-319-49847-8.

[KL21] David Kerr and Hanfeng Li. "Entropy, Shannon Orbit Equivalence, and Sparse Connectivity". In: *Mathematische Annalen* 380.3 (2021), pp. 1497–1562. ISSN: 1432-1807. DOI: 10.1007/s00208-021-02190-x.

[KL24] David Kerr and Hanfeng Li. "Entropy, Virtual Abelianness and Shannon Orbit Equivalence". In: *Ergodic Theory and Dynamical Systems* (2024), pp. 1–20. ISSN: 0143-3857, 1469-4417. DOI: 10.1017/etds.2024.26.

[Kid08] Yoshikata Kida. "The Mapping Class Group from the Viewpoint of Measure Equivalence Theory". In: *Memoirs of the American Mathematical Society* 196.916 (2008), pp. 0–0. DOI: 10.1090/memo/0916.

[Kid10] Yoshikata Kida. "Measure Equivalence Rigidity of the Mapping Class Group". In: *Annals of Mathematics* 171.3 (2010), pp. 1851–1901. DOI: 10.4007/annals.2010.171.1851.

[Man11] Avinoam Mann. *How Groups Grow*. London Mathematical Society Lecture Note Series. Cambridge: Cambridge University Press, 2011. ISBN: 978-1-107-65750-2. DOI: 10.1017/CBO9781139095129.

[NP25] Petr Naryshkin and Spyridon Petrakos. *Quantitative Orbit Equivalence for $\mathbb{Z}$-Odometers*. 2025. DOI: 10.48550/ARXIV.2510.16749.

[New71] D. Newton. "Coalescence and Spectrum of Automorphisms of a Lebesgue Space". In: *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* 19.2 (1971), pp. 117–122. ISSN: 1432-2064. DOI: 10.1007/BF00536902.

[Orn70] Donald Ornstein. "Bernoulli Shifts with the Same Entropy Are Isomorphic". In: *Advances in Mathematics* 4.3 (1970), pp. 337–352. ISSN: 00018708. DOI: 10.1016/0001-8708(70)90029-0.

[Orn72] Donald S. Ornstein. "On the Root Problem in Ergodic Theory". In: *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, Volume 2: Probability Theory*. Vol. 6.2. Berkeley Symp. on Math. Statist. and Prob. Berkeley: University of California Press, 1972, pp. 347–357.

[ORW82] Donald S. Ornstein, Daniel J. Rudolph, and Benjamin Weiss. "Equivalence of Measure Preserving Transformations". In: *Memoirs of the American Mathematical Society* 37.262 (1982), xii+116 pp. ISSN: 0065-9266, 1947-6221. DOI: 10.1090/memo/0262.

[OW80] Donald S. Ornstein and Benjamin Weiss. "Ergodic Theory of Amenable Group Actions. I: The Rohlin Lemma". In: *Bulletin (New Series) of the American Mathematical Society* 2.1 (1980), pp. 161–164. ISSN: 0273-0979, 1088-9485.

[OW87] Donald S. Ornstein and Benjamin Weiss. "Entropy and Isomorphism Theorems for Actions of Amenable Groups". In: *Journal d'Analyse Mathématique* 48.1 (1987), pp. 1–141. ISSN: 0021-7670, 1565-8538. DOI: 10.1007/BF02790325.

[Put89] Ian Putnam. "The C^*-Algebras Associated with Minimal Homeomorphisms of the Cantor Set". In: *Pacific Journal of Mathematics* 136.2 (1989), pp. 329–353. ISSN: 0030-8730, 0030-8730. DOI: 10.2140/pjm.1989.136.329.

[SZ15] Laurent Saloff-Coste and Tianyi Zheng. "Random Walks on Free Solvable Groups". In: *Mathematische Zeitschrift* 279.3-4 (2015), pp. 811–848. DOI: 10.1007/s00209-014-1395-2.

[SZ16] Laurent Saloff-Coste and Tianyi Zheng. "Random Walks and Isoperimetric Profiles under Moment Conditions". In: *The Annals of Probability* 44.6 (2016). DOI: 10.1214/15-AOP1070.

[SZ18] Laurent Saloff-Coste and Tianyi Zheng. "Isoperimetric Profiles and Random Walks on Some Permutation Wreath Products". In: *Revista Matemática Iberoamericana* 34.2 (2018), pp. 481–540. DOI: 10.4171/rmi/994.

[Sha04] Yehuda Shalom. "Harmonic Analysis, Cohomology, and the Large-Scale Geometry of Amenable Groups". In: *Acta Mathematica* 192.2 (2004), pp. 119–185. DOI: 10.1007/BF02392739.

[VO16] Marcelo Viana and Krerley Oliveira. *Foundations of Ergodic Theory*. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 2016. ISBN: 978-1-107-12696-1. DOI: 10.1017/CBO9781316422601.

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