Université de Paris

GEOMETRIC GROUP THEORY ON THE COMMENSURATING SYMMETRIC GROUP

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Contents

Introduction

We study in this thesis two Polish groups S_∞ and $\mathcal{S}(\mathbb{Z},\mathbb{N})$.

Definition 1. *A* Polish group *is a topological group, i.e. a group* (G,τ) *such that*

(i) τ *is an Hausdorff topology,*

(ii) the map from $G\times G$ *to* G *that sends* (g,h) *to* gh *is continuous,*

(iii) the map from G *to* G *that sends* g *to* g^{-1} *is continuous,*

whose topology is Polish, i.e. its topology admits a complete compatible metric and is separable which means that there exists a countably dense subset.

S_∞ is the infinite symmetric group and $\mathcal{S}(\mathbb{Z},\mathbb{N})$ is the group of permutations of $\mathbb{Z}$ commensurating $\mathbb{N}$, i.e. the group of $\sigma\in\mathrm{Sym}(\mathbb{Z})$ such that

$$|\mathbb{N}\bigtriangleup\sigma\mathbb{N}|<+\infty.$$

S_∞ has been studied in various papers especially by George M. Bergman in [Ber06]. Indeed in this paper, Bergman discovers that S_∞ has the Bergman property.

Definition 2. *A group* G *has the* Bergman property *or is* Bergman *if whenever* $W_0\subseteq W_1\subseteq\ldots\subseteq G=\bigcup_n W_n$*, there are* n *and* k *such that* $G=W_n^k$.

Thanks to the work done by Christian Rosendal in [Ros09], we are able to prove that a group G has the Bergman property implies that G has property (OB) which is a geometric property. This result holds mainly due to Theorem 4.

Definition 3. *A topological group* G *has* property (OB) *if whenever* G *acts by continuous isometries on a metric space (X,d), then every orbit is bounded.*

Theorem 4. *The following are equivalent for a group G:*

(i) Whenever G *acts by isometries on a metric space (X,d), every orbit is bounded;*

(ii) Any left-invariant metric on G *is bounded;*

(iii) G *has the Bergman property.*

S_∞ is bounded whereas $\mathcal{S}(\mathbb{Z},\mathbb{N})$ is not. Thus $\mathcal{S}(\mathbb{Z},\mathbb{N})$ cannot have the property (OB). But we will show that $\mathcal{S}(\mathbb{Z},\mathbb{N})$ is locally bounded.

Definition 5. *A topological group* G *is* locally bounded *if and only if it has a coarsely bounded identity neighborhood.*

Following Rosendal, our aim is to show that $\mathcal{S}(\mathbb{Z},\mathbb{N})$ admits a left-invariant pseudometric which solely depends on its group topology, and which is well-defined up to quasi-isometry. Such a pseudometric on $\mathcal{S}(\mathbb{Z},\mathbb{N})$ is called maximal. This is a generalization for finitely generated groups with the word metric, with respect to a finite generating set, as a maximal metric.

Definition 6. *Let* (X,d_X) *and* (Y,d_Y) *be pseudometric spaces.*
A map $\Phi: X\mapsto Y$ *is said to be a* quasi-isometric embedding *if there are positive constants K, C such that*

$$\frac{1}{K}\cdot d_X(x_1,x_2)-C\leqslant d_Y(\Phi(x_1),\Phi(x_2))\leqslant K\cdot d_X(x_1,x_2)+C.$$

Also Φ *is a* quasi-isometry *if, moreover there exists a positive constant* C *that for any* $y \in Y$*, there exists* $x \in X$ *such that*

$$d_Y(\Phi(x), y) \leqslant C.$$

$\mathcal{S}(\mathbb{Z}, \mathbb{N})$ already admits a pseudometric which is defined by Yves De Cornulier in [Cor16].

Definition 7. *For* $g, h \in \mathcal{S}(\mathbb{Z}, \mathbb{N})$*,* $d_{\mathbb{N}}(g, h) = |g_{\mathbb{N}} \triangle h_{\mathbb{N}}|$ *is a left-invariant pseudometric on* $\mathcal{S}(\mathbb{Z}, \mathbb{N})$*.*

So we show that this pseudometric is maximal on $\mathcal{S}(\mathbb{Z}, \mathbb{N})$ thanks to the following proposition.

Proposition 8. *For a continuous left-invariant pseudometric* d *on a topological group* G*, the following are equivalent:*

(i) *d is maximal;*
(ii) *d is coarsely proper and (G,d) is large scale geodesic;*
(iii) *d is quasi-isometric to the word metric* ρ_A *given by a coarsely bounded symmetric generating set* $A \subseteq G$*.*

To show this, we need a characterization of the notion of being locally bounded which is done by Rosendal in [Ros]. We will show that $\mathcal{S}(\mathbb{Z}, \mathbb{N})$ is locally bounded and even generated by a coarsely bounded set. The fact that $\mathcal{S}(\mathbb{Z}, \mathbb{N})$ is locally bounded is then a corollary of the following result:

Theorem 9. *For a European topological group* G*, the following are equivalent:*

(i) *G admits a continuous left-invariant maximal pseudometric d;*
(ii) *G is generated by a coarsely bounded set;*
(iii) *G is locally bounded and not the union of a countable chain of proper open subgroups;*
(iv) *the coarse structure is monogenic.*

We finally show that for every $k \geqslant 1$, the group $\mathbb{Z}^k$ embeds into $\mathcal{S}(\mathbb{Z}, \mathbb{N})$ isometrically for its natural word metric. This shows that $\mathcal{S}(\mathbb{Z}, \mathbb{N})$ has infinite asymptotic dimension, although we did not have the time to consider this notion in details (see for instance the third section of [BD08]).

Let us finally present the plan of this thesis. In the first section we show that S_∞ has property (OB). We then present some basic results on the commensurating symmetric group $\mathcal{S}(\mathbb{Z}, \mathbb{N})$ which will be needed later on. In the third section, we prove Theorem 9. Finally in the fourth section, we prove that $d_{\mathbb{N}}$ is maximal and in the fifth section, we show that $\mathbb{Z}^k$ is isometrically embedded into $\mathcal{S}(\mathbb{Z}, \mathbb{N})$.

1 S_∞ has property (OB)

S_∞ is the symmetric group of $\mathbb{N}$. It is a Polish group. In this section, we will prove that any symmetric group of an infinite group has property (OB) using the Bergman property. Indeed being Bergman is stronger than having the property (OB). In particular, S_∞ has property (OB). This section is mainly from [Ber06].

1.1 Definitions and structure of the proof

H stands for the symmetric group Ω which is an infinite set, i.e. $H = \mathrm{Sym}(\Omega)$. First we will clarify some basic definitions that will be needed throughout the proof.

Definition 10. *A subset $\Delta \subseteq \Omega$ is a* moiety *if $|\Delta| = |\Omega| = |\Omega \setminus \Delta|$.*

Notation 11. *For subsets $\Delta \subseteq \Omega$ and $U \subseteq H$, $U_{(\Delta)}$ denotes the set of elements of U that stabilizes Δ pointwise.*

Definition 12. *For two sets A, B, we say that A and B are* commensurated, *and we write $A \sim B$ if, $|A \bigtriangleup B| < \infty$. Here $\bigtriangleup$ is the symmetric difference. Notice that $\sim$ is an equivalence relation.*

Definition 13. *For Ω an infinite set, an element $\sigma \in Sym(\Omega)$ is* replete *if it has $|\Omega|$ orbits of each positive cardinality smaller than $\aleph_0$. For a subset $\Delta \subseteq \Omega$ of cardinality $|\Omega|$, σ is* replete on Δ *if $\sigma\Delta = \Delta$ and the restriction of σ to Δ is a replete permutation of Δ.*

Definition 14. *A group G has the* Bergman property *or is* Bergman *if whenever $W_0 \subseteq W_1 \subseteq ... \subseteq G = \bigcup_n W_n$, there are n and k such that $G = W_n^k$.*

Definition 15. *A group G* acts by continuous isometries *on a metric space (X,d) if for all $x \in X$ the function from G to X, $g \mapsto gx$ is continuous.*

Definition 16. *A topological group G has* property (OB) *if whenever G acts by continuous isometries on a metric space (X,d), then every orbit is bounded.*

To complete the proof that S_∞ has property (OB), we will show in section 1.2 that it has uncountable cofinality and is Cayley-bounded, and then we will show and use the following connections between properties of a topological group G.

G is Cayley-bounded and has uncountable cofinality,
$\Leftrightarrow$ G has the Bergman property (Section 1.3),
$\Leftrightarrow$ whenever G acts by isometries on a metric space (X, d), every orbit is bounded (Section 1.4),
$\Rightarrow$ G has property (OB) (Section 1.5).

1.2 S_∞ is Cayley-bounded and has uncountable cofinality

We recall that H stands for the symmetric group of Ω, an infinite set. We are going to show that any H is Cayley-bounded and has uncountable cofinality. To show these two properties, we need two theorems from [Ber06]:

Theorem 23. *If U generates H then there exists n such that $H \subseteq (U \cup U^{-1})^n$. We say that H is Cayley-bounded.*

Theorem 24. *Let $(H_n)_{n\in I}$ be a chain of subgroups of H with $|I| \leqslant |\Omega|$ such that*

$$H = \bigcup_{n\in I} H_n.$$

Then there exists n such that $H = H_n$. We say that H has uncountable cofinality.

To be able to prove these theorems, a few lemmas are required.

Lemma 17. *For every permutation σ of Ω, there exists two replete permutations σ_1, σ_2 such that $\sigma = \sigma_1\sigma_2$.*

Proof. For σ a permutation, we choose Δ_0 a moiety of Ω such that σ moves finitely many elements from Δ_0 to $\Omega \setminus \Delta_0$ or from $\Omega \setminus \Delta_0$ to Δ_0. If Ω is uncountable, there exists such a Δ_0 because Ω contains $|\Omega|$ orbits, and thus can be written as the disjoint union of two sets, each of which contains $|\Omega|$ orbits. We can thus define Δ_0 as one of theses two sets.
If Ω is countable:

* If σ has infinitely many orbits or if σ has more than one infinite orbit, then we do the same as above.
* If σ has exactly one infinite orbit $\alpha\langle\sigma\rangle$ and finitely many finite orbits, then we take $\Delta_0 = \{\alpha\sigma^n : n \geqslant 0\}$. We can see that σ moves one element out of Δ_0 and none into it.

Now we split $\Omega \setminus \Delta_0$ into two disjoint moieties Δ_1 and Δ_2 such that

$$(\sigma\Delta_0 \cup \sigma^{-1}\Delta_0) \setminus \Delta_0 \subseteq \Delta_1.$$

We claim that for any permutation τ_0 of Δ_0 and any permutation τ_2 of Δ_2, there exists a permutation ρ of Ω such that: $\sigma\rho|_{\Delta_0} = \tau_0$ and $\rho|_{\Delta_2} = \tau_2$.
Indeed, suppose τ_0 and τ_2 as above. Thanks to the two conditions, the values of ρ are specified on $\sigma\Delta_0$ and on Δ_2. ρ has not yet been defined on $\Omega \setminus (\sigma\Delta_0 \cup \Delta_2)$. Since $\Delta_0 \sim \sigma\Delta_0$, we have

$$\Delta_0 \cup \Delta_2 \sim \sigma\Delta_0 \cup \Delta_2.$$

By taking the complement of the latter and since $\Omega = \Delta_0 \sqcup \Delta_1 \sqcup \Delta_2$, we have

$$\Omega \setminus (\sigma\Delta_0 \cup \Delta_2) \sim \Delta_1.$$

Since $|\Delta_1| = |\Omega|$, we also have $|\Omega \setminus (\sigma\Delta_0 \cup \Delta_2)| = |\Omega|$.
Now we look at the set in the image of ρ which has not been defined. Indeed the set of values for ρ that has not been specified is

$$\Omega \setminus (\tau_0\sigma\Delta_0 \cup \tau_2\Delta_2).$$

Since τ_0 and τ_2 are permutations, they are bijections then $|\tau_0\sigma\Delta_0| = |\sigma\Delta_0|$ and $|\tau_2\Delta_2| = |\Delta_2|$. Moreover $\sigma\Delta_0$ is equal to Δ_0. Hence

$$\tau_0\sigma\Delta_0 \cup \tau_2\Delta_2 = \sigma\Delta_0 \cup \Delta_2 \sim \Delta_0 \cup \Delta_2.$$

Since $\Omega \setminus (\Delta_0 \cup \Delta_2) = \Delta_1$ which is infinite, the set

$$\Omega \setminus (\tau_0 \sigma \Delta_0 \cup \tau_2 \Delta_2)$$

is also infinite. Hence

$$\Omega \setminus (\tau_0 \sigma \Delta_0 \cup \tau_2 \Delta_2)$$

can be mapped bijectively into $\Omega \setminus (\sigma\Delta_0 \cup \Delta_2)$. We call this bijection ρ which is then well-defined on Ω.

Now we take two replete permutations for τ_0 and τ_2, then ρ is replete on Δ_2 and $\sigma\rho$ is replete on Δ_0 so they are both replete. Then $\sigma = (\sigma\rho)\rho^{-1}$ is the product of two replete permutations which is what we wanted to show. $\square$

Lemma 18. *For $\sigma \in H$, there exists $\tau_1, \tau_2 \in H$ such that*

$$\sigma = \tau_1^{-1} \tau_2^{-1} \tau_1 \tau_2.$$

Then any element of H is a commutator.

Proof. By Lemma 17, there exists σ_1, σ_2 two replete permutations such that $\sigma = \sigma_1 \sigma_2$.
Since any two permutations σ_1, σ_2 in H are conjugate if and only if for any $n = 1, ..., \aleph_0$, σ_1 and σ_2 have the same number of orbits of cardinal n, then σ_1 is conjugate of σ_2. Moreover for any $\sigma \in H$, σ is conjugate to its inverse. So σ_1 is conjugate of σ_2^{-1}. Indeed there exists $\rho \in H$ such that $\sigma_1 = \rho \sigma_2^{-1} \rho^{-1}$.
Since, $\sigma = \sigma_1 \sigma_2$, $\sigma = \rho \sigma_2^{-1} \rho^{-1} \sigma_2$. In particular, we have $\tau_1 = \rho^{-1}$ and $\tau_2 = \sigma_2$. $\square$

Definition 19. *A subset $\Delta \subseteq \Omega$ is* full with respect to *a subset $U \subseteq H$ if the set of permutations of Δ induced by members of $U_{\{\Delta\}} := \{\sigma \in U : \sigma\Delta = \Delta\}$ is all of $Sym(\Delta)$.*

Lemma 20. *Let Δ_1 and Δ_2 be moieties of Ω such that $\Delta_1 \cap \Delta_2$ is also a moiety and $\Delta_1 \cup \Delta_2 = \Omega$. Let $U, V \subseteq H$. If U and V closed under inverses such that Δ_1 is full with respect to U and Δ_2 is full with respect to V, then*

$$H = (UV)^4 V \cup (VU)^4 U.$$

Proof. First we notice that the set of elements of H that stabilize $\Omega \setminus (\Delta_1 \cap \Delta_2)$ pointwise

$$H_{(\Omega \setminus (\Delta_1 \cap \Delta_2))} = \{u \in H \mid u.s = s \ \forall s \in \Omega \setminus (\Delta_1 \cap \Delta_2)\}$$

is isomorphic to $\mathrm{Sym}(\Delta_1 \cap \Delta_2)$ thanks to the following isomorphism

$$\begin{array}{ccc} H_{(\Omega \setminus (\Delta_1 \cap \Delta_2))} & \to & H_{(\Delta_1 \cap \Delta_2)} \\ g & \mapsto & g_{|\Delta_1 \cap \Delta_2} \end{array}$$

which is injective. By Lemma 18, every element $\sigma \in H$ can be written as a commutator: $\sigma = \tau_1^{-1} \tau_2^{-1} \tau_1 \tau_2$ with $\tau_1, \tau_2 \in H$.

We assumed that Δ_1 is full with respect to U, so we can find an element $\rho \in U_{\{\Delta_1\}}$ such that $\rho|_{\Delta_1 \cap \Delta_2}$ acts like τ_1 and $\rho|_{\Delta_1 \setminus \Delta_2} = \mathrm{id}$. Similarly, we can find an element $\gamma \in V_{\{\Delta_2\}}$ such that $\gamma|_{\Delta_1 \cap \Delta_2}$ acts like τ_2 and $\gamma|_{\Delta_2 \setminus \Delta_1} = \mathrm{id}$.

So the commutator $\rho^{-1}\gamma^{-1}\rho\gamma$ acts like σ on $\Delta_1 \cap \Delta_2$ and is the identity on $\Omega \setminus (\Delta_1 \cap \Delta_2)$. Hence

$$\rho^{-1}\gamma^{-1}\rho\gamma \in H_{(\Omega\setminus(\Delta_1\cap\Delta_2))}.$$

So we have:

$$H_{(\Omega\setminus(\Delta_1\cap\Delta_2))} \subseteq U^{-1}V^{-1}UV = UVUV,$$

since U and V are closed under inverses.

$\Delta_1 \cap \Delta_2$ is a moiety so $|\Delta_1 \cap \Delta_2| = |\Omega|$. Moreover Δ_1 is a moiety so

$$|\Delta_1| = |\Omega \setminus (\Delta_2 \setminus \Delta_1)| = |\Delta_2 \setminus \Delta_1| = |\Omega|.$$

So $\Delta_1 \cap \Delta_2$ and $\Delta_2 \setminus \Delta_1$ are of cardinality $|\Omega|$. Hence we can find an element λ of $\mathrm{Sym}(\Delta_2)$ that interchanges the two sets. Since Δ_2 is full with respect to V, λ is actually in V. We now have:

$$\lambda^{-1}H_{(\Omega\setminus(\Delta_1\cap\Delta_2))}\lambda \subseteq \lambda^{-1}UVUV\lambda \subseteq V^{-1}UVUVV = VUVUVV. \tag{1}$$

Since λ is interchanging $\Delta_1 \cap \Delta_2$ with $\Delta_2 \setminus \Delta_1$,

$$\lambda^{-1}H_{(\Omega\setminus(\Delta_1\cap\Delta_2))}\lambda = H_{(\Omega\setminus(\Delta_1\cap\Delta_2)\lambda)} = H_{(\Omega\setminus(\Delta_2\setminus\Delta_1))} = H_{(\Delta_1)}. \tag{2}$$

Combining the two equations 1 and 2, we get

$$H_{(\Delta_1)} = \lambda^{-1}H_{(\Omega\setminus(\Delta_1\cap\Delta_2))}\lambda \subseteq VUVUVV. \tag{3}$$

Since we could exchange Δ_1 and Δ_2 in the previous reasoning, we then have a similar result for $H_{(\Delta_2)}$:

$$H_{(\Delta_2)} \subseteq UVUVUU.$$

Let $\sigma \in H$. We notice that $\sigma^{-1}(\Delta_1 \cap \Delta_2)$ has either $|\Omega|$ elements of Δ_1 or $|\Omega|$ elements of Δ_2. Without loss of generality, we assume that $\sigma^{-1}(\Delta_1 \cap \Delta_2)$ has $|\Omega|$ elements from Δ_1. So in particular, $\sigma^{-1}\Delta_1$ has $|\Omega|$ elements of Δ_1 since it cannot have more than $|\Omega|$. Since Δ_1 is full with respect to U, we can find a permutation $\delta \in U_{\{\Delta_1\}}$ such that all the elements of $\Delta_1 \setminus \sigma^{-1}\Delta_1$ are mapped into $\Delta_1 \cap \Delta_2$ and the $|\Omega|$ elements of $\Delta_1 \cap \sigma^{-1}\Delta_1$ into $\Delta_1 \cap \Delta_2$. Moreover δ maps all the elements of $\Omega \setminus \Delta_1$ onto itself. Then δ maps all elements of

$$\sigma^{-1}(\Omega \setminus \Delta_1) = \sigma^{-1}\Delta_2$$

into Δ_2.

We want to find $\theta \in V_{\{\Delta_2\}}$ such that

$$\theta\delta\sigma^{-1}(\Omega \setminus \Delta_1) = \Omega \setminus \Delta_1.$$

We construct the following permutation:
If $x \in \Omega \setminus \sigma^{-1}\Delta_1$, either $x \in \Delta_1$, so $x \in \Delta_1 \setminus \sigma^{-1}\Delta_1$. Or since δ maps the elements of $\Delta_1 \setminus \sigma^{-1}\Delta_1$ into $\Delta_1 \cap \Delta_2$, we have that $\delta x \in \Delta_1 \cap \Delta_2$. Either $x \notin \Delta_1$, since δ maps the elements of $\Omega \setminus \Delta_1$ into $\Omega \setminus \Delta_1$, we have that $\delta x \notin \Delta_1$. In particular, since $\Omega = \Delta_1 \cap \Delta_2$, $\delta x \in \Delta_2$. Now if

$$y \in (\Omega \setminus \Delta_1)\delta\sigma^{-1} \subseteq \Delta_2,$$

$y = \delta\sigma^{-1}x$ where $x \in \Omega \setminus \Delta_1 \subseteq \Delta_2$, so we fix $\theta y = x$. Our θ is only a partial bijection of Δ_2 for now. We still need to show that $\mathrm{dom}(\theta)$ and $\mathrm{im}(\theta)$ have infinite complements in Δ_2 to be able to put them in bijection.
We have that

$$\mathrm{dom}(\theta) = \delta(\Omega \setminus \sigma^{-1}\Delta_1).$$

We denote by K the set of elements of $\Delta_1 \cap \Delta_2$ that are mapped by δ in $\Delta_1 \cap \Delta_2$. Our aim is to show that δK is included in the complement of $\delta(\Omega \setminus \sigma^{-1}\Delta_1)$ in Δ_2 and then that δK has an infinite complement.
First we have that $K \subseteq \sigma^{-1}\Delta_1$ so K is disjoint from $\Omega \setminus \sigma^{-1}\Delta_1$. Thus δK is disjoint from $\delta(\Omega \setminus \sigma^{-1}\Delta_1)$. Therefore $\delta(\Omega \setminus \sigma^{-1}\Delta_1)$ is included in the complement of δK in Δ_2. Moreover $\delta(\Omega \setminus \sigma^{-1}\Delta_1)$ is infinite so δK has an infinite complement. Besides

$$\delta K \subseteq \delta(\Omega \setminus \sigma^{-1}\Delta_1),$$

hence $\mathrm{dom}(\theta)$ has infinite complement in Δ_2.
Now we have $\theta y = x$ for $x \in \Omega \setminus \Delta_1$ and $y \in \delta\sigma^{-1}(\Omega \setminus \Delta_1)$, so $\mathrm{im}(\theta)$ is included in $\Omega \setminus \Delta_1$. Moreover $\Delta_1 \cap \Delta_2$ is infinite and disjoint from $\mathrm{im}(\theta)$, thus $\Delta_1 \cap \Delta_2$ is included in the complement of $\mathrm{im}(\theta)$ in Δ_2. Therefore $\mathrm{im}(\theta)$ has infinite complement in Δ_2.
On the remaining part of Ω, we define $\theta = \mathrm{id}$. So we have $\theta\Delta_2 = \Delta_2$, therefore $\theta \in V_{\{\Delta_2\}}$. This concludes the definition of θ.

Taking the complements of $\delta\sigma^{-1}(\Omega \setminus \Delta_1)$ and $\Omega \setminus \Delta_1$, we have that

$$\theta\delta\sigma^{-1}\Delta_1 = \Delta_1.$$

Since Δ_1 is full with respect to U, we can find $\alpha \in U_{\{\Delta_1\}}$ such that

$$\alpha|_{\Delta_1} = (\theta\delta\sigma^{-1})^{-1} \text{ i.e. } \alpha\theta\delta\sigma^{-1} \in H_{(\Delta_1)}.$$

Then $\sigma(\alpha\theta\delta)^{-1} \in H_{(\Delta_1)}$. By equation (3), we have that

$$\sigma(\alpha\theta\delta)^{-1} \in VUVUVV.$$

Thus

$$\sigma \in UVU\ VUVUVV = (UV)^4V.$$

Since we can exchange the roles of U and V, then there is also the following alternative: $\sigma \in (VU)^4U$. Hence

$$H = (UV)^4V \cup (VU)^4U.$$

□

Lemma 21. *If $U \subseteq H$ is closed under inverses has a full moiety, then there exists $x \in H$ of order 2 such that*

$$H = (Ux)^7U^2x \cup (xU)^7xU^2.$$

Proof. Assume Δ_1 is a full moiety with respect to U. We choose a moiety $\Delta_2 \subseteq \Omega$ such that $\Delta_1 \cap \Delta_2$ is a moiety and $\Delta_1 \cup \Delta_2 = \Omega$.
First we have

$$\Omega \setminus \Delta_1 = (\Delta_1 \cup \Delta_2) \setminus \Delta_1 = \Delta_2 \setminus \Delta_1.$$

Similarly we have $\Omega \setminus \Delta_2 = \Delta_1 \setminus \Delta_2$. So $\Omega \setminus \Delta_1 \cap \Omega \setminus \Delta_2$ is empty. Moreover Δ_1 and Δ_2 are moieties, so

$$|\Omega \setminus \Delta_1| = |\Omega \setminus \Delta_2| = |\Omega|.$$

Thanks to the two last results, we can find an element of order 2 in H that interchanges the two sets $\Omega \setminus \Delta_1$ and $\Omega \setminus \Delta_2$. It also interchanges their complements Δ_1 and Δ_2. We call this element x. Since Δ_1 is full with respect to U, we have that the set of permutations of Δ_1 induced by members of $U_{\{\Delta_1\}} = \{\sigma \in U : \sigma\Delta_1 = \Delta_1\}$ is exactly $\mathrm{Sym}(\Delta_1)$. Since $\Delta_2 = x\Delta_1$, we have

$$U_{\{x^{-1}\Delta_2\}} = \{\sigma \in U :^{-1} \Delta_2 = x^{-1}\Delta_2\} = \{\sigma \in U : x\sigma x^{-1}\Delta_2 = \Delta_2\}.$$

Then we get

$$xUx^{-1}_{\{\Delta_2\}} = \{\tau \in xUx^{-1} : \tau\Delta_2 = \Delta_2\}.$$

So the set of permutations of Δ_2 induced by members of $U_{\{\Delta_2\}}$ is exactly $\mathrm{Sym}(\Delta_2)$. Thus Δ_2 is a full moiety with respect to $xUx^{-1} = xUx$. We set $V = xUx$. By Lemma 20, we have $H = (UV)^4V \cup (VU)^4U$. Then we have the two following expressions:

$$(UV)^4V = (UxUx)^4xUx = (Ux)^8xUx = (Ux)^7UxxUx = (Ux)^7U^2x,$$
$$(VU)^4U = (xUxU)^7U = (xU)^8U = (xU)^7xUU = (xU)^7xU^2.$$

Thus $H = (Ux)^7U^2x \cup (xU)^7xU^2$. □

Lemma 22. *Let $(U_i)_{i\in I}$ be a family of subsets of H with $|I| \leqslant |\Omega|$ such that*

$$\bigcup_{i\in I} U_i = H.$$

Then Ω contains a full moiety with respect to at least one of the U_i.

Proof. We show the lemma by contradiction.
Since Ω is infinite and $|I| \leqslant |\Omega|$, we can write Ω as an union of disjoint moieties Δ_i for $i \in I$. So we have

$$\Omega = \bigcup_{i\in I} \Delta_i$$

and $\Delta_i \cap \Delta_j = \varnothing$ for $i, j \in I$.
If there are no full moiety with respect to any of the U_i, then in particular for any $i \in I$ Δ_i is non-full with respect to U_i. By contradiction of the definition of full, we can choose a permutation $\sigma_i \in \mathrm{Sym}(\Delta_i)$ which is not the restriction to Δ_i of a member of $(U_i)_{i\in I}$.
Let $\sigma \in H$ be the permutation such that $\forall i\ \sigma|_{\Delta_i} = \sigma_i$. Then $\forall i\ \sigma \notin U_i$. This leads to a contradiction with $\bigcup_{i\in I} U_i = H$. □

We can now prove Theorem 23 and Theorem 24, using the previous lemmas.

Theorem 23. *If U generates H then there exists n such that $H \subseteq (U \cup U^{-1})^n$. In other words H is Cayley-bounded.*

Proof. Assume U generates H as a monoid. Without loss of generality, assume $1 \in U$ and $1 \in U^{-1}$. Indeed if not, we set $V = U \cup \{1\}$. For $i \in \mathbb{N}^*$, let $U_i = U^i \cap (U^{-1})^i$.
Since U generates H, $H = \bigcup_i U^i$ which implies that $H = \bigcup_i (U^{-1})^i$.
Thus

$$H = \bigcup_i U^i \bigcap \bigcup_i (U^{-1})^i = \bigcup_i (U^i \cap (U^{-1})^i) = \bigcup_i U_i.$$

Since Ω is infinite, by Lemma 22, Ω contains a full moiety with respect to some U_i. Since $U_i \subseteq H$ is closed under inverses has a full moiety, we can use Lemma 21:

$$\text{there exists } x \in H \text{ such that } H = (U_i x)^7 U_i^2 x \cup (xU_i)^7 x U_i^2.$$

We notice that $(U_i x)^7 U_i^2 x \cup (xU_i)^7 x U_i^2 \subseteq (U_i \cup \{x\})^{17}$ so $H = (U_i \cup \{x\})^{17}$.
We take a $j \geqslant i$ such that $x \in U_j$. Since $x \in U_j$, we obtain

$$U^{17j} = (U^j)^{17} \supseteq U_j^{17} = (U_j \cup \{x\})^{17}.$$

Moreover $j \geqslant i$, so $U_i \cup \{x\} \subseteq U_j \cup \{x\}$. Thus we obtain

$$H = (U_i \cup \{x\})^{17} \subseteq U^{17j}.$$

Now if U generates H as a group, only the inverses are missing so

$$H \subseteq (U \cup U^{-1})^{17j}.$$

□

Theorem 24. *Let $(H_n)_{n \in I}$ be a chain of subgroups of H with $|I| \leqslant |\Omega|$ such that*

$$H = \bigcup_{n \in I} H_n.$$

Then there exists n such that $H = H_n$. In other words H has uncountable cofinality.

Proof. By Lemma 22, Ω has a full moiety with respect to some H_n. The H_n are subgroups so by Lemma 21, we have that

$$H = (H_n x)^7 H_n^2 x \cup (xH_n)^7 x H_n^2 \text{ for some } x \in H.$$

Then $H = \langle H_n \cup \{x\} \rangle$.
Since $H = \cup_{n \in I} H_n$ and the H_n form a chain of subgroups of H, there exists $k \geqslant n$ such that $x \in H_k$. Thus

$$H_n \subseteq H_n \cup \{x\} \subseteq H_k.$$

Since $H = \langle H_n \cup \{x\} \rangle \subseteq H_k$, then $H = H_k$. □

We have now proven that H is Cayley-bounded and has uncountable cofinality.

1.3 Characterization of the Bergman property

In this part, we are going to show the following theorem that has been shown by Manfred Droste and W. Charles Holland in [DH05]:

Theorem 25. *G is Cayley-bounded and has uncountable cofinality if and only if S has the Bergman property.*

Proof. If $U_1 \subseteq U_2 \subseteq ... \subseteq G$ subsets such that $\cup_i U_i = G$. Without loss of generality, we suppose the U_i to be symmetric. Indeed if they are not symmetric, the we set $V_i = U_i \cup U_i^{-1}$. Then the V_i are symmetric.
Let $G_i = < U_i >$ be the subgroup generated by U_i for any i. Then

$$G_1 \subseteq G_2 \subseteq ... \subseteq S \text{ and } \bigcup_i G_i = G.$$

Since G has uncountable cofinality, there exists n such that $G_n = G$. Therefore U_n is a set of generators for G. Since G is Cayley-bounded, there exists k such that $G \subseteq (U_n \cup U_n^{-1})^k = U_n^k$. Hence $G = U_n^k$.

Let $(G_n)_{n \in I}$ be a chain of subgroups of G with $|I| \leqslant |\Omega|$ such that $G = \cup_{n \in I} G_n$. Since G is Bergman, there exists n and k such that $G = G_n^k$ so in particular $G = G_n$.
Assume U generates G and without loss of generality, assume U is symmetric and contains the identity. Take $W_1 \subseteq W_2 \subseteq ... \subseteq G$ by setting $W_i = U^i$. Then

$$\bigcup_i W_i = G$$

since U generates G and is symmetric. Since G is Bergman, there exists n and k such that $G = W_n^k = (U^n)^k = U^{nk}$. □

An alternative to the proofs of the theorems would have been by using the lemmas directly and show that if the assumptions of the lemmas are satisfied, then the group S has the Bergman property. The proof would then be the following:

Proof. Let $G = \cup_n G_n$ with U_n symmetric and $U_n \subseteq U_{n+1}$. By Lemma 22, Ω contains a full moiety with respect to one of U_n, say U_k. By Lemma 21, there exists $x \in G$ of order 2 such that

$$G = (U_k x)^7 U_k^2 x \cup (x U_k)^7 x U_k^2.$$

Moreover $G = \bigcup_n G_n$, so there exists $n \geqslant k$ such that $x \in U_n$. Hence we have

$$G = (U_k x)^7 U_k^2 x \cup (x U_k)^7 x U_k^2 \subseteq (U_n)^{14} U_n^3 \cup (U_n)^{14} U_n^3 \subseteq U_n^{17}.$$

Therefore $F = G \subseteq U_n^{17}$. Hence $G = U_n^{17}$. □

1.4 End of the proof

In this part, we will show that a group G has the Bergman property if and only if whenever G acts by isometries on a metric space (X, d), every orbit is bounded.
This equivalence comes from the following theorem which is from [Ros09], Theorem 2.2:

Theorem 26. *The following are equivalent for a group G:*

(i) Whenever G acts by isometries on a metric space (X,d), every orbit is bounded;

(ii) Any left-invariant metric on G is bounded;

(iii) G has the Bergman property.

Proof. $(i) \implies (ii)$: We take G acting on (G, d) with d any left-invariant metric. Since d is left-invariant, G acts by isometries of (G, d). So by (i), every orbit is bounded. Then for any $x \in G$, there exists M such that for all $g \in G$, $d(x, gx) \leqslant M$. So in particular, for all $g \in G$, $d(e, g) \leqslant M$. Therefore for $g_1, g_2 \in G$,

$$d(g_1, g_2) \leqslant d(g_1, e) + d(e, g_2) \leqslant 2M.$$

Hence d is bounded.
$(ii) \implies (iii)$: Assume we have $W_0 \subseteq W_1 \subseteq \cdots \subseteq W_n \subseteq \cdots \subseteq G$ an exhaustive sequence of subsets of G. Moreover we also have

$$W_0 \cap W_0^{-1} \subseteq W_1 \cap W_1^{-1} \subseteq \cdots \subseteq W_n \cap W_n^{-1} \subseteq \cdots \subseteq G$$

which is an exhaustive sequence of subsets of G. So without loss of generality, we can suppose the W_i to be symmetric. We also suppose that $W_0 = \{1\}$.
We define the following left-invariant metric on G for some f, g:

$$d(f, g) = \min(k_1 + k_2 + \cdots + k_n \mid \exists h_i \in W_{k_i}\ fh_1 \ldots h_n = g).$$

Our aim is to prove that d is bounded if and only if $G = W_n^k$ for some n and k.
If d is bounded then there exists an M such that for any $f, g \in G$, $d(f, g) \leqslant M$.
In particular, for any $g \in G$,

$$d(e, g) = \min(k_1 + k_2 + \cdots + k_n \mid \exists h_i \in W_{k_i}\ h_1 \ldots h_n = g) \leqslant M.$$

It means that there exists $k_1, \ldots, k_j$ and $h_1, \ldots, h_j$ with $j \leqslant n$ and $h_i \in W_{k_i}$ such that $\sum_i k_i \leqslant M$ which means that all the h_i are in $W_{\lceil M \rceil}$. This holds for any $g \in G$, so

$$G = W_{\lceil M \rceil}^j.$$

Now if $G = W_n^k$ for some n and k, for $f, g \in G$, since $f^{-1}g \in G = W_n^k$, there exists $h_i \in W_n$ such that $f^{-1}g = h_1 \ldots h_k$. All the h_i are in W_n so

$$\sum_{i=1}^{k} h_i = \sum_{i=1}^{k} n = n \cdot k.$$

Since for $f, g \in G$ we have

$$d(f, g) = \min(k_1 + k_2 + \cdots + k_n \mid \exists\ h_i \in W_{k_i}\ fh_1 \ldots h_n = g),$$

this minimum is smaller than or equal to $n \cdot k$. Thus d is bounded.
$(iii) \implies (i)$: Assume G has the Bergman property and acts by isometries on a metric space (X,d).
Fix an $x_0 \in X$ and let for $n \geqslant 1$,

$$W_n := \{g \in G \mid d(x_0, g \cdot x_0) \geqslant n\}.$$

(W_n) is an increasing exhaustive sequence of subsets of G. Since G has the Bergman property, $G = W_M^k$ for some M and k. Then there exists $g_i \in W_M$ such that $g = g_1 \dots g_k \in G$ and

$$\begin{aligned}
d(x_0, g \cdot x_0) &= d(x_0, g_1 \dots g_k \cdot x_0) \\
&\leqslant d(x_0, g_1 \cdot x_0) + d(g_1 x_0, g_1 g_2 \cdot x_0) + \dots \\
&\quad + d(g_1 ... g_{k-1} x_0, g_1 \dots g_{k-1} \cdot x_0) \\
&\leqslant d(x_0, g_1 \cdot x_0) + d(x_0, g_2 \cdot x_0) + \dots + d(x_0, g_k \cdot x_0) \\
&\leqslant kM.
\end{aligned}$$

Furthermore for $x \in X$,

$$\begin{aligned}
d(x, g.x) &\leqslant d(x, x_0) + d(x_0, g \cdot x_0) + d(g \cdot x_0, g \cdot x) \\
&\leqslant d(x, x_0) + kM + d(x, x_0) \\
&\leqslant 2d(x, x_0) + kM.
\end{aligned}$$

So for any $x \in X$, the orbit of x is bounded. □

1.5 Last part

If we have that whenever H acts by isometries on a metric space (X,d), every orbit is bounded, then the latter is also true when H acts continuously by isometries on a metric space (X,d) which is exactly what having property (OB) means.

In the end, the following result has been proved:
H is Cayley-bounded and has uncountable cofinality implies that H has property (OB) for $H = \mathrm{Sym}(N)$ with N any infinite set.
Hence we have proved that the symmetric group of any infinite set has the property (OB). Thus in particular, $\mathbf{S}_\infty$ **has property (OB)** since $S_\infty = \mathrm{Sym}(\mathbb{N})$.

2 Results on $\mathcal{S}(X, M)$ and $\mathcal{S}^o(X, M)$

We now look at another Polish group which is $\mathcal{S}(X, M)$:

Definition 27. *For a set X and a subset M of X, let $\mathcal{S}(X, M)$ be* the group of permutations of X commensurating M*, i.e. the group of $\sigma \in Sym(X)$ such that $|M \triangle \sigma M| < +\infty$.*

In this section, we will get some results about $\mathcal{S}(X, M)$ that have been proven in [Cor16] by Yves De Cornulier.

2.1 Definitions

Let X be a set and M a subset of X. First we define the following map that will allow us to define $\mathcal{S}^o(X, M)$.

Definition 28. *We define* the transfer character map*:*

$$\begin{array}{cccc} \mathrm{tr}_M : & \mathcal{S}(X, M) & \to & \mathbb{Z} \\ & g & \mapsto & |g^{-1}M \setminus M| - |M \setminus g^{-1}M| \\ & & & = \sum\limits_{x \in X} \mathbb{1}_{g^{-1}M}(x) - \mathbb{1}_M(x) \end{array}$$

A few more denotations:
∗ $\mathcal{S}_0(X)$ is the group of finitely supported permutations of X;
∗ $\mathcal{S}_0^+(X)$ is its subgroup of index of alternating permutations;
∗ $\mathcal{S}^o(X, M)$ is the kernel of tr_M.

Definition 29. *The length $\mathcal{L}_M$ is defined by $\mathcal{L}_M = |M \triangle gM|$ for $g \in \mathcal{S}(X, M)$.*

Definition 30. *A group G is called* perfect *if it equals its own commutator subgroup, i.e. if the group has no non-trivial abelian quotients. A group G is called* simple *if it is a nontrivial group whose only normal subgroups are the trivial group and G itself.*

2.2 Results

Thanks to the transfer character map, we get more information and results on $\mathcal{S}(X, M)$ and $\mathcal{S}^o(X, M)$.

Proposition 31. *The function tr_M is a continuous homomorphism from $\mathcal{S}(X, M)$ to $\mathbb{Z}$ and is bounded above by $\mathcal{L}_M$. It is surjective, unless M or M^c is finite (in which case it is zero). It does not depend on the choice of M within its commensurability class and tr_{M^c} = -tr_M. If X is infinite, its kernel $\mathcal{S}^o(X, M)$ is a perfect group and is generated by $\mathcal{S}(M) \cup \mathcal{S}(M^c) \cup \mathcal{S}_0^+(X)$.*

Proof. For $g \in \mathcal{S}(X, M)$, we have

$$\begin{aligned} \mathcal{L}_M(g) &= |M \triangle gM| \\ &= |gM \setminus M| + |M \setminus gM| \\ &= |M \setminus g^{-1}M| + |g^{-1}M \setminus M| \\ &\geqslant |g^{-1}M \setminus M| - |M \setminus g^{-1}M| \\ &\geqslant \mathrm{tr}_M(g). \end{aligned}$$

Therefore tr_M is bounded above by $\mathcal{L}_M$.
Let M, N be such that $|M \triangle N| < \infty$. Suppose that there exists F finite subset such that $N = M \sqcup F$. Let $g \in \mathcal{S}(X, M)$, then

$$\begin{aligned}
\mathrm{tr}_N(g) - \mathrm{tr}_M(g) &= \mathrm{tr}_{M \sqcup F}(g) - \mathrm{tr}_M(g) \\
&= \left(\sum_{x \in X} \mathbb{1}_{g^{-1}M \sqcup F}(x) - \mathbb{1}_{M \sqcup F}(x) \right) - \left(\sum_{x \in X} \mathbb{1}_{g^{-1}M}(x) - \mathbb{1}_M(x) \right) \\
&= \sum_{x \in X} \mathbb{1}_{g^{-1}M \sqcup F}(x) - \mathbb{1}_{M \sqcup F}(x) - \mathbb{1}_{g^{-1}M}(x) + \mathbb{1}_M(x) \\
&= \sum_{x \in X} \mathbb{1}_{g^{-1}M}(x) + \mathbb{1}_{g^{-1}F}(x) - \mathbb{1}_M(x) - \mathbb{1}_F(x) - \mathbb{1}_{g^{-1}M}(x) + \mathbb{1}_M(x) \\
&= \sum_{x \in X} \mathbb{1}_{g^{-1}F}(x) - \mathbb{1}_F(x) \\
&= \mathrm{tr}_F(g) \\
&= 0,
\end{aligned}$$

since F is finite.
Now let $N' = M \cap N$. Therefore

$$N' \triangle M \subseteq (M \triangle M) \cap (N \triangle M),$$

which is finite. Thus N' is commensurated to M. A similar result holds for N, so N' is also commensurated to N. Since $N' \subseteq M$, there exists F_1 finite subset such that $M = N' \sqcup F_1$. Indeed $F_1 = M \setminus N'$ is finite since $N' \triangle M = M \setminus N'$. Similarly since $N' \subseteq N$, there exists F_2 finite subset such that $N = N' \sqcup F_2$. Applying the previous result on M and $M \sqcup F$, we obtain

$$\mathrm{tr}_M = \mathrm{tr}_{N'} = \mathrm{tr}_N.$$

Thus tr_M does not depend on the choice of M within its commensurability class. For $g, h \in \mathcal{S}(X, M)$, one has

$$\begin{aligned}
\mathrm{tr}_M(gh) &= \sum_{x \in X} \mathbb{1}_{(gh)^{-1}M}(x) - \mathbb{1}_M(x) \\
&= \sum_{x \in X} \mathbb{1}_{h^{-1}g^{-1}M}(x) - \mathbb{1}_{h^{-1}M}(x) + \sum_{x \in X} \mathbb{1}_{h^{-1}M}(x) - \mathbb{1}_M(x) \\
&= \sum_{x \in X} \mathbb{1}_{g^{-1}M}(hx) - \mathbb{1}_M(hx) + \mathrm{tr}_M(h) \\
&= \mathrm{tr}_M(g) + \mathrm{tr}_M(h).
\end{aligned}$$

Thus tr_M is a homomorphism from $\mathcal{S}(X, M)$ to $\mathbb{Z}$. Moreover

$$\mathrm{tr}_M(g) = 0 \implies |g^{-1}M \setminus M| - |M \setminus g^{-1}M| = 0.$$

This implies that g stabilizes M. Thus the stabilizer of M is contained in $\ker(\mathrm{tr}_M)$. Furthermore the stabilizer of M is open by definition of the topology

of $\mathcal{S}(X, M)$. Therefore tr_M is continuous.
Moreover for $n \in \mathbb{Z}$, there exists g such

$$|g^{-1}M \setminus M| = n + |M \setminus g^{-1}M|,$$

then $|M \triangle gM| < +\infty$. Thus $g \in \mathcal{S}(X, M)$. Hence tr_M is surjective.
Let $g \in \mathrm{Ker}(\mathrm{tr}_M)$, it stabilizes M. Then the finite sets $g^{-1}M \setminus M$ and $M \setminus g^{-1}M$ have the same cardinal. So there exists a permutation σ with finite support that exchanges the two sets and is the identity on the complement of the two sets. Let τ be either the identity when σ is even or a transposition with support M or M^c when σ is odd. Thus $\tau\sigma$ is an even permutation and $\tau\sigma g$ also stabilizes M. If X is infinite, then M and M^c are infinite. Then $\mathcal{S}(M), \mathcal{S}(M^c)$ and $\mathcal{S}_0^+(X)$ are perfect groups. Thus $\ker(\mathrm{tr}_M) = \mathcal{S}^o(X, M)$ is a perfect group. If either M or M^c is infinite, then $\ker(\mathrm{tr}_M)$ is equal to $\mathcal{S}(X)$ which is perfect. ☐

Proposition 32. *Some normal subgroups of $\mathcal{S}(\mathbb{Z}, \mathbb{N})$ are the following:*
- $\{1\}, \mathcal{S}_0(\mathbb{Z}), \mathcal{S}_0^+(\mathbb{Z})$*;*
- $\mathcal{S}^o(\mathbb{Z}, \mathbb{N})$ *and the subgroups which have finite index in* $\mathcal{S}(\mathbb{Z}, \mathbb{N})$*.*

Proof. $\mathcal{S}_0^+(\mathbb{Z})$ is generated by the 3-cycles and by the transpositions with disjoint support. $\mathcal{S}_0(\mathbb{Z})$ is generated by the transpositions. All the transpositions are conjugated in $\mathcal{S}_0(\mathbb{Z})$. So if $N \triangleleft \mathcal{S}(\mathbb{Z}, \mathbb{N})$ and N contains a transposition, then $N \geqslant \mathcal{S}_0(\mathbb{Z})$.

First $\mathcal{S}_0^+(\mathbb{Z})$ is dense in $\mathcal{S}_0(\mathbb{Z})$. Moreover $\mathcal{S}_0^+(\mathbb{Z})$ is simple. Indeed let N be such that $N \triangleleft \mathcal{S}_0^+(\mathbb{Z})$, $N \neq \{1\}$. Let $\sigma \in N \setminus \{1\}$, then σ is not a transposition.
- if σ is a 3-cycle, then we have the result since $\mathcal{S}_0^+(\mathbb{Z})$ is generated by the 3-cycles;
- otherwise there exists i, j, k, l two by two different such that $\sigma(i) = j$ and $\sigma(k) = l$. We have

$$\sigma(ik)\sigma^{-1}(ik) = \sigma(ik)\sigma^{-1}(ik)^{-1} = (jl)(ik) \in \mathbb{N},$$

  since $(ik)\sigma^{-1}(ik)^{-1} \in \mathbb{N}$ is a commutator.

Let $N \trianglelefteq \mathcal{S}(\mathbb{Z}, \mathbb{N})$ be closed and $N \neq \{1\}$. Since $N \neq \{1\}$, $N \geqslant \mathcal{S}_0^+(\mathbb{Z})$. Let $\sigma \in \mathbb{N} \setminus \{1\}$. Then there exists i such that $\sigma(i) \neq i$.
- if σ is a transposition then $N \geqslant \mathcal{S}_0(\mathbb{Z})$;
- otherwise either σ is a 3-cycle. In this case, $N \geqslant \mathcal{S}_0(\mathbb{Z})$. Or there exists i, j, k and l such that $(jl)(ik) \in \mathbb{N}$. Therefore $\mathcal{S}_0(\mathbb{Z}) \leqslant N$.

Second, $\mathcal{S}_0(\mathbb{Z})$ is dense in $\mathcal{S}^o(\mathbb{Z}, \mathbb{N})$. Indeed let $\sigma \in \mathcal{S}_0(\mathbb{Z})$ and let U be a neighborhood of σ. Then we can find $P_1, ..., P_n$ commensurated to $\mathbb{N}$ such that

$$U \supseteq \{\tau \ : \ \tau(P_i) = \sigma(P_i)\}.$$

Since the P_i are commensurated to $\mathbb{N}$, there exists $K \in \mathbb{N}$ such that

$$\text{for any } i, \ P_i \triangle \mathbb{N} \subseteq [\![-K, K]\!] \text{ and } \sigma(P_i) \triangle \mathbb{N} \subseteq [\![-K, K]\!].$$

Thus $\sigma(\mathbb{N}) \triangle \mathbb{N} \subseteq [\![-K, K]\!]$ and $\sigma^{-1}(\mathbb{N}) \triangle \mathbb{N} \subseteq [\![-K, K]\!]$.
Thus $\mathcal{S}_0^+(\mathbb{Z})$ is also dense in $\mathcal{S}^o(\mathbb{Z}, \mathbb{N})$.

By Proposition 31, we have that

$$\mathcal{S}^o(\mathbb{Z}, \mathbb{N}) = \left\langle \mathcal{S}_0^+(\mathbb{Z}), \mathcal{S}(\mathbb{N}), \mathcal{S}(\mathbb{Z} \setminus \mathbb{N}) \right\rangle.$$

The topology induced on $\mathcal{S}(\mathbb{N})$ is the usual topology. Indeed $\mathcal{S}(\mathbb{N})$ has an unique topology of a separable group. (Following from Corollary 1.5 in [BYT16]) We know that $\mathcal{S}_0(\mathbb{N})$ is dense in $\mathcal{S}(\mathbb{N})$ for the usual topology.
Let $\Phi : \mathcal{S}(\mathbb{Z}, \mathbb{N}) \to \mathcal{S}(\mathbb{Z})$. Φ is continuous if and only if it is continuous at identity. Indeed if the latter is true, then for $g_n \in \mathcal{S}(\mathbb{Z}, \mathbb{N})$,

$$\text{if } g_n \to g \text{ then } g_n g^{-1} \to 1.$$

Therefore $\Phi(g_n g^{-1}) \to 1$ which implies that $\Phi(g_n)\Phi(g)^{-1} \to 1$ since Φ is a morphism. Thus $\Phi(g_n) \to \Phi(g)$. Now to show that Φ is continuous at identity, we need to show that if U is an open identity neighborhood in $\mathcal{S}(\mathbb{Z})$ then so is $\Phi^{-1}(U)$. Moreover any identity neighborhood is included in

$$U' = \{\sigma \ : \ \sigma(n) = n\} \text{ where } n \in \mathbb{Z} \text{ is fixed.}$$

So it is enough to show that U' is an open identity neighborhood in $\mathcal{S}(\mathbb{Z})$. Let

$$V = \{\sigma \in \mathcal{S}(\mathbb{Z}) \mid \sigma(\mathbb{N}) = \mathbb{N}, \sigma(\mathbb{N} \setminus \{n\}) = \mathbb{N} \setminus \{n\} \text{ and } \sigma(\mathbb{N} \cup \{n\}) = \mathbb{N} \cup \{n\}\}.$$

V is open in $\mathcal{S}(\mathbb{Z}, \mathbb{N})$. Also $V \subseteq U'$,
* if $n \in \mathbb{N}$, then $\sigma(\mathbb{N} \setminus \{n\}) = \mathbb{N} \setminus \{\sigma(n)\}$,
* if $n < 0$, then $\sigma(\mathbb{N} \cup \{n\}) = \{n\} \cup \mathbb{N}$.

Therefore

$$U' = \bigcup_{u \in U'} uV,$$

thus U' is open. Hence the morphism $\Phi : \mathcal{S}(\mathbb{Z}, \mathbb{N}) \to \mathcal{S}(\mathbb{Z})$ is continuous. Thus $\mathcal{S}_0(\mathbb{N}) \leqslant \mathcal{S}_0(\mathbb{Z})$ is dense in $\mathcal{S}(\mathbb{N})$. Similarly $\mathcal{S}_0(\mathbb{Z} \setminus \mathbb{N}) \leqslant \mathcal{S}_0(\mathbb{Z})$ is dense in $\mathcal{S}(\mathbb{Z} \setminus \mathbb{N})$. Furthermore $\mathcal{S}_0^+(\mathbb{Z})$ is also dense in $\mathcal{S}^o(\mathbb{Z}, \mathbb{N})$. Therefore $\mathcal{S}^o(\mathbb{Z}, \mathbb{N})$ is dense in $\mathcal{S}(\mathbb{Z}, \mathbb{N})$.
So $N \geqslant \ker \mathrm{tr}_{\mathbb{N}} = \mathcal{S}^o(\mathbb{Z}, \mathbb{N})$. Either $N = \mathcal{S}^o(\mathbb{Z}, \mathbb{N})$, or $\mathrm{tr}_{\mathbb{N}}$ is a non-trivial subgroup of $\mathbb{Z}$ of finite index. Then $\mathcal{S}^o(\mathbb{Z}, \mathbb{N})$ and all its subgroups contain it. Thus N has finite index in $\mathcal{S}(\mathbb{Z}, \mathbb{N})$. □

So we have shown that $\mathcal{S}^o(\mathbb{Z}, \mathbb{N})$ is a normal subgroup of $\mathcal{S}(\mathbb{Z}, \mathbb{N})$. Therefore when we are going to show some results on $\mathcal{S}(\mathbb{Z}, \mathbb{N})$, we will first show it on $\mathcal{S}^o(\mathbb{Z}, \mathbb{N})$ and then we will be able to show it on $\mathcal{S}(\mathbb{Z}, \mathbb{N})$.

3 A characterization of local boundedness

In this section, our aim is the following theorem from [Ros]. All of this section is originally from the latter. This theorem will give us a characterization of the notion of being locally bounded. This characterization holds for a larger claim than Polish groups, namely European groups (cf. Definition 47).

Theorem 68. *For a European topological group G, the following are equivalent:*
(i) G admits a continuous left-invariant maximal pseudometric d;
(ii) G is generated by a coarsely bounded set;
(iii) G is locally bounded and not the union of a countable chain of proper open subgroups;
(iv) the coarse structure is monogenic.

Since $\mathcal{S}(\mathbb{Z}, \mathbb{N})$ is a European group, we will be able to apply this result in the next section.

3.1 Definitions

To start, we will see a few definitions to be able to understand the notion of coarse structure.

Definition 33. *A* coarse space *is a set X equipped with a condition $\mathcal{E}$ of subsets $E \subseteq X \times X$ called* entourages *satisfying the following conditions:*
- *The diagonal $\nabla = \{(x, x); x \in X\}$ belongs to $\mathcal{E}$;*
- *if $E \subseteq F \in \mathcal{E}$, then $E \in \mathcal{E}$;*
- *if $E, F \in \mathcal{E}$, then $E \cup F$, E^{-1}, $E \circ F \in \mathcal{E}$.*

The condition $\mathcal{E}$ is also called a coarse structure *on X.*

Definition 34. *A* pseudometric space *is a set X equipped with an* pseudometric, *i.e. a map $d : X \times X \longrightarrow \mathbb{R}_+$ such that d is symmetric, satisfies the triangle inequality and $d(x, x) = 0$ for all $x \in X$.*

Definition 35. *For a topological group G, we define its* left-coarse structure *$\mathcal{E}_L$ by*

$$\mathcal{E}_L = \bigcap \{\mathcal{E}_d \mid d \text{ is a continuous left-invariant pseudometric on } G\}.$$

Definition 36. *A subset $A \subseteq X$ of a coarse space $(X,\mathcal{E})$ is said to be* coarsely bounded *if $A \times A \in \mathcal{E}$.*

3.2 First results

The metrisation theorem of Birkhoff and Kakutani stated and proved in the book from Su Gao [Inv], has inspired the next lemma. The latter will be used several times in the next results.

Lemma 37. *Let G be a metrisable topological group and $(V_n)_{n\in\mathbb{Z}}$ an increasing chain of symmetric open identity neighborhoods satisfying $G = \cup_{n\in\mathbb{Z}} V_n$ and $V_n^3 \subseteq V_{n+1}$. Defining for $f, g \in G$,*

$$\delta(f, g) := \mathit{inf}(2^n \mid f^{-1}g \in V_n) \text{ and}$$

$$d(f, g) := \mathit{inf}\left(\sum_{i=0}^{k-1} \delta(h_i, h_{i+1}) \mid h_0 = f, h_k = g\right),$$

we get that

$$\frac{1}{2}\delta(f,g) \leqslant d(f,g) \leqslant \delta(f,g)$$

and d is a continuous and compatible left-invariant metric on G.

Proof. Since the V_i are symmetric, δ is symmetric. Moreover $\delta(f,g) \geqslant 0$ for any f, g. Also since for $f \neq g$, $d(f,g) \geqslant 2$. So $\delta(f,g) > 0$. Thus $\delta(f,g) = 0$ if and only if $f = g$ since $f^{-1}f = 1_G \in V_0$.
Let $f_0, f_1, f_2, f_3 \in G$ such that

$$\delta(f_0,f_1), \delta(f_1,f_2), \delta(f_2,f_3) \leqslant \varepsilon.$$

Let p be such that

$$2^p = \max\{\delta(f_0,f_1), \delta(f_1,f_2), \delta(f_2,f_3)\}.$$

Moreover

$$f_0^{-1}f_3 = f_0^{-1}f_1 \cdot f_1^{-1}f_2 \cdot f_2^{-1}f_3 \in V_p^3 \subseteq V_{p+1}.$$

Thus

$$\delta(f_0,f_3) = \inf(2^n \mid f_0^{-1}f_3 \in V_n) \leqslant 2^{p+1} = 2 \times 2^p \leqslant 2\varepsilon.$$

We check now that d is a compatible left-invariant metric on G.
Since $\delta(f,g) \geqslant 0$, we have $d(f,g) \geqslant 0$ for any $f, g \in G$. Moreover

$$d(f,f) = \inf\left(\sum_{i=0}^{k-1} \delta(h_i,h_{i+1}) \mid h_0 = f, h_k = f\right) = \delta(f,f) = 0.$$

Also since δ is symmetric, d is also symmetric. Moreover for $f, g, h \in G$, let h_i for $i \in \{0, ..., k\}$ be such that $h_0 = f$ and $h_k = g$. Similarly let h'_i for $i \in \{0, ..., k'\}$ be such that $h'_0 = f = h_0$ and $h'_{k'} = h$ and h''_i for $i \in \{0, ..., k''\}$ be such that $h''_0 = h = h'_{k'}$ and $h''_{k''} = g = h_k$. Since we add more elements to the initial sum, we have

$$\sum_{i=0}^{k-1} \delta(h_i,h_{i+1}) \leqslant \sum_{i=0}^{k'-1} \delta(h'_i,h'_{i+1}) + \sum_{i=0}^{k''-1} \delta(h''_i,h''_{i+1})$$

Therefore by taking the infimum of each sum, we have

$$d(f,g) \leqslant d(f,h) + d(h,g).$$

Hence d verifies the triangle inequality. For d to be a metric, $f \neq g$ implies that $d(f,g) \neq 0$ is left to show. First we claim that

$$\frac{1}{2}\delta(f,g) \leqslant d(f,g) \text{ for } f \neq g. \tag{4}$$

To obtain this, we show by induction on $k \in \mathbb{N}$ that

$$\sum_{i=0}^{k+1} \delta(h_i,h_{i+1}) \geqslant \frac{1}{2}\delta(h_0,h_{k+2}). \tag{5}$$

For $k \leqslant 1$, we mainly use the fact that

$$\text{for } \varepsilon > 0, \text{ if } \delta(f_0, f_1), \delta(f_1, f_2), \delta(f_2, f_3) \leqslant \varepsilon, \text{ then } \delta(f_0, f_3) \leqslant 2\varepsilon. \tag{6}$$

We have that

$$\delta(h_0, h_1) \leqslant \sum_{i=0}^{2} \delta(h_i, h_{i+1}).$$

Moreover by fact (6),

$$\frac{\delta(h_0, h_3)}{2} \leqslant \delta(h_0, h_1).$$

Hence

$$\frac{\delta(h_0, h_3)}{2} \leqslant \sum_{i=0}^{2} \delta(h_i, h_{i+1}).$$

For $k \geqslant 2$, we assume that inequality (5) holds for all $l < k$. Let

$$S = \sum_{i=0}^{k+1} \delta(h_i, h_{i+1}).$$

* if $\delta(h_0, h_1) \geqslant \frac{1}{2}S$, then by induction hypothesis

$$S - \delta(h_0, h_1) \geqslant \frac{1}{2}\delta(h_1, h_{k+2}) \Leftrightarrow 2S - 2\delta(h_0, h_1)) \geqslant \delta(h_1, h_{k+2}).$$

  Since $\delta(h_0, h_1) \geqslant \frac{1}{2}S$, we have $\delta(h_1, h_{k+2}) \leqslant S$. By fact (6),

$$\delta(h_0, h_{k+2}) \leqslant S.$$

* if $\delta(h_k, h_{k+1}) \geqslant \frac{1}{2}S$, we use a symmetric argument.

* if $\delta(h_0, h_1), \delta(h_k, h_{k+1}) < \frac{1}{2}S$, let m be the largest such that

$$\sum_{i=0}^{m} \delta(h_i, h_{i+1}) \leqslant \frac{1}{2}S. \tag{7}$$

  Then $1 \leqslant m < k + 1$. By inductive hypothesis, we have that

$$\delta(h_0, h_{m+1}) \leqslant 2\sum_{i=0}^{m} \delta(h_i, h_{i+1}) \leqslant S.$$

  Since m is the largest such that inequality (7) holds,

$$\sum_{i=0}^{m+1} \delta(h_i, h_{i+1}) > \frac{1}{2}S.$$

  Thus

$$\sum_{i=m+2}^{k+1} \delta(h_i, h_{i+1}) \leqslant \frac{1}{2}S.$$

Applying the inductive hypothesis, we have

$$\delta(h_{m+2}, h_{k+2}) \leqslant 2 \sum_{i=m+2}^{k+1} \delta(h_i, h_{i+1}) \leqslant S.$$

Moreover $\delta(h_{m+1}, h_{m+2}) \leqslant S$. By fact (6), we have that $\delta(h_0, h_{k+2}) \leqslant 2S$.

Hence

$$\frac{1}{2}\delta(f,g) \leqslant d(f,g) \text{ for } f \neq g.$$

Since for $f \neq g$, $\delta(f,g) > 0$, then also $d(f,g) > 0$. This implies that d is a metric.
We show now that δ is left-invariant: for $f, g, h \in G$, one has

$$\begin{aligned}
\delta(hf, hg) &= \inf(2^n \mid (hf)^{-1}(hg) \in V_n) \\
&= \inf(2^n \mid f^{-1}h^{-1}hg \in V_n) \\
&= \inf(2^n \mid f^{-1}g \in V_n) \\
&= \delta(f,g).
\end{aligned}$$

Thus d is also left-invariant.

We show finally that d is compatible with the topology of G. Let U be open in G and $g \in U$. Then for some $n \in \mathbb{N}$, $gV_n \subseteq U$. Let

$$f \in B_d\left(g, 2^{n-1}\right) = \left\{h \in G \mid d(g,h) < 2^{n-1}\right\},$$

then $d(f,g) < 2^{n-1}$. So by using claim (4),

$$\delta(f,g) \leqslant 2d(f,g) < 2^n.$$

Thus $g^{-1}f \in V_n$. Hence $h \in gV_n \subseteq U$. Therefore

$$B_d\left(g, 2^{n-1}\right) \subseteq U.$$

Now let U be open in the topology given by d and $g \in U$. There exists $n \in \mathbb{N}$ such that $B_d(g, 2^n) \subseteq U$. Let $f \in gB_{n+1}$, then $\delta(f,g) \leqslant 2^{n-1}$. Moreover

$$d(f,g) \leqslant \delta(f,g) \leqslant 2^{n-1} < 2^n,$$

by the definitions of δ and d. Thus $f \in B_d(g, 2^n)$. So $f \in U$. Therefore

$$gV_{n+1} \subseteq U.$$

□

Thanks to the last lemma, we have the following proposition:

Proposition 38. *Let G be a topological group equipped with its left-coarse structure. Then the following conditions are equivalent for a subset $A \subseteq G$,*
(i) A is coarsely bounded,
(ii) for every continuous left-invariant pseudometric d on G, $\mathrm{diam}_d(A) < +\infty$,

(iii) *for every continuous isometric action on a metric space, $G \circlearrowleft (X, d)$, and every $x \in X$, $\operatorname{diam}_d(A \cdot x) < +\infty$,*

(iv) *for every increasing exhaustive sequence $V_1 \subseteq V_2 \subseteq ... \subseteq G$ of open subsets with $V_n^2 \subseteq V_{n+1}$, we have $A \subseteq V_n$ for some n.*

Moreover, suppose G is countably generated over every identity neighborhood, i.e for every identity neighborhood V there is a countable set $C \subseteq G$ such that $G = \langle V \cup C \rangle$. Then (i)-(iv) are equivalent to:

(v) *for every identity neighborhood V, there is a finite set $F \subseteq G$ and $k \geqslant 1$ such that $A \subseteq (FV)^k$.*

Proof. $(i) \iff (ii)$: By definition, A is coarsely bounded is equivalent to $A \times A \in \mathcal{E}_L$. Moreover:

$$
\begin{aligned}
A \times A \in \mathcal{E}_L \Leftrightarrow & \text{ for any continuous left-invariant pseudometric } d \text{ on } G, \\
& A \times A \in \mathcal{E}_d, \\
\Leftrightarrow & \text{ for any continuous left-invariant pseudometric } d \text{ on } G, \\
& \sup\nolimits_{(f,g) \in A \times A} d(f, g) < +\infty, \\
\Leftrightarrow & \text{ for any continuous left-invariant pseudometric } d \text{ on } G, \\
& \operatorname{diam}_d(A) < +\infty.
\end{aligned}
$$

$(ii) \implies (iii)$: Let $G \circlearrowleft (X, d)$ be a continuous isometric action on a metric space. Let $x \in X$. For $f, g \in G$, we define

$$\partial(f, g) = d(f \cdot x, g \cdot x),$$

which is a continuous left-invariant pseudometric on G. If $\operatorname{diam}_\partial(A) < +\infty$, then

$$\operatorname{diam}_d(A \cdot x) < +\infty.$$

The same applies for any $x \in X$.

$(iii) \implies (ii)$: Let d be a continuous left-invariant pseudometric on G and X be the corresponding metric quotient of G. The isometry of the pseudometric space (G, d) factors through a metric space with the following equivalence relation

$$x \sim y \text{ if } d(x, y) = 0.$$

Then the left-shift action of G onto itself factors through to a continuous transitive isometric action on X. Thus if every A-orbit is bounded then A is d-bounded on G.

$(ii) \implies (iv)$: We are showing the contraposition, i.e. $(\neg iv) \implies (\neg ii)$: Suppose there exists an increasing exhaustive chain of symmetric open subsets

$$W_1 \subseteq W_2 \subseteq ... \subseteq G,$$

such that $W_n^2 \subseteq W_{n+1}$ and $A \nsubseteq W_n$ for all n. Without loss of generality, we suppose that $1 \in W_0$. We take symmetric open identity neighborhoods $V_k \subseteq W_0$ for all $k < 0$ such that $V_k^3 \subseteq V_{k+1}$ and for all $k \geqslant 0$, $V_k = W_{2k+2}$. Then the chain V_k for $k \in \mathbb{Z}$ satisfy the conditions of Lemma 37. Thus there exists a continuous left-invariant pseudometric d on G such that its open n-ball is contained in V_{2^n} since

$$d(f, g) \leqslant \delta(f, g) = \inf(2^n \mid f^{-1}g \in V_n).$$

Therefore $\text{diam}_d(A) = +\infty$.

$(iv) \implies (ii)$: For d a continuous left-invariant pseudometric on G, we set

$$V_n = \{f \in G \mid d(1, f) < 2^n\}.$$

For all n, $V_n^2 \subseteq V_{n+1}$ since for $f, g \in V_n$,

$$d(1, fg) \leqslant d(1, f) + d(f, fg) = d(1, f) + d(1, g) < 2^n + 2^n = 2^{n+1}.$$

Moreover V_n forms an increasing exhaustive sequence of open subsets of G. By (iv), there exists k such that $A \subseteq V_k$. Furthermore notice that the d-bounded sets are exactly those contained in some V_n. Thus

$$\text{diam}_d(A) < +\infty.$$

This is for any continuous left-invariant pseudometric on G, so we get (ii).

$(v) \implies (iv)$: Suppose

$$V_1 \subseteq V_2 \subseteq ... \subseteq G,$$

is an increasing exhaustive sequence of open subsets with $V_n^2 \subseteq V_{n+1}$ for any n. Then V_1 is an identity neighborhood. Therefore there exists a finite set $F \subseteq G$ and $k \geqslant 1$ such that $A \subseteq (FV_1)^k$. Since $F \subseteq G$, there exists p such that $F \subseteq V_p$. Hence

$$A \subseteq (FV_1)^k \subseteq (V_pV_1)^k \subseteq V_{p+k+1},$$

since $V_n^2 \subseteq V_{n+1}$ for any n.

$(iv) \implies (v)$: Suppose G is countably generated over every identity neighborhood. Let A be a coarsely bounded set and V an identity neighborhood. Take a countable set $C = \{x_n\}_n$ such that $G = \langle V \cup C \rangle$. Let

$$V_n = (V \cup \{x_1, ..., x_n\})^{2n}.$$

Then $V_1 \subseteq V_2 \subseteq ... \subseteq G$ is an increasing exhaustive sequence of open subsets with $V_n^2 \subseteq V_{n+1}$. Therefore there exists p such that

$$A \subseteq V_p = (V \cup \{x_1, ..., x_p\})^{2p} = (FV)^{2p},$$

where F is finite. □

We now need a new definition, the ideal $\mathcal{OB}$. This name has not been chosen by hazard. We have seen in the first section the property (OB). The two notions are connected. Indeed G has the property (OB) is equivalent to $G \in \mathcal{OB}$.

Definition 39. *The ideal $\mathcal{OB}$ of a group G is the ideal of closed coarsely bounded sets in G.*

The notion of left-invariant coarse structures on groups can be reformulated as ideals of subsets, which will help us in the next results.

Proposition 40. *Let G be a group. Then the map Φ sending $\mathcal{E}$ onto*

$$\mathcal{A}_\mathcal{E} = \{A \mid A \subseteq A_E \textit{ for some } E \in \mathcal{E}\}$$

with inverse map sending $\mathcal{A}$ onto

$$\mathcal{E}_{\mathcal{A}} = \{E \mid E \subseteq E_A \textit{ for some } A \in \mathcal{A}\}$$

defines a bijection between the collection of left-invariant coarse structures $\mathcal{E}$ on G and the collection of ideals $\mathcal{A}$ on G, containing $\{1\}$ and closed under inversion $A \mapsto A^{-1}$ and products $(A, B) \mapsto AB$.

Proof. Let G be a group and $E \subseteq G \times G$ left-invariant. The corresponding set

$$A_E = \{x \in G \mid (1, x) \in E\}$$

is covering all of E writing $E = \{(x, y) \in G \times G \mid x^{-1}y \in A\}$. This is similar for the converse. Therefore the map that sends E to A_E is a bijection between left-invariant subsets of $G \times G$ and subsets of G with inverse

$$A \mapsto E_A = \{(x, y) \in G \times G \mid x^{-1}y \in A\}.$$

Moreover for $A \subseteq G$,

$$\begin{aligned} E_A^{-1} &= \{(y, x) \in G \times G \mid x^{-1}y \in A\} \\ &= \{(y, x) \in G \times G \mid y^{-1}x \in A^{-1}\} \\ &= E_{A^{-1}}. \end{aligned}$$

Also for $A, B \subseteq G$,

$$\begin{aligned} E_A \circ E_B &= \{(x, y) \in G \times G \mid x^{-1}y \in A\} \circ \{(x, y) \in G \times G \mid x^{-1}y \in B\} \\ &= \{(x, z) \in G \times G \mid \exists y \in G\ (x, y) \in E_A, (y, z) \in E_B\} \\ &= \{(x, z) \in G \times G \mid \exists y \in G\ x^{-1}y \in A \text{ and } y^{-1}z \in B\} \\ &= \{(x, z) \in G \times G \mid x^{-1}z \in AB\} \\ &= E_{AB}. \end{aligned}$$

Another property is

$$E_A[B] := \{x \in G \mid \exists b \in B\ (x, b) \in \mathcal{E}_A\} = BA^{-1}.$$

The coarse structure generated by a collection of left-invariant sets has a cofinal basis consisting of left-invariant sets. So $\mathcal{E}$ is left-invariant. Moreover the collection of ideals $\mathcal{A}$ on G are closed under inversion and products. $\square$

Proposition 41. *For every topological group G, we have*

$$\mathcal{E}_L = \mathcal{E}_{\mathcal{OB}} = \{E \mid E \subseteq E_A \textit{ for some } A \in \mathcal{OB}\}.$$

Proof. Suppose $E \in \mathcal{E}_{\mathcal{OB}}$. Then there exists $A \in \mathcal{OB}$ such that $E \subseteq E_A$. Let d be a continuous left-invariant pseudometric on G. Since A is coarsely bounded, by Proposition 38, $\mathrm{diam}_d(A) < +\infty$. Then there exists a such that $d(1, x) < a$ for all $x \in A$. Hence

$$E \subseteq E_A \subseteq \{(x, y) \mid d(x, y) < a\}, \text{ i.e. } E \in \mathcal{E}_d.$$

Since it holds for any d continuous left-invariant pseudometric on G, we have $E \in \mathcal{E}_L$.

Suppose $E \in \mathcal{E}_L$. Since $\mathcal{E}_L$ is left-invariant,

$$E' = \{(zx, zy) \mid z \in G \text{ and } (x, y) \in E\}$$

is also in $\mathcal{E}_L$. Moreover E' is also left-invariant so there exists $A \subseteq G$ such that $E' = E_A$ and A has finite diameter with respect to every continuous left-invariant pseudometric on G. Hence by Proposition 38, A is coarsely bounded. Thus $A \in \mathcal{OB}$. Therefore $E \subseteq E_A \in \mathcal{OB}$, i.e. $E \in \mathcal{OB}$. □

3.3 Intermediate theorems

In this part, our aim is to show the following intermediate theorem:

Theorem 55. *For an European topological group G, the following are equivalent:*

(i) the left-coarse structure $\mathcal{E}_L$ is monogenic;

(ii) G is generated by a coarsely bounded set, i.e. there is some $A \in \mathcal{OB}$ algebraically generating G;

(iii) G is locally bounded and not the union of a countable chain of proper open subgroups.

We need some new definitions:

Definition 42. *A coarse space $(X, \mathcal{E})$ is* metrisable *if it is of the form $\mathcal{E}_d$ for some generalised metric $d : X \times X \longrightarrow \mathbb{R}_+$.*

Definition 43. *A topological group G is* locally bounded *if and only if it has a coarsely bounded identity neighborhood.*

Definition 44. *A topological group G is a* Baire *if it satisfies the Baire category theorem, i.e., if the intersection of a countable family of dense open sets is dense in G.*

Definition 45. *A family $\{B\}_n \subseteq A$ is said to be* cofinal *in A if for every $A \subseteq A$, there exists B_n such that $A \subseteq B_n$.*

Definition 46. *A subset $B \subseteq X$ of a topological space X is* nowhere dense *if its closure has empty interior, i.e. if for each open set $U \subseteq$, the set $B \cap U$ is not dense in U.*

A subset $B \subseteq X$ is somewhere dense *if it is not nowhere dense.*

A subset of a topological space X is said to be meager *in X if it is a countable union of nowhere dense subsets of X.*

A subset is said to be non-meager *if it is not meager.*

Definition 47. *A topological group G is* European *if it is Baire and countably generated over every identity neighborhood.*

The next lemmas are going to be directly used to prove Theorem 53. The latter will help proving one of the equivalences of Theorem 55.

Lemma 48. *Suppose G is a topological group countably generated over every identity neighborhood. Then, for every symmetric open identity neighborhood V, there is a continuous left-invariant pseudometric d so that a subset $A \subseteq G$ is d-bounded if and only if there are a finite set F and a natural k such that $A \subseteq (FV)^k$.*

Proof. Let V be a symmetric open identity neighborhood. Since G is countably generated over V, there exists $x_1, x_2, x_3, ... \in G$ such that

$$G = \langle V \cup \{x_1, x_2, x_3, ...\} \rangle.$$

Now let

$$V_n = (V \cup \{x_1, x_1^{-1}, ..., x_n, x_n{-}1\})^{3^n}.$$

We have $G = \cup_n V_n$ and the V_n is an increasing exhaustive chain of open symmetric identity neighborhood such that $V_n^3 \subseteq V_{n+1}$ for any n. Then adding the negative indexes of the chain V_n with symmetric open identity neighborhoods V_i such that $V_i^3 \subseteq V_{i+1}$ for any i, we get $V_n \subseteq V_{n+1}$ for $n \in \mathbb{Z}$. By Lemma 37, we obtain a continuous left-invariant pseudometric d on G. Moreover each d-ball is contained in some V_n and each V_n has finite d-diameter. Then a subset $A \subseteq G$ is d-bounded if and only if $A \subseteq V_n$ for some n. Let $F \subseteq G$ be a finite subset. Then there exists $n \geqslant 1$ such that $F \subseteq V_n$. Thus

$$(FV)^k \subseteq V_{n+k}$$

has finite d-diameter for all $k \geqslant 1$. This shows the equivalence. □

Lemma 49. *Let G be a locally bounded topological group and assume that G is countably generated over every identity neighborhood. Then $\mathcal{E}_L$ is induced by a continuous left-invariant pseudometric d on G.*

Proof. Let V be a symmetric open identity neighborhood coarsely bounded in G. Let d be a continuous left-invariant pseudometric defined like in Lemma 48. Then a subset $A \subseteq G$ is d-bounded if and only if there are a finite set F and a natural n such that $A \subseteq (FV)^n$. Thus A is coarsely bounded in G. Therefore d induces the left-coarse structure $\mathcal{E}_L$ on G. □

Lemma 50. *For a topological group G, the following are equivalent:*

(i) the left-coarse structure $\mathcal{E}_L$ is metrisable;

(ii) the ideal $\mathcal{OB}$ is countably generated, i.e. the ideal $\mathcal{OB}$ has a countable cofinal subfamily;

(iii) $\mathcal{E}_L$ is metrised by a left-invariant metric d on G.

Proof. $(i) \implies (ii)$: Recall that

$$\mathcal{E}_{\mathcal{OB}} = \{E \mid E \subseteq E_A \text{ for some } A \in \mathcal{OB}\},$$

where

$$E_A = \{(x, y) \in G \times G \mid (1, x) \in E\}.$$

Since by Lemma 41 $\mathcal{E}_{\mathcal{OB}} = \mathcal{E}_L$, then the ideal $\mathcal{OB}$ is countably generated.

$(ii) \implies (iii)$: Suppose $\mathcal{OB}$ is generated countably by a cofinal family $\{A_n\}_n \subseteq \mathcal{OB}$. Let $A'_n = \{1\} \cup A_n A_n^{-1}$ and define the sequence $\{B_n\}_n$ by

$$B_0 = \{1\}$$
$$B_{n+1} = A'_{n+1} \cup B_n B_n.$$

Then $\{B_n\}_n$ is an increasing cofinal sequence in $\mathcal{OB}$. Each B_n is symmetric and $B_n^2 \subseteq B_{n+1}$. Now define a metric d such that for $x, y \in G$,

$$d(x, y) = \min(k \mid x^{-1}y \in B_k).$$

Let A be a d-bounded set. Then for any $x, y \in A$, there exists $C \in \mathbb{R}_+^*$ such that $d(x, y) \leqslant C$. Then $A \times A \in \mathcal{E}_d$. There exists $K > C$ such that for any $x, y \in A$ such that $x^{-1}y \in B_K$. Hence $A \times A \in \mathcal{E}_{\mathcal{OB}} = \mathcal{E}_L$ by Lemma 41. Thus A is coarsely bounded. Therefore $\mathcal{E}_L$ is metrised by d on G.

(iii) $\implies$ *(i)* : This follows by definition. □

Lemma 51. *Let G be a Baire topological group with metrisable left-coarse structure $\mathcal{E}_L$. Then G is locally bounded.*

Proof. $\mathcal{E}_L$ is metrisable, then by Lemma 50, the ideal $\mathcal{OB}$ is countably generated. So there is a countable cofinal family $\{A_n\}_n$. Thus $\{\overline{A_n}\}_n$ is cofinal in $\mathcal{OB}$. Moreover $\mathcal{OB}$ contains all the singletons, so

$$G = \bigcup_n \overline{A_n}.$$

Also G is Baire, so G is non-empty and open. Thus G is non-meager. Since $G = \cup_n \overline{A_n}$, there exists at least one of the $\overline{A_n}$, say $\overline{A_k}$ which is not meager. Therefore $\overline{A_k}$ has non-empty interior W. Moreover $V = WW^{-1}$ is an identity neighborhood. Furthermore since $W \in \mathcal{OB}$, $WW^{-1} \in \mathcal{OB}$. Thus $V \in \mathcal{OB}$. Hence V is coarsely bounded implying that G is locally bounded. □

Lemma 52. *Let G be a topological group and suppose that $\mathcal{E}_L$ is induced by a continuous left-invariant pseudometric d on G. Then G is locally bounded.*

Proof. Since the left-invariant pseudometric d is continuous, d is bounded on an identity neighborhood V. Moreover $V \times V \in \mathcal{E}_d = \mathcal{E}_L$, thus $V \times V \in \mathcal{E}_L$. Hence V is coarsely bounded and so G is locally bounded. □

Theorem 53. *For a European topological group G, the following are equivalent:*

(i) the left-coarse structure $\mathcal{E}_L$ is metrisable;

(ii) G is locally bounded;

(iii) $\mathcal{E}_L$ is induced by a continuous left-invariant pseudometric d on G.

Proof. *(i)* $\implies$ *(ii)*: follows from Lemma 51.

(ii) $\implies$ *(iii)*: follows from Lemma 49.

(iii) $\implies$ *(i)*: Let d be a continuous left-invariant pseudometric on G which induces $\mathcal{E}_L$. By Lemma 37, there exists ∂ that is a compatible and continuous left-invariant metric on G. Thus the left-coarse structure $\mathcal{E}_L$ is metrisable. □

Definition 54. *A coarse structure $(X, \mathcal{E})$ is* monogenic *if $\mathcal{E}$ is generated by a single entourage E.*

Theorem 55. *For a European topological group G, the following are equivalent:*

(i) the left-coarse structure $\mathcal{E}_L$ is monogenic;

(ii) G is generated by a coarsely bounded set, i.e. there is some $A \in \mathcal{OB}$ algebraically generating G;

(iii) G is locally bounded and not the union of a countable chain of proper open subgroups.

Proof. *(i)* $\implies$ *(iii)* : $\mathcal{E}_L$ is monogenic, so it is countably generated. Hence $\mathcal{E}_L$ is metrisable. By Theorem 53, G is locally bounded. Now suppose $G = \cup_n G_n$ where for all n, $G_n \subseteq G$ which are open subgroups. By Proposition 38, each coarsely bounded set is included in one the G_n. So there exists n such that $A \subseteq G_n$ and since G is generated by A,

$$G = \langle A \rangle \subseteq G_n.$$

Hence $A = G_n$.

$(iii) \implies (ii)$: Suppose G is locally bounded. Let V be an identity neighborhood coarsely bounded and $\{x_n\}_n$ countable set which generates G on V,

$$\text{i.e. } G = V \cup \{x_1, ..., x_n, ...\}.$$

Moreover G is not the union of a countable chain of proper open subgroups, thus G is generated by $V \cup \{x_1, ...x_n\}$. Let for $n \in \mathbb{N}$,

$$G_n = \langle V \cup \{x_1, ...x_n\} \rangle.$$

Each G_n is an open subgroup of G and the G_n's form an increasing exhaustive chain. Furthermore for any $g \in G$, there exists an n such that $g \in G_n$. Now $G = \cup_n G_n$ is possible only if there exists n such that

$$G = G_n = \langle V \cup \{x_1, ...x_n\} \rangle.$$

The latter is coarsely bounded. Hence G is generated by a coarsely bounded set.

$(ii) \implies (i)$: Suppose there exists $A \in \mathcal{OB}$ algebraically generating G, then $A \subseteq G$,

$$\text{i.e. } G = \bigcup_n A^n.$$

By the Baire theorem, some A^n must be somewhere dense and thus $B = \overline{A^n}$ is coarsely bounded with non-empty interior and it is generating G. If $C \subseteq G$ coarsely bounded, by Proposition 38, since $\text{int}(B) \neq \emptyset$, i.e B is an identity neighborhood, there exists F finite set such that $F \subseteq G$ and $k \geqslant 1$ such that $C \subseteq (FB)^k$. Since B generates G, $C \subseteq B^m$ for some $m \geqslant k$. Therefore $\{B^n\}_n$ is cofinal in $\mathcal{OB}$.

Hence $\mathcal{E}_{\mathcal{OB}}$ is monogenic. Indeed by replacing the generator $E \in \mathcal{E}$ in the definition of monogenic, by $E \cup \Delta$, we have that $\mathcal{E}$ is monogenic if and only if there is some entourage $E \in \mathcal{E}$ such that $\{E^n\}_n$ is cofinal in $\mathcal{E}$. Moreover $\mathcal{E}_{\mathcal{OB}}$ is a left-coarse structure, so the latter E is of the form E_A with A a coarsely bounded set. Since $E_A^n = E_{A^n}$, we have the following equivalence: there exists a coarsely bounded set B such that $\{B^n\}_n$ is cofinal in $\mathcal{OB}$ if and only if $\mathcal{E}_{\mathcal{OB}}$ is monogenic. Then by Lemma 41, $\mathcal{E}_L$ is monogenic. □

3.4 Final theorem

In this section, our aimed theorem will be proved thanks mainly to Theorem 55.

First we need a few geometric notions on coarse spaces and pseudometric spaces:

Definition 56. *Let $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ be coarse spaces and $\Phi : X \to Y$. Φ is called* bornologous *if $(\Phi \times \Phi)[\mathcal{E}] \subseteq \mathcal{F}$.*

Definition 57. *A continuous left-invariant pseudometric d on a topological group G is* coarsely proper *if d induces the left-coarse structure on G, i.e.* $\mathcal{E}_L = \mathcal{E}_d$.

Definition 58. *Let* (X, d_X) *and* (Y, d_Y) *be pseudometric spaces.*
A map $\Phi : X \to Y$ *is said to be a* quasi-isometric embedding *if there are positive constants K, C such that*

$$\frac{1}{K} \cdot d_X(x_1, x_2) - C \leqslant d_Y(\Phi x_1, \Phi x_2) \leqslant K \cdot d_X(x_1, x_2) + C.$$

Also Φ *is a* quasi-isometry *if, moreover there exists a positive C that for any* $y \in Y$*, there exists* $x \in X$ *such that*

$$d_Y(\Phi(x), y) \leqslant C.$$

A map $\Phi : X \to Y$ *is* Lipschitz for large distances *if there are positive constants K, C such that*

$$d_Y(\Phi x_1, \Phi x_2) \leqslant K \cdot d_X(x_1, x_2) + C.$$

Definition 59. *A* quasimetric space *is a set X equipped with a quasi-isometric equivalence class* $\mathcal{D}$ *of pseudometrics d on X which is defined by the quasi-isometry between two spaces. Moreover two pseudometrics d and* ∂ *on a set X are* quasi-isometric *if the identity map id :* $(X, d) \to (X, \partial)$ *is a quasi-isometry.*

Definition 60. *A pseudometric space (X,d) is* large scale geodesic *if there is* $K \geqslant 1$ *such that, for all* $x, y \in X$*, there are* $z_0 = x, z_1, z_2, ..., z_n = y$ *such that* $d(z_i, z_{i+1}) \leqslant K$ *and*

$$\sum_{i=0}^{n-1} d(z_i, z_{i+1}) \leqslant K \cdot d(x, y).$$

Definition 61. *A continuous left-invariant pseudometric d on a topological group G is* maximal *if for every other continuous left-invariant pseudometric* ∂*, there are constants K,C such that* $\partial \leqslant K \cdot d + C$.

Definition 62. *Let G be a topological group admitting a maximal pseudometric. The* quasimetric structure *on G is the quasi-isometric equivalence class of its maximal pseudometrics.*

Definition 63. *If* Σ *is a symmetric generating set for a topological group G, then its associated* word metric $\rho_\Sigma : G \longrightarrow \mathbb{N}$ *is defined by*

$$\rho_\Sigma(g, h) = \min(k \geqslant 0 \mid \exists\ s_1, ..., s_k \in \Sigma\ g = hs_1...s_k).$$

The next lemma has been adapted from Theorem 1.4.13 (p.48) of the following paper [Han14] written by Bernhard Hanke, Piotr Nowak and Guoliang Yu.

Lemma 64. *Let* $\Phi : X \longrightarrow Y$ *be a bornologous map between quasi-metric spaces* (X, d_X) *and* (Y, d_Y) *and assume* (X, d_X) *is large scale geodesic. Then* Φ *is Lipschitz for large distances.*

Proof. Since (X, d_X) is large scale geodesic, there exists $K \geqslant 1$ such that for any $x, y \in X$, there exists $z_0 = x, z_1, ..., z_n = y$ such that

$$d_X(x_i, x_{i+n}) \leqslant K \text{ and } \sum_{i=0}^{n-1} d_X(x_i, x_{i+1}) \leqslant K \cdot d_X(x, y).$$

First by triangle inequality, we have for $x, y \in X$ that

$$d_Y(\Phi(x), \Phi(y)) \leqslant \sum_{i=0}^{n-1} d_Y(\Phi(x_i), \Phi(x_{i+1})).$$

Moreover since Φ is bornologous, for $E \in \mathcal{E}_{d_X}$,

$$E = \{(x_1, x_2) \in X^2 \mid d_X(x_1, x_2) \leqslant K\}$$

and then $\Phi(E) \in \mathcal{E}_{d_Y}$. Thus there exists $K \geqslant 1$ such that

$$\Phi(E) \subseteq \{(y_1, y_2) \in Y^2 \mid d_Y(y_1, y_2) \leqslant K'\}.$$

Therefore

$$d_Y(\Phi(x), \Phi(y)) \leqslant K' \sum_{i=0}^{n-1} d_X(x_i, x_{i+1}) \leqslant K'K \cdot d_X(x, y).$$

□

Lemma 65. *For a continuous left-invariant pseudometric d on a topological group G, the following are equivalent:*
(i) d is coarsely proper;
(ii) a set $A \subseteq G$ is coarsely bounded if and only if it is d-bounded;
(iii) for every left-invariant pseudometric ∂ on G, the map

$$\text{id} : (G, d) \longrightarrow (G, \partial)$$

is bornologous.

Proof. $(i) \implies (iii)$: Since d is coarsely proper,

$$\mathcal{E}_d = \mathcal{E}_L = \bigcap\{\mathcal{E}_\partial \mid \partial \text{ continuous left-invariant pseudometric on } G\}$$

So for any ∂ continuous left-invariant on G, $\mathcal{E}_d \subseteq \mathcal{E}_\partial$. Thus

$$(\text{id} \times \text{id})[\mathcal{E}_d] \subseteq \mathcal{E}_\partial.$$

Hence

$$\text{id} : (G, d) \longrightarrow (G, \partial)$$

is bornologous.

$(ii) \implies (iii)$: Let ∂ be a continuous left-invariant pseudometric. To show that $\text{id} : (G, d) \longrightarrow (G, \partial)$ is bornologous. Let $A \subseteq G$ d-bounded, then by (ii) A is coarsely bounded, so A is ∂-bounded. By Lemma 40, the coarse structure induced by d is included in the one induced by ∂. Then $\mathcal{E}_d \subseteq \mathcal{E}_\partial$. Hence the identity map from (G, d) to (G, ∂) is bornologous.

$(iii) \implies (ii)$: Suppose that for any continuous left-invariant pseudometric ∂, the identity map from (G, d) to (G, ∂) is bornologous Then in particular, for $A \subseteq G$ d-bounded and for a continuous left-invariant pseudometric ∂,

$$A \ d\text{-bounded} \Leftrightarrow A \times A \in \mathcal{E}_d \Rightarrow A \times A \in \mathcal{E}_\partial \Rightarrow A \ \partial\text{-bounded}.$$

Thus we have shown that if A is d-bounded then $A \times A \in \mathcal{E}_\partial$ for any continuous left-invariant pseudometric ∂ which is equivalent to $A \times A \in \mathcal{E}_L$ i.e. A is coarsely bounded.

$(i) \iff (ii)$: if a d-ball of radius R is contained in a ∂-ball of radius S, then

$$d(x, y) = d(1, x^{-1}y) < R \Rightarrow \partial(x, y) = \partial(1, x^{-1}y) < S$$

By Lemma 40, d is coarsely proper is equivalent to a set $A \subseteq G$ is coarsely bounded when it is d-bounded. □

Lemma 66. *Suppose d is a compatible left-invariant metric on a topological group G and V is a symmetric open identity neighborhood generating G containing 1 and having finite d-diameter. Define*

$$\partial(f, h) = \inf\left(\sum_{i=1}^{n} d(g_i, 1) \mid g_i \in V, f = hg_1...g_n\right).$$

Then ∂ is a compatible left-invariant metric, quasi-isometric to the word metric ρ_V.

Proof. Firstly ∂ is left-invariant, moreover V is open and d is continuous, thus ∂ is also continuous. Since $\partial \geqslant d$, ∂ is a compatible metric on G.
Then we show the last part: ∂ is quasi-isometric to the word metric ρ_V.
For $f, h \in G$, let $n = \rho_V(f, h)$, $f = hg_1...g_n$ with $g_i \in V$. Since $g_i \in V$ and $1 \in V$, $d(g_i, 1) \leqslant \text{diam}_d(V)$. So,

$$\sum_{i=1}^{n} d(g_i, 1) \leqslant n \cdot \text{diam}_d(V),$$

$$\text{i.e. } \partial(f, h) \leqslant \sum_{i=1}^{n} d(g_i, 1) \leqslant \rho_V(f, h) \cdot \text{diam}_d(V).$$

Now pick $\varepsilon > 0$ such that

$$\{g \in G \mid d(g, 1) < 2\varepsilon\} \subseteq V.$$

We fix $f, h \in G$ and take the shortest sequence such that for $g_i \in V$ with $i \in [\![0, n]\!]$,

$$f = hg_1...g_n \text{ and } \sum_{i=1}^{n} d(g_i, 1) \leqslant \partial(f, h) + 1.$$

Then we have $g_i g_{i+1} \notin V$. Otherwise a sequence with $g_i g_{i+1}$ instead of g_i and g_{i+1} would be a shorter sequence since $d(g_i g_{i+1}, 1) \leqslant d(g_i, 1) + d(g_{i+1}, 1)$.
Let $d(g_i, 1) \geqslant \varepsilon$ and $d(g_{i+1}, 1) \geqslant \varepsilon$. Then there are at least $\frac{n-1}{2}$ g_i such that $d(g_i, 1) \geqslant \varepsilon$. Therefore

$$\frac{n-1}{2} \cdot \varepsilon \leqslant \sum_{i=1}^{n} d(g_i, 1) \leqslant \partial(f, h) + 1.$$

Since $\rho_V(f,h) \leqslant n$,

$$\frac{\rho_V(f,h)-1}{2}\cdot\varepsilon \leqslant \sum_{i=1}^{n} d(g_i,1) \text{ and so}$$
$$\frac{\varepsilon\rho_V(f,h)}{2} - \frac{\varepsilon}{2} - 1 \leqslant \partial(f,h) \leqslant \mathrm{diam}_d(V)\cdot\rho_V(f,h).$$

Hence ∂ and ρ_V are quasi-isometric. □

Proposition 67. *For a continuous left-invariant pseudometric d on a topological group G, the following are equivalent:*

(i) d is maximal;

(ii) d is coarsely proper and (G,d) is large scale geodesic;

(iii) d is quasi-isometric to the word metric ρ_A given by a coarsely bounded symmetric generating set $A \subseteq G$.

Proof. $(ii) \implies (i)$: Suppose $\partial \neq d$ is a continuous left-invariant pseudometric on G. By Lemma 65, $\mathrm{id} : (G,d) \longrightarrow (G,\partial)$ is bornologous. Then by Lemma 64, id is Lipschitz for large distances. Thus d is maximal.

$(i) \implies (iii)$: Claim: G is generated by a closed ball

$$B_k = \{g \in G \mid d(g,1) \leqslant k\}.$$

Suppose it is false, then G is the union of an increasing chain of proper open sub-groups $V_n = \langle B_n \rangle$ for $n \geqslant 1$. We now add the negative indexes to the chain $V_n = \langle B_n \rangle$ for $n \geqslant 1$ with

$$V_0 \supseteq V_{-1} \supseteq V_{-1} \supseteq ... \ni 1,$$

where the V_{-1} are symmetric and open such that $V_n^3 \subseteq V_{n+1}$. From Lemma 37, we get

$$\partial(f,g) = \inf\left(\sum_{i=0}^{k-1}\delta(h_i,h_{i+1}) \mid h_0 = f, h_k = g\right).$$

Since d is maximal, there exists $K, C > 0$ such that

$$\partial(f,g) \leqslant Kd(f,g) + C \text{ for all } f,g.$$

Since $B_n \setminus V_{n-1} \neq \emptyset$ for infinitely many $n \geqslant 1$, for $g \in B_n \setminus V_{n-1} \subseteq V_n \setminus V_{n-1}$, there exists an infinity of n such that $\partial(g,1) \geqslant 2^{n-1}$ and $d(g,1) \leqslant n$. Then

$$2^{n-1} \leqslant \partial(g,1) \leqslant Kn + C,$$

for infinitely many n. This cannot be. Therefore

$$G = V_k = \langle B_k \rangle \text{ for a } k \geqslant 1.$$

Let

$$\partial'(f,g) = \inf\left(\sum_{i=1}^{n} d(g_i,1) \mid g_i \in B_k, f = hg_1...g_n\right)$$

from Lemma 66 where $V = B_k$. Thus $d \leqslant \partial'$. Moreover since d is maximal, there exists $K, C > 0$ such that $\partial' \geqslant Kd + C$. Therefore d and ∂' are quasi-isometric. By Lemma 66, ∂' is quasi-isometric to the word metric ρ_{B_k}. Thus d

and ρ_{B_k} are quasi-isometric.
We need to check that B_k is coarsely bounded. Let d' a continuous left-invariant pseudometric, then there exists $K, C > 0$ such that $d' \leqslant Kd + C$. If $x, y \in B_k$, then $d(x, y) \leqslant 2k$. So

$$d'(x, y) \geqslant 2kK + C$$

and in particular, B_k is d-bounded. Therefore B_k is coarsely bounded.

$(iii) \implies (ii)$: ρ_A is the shortest path on the Cayley graph of G with respect to the symmetric generating $A \subseteq G$. Indeed the Cayley graph has vertexes G and for $g, h \in G$, g is related to h by an edge if and only if there exists $a \in A$ such that $g = ha$.
For $x, y \in G$, let

$$d_c(x, y) = \min\{n \in \mathbb{N} \mid \exists x_0 = x, x_1, ..., x_{n-1}, x_n = y \text{ where } (x_i, x_{i+1}) \text{ is an edge}\}.$$

Then since (x_i, x_{i+1}) is an edge, $d_c(x_i, x_{i+1}) \leqslant 1$. Thus

$$\sum_{i=0}^{n-1} d_c(x_i, x_{i+1}) \leqslant d_c(x, y).$$

Therefore (G, ρ_A) is large scale geodesic. Moreover d is quasi-isometric to ρ_A, so (G, d) is large scale geodesic.
Moreover each d-bounded set is ρ_A-bounded. Therefore these sets are in A^n for an n. Then they are coarsely bounded. Hence d is coarsely proper. □

Using the previous results, we are now able to prove our main theorem.

Theorem 68. *For a European topological group G, the following are equivalent:*
- *(i) G admits a continuous left-invariant maximal pseudometric d;*
- *(ii) G is generated by a coarsely bounded set;*
- *(iii) G is locally bounded and not the union of a countable chain of proper open subgroups;*
- *(iv) the coarse structure is monogenic.*

Proof. Thanks to Theorem 55, we already have $(ii) \iff (iii) \iff (iv)$.

$(ii) \implies (i)$: Let d be a continuous left-invariant pseudometric admitted by G. G is generated by a coarsely bounded set, so it is generated by a d-bounded subset. Then this subset has finite d-dimension. So there exists $k \in \mathbb{R}$ such that G is generated by the open d-ball:

$$V = \{x \in G \mid d(1, x) < k\}.$$

Taking the ∂ of Lemma 66, we have ∂ is quasi-isometric to the word metric ρ_V. Thus by Lemma 66, V is coarsely bounded generating G. Therefore by Proposition 67, ∂ is maximal.

$(i) \implies (ii)$: Let d be a maximal pseudometric on G. By Proposition 67, d is quasi-isometric to the word metric ρ_A with A coarsely bounded set generating G. □

4 $\mathcal{S}(\mathbb{Z},\mathbb{N})$ is locally bounded

In this section, we will show that $\mathcal{S}(\mathbb{Z},\mathbb{N})$ is locally bounded. Thanks to Theorem 68, we have a characterization of such a result. Thanks to this theorem, we need to show that $\mathcal{S}(\mathbb{Z},\mathbb{N})$ admits a continuous left-invariant maximal pseudometric d. The perfect candidate for d is the pseudometric defined in [Cor16] at the top of page 24. For a set X and M a subset of X, we have the following general definition.

Definition 69. *For $g, h \in \mathcal{S}(X, M)$, $d_M(g,h) = |gM \triangle hM|$ is a left-invariant pseudometric on $\mathcal{S}(X,M)$.*

We are only interested in $\mathcal{S}(\mathbb{Z},\mathbb{N})$. So let us show that

Theorem 70. *$d_{\mathbb{N}}(g,h) = |g\mathbb{N} \triangle h\mathbb{N}|$ is maximal on $\mathcal{S}(\mathbb{Z},\mathbb{N})$.*

Thanks to Proposition 67, we need to show that $d_{\mathbb{N}}$ is coarsely proper and $(\mathcal{S}(\mathbb{Z},\mathbb{N}), d_{\mathbb{N}})$ is large scale geodesic.

4.1 Proof on $\mathcal{S}^o(\mathbb{Z},\mathbb{N})$

First we are going to show it on $\mathcal{S}^o(\mathbb{Z},\mathbb{N})$ because it is an easier case. We will then use it to prove the result on $\mathcal{S}(\mathbb{Z},\mathbb{N})$.

To show that $(\mathcal{S}^o(\mathbb{Z},\mathbb{N}), d_{\mathbb{N}})$ is large scale geodesic, it is enough to show that for k and e the neutral element,

$$B_d(e,k) \subseteq B_d(e,2)^{2k}.$$

Since $\mathcal{S}_0^+(\mathbb{Z})$ the set of finite support permutations is dense in $\mathcal{S}^o(\mathbb{Z},\mathbb{N})$ and $B_d(e,0)$ is open, it is enough to show

$$B_d(e,k) \cap \mathcal{S}_0^+(\mathbb{Z}) \subseteq B_d(e,2)^{2k}.$$

Let $\sigma \in \mathcal{S}_0^+(\mathbb{Z})$ and $k \in \mathbb{R}_+$ such that $d_{\mathbb{N}}(\sigma, e) = k$. Then $|\sigma\mathbb{N} \triangle \mathbb{N}| = k$. Since σ has finite support, there exists $\sigma_1, \ldots, \sigma_p$ cyclic permutations with disjoint supports such that $\sigma = \sigma_1 \cdots \sigma_p$. For $j \in \{1, \ldots, p\}$,

∗ if σ_j has its support included in $\mathbb{N}^c$ or $\mathbb{N}$ then $d_{\mathbb{N}}(\sigma_j, e) = 0$,

∗ otherwise $\sigma_j = (a_1 \cdots a_n)$ where $a_i \in \mathbb{Z}$. Let

$$\begin{aligned} F &= \{b_1, \ldots, b_k \in \mathbb{N} \text{ such that } \sigma(b_i) \leqslant 0\} \cup \{c_1, \ldots, c_k \in \mathbb{N}^c \text{ such that } \sigma(c_i) \geqslant 0\} \\ &= \{f_1, \ldots, f_{2k}\}. \end{aligned}$$

Then there is only a finite number of those f_i in the σ_j, say $l \leqslant 2k$. Then

$$\begin{aligned} \sigma_j &= (\text{ — } f_1 \text{ — } f_2 \text{ — } f_3 \text{ — } \cdots \text{ — } f_l \text{ — }) \\ &= (\text{ — } f_1)(f_1 \text{ — } f_2)(f_2 \text{ — } f_3) \ \cdots \ (f_l \text{ — }) \end{aligned}$$

where each — means that there are some $a_i \neq f_m$ for any $m \in \{1, \ldots, 2k\}$. So first we have that

$$d_{\mathbb{N}}((\text{ — } f_1), e) = d_{\mathbb{N}}((f_l \text{ — }, e) = 0.$$

Let us compute it for the cycle $(— f_1)$. In the latter, we know that all the $a_i \neq f_1$ are all of the same sign. Also σ sends the element before f_1 onto f_1 so f_1 is of the same sign too. Thus all the elements of $(— f_1)$ are of the same sign. Therefore $d_{\mathbb{N}}((— f_1), e) = 0$. Second we have the following

$$d_{\mathbb{N}}((f_i — f_{i+1}), e) = 2,$$

since two elements are sent from $\mathbb{N}$ to $\mathbb{N}^c$ or the other direction.

Since σ is the product of σ_j, we have:

$$B_d(e, k) \subseteq B_d(e, 2)^{2k}. \tag{8}$$

By Lemma 65, $d_{\mathbb{N}}$ is coarsely proper if and only if every $d_{\mathbb{N}}$-bounded set A is coarsely bounded. Now thanks to Proposition 38, it is equivalent to for every identity neighborhood V, there is a finite set $F \subseteq G$ and $k \geqslant 1$ such that $A \subseteq (FV)^k$. Equation 8 is used to reduce to the case $A = B_d(e, 2)$ which is a subset of G. Let

$$V = \{\sigma \in \mathcal{S}^o(\mathbb{Z}, \mathbb{N}) \ : \ \sigma(P_i) = P_i \text{ where } i \in \{1, ..., k\},\ P_i \mid |P_i \triangle \mathbb{N}| < \infty\}.$$

There exists K such that $|P_i \triangle \mathbb{N}| \subseteq \{-K, ..., K\}$. Hence

$$V \supseteq \widetilde{V} = \{\sigma \in V \ : \ \sigma(i) = i \ \forall i \in \{-K, ..., K\}\}.$$

Moreover

$$\widetilde{V} \cong \operatorname{Sym}(]\!] - \infty, -K-1]\!]) \times \operatorname{Sym}([\![K+1, +\infty[\![)$$

since the values outside of these two intervals do not matter as long as they stay either in $\mathbb{N}$ or $\mathbb{N}^c$.

Furthermore $\operatorname{Sym}(]\!] - \infty, -K-1]\!])$, respectively $\operatorname{Sym}([\![K+1, +\infty[\![)$ is an open subgroup of $\operatorname{Sym}(\mathbb{N}^c)$, respectively of $\operatorname{Sym}(\mathbb{N})$.

We start with an easier case: let $\sigma \in B_d(e, 0)$ and $A = \sigma([0, K[)$. Then there exists $\tau \in V$ such that $\tau(A \subseteq [0, 2K[)$. Thus

$$B = \tau\sigma\left([0, K[\right) \subseteq [0, 2K[.$$

Now we construct $\tau' \in \operatorname{Sym}([0, 2K[)$ such that $\tau'(B) = [0, K[$.

Since for $x \in [0, K[$, $\tau\sigma(x) \in [0, 2K[$, we define τ' by $\tau'(\tau\sigma(x)) = x$ for all $x \in [0, 2K[$. Then we extend τ' arbitrarily such that $\tau' \in \operatorname{Sym}([0, 2K[)$. We have $\tau'\tau\sigma \in \operatorname{Sym}([0, 2K[)$ where $\tau' \in \operatorname{Sym}([0, 2K[)$, $\tau \in V$. So $\tau\sigma \in \operatorname{Sym}([0, 2K[)\ V$. Thus $\sigma \in V\ \operatorname{Sym}([0, 2K[)\ V$. Hence

$$B_d(e, 0) \subseteq V\ \operatorname{Sym}([0, 2K[)\ V.$$

Since $\operatorname{Sym}([0, 2K[)$ is a finite subset, we found F such that

$$B_d(e, 0) \subseteq VFV.$$

Then we show the case which is the one we need. Let $\sigma \in B_d(e, 2)$. Let $a < 0$ such that $\sigma(a) \geqslant 0$ and $b \geqslant 0$ such that $\sigma(b) < 0$. Moreover let $A = \sigma(]-K, K[)$. Then there exists $\tau \in \widetilde{V}$ such that $\tau(A) \subseteq]-2K, 2K[$. Our aim is to have for $K \geqslant 0$ such that

$$\tau \in \widetilde{V},\ \tau(\{a, b, \sigma(a), \sigma(b)\}) \subseteq]-2K, 2K[.$$

We set $\sigma' = \tau\sigma\tau^{-1}$. Then $\tau(a) \mapsto \tau\sigma(a)$ and $\tau(b) \mapsto \tau\sigma(b)$ by σ'. By composing by σ' if needed, we can suppose that $a, \sigma(a), b$ and $\sigma(b)$ are in $]-2K, 2K[$. Thus

$$\text{there exists } \tau \in \text{Sym}(]-2K, 2K[) \text{ such that } \tau\sigma(\mathbb{N}) = N.$$

Hence $\tau\sigma \in B_d(e, 0)$. Since $B_d(e, 0) \subseteq VFV$ as shown above, $\tau\sigma \in VFV$. Moreover since $\tau \in \widetilde{V} \subseteq V$, $\sigma \in VFV$. Thus

$$B_d(e, 2) \subseteq VFV.$$

Therefore $d_{\mathbb{N}}$ is maximal on $\mathcal{S}^o(\mathbb{Z}, \mathbb{N})$.

□

4.2 Proof on $\mathcal{S}(\mathbb{Z}, \mathbb{N})$

We have proven that $d_{\mathbb{N}}$ is maximal on $\mathcal{S}^o(\mathbb{Z}, \mathbb{N})$. So now we are going to prove it on $\mathcal{S}(\mathbb{Z}, \mathbb{N})$.

We first prove that $(\mathcal{S}(\mathbb{Z}, \mathbb{N}), d_{\mathbb{N}})$ is large scale geodesic. Let $\sigma \in \mathcal{S}(\mathbb{Z}, \mathbb{N})$, $k = |\sigma\mathbb{N} \triangle \mathbb{N}|$ and $t : n \mapsto n+1$. Then $\text{tr}(t) = 1$ and $\text{d}_{\mathbb{N}}(t, \text{id}) = 1$. Let $i = \text{tr}(\sigma)$ and $\tau = \sigma t^{-i}$. Then since tr is a morphism, we have

$$\text{tr}(\tau) = \text{tr}(\sigma) + \text{tr}(t^{-i}) = i - i = 0.$$

Thus $\tau \in \mathcal{S}^o(\mathbb{Z}, \mathbb{N})$. Moreover

$$\begin{aligned} d_{\mathbb{N}}(\tau, \text{id}) &\leqslant d_{\mathbb{N}}(\tau, \sigma) + d_{\mathbb{N}}(\sigma, \text{id}) \\ &\leqslant d_{\mathbb{N}}(\sigma, \text{id}) + d_{\mathbb{N}}(t^{-i}, \text{id}) \\ &\leqslant k + |i| \\ &\leqslant k + k \\ &\leqslant 2k. \end{aligned}$$

Thus τ is the product of at most $2k$ elements with distance smaller or equal than 2 to the identity. Since $\sigma = \tau t^i$, σ is the product of at most $3k$ elements with distance smaller or equal than 2 to the identity.

Second we prove that $d_{\mathbb{N}}$ is coarsely proper. Let $r \geqslant 1$. The aim is to show that $B_d(\text{id}, r)$ is bounded, i.e. for any open identity neighborhood V there exists a finite subset $F \subseteq G$ and an n such that $B_d(\text{id}, r) \subseteq (VF)^n$.
Since $\mathcal{S}^o(\mathbb{Z}, \mathbb{N})$is an open subgroup of $\mathcal{S}(\mathbb{Z}, \mathbb{N})$, we can suppose that $V \subseteq \mathcal{S}^o(\mathbb{Z}, \mathbb{N})$. From what we have done above, we already know that there exists n and a finite subset $F \in \mathcal{S}^o(\mathbb{Z}, \mathbb{N})$ such that

$$B_d(\text{id}, r) \cap \mathcal{S}^o(\mathbb{Z}, \mathbb{N}) \subseteq (VF)^n.$$

By letting $\widetilde{F} = F \cup \{t^{-r}, ..., t^r\}$, we have that

$$B_d(\text{id}, r) \subseteq (V\widetilde{F})^{n+1}.$$

Indeed, if $\sigma \in B_d(\text{id}, r)$,

$$|\text{tr}(\sigma)| \leqslant d(r, \text{id}) \leqslant r.$$

Then $t^{\mathrm{tr}(\sigma)} \in \widetilde{F}$ and $\sigma = \tau t^{\mathrm{tr}(\sigma)}$ by letting $\tau = \sigma t^{-\mathrm{tr}(\sigma)}$. So $\mathrm{tr}(\tau) = 0$. Thus $\tau \in \mathcal{S}^o(\mathbb{Z}, \mathbb{N})$. Therefore $\tau \in (VF)^n$. Hence

$$\sigma \in (VF)^n \widetilde{F} \subseteq (VF)^{n+1}.$$

Therefore $d_{\mathbb{N}}$ is maximal on $\mathcal{S}(\mathbb{Z}, \mathbb{N})$.

Thus we have proven that $\mathcal{S}(\mathbb{Z}, \mathbb{N})$ is locally bounded.

□

5 Embedding in $\mathcal{S}(\mathbb{Z}, \mathbb{N})$

We want to know what can be embedded in $\mathcal{S}(\mathbb{Z}, \mathbb{N})$.
The aim of this part is to show that $\mathbb{Z}^k$ can be embedded in $\mathcal{S}(\mathbb{Z}, \mathbb{N})$ where $\mathbb{Z}^k =< e_1, ..., e_k >$ vectors of the canonical basis of $\mathbb{Z}^k$.

On $\mathbb{Z}^k$, there is the l^1 metric which is the distance to the generators defined for $(n_1, ..., n_k), (m_1, ..., m_k) \in \mathbb{Z}^k$ by

$$d_{l^1}((n_1, ..., n_k), (m_1, ..., m_k)) = \sum_{i=1}^{k} |n_i - m_i| .$$

We want to find a map $\rho : \mathbb{Z}^k \to \mathcal{S}(\mathbb{Z}, \mathbb{N})$ such that

$$d_{l^1}((n_1, ..., n_k), (m_1, ..., m_k)) = d_{\mathbb{N}}(\rho(n_1, ..., n_k), \rho(m_1, ..., m_k)).$$

Let $\rho(e_i) = \tau_i$ for $1 \leqslant i \leqslant k$ where

$$\tau_i(x) = \begin{cases} x + k \text{ if } x \equiv i[k], \\ x \text{ otherwise.} \end{cases}$$

For $(n_1, ..., n_k) \in \mathbb{Z}^k$ and $p \in \mathbb{N}$,

$$\begin{aligned} \rho(n_1, ..., n_k)(p) &= (\tau_1^{n_1} \circ \cdots \circ \tau_k^{n_k})(p) \\ &= p + n_i k \text{ where } i \text{ is the only element of } \{1, ..., k\} \text{ such that} \\ &\quad p \equiv i[k]. \end{aligned}$$

Now we prove that d_{l^1} is also left-invariant. Indeed for $m = (m_1, ..., m_k)$, $n = (n_1, ..., n_k), p = (p_1, ..., p_k) \in \mathbb{Z}^k$,

$$\begin{aligned} d_{l^1}(p + n, p + m) &= \sum_{i=1}^{k} |(p_i + n_i) - (p_i + m_i)| \\ &= \sum_{i=1}^{k} |p_i + n_i - p_i - m_i| \\ &= \sum_{i=1}^{k} |n_i - m_i| \\ &= d_{l^1}(n, m). \end{aligned}$$

Since d_{l^1} and $d_{\mathbb{N}}$ are left-invariant, it is enough to show for any $(n_1, ..., n_k) \in \mathbb{Z}^k$ that

$$d_{l^1}((n_1, ..., n_k), (0, ..., 0)) = d_{\mathbb{N}}(\rho(n_1, ..., n_k), \mathrm{id}).$$

Indeed for any $(m_1, ..., m_k) \in \mathbb{Z}^k$,

$$d_{l^1}((n_1, ..., n_k), (0, ..., 0)) = d_{l^1}((m_1, ..., m_k) + (n_1, ..., n_k), (m_1, ..., m_k)).$$

Moreover

$$\begin{aligned} & d_{l^1}((n_1, ..., n_k), (0, ..., 0)) = d_{\mathbb{N}}(\rho(n_1, ..., n_k), \mathrm{id}) \\ \Leftrightarrow & \sum_{i=1}^{k} |n_i| = |\rho(n_1, ..., n_k)\mathbb{N} \triangle \mathbb{N}|. \end{aligned}$$

So we need to show the above. To do that we fix $i \in \{0, ..., k-1\}$. Let

$$A_i = \{p \in \mathbb{Z} \ : \ p \equiv i[k]\}.$$

Each A_i is ρ-invariant: for any $q \in \mathbb{Z}$,

$$\rho(n_1, ..., n_k)(i + qk) = i + (q + n_i)k.$$

Hence $A_i = \{i + qk \ : \ q \in \mathbb{Z}\}$. Let $B_i = \{i + qk \ : \ q \in \mathbb{N}\} = A_i \cap \mathbb{N}$. Thus

$$\mathbb{N} = B_o \sqcup \cdots \sqcup B_{k-1}.$$

Clearly $\rho(n_1, ..., n_k)(B_i) = n_i k + B_i \subseteq A_i$. Moreover we have

$$\begin{aligned}\rho(n_1, ..., n_k)\mathbb{N} \triangle \mathbb{N} &= \bigsqcup_i \rho(n_1, ..., n_k)B_i \triangle \bigsqcup_j B_j \\ &= \bigsqcup_{i,j=0}^{k-1} \rho(n_1, ..., n_k)B_i \triangle B_j.\end{aligned}$$

If $i \neq j$ then $\rho(n_1, ..., n_k)B_i \cap B_j = \emptyset$. Therefore $\rho(n_1, ..., n_k)B_i \triangle B_j = \emptyset$. Hence

$$\rho(n_1, ..., n_k)\mathbb{N} \triangle \mathbb{N} = \bigsqcup_{i=0}^{k-1} \rho(n_1, ..., n_k)B_i \triangle B_i.$$

Now we look at the n_i's for each B_i.

The first case is if $n_i \geqslant 0$:
$\rho(n_1, ..., n_k)B_i = \{i + (n_i + q)k \ : \ q \in \mathbb{N}\} \subseteq B_i$.

$$\text{So } \rho(n_1, ..., n_k)B_i \triangle B_i = B_i \setminus \rho(n_1, ..., n_k)B_i.$$

If $x \in B_i$, then $x = i + qk$ where $q \in \mathbb{N}$. So if $x \in \rho(n_1, ..., n_k)B_i$, then $x = i + (q' + n_i)k$ with $q' \in \mathbb{N}$. Thus

$$i + qk = i + (q' + n_i)k.$$

Hence $q = q' + n_i$. The reasoning also goes from bottom to top.
So $x \in \rho(n_1, ..., n_k)B_i$ if and only if $q' = q - n_i > 0$. Thus

$$x \in B_i \setminus \rho(n_1, ..., n_k)B_i \Leftrightarrow q - n_i < 0 \Leftrightarrow q < n_i.$$

Therefore $\rho(n_1, ..., n_k)B_i \triangle B_i = \{i + qk \ : \ q \in \{0, ..., n-1\}\}$. Hence

$$|\rho(n_1, ..., n_k)B_i \triangle B_i| = |n_i|.$$

The second case is if $n_i < 0$:
$\rho(n_1, ..., n_k)B_i = \{i + (n_i + q)k \ : \ q \in \mathbb{N}\} \supseteq B_i$. So

$$\begin{aligned}\rho(n_1, ..., n_k)B_i \triangle B_i &= \rho(n_1, ..., n_k)B_i \setminus B_i \\ &= \{i + (n_i + q)k \ : \ q \in \mathbb{N}, \ n_i + q < 0\} \\ &= \{i + (n_i + q)k \ : \ q \in \{0, ..., -n-1\}\}.\end{aligned}$$

Therefore we also have $|\rho(n_1, ..., n_k)B_i \bigtriangleup B_i| = |n_i|$. Hence

$$|\rho(n_1, ..., n_k)\mathbb{N} \bigtriangleup \mathbb{N}| = \sum_{i=1}^{k} |n_i|.$$

Therefore $\rho : \mathbb{Z}^k \to \mathcal{S}(\mathbb{Z}, \mathbb{N})$ is an isometric embedding. Hence $\mathbb{Z}^k \leqslant \mathcal{S}(\mathbb{Z}, \mathbb{N})$. This embedding would then allow us to find the asymptotic dimension of $\mathcal{S}(\mathbb{Z}, \mathbb{N})$. This is done in the third section by Bell and in Dranishnikov [BD08].

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