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Model theory for pmp hyperfinite actions

by

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Abstract

In this paper we generalize the work of Berenstein and Henson about model theory of probability spaces with an automorphism in [BH04] by studying model theory of probability spaces with a countable group acting by automorphisms. We use continuous model theory to axiomatize the class of probability algebras endowed with an action of the countably generated free group F_∞ by automorphisms, and we can then study probability spaces through the correspondence between separable probability algebras and standard probability spaces.

Given an IRS (invariant random subgroup) θ on F_∞, we can furthermore axiomatize the class of probability algebras endowed with an action of the countably generated free group F_∞ by automorphisms whose IRS is θ. We prove that if θ is hyperfinite, then this theory is complete, model complete and stable and we show that the stable independence relation is the classical probabilistic independence of events.

In order to do so, we give a shorter proof of the result of Gábor Elek stating that given a hyperfinite IRS θ on a countable group Γ, every orbit of the conjugacy relation on the space of actions of Γ with IRS θ is dense for the uniform topology.

Contents

1 Introduction

Section 2 deals with graphs. We recall the definition of a graphing and prove some basic properties about it. In particular, we give a proof of Aldous and Lyons that for every convergent sequence of finite graphs, there is a graphing which is a limit of this sequence, using the Bernoullization of a measure.

In Section 3 we focus our attention on the notion of hyperfiniteness of graphings. First we recall a well known result about hyperfinite graphings : every hyperfinite graphing is a limit of a convergent sequence of finite graphs. Then we present a version of a theorem of Gábor Elek ([Ele12]) stating that hyperfiniteness is an invariant of statistical equivalence (see Definition 2.9), using the general notion of Bernoullization of a graphing.

In Section 4 we present the corollary of Rokhlin Lemma which is the key point in the study of probability spaces with an automorphism conducted in [BH04]. The goal of this section is to prove a generalization to hyperfinite pmp actions of this corollary that we call the big theorem of this paper. Namely we prove that if θ is a hyperfinite IRS on a countable group Γ, then for any two pmp action $\alpha : \Gamma \hookrightarrow (X, \mu)$ and $\beta : \Gamma \hookrightarrow (Y, \nu)$ on standard probability spaces, $F \subseteq \Gamma$ finite and $\varepsilon > 0$ there is a pmp bijection $\rho : X \to Y$ such that

$$\mu(\{x \in X : \forall \gamma \in F,\ \gamma^{\beta} \circ \rho(x) = \rho \circ \gamma^{\alpha}(x)\}) > 1 - \varepsilon$$

In other words, the relation of conjugacy on the Polish space of pmp actions of Γ on a standard probability space having IRS θ has dense orbits.

Moreover we prove a stronger version of the latter theorem involving the stabilization of set of Borel parameters, when α is a factor of β (see Definition 4.11).

Finally in Section 5 we use the tools of Section 4 to study model theory of atomless probability algebras with a countable group Γ acting by automorphisms. In fact, without loss of generality, we restrict our study to actions of F_∞ the countably generated free group as any action of a countable group can be seen as an action of F_∞.

First we recall the correspondence between separable models of the latter theory and pmp actions of F_∞ on standard probability spaces using the classical correspondence between separable atomless probability algebras and standard probability spaces and then applying lifting theorems. For any IRS θ on F_∞ we define a theory $\mathfrak{A}_\theta$ representing pmp actions with IRS θ, after proving that the IRS is indeed expressible in the first order.

Then the big theorem of Section 4 and the stronger version we already talked about allow us to prove that these theories for θ hyperfinite are complete and model complete by repeating the proofs found in [BH04]. However, there is a small subtlety for quantifier elimination. Indeed in general the theories of the form $\mathfrak{A}_\theta$ for θ hyperfinite do not admit quantifier elimination. We prove that we can still get elimination of quantifiers by adding the supports (see Definition 5.10) of the automorphisms to the signature of the theory, and we use this to prove that our theories are stable and to describe the stable independence relation given by non dividing.

2 Preliminaries

Throughout the paper, the abbreviation pmp stands for ”probability measure preserving”.

2.1 About graphs

Definition 2.1. A **graph** G is a couple $(V(G), E(G))$ where $V(G)$ is a set and $E(G)$ is an irreflexive and symmetric relation on $V(G)$. Elements of $V(G)$ are called **vertices** of G and elements of $E(G)$ are called **edges** of G.

For G a graph, for each $v \in V(G)$ we let $\deg_G(v) = |\{u \in V(G) : (v,u) \in E(G)\}|$ and we call $\sup_{v \in V(G)} \deg_G(v) \in \mathbb{N} \cup \{\infty\}$ the degree bound of G. In the first two sections, we fix $d \in \mathbb{N}$ and unless it is precised otherwise, we consider only graphs of degree bound less than d.

Definition 2.2. An **isomorphism of graphs** between G and H is a bijection $f : V(G) \to V(H)$ such that $\forall x, y \in V(G)$, $(x,y) \in E(G) \Leftrightarrow (f(x), f(y)) \in E(H)$.

Definition 2.3. Let G be a graph, $A \subseteq V(G)$ and $B \subseteq E(G)$, then we define :

- $V^G_{adj}(A) = \{v \in V(G) : \exists a \in A \ (a,v) \in E(G)\}$ the set of **vertices adjacent to** A.
- $V^G_{inc}(B) = \{v \in V(G) : \exists u \in V(G),\ (u,v) \in B \vee (v,u) \in B\}$ the set of **vertices incident to** B.
- $E^G_{inc}(A) = \{(a,v) \in E(G) : a \in A\}$ the set of **edges incident to** A.

We will write $V_{adj}(A)$, $V_{inc}(B)$ and $E_{inc}(A)$ when by the context it is clear which graph G is considered.

Definition 2.4. Let G be a graph, a **subgraph** of G is a graph H such that $V(H) = V(G)$ and $E(H) \subseteq E(G)$. In this case, we write $H \subseteq G$.

If $V \subseteq V(G)$, the **subgraph of** G **induced by** V is the graph $(V(G), E(G) \cap V \times V)$. Nevertheless, in many cases it will be convenient to identify the induced graph on V and the graph $(V, E(G) \cap V \times V)$ and therefore see the induced graph on V as a graph on the set of vertices V.

Whenever G is a graph and $x \in V(G)$, we denote $B^G_r(x)$ and we call the r**-ball around** x **in** G the subgraph induced by G on vertices that are accessible from x with a walk of distance at most r in $E(G)$.

We consider three classes of graphs :

- $\mathbf{G}$ is the set of finite graphs, up to isomorphism.
- For $r \in \mathbb{N}$, $\mathbf{G}_r$ is the set of rooted graphs of radius at most r, up to rooted isomorphism.
- $\mathscr{G}_*$ is the set of rooted connected graphs up to rooted isomorphism.

In what follows, we will also consider graphs colored on vertices by the Cantor set
$\mathbf{K} := \{0,1\}^{\mathbb{N}}$:

- $\mathbf{G}^{\mathbf{K}}$ is the set of finite graphs colored on vertices by $\mathbf{K}$, up to colored isomorphism.

- For $r, s \in \mathbb{N}$, $\mathbf{G}^{\mathbf{K}}_{r,s}$ is the set of rooted graphs of radius at most r and colored on vertices by $\mathbf{K}$, up to rooted isomorphism of graphs preserving the first s digits of the coloring of any vertex. $\mathbf{G}^{\mathbf{K}}_{r,s}$ can also be seen as the set of rooted graphs of radius at most r and colored by $\{0,1\}^s$, up to rooted colored isomorphism. When $r = s$, we write $\mathbf{G}^{\mathbf{K}}_r$ for $\mathbf{G}^{\mathbf{K}}_{r,s}$.
- $\mathscr{G}^{\mathbf{K}}_*$ is the set of rooted connected graphs colored on vertices by $\mathbf{K}$ up to rooted colored isomorphism.

In general, we write $G \sim H$ to indicate that G and H are isomorphic. The type of isomorphism depends implicitly on the graphs G and H considered. For example if both G and H are rooted graphs, the isomorphism is supposed to be a rooted isomorphism. If both G and H are colored graphs, the isomorphism is supposed to be a colored isomorphism.

For G and H graphs colored by $\mathbf{K}$, we write $G \sim_s H$ when G and H are isomorphic as non-colored graphs by an isomorphism of graphs that preserves the first s digits of the coloring of any vertex. For G colored by $\mathbf{K}$ and $\alpha \in \mathbf{G}^{\mathbf{K}}_{r,s}$, we just write $G \sim \alpha$ for $G \sim_s \alpha$. Note that this definition makes sense as there cannot be a stronger form of isomorphism than $\sim_s$ between G and α.

In this paper we will have to consider many sequences, if $(x_n)_{n\in\mathbb{N}}$ is a sequence, we use the notation $\widehat{x}$ to refer to $(x_n)_{n\in\mathbb{N}}$.

For $\alpha \in \mathbf{G}_r$ (resp. $\mathbf{G}^{\mathbf{K}}_{r,s}$) we let $N_\alpha \subseteq \mathscr{G}_*$ (resp. $\mathscr{G}^{\mathbf{K}}_*$) be the set of $x \in \mathscr{G}_*$ (resp. $\mathscr{G}^{\mathbf{K}}_*$) with root u such that $B^x_r(u) \sim \alpha$. It is easy to see that $\mathscr{G}_*$ (resp. $\mathscr{G}^{\mathbf{K}}_*$) is a zero-dimensional Polish space, whose basic clopen sets are the N_α for $\alpha \in \mathbf{G}_r$ (resp. $\mathbf{G}^{\mathbf{K}}_{r,s}$). Notice that the N_α for $\alpha \in \mathbf{G}^{\mathbf{K}}_r$ already form a basis for the topology on $\mathscr{G}^{\mathbf{K}}_*$. For $A \subseteq \mathbf{G}_r$ (resp. $\mathbf{G}^{\mathbf{K}}_{r,s}$) we also write N_A for $\bigcup\limits_{\alpha\in A} N_\alpha$.

2.2 Graphings

We now introduce the notion of graphing, which is at the intersection of the notions of graphs and of pmp equivalence relations :

Definition 2.5. Let X be a standard Borel space and R be a Borel equivalence relation on X. We let $[R]$ be the group of Borel automorphisms of X whose graphs are contained in R. We say that a Borel probability measure μ on X is R-**invariant** if every element of $[R]$ preserves the measure μ, namely, $\forall f \in [R]\ f_*\mu = \mu$.

From now on, for X any measurable space, we denote by $\mathfrak{P}(X)$ the set of probability measures on X. We will only consider Borel σ-algebras so in the following every element of $\mathfrak{P}(X)$ is a Borel probability measure on X.

Proposition 2.6 (Admitted, [Kec04])**.** *With the same notations as above, for any $\mu \in \mathfrak{P}(X)$, we can define two measures μ_l and μ_r on R by*

- *For all non-negative Borel $f : R \to [0,\infty]$, $\int_R f\ d\mu_l = \int_X \sum\limits_{y\in[x]_R} f(x,y)\ d\mu(x)$*
- *For all non-negative Borel $f : R \to [0,\infty]$, $\int_R f\ d\mu_r = \int_X \sum\limits_{y\in[x]_R} f(y,x)\ d\mu(x)$*

Then $\mu_l = \mu_r$ if and only if μ is R-invariant.

Definition 2.7. Let $\mathcal{G}$ a Borel graph on a standard probability space (X, μ). Then the equivalence relation $R_{\mathcal{G}}$ induced by $\mathcal{G}$ is the Borel equivalence relation on (X, μ) whose classes are the connected components of $\mathcal{G}$. We say that $\mathcal{G}$ is a **graphing** when μ is $R_{\mathcal{G}}$-invariant.

We can define a measure on the set of edges of a graphing by :

Definition 2.8. Let $\mathcal{G}(X, \mu)$ be a graphing and $Z \subseteq E(\mathcal{G})$ Borel, the edge measure of the set Z is defined by $\mu_E(Z) := \mu_l(Z) = \mu_r(Z) = \int_X deg_Z(x) d\mu(x)$, where μ_l and μ_r are defined with respect to the Borel equivalence relation $R_{\mathcal{G}}$ and $deg_Z(x)$ is the number of edges in Z incident to x.

In this paper, any graphing is supposed of degree bound d, so the edge measure of a set of edges is bounded by the measure of the vertices incident to this set. Namely, for all Borel $Z \subseteq E(\mathcal{G})$ we have $\mu(V_{inc}(Z)) \leqslant \mu_E(Z) \leqslant d.\mu(V_{inc}(Z))$.

2.3 Convergence of graphs

To any finite graph and to any graphing we can associate a probability measure on $\mathscr{G}_*$ (resp. $\mathscr{G}_*^{\mathbf{K}}$) :

- If $G \in \mathbf{G}$ (resp. $\mathbf{G}^{\mathbf{K}}$), then let μ_C be the probability counting measure on $V(G)$ and $\pi_G : V(G) \to \mathscr{G}_*$ (resp. $V(G) \to \mathscr{G}_*^{\mathbf{K}}$) defined by $\pi_G(v) = [G, v]$. Now let $\mu_G = \pi_{G*}\mu_C$.
- If $\mathcal{G}$ is a graphing on a probability space (X, μ), then $\pi_{\mathcal{G}} : V(G) \to \mathscr{G}_*$ (resp. $V(G) \to \mathscr{G}_*^{\mathbf{K}}$) defined by $\pi_{\mathcal{G}}(x) = [G, x]$. Now let $\mu_{\mathcal{G}} = \pi_{\mathcal{G}*}\mu$.

For G a graph, $r \in \mathbb{N}$ and $\alpha \in \mathbf{G}_r$ (resp. $r, s \in \mathbb{N}$ and $\mathbf{G}_{r,s}$ let $V_\alpha(G) = \{v \in V(G) : B_r^G(v) \sim \alpha\}$.

Note that since measures on $\mathscr{G}_*$ (resp. $\mathscr{G}_*^{\mathbf{K}}$) are determined by their values on the clopen sets N_α for $\alpha \in \bigcup_r \mathbf{G}_r$ (resp. $\bigcup_{r,s} \mathbf{G}_{r,s}$), the measures μ_G and $\mu_{\mathcal{G}}$ are totally determined by the equalities $\mu_G(N_\alpha) = \mu_C(V_\alpha(G))$ and $\mu_{\mathcal{G}}(N_\alpha) = \mu(V_\alpha(\mathcal{G}))$.

Definition 2.9. For G a finite graph and $\mathcal{G}$ a graphing, we call μ_G and $\mu_{\mathcal{G}}$ the **random graphs** associated respectively to G and $\mathcal{G}$.

We say that two graphings $\mathcal{G}$ and $\mathcal{H}$ are **statistically equivalent** if $\mu_{\mathcal{G}} = \mu_{\mathcal{H}}$.

Definition 2.10. Let X be a Polish space along with its Borel σ-algebra, then we call the **weak topology on** $\mathfrak{P}(X)$ the topology generated by the applications $\mu \mapsto \int f d\mu$, for $f : X \to [0, \infty]$ continuous and bounded.

For this topology, (μ_n) converges to μ if for every bounded continuous function $f : X \to [0, \infty]$, we have $\lim_{n\to\infty} \int f d\mu_n = \int f d\mu$. In this case we say that (μ_n) **weakly converges to** μ and we write $(\mu_n) \Rightarrow \mu$.

Proposition 2.11 (Portmanteau theorem, Admitted, [Kec10])**.** *The following are equivalent :*

1. $(\mu_n) \Rightarrow \mu$
2. $\forall C \subseteq X$ *closed,* $\limsup_{n\to\infty} \mu_n(C) \leqslant \mu(C)$
3. $\forall U \subseteq X$ *open,* $\liminf_{n\to\infty} \mu_n(U) \geqslant \mu(U)$

We now want to describe weak convergence in the spaces $\mathfrak{P}(\mathscr{G}_*)$ and $\mathfrak{P}(\mathscr{G}_*^{\mathbf{K}})$. For this we have the following lemma :

Lemma 2.12. *Let X be a zero-dimensional Polish space along with its Borel σ-algebra and let (U_k) be a basis of X consisting of clopen sets. Then the weak topology on $\mathfrak{P}(X)$ is the topology generated by the family of applications $(\mu \mapsto \mu(U_k))_{k\in\mathbb{N}}$.*

Proof. Suppose $(\mu_n) \Rightarrow \mu$, then by Proposition 2.11, $\forall k \in \mathbb{N}$, $\limsup\limits_{n\to\infty} \mu_n(U_k) \leqslant \mu(U_k) \leqslant \liminf\limits_{n\to\infty} \mu_n(U_k)$ so $\lim\limits_{n\to\infty} \mu_n(U_k) = \mu(U_k)$.

Conversely, suppose that $\forall k \in \mathbb{N}$ $\lim\limits_{n\to\infty} \mu_n(U_k) = \mu(U_k)$. Take any open set $U \subseteq X$ and (k_i) a finite or countable sequence in $\mathbb{N}$ such that $U = \bigsqcup_i U_{k_i}$. Using discrete Fatou's Lemma, we then have

$$\begin{aligned}\liminf_{n\to\infty} \mu_n(U) &= \liminf_{n\to\infty} \sum_i \mu_n(U_{k_i}) \\ &\geqslant \sum_i \liminf_{n\to\infty} \mu_n(U_{k_i}) \\ &= \sum_i \mu(U_{k_i}) \\ &= \mu(U)\end{aligned}$$

So by Proposition 2.11, $(\mu_n) \Rightarrow \mu$. ☐

Thus, if X be a zero-dimensional Polish space and (U_n) is a basis of X consisting of clopen sets, then the weak topology on $\mathfrak{P}(X)$ is induced by the distance $d_w(\mu,\nu) = \sum\limits_{n\in\mathbb{N}} \frac{1}{2^n}|\mu(U_n) - \nu(U_n)|$.

For $\widehat{G} = (G_n) \in \mathbf{G}^{\mathbb{N}}$ (resp. $\mathbf{G}^{\mathbf{K}^{\mathbb{N}}}$), we say that $\widehat{G}$ converges (or is a convergent sequence) when (μ_{G_n}) weakly converges in $\mathfrak{P}(\mathscr{G}_*)$ (resp. $\mathfrak{P}(\mathscr{G}_*^{\mathbf{K}})$). Without loss of generality, we may suppose that any convergent sequence of graphs $\widehat{G} = (G_n)$ verifies $\lim\limits_{n\to\infty} |V(G_n)| = \infty$, since if G is a finite graph and H is a graph composed of finitely many disconnected copies of G, then $\mu_G = \mu_H$. From now on, every convergent graph sequence considered is supposed to satisfy the latter property.

If $\widehat{G}$ is a convergence sequence, we denote by $\mu_{\widehat{G}}$ the weak limit of the sequence (μ_{G_n}) and we call it the **Benjamini-Schramm limit of** $\widehat{G}$.

As $\mathscr{G}_*$ (resp. $\mathscr{G}_*^{\mathbf{K}}$) is a zero-dimensional Polish space and the N_α for $r \in \mathbb{N}$ and $\alpha \in \mathbf{G}_r$ (resp. $r, s \in \mathbb{N}$ and $\alpha \in \mathbf{G}_{r,s}^{\mathbf{K}}$) is a basis consisting of clopen sets, $\widehat{G}$ converges if and only if for any α, $(\mu_{G_n}(N_\alpha))_{n\in\mathbb{N}}$ converges and in this case, $\mu_{\widehat{G}}(N_\alpha) = \lim\limits_{n\to\infty} \mu_{G_n}(N_\alpha)$.

Finally, if $\mathcal{G}$ is a graphing and $\widehat{G} = (G_n) \in \mathbf{G}^{\mathbb{N}}$ is a sequence of finite graphs, we say that $\widehat{G}$ converges to $\mathcal{G}$ if $(\mu_{G_n}) \Rightarrow \mu_{\mathcal{G}}$. If $\widehat{\mathcal{G}} = (\mathcal{G}_n)$ is a sequence of graphings, we say that $\widehat{\mathcal{G}}$ converges to $\mathcal{G}$ if $(\mu_{\mathcal{G}_n}) \Rightarrow \mu_{\mathcal{G}}$.

2.4 Realization of a limit of graphs by a graphing

Two questions naturally arise from the definitions of the latter paragraph :

1. *Is every graphing a limit of finite graphs ?*
2. *Does every convergent sequence of finite graphs converge to a graphing ?*

While the first one is an open conjecture of Aldous and Lyons and will be discussed partially in Section 3, the answer to the second one is positive. We devote this subsection to the construction of a suitable graphing.

2.4.1 Unimodularity

First, one can define a Borel graph structure on $\mathscr{G}_*$ (resp. $\mathscr{G}_*^{\mathbf{K}}$) by letting $(x,y) \in E(\mathscr{G}_*)$ (resp. $E(\mathscr{G}_*^{\mathbf{K}})$) if and only if there is a connected graph (resp. a connected graph colored by $\mathbf{K}$) G and $(u,v) \in E(G)$ such that $x = [G,u]$ and $y = [G,v]$. In other words, $(x,y) \in E(\mathscr{G}_*)$ if and only if y can be obtained from x by changing the root according to an edge of x. From now on we write $\mathscr{G}_*$ (resp. $\mathscr{G}_*^{\mathbf{K}}$) either to talk about the graph we just defined or its underlying space.

Under certain conditions, the latter graph is locally similar to its elements, as witnesses the Lemma below :

Lemma 2.13. *If a graph G has no automorphism, then the graph with vertices $\{[G,v] : v \in V(G)\}$ and edges induced by changing the root by an adjacent vertex is isomorphic to G.*

Proof. By construction the map $v \mapsto [G,v]$ is a surjective graph morphism. If it was not injective, there would be at least two different vertices v_1, v_2 such that $[G,v_1] = [G,v_2]$, in other words, there would be an automorphism of G sending v_1 to v_2, hence the result. $\square$

Thus, if $\mu \in \mathfrak{P}(\mathscr{G}_*)$ is concentrated on graphs with no automorphism, then for $\alpha \in \mathbf{G}_r$

$$\mu_{\mathscr{G}_*}(N_\alpha) = \mu(\{x \in \mathscr{G}_* : B_r^{\mathscr{G}_*}(x) \sim \alpha\}) = \mu(N_\alpha)$$

However, even in this case, we cannot just take the graph on $(\mathscr{G}_*, \mu_{\widehat{G}})$ to answer the question because $\mu_{\widehat{G}}$ may not be $R_{\mathscr{G}_*}$-preserving. Nevertheless, $\mu_{\widehat{G}}$ is always a unimodular measure :

Definition 2.14. Let $\mathscr{G}_{**}$ be the set of connected birooted graphs, namely of connected graphs with an ordered pair of roots, up to birooted isomorphism.

Definition 2.15. Let $\mu \in \mathfrak{P}(\mathscr{G}_*)$ (resp. $\mathfrak{P}(\mathscr{G}_*^{\mathbf{K}})$). We define two measures μ_L and μ_R on $\mathscr{G}_{**}$ (resp. $\mathscr{G}_{**}^{\mathbf{K}}$) by

- For all non-negative Borel f, $\int_{\mathscr{G}_{**}} f \, d\mu_L = \int_{\mathscr{G}_*} \sum_{v \in V(x)} f(x,u,v) \, d\mu([x,u])$
- For all non-negative Borel f, $\int_{\mathscr{G}_{**}} f \, d\mu_R = \int_{\mathscr{G}_*} \sum_{v \in V(x)} f(x,v,u) \, d\mu([x,u])$

We call μ **unimodular** if $\mu_L = \mu_R$.

Of course for every finite graph G, μ_G is unimodular as a linear combination of Dirac measures, so the fact that $\mu_{\widehat{G}}$ is unimodular follows from :

Lemma 2.16. *Every weak limit of unimodular measures on $\mathfrak{P}(\mathscr{G}_*)$ is unimodular.*

Proof. Let (μ_n) be a sequence of unimodular measures in $\mathfrak{P}(\mathscr{G}_*)$ (resp. $\mathfrak{P}(\mathscr{G}_*^{\mathbf{K}})$) weakly converging to μ. Let's show that $\mu_L = \mu_R$.

Like $\mathscr{G}_*$, $\mathscr{G}_{**}$ is a zero-dimensional space Polish space. For $r \in \mathbb{N}$, define $\mathbf{G}_r^{**}$ to be the set of birooted graphs (x, u, v) such that $x = B_r^x(u) \cup B_r^x(v)$, up to birooted isomorphism. Then a base for the topology on $\mathscr{G}_{**}$ composed of clopen sets is given by the
$N_\beta = \{[x, u, v] \in \mathscr{G}_{**} : B_r^x(u) \cup B_r^x(v) \sim \beta\}$ for $r \in \mathbb{N}$ and $\beta \in \mathbf{G}_r^{**}$. Moreover, the N_β for β connected already form a base for this topology. Thus letting $C\mathbf{G}_r^{**}$ be the subset of $\mathbf{G}_r^{**}$ consisting of the connected elements, we only need to prove that μ_L and μ_R coincide on the N_β for $\beta \in \bigcup_r C\mathbf{G}_r^{**}$.

The maps $u \mapsto \sum_{v \in V(x)} \mathbb{1}_{N_\beta}([x, u, v])$ and $u \mapsto \sum_{v \in V(x)} \mathbb{1}_{N_\beta}([x, v, u])$ are continuous and bounded (indeed, $[x, u, v] \in N_\beta \Rightarrow d_x(u, v) \leqslant 2n$) so by weak convergence of the sequence (μ_n).

$$
\begin{aligned}
\mu_L(N_\beta) &= \int_{\mathscr{G}_*} \sum_{v \in V(x)} \mathbb{1}_{N_\beta}([x, u, v]) \, d\mu([x, u]) \\
&= \lim_{n \to \infty} \int_{\mathscr{G}_*} \sum_{v \in V(x)} \mathbb{1}_{N_\beta}([x, u, v]) \, d\mu_n([x, u]) \\
&= \lim_{n \to \infty} \int_{\mathscr{G}_{**}} \mathbb{1}_{N_\beta} \, d\mu_{n\,L} \\
&= \lim_{n \to \infty} \int_{\mathscr{G}_{**}} \mathbb{1}_{N_\beta} \, d\mu_{n\,R} \\
&= \lim_{n \to \infty} \int_{\mathscr{G}_*} \sum_{v \in V(x)} \mathbb{1}_{N_\beta}([x, v, u]) \, d\mu_n([x, u]) \\
&= \int_{\mathscr{G}_*} \sum_{v \in V(x)} \mathbb{1}_{N_\beta}([x, v, u]) \, d\mu([x, u]) \\
&= \mu_R(N_\beta)
\end{aligned}
$$

Hence μ is unimodular. $\square$

Now there are natural applications $\Phi : \mathscr{G}_{**} \to R_{\mathscr{G}_*}$ and $\Phi^{\mathbf{K}} : \mathscr{G}_{**}^{\mathbf{K}} \to R_{\mathscr{G}_*^{\mathbf{K}}}$ (recall that $R_{\mathscr{G}_*}$
(resp. $R_{\mathscr{G}_*^{\mathbf{K}}}$) is the equivalence relation on $\mathscr{G}_*$ (resp. $\mathscr{G}_*^{\mathbf{K}}$) induced by its graph structure), defined by $[x, u, v] \mapsto ([x, u], [x, v])$. Φ (resp. $\Phi^{\mathbf{K}}$) is a bijection when restricted to the graphs that have no automorphism.

Thus, if $\mu \in \mathfrak{P}(\mathscr{G}_*)$ (resp. $\mathfrak{P}(\mathscr{G}_*^{\mathbf{K}})$) is unimodular and concentrated on the elements $[x, u]$ of $\mathscr{G}_*$ (resp. $\mathscr{G}_*^{\mathbf{K}}$) such that x has no automorphism, then Φ (resp. $\Phi^{\mathbf{K}}$) is a bijection and it is easy to see that Φ (resp. $\Phi^{\mathbf{K}}$) sends μ_L to μ_l and μ_R to μ_r. By Proposition 2.6, μ is then $R_{\mathscr{G}_*}$-invariant (resp. $R_{\mathscr{G}_*^{\mathbf{K}}}$-invariant), and so $(\mathscr{G}_*, \mu)$ (resp. $(\mathscr{G}_*^{\mathbf{K}}, \mu)$) is a graphing.

Combining the latter results, we see that if $\mu_{\widehat{G}}$ is concentrated on the elements $[x, u]$ of $\mathscr{G}_*$ such that x has no automorphism, then $\widehat{G}$ converges to the graphing $(\mathscr{G}_*, \mu_{\widehat{G}})$. In the general case, we use the Bernoullization of $\mu_{\widehat{G}}$ to break symmetries.

2.4.2 Bernoullization of a measure

Definition 2.17. Let $\mu \in \mathfrak{P}(\mathscr{G}_*)$, we define the Bernoullization of μ and we denote by $\mu^{\mathbf{K}}$ to be a Borel probability measure on $\mathscr{G}_*^{\mathbf{K}}$ extending μ and such that the probabilities of colorings follow the Lebesgue measure on $\mathbf{K}$.

Namely for $r, s \in \mathbb{N}$ and $\alpha \in \mathbf{G}_{r,s}^{\mathbf{K}}$, letting $\beta \in \mathbf{G}_r$ be the underlying colorless graph of α, we let A_β^α be the coefficient $\frac{|\mathrm{Aut}(G_2)|}{|\mathrm{Aut}(G_1)|}$ where G_1, G_2 are respective representatives for the classes β and α. Then we have $\mu^{\mathbf{K}}(N_\alpha) = \frac{A_\beta^\alpha}{2^{s.|V(\alpha)|}}\mu(N_\beta)$, since $\frac{A_\beta^\alpha}{2^{s.|V(\alpha)|}}$ is indeed the Lebesgue probability that a random coloring of an element of N_β induces an element of N_α.

The Bernoullization of $\mu_{\widehat{G}}$ can be obtained with probability 1 by coloring randomly independently the graphs G_n and taking the Benjamini-Schramm limit :

Lemma 2.18 ([Ele10])**.** *Let $\widehat{G} = (G_n) \in \mathbf{G}^{\mathbb{N}}$ be a convergent sequence of graphs and $\mu_{\widehat{G}} \in \mathfrak{P}(\mathscr{G}_*)$ be its Benjamini-Schramm limit measure. Let $\mu_{\widehat{G}}^{\mathbf{K}} \in \mathfrak{P}(\mathscr{G}_*^{\mathbf{K}})$ be the Bernoullization of $\mu_{\widehat{G}}$. Let $\mathcal{C}_{\widehat{G}}$ be the product of the spaces $\mathcal{C}_{G_n}$ for $n \in \mathbb{N}$ endowed with the product measure ν. $\mathcal{C}_{\widehat{G}}$ is the space of random and independent colorings on the sequence $\widehat{G}$. For $c \in \mathcal{C}_{\widehat{G}}$ let us denote the graphs G_n colored with c by G_n^c.*

Then $\forall^* c \in \mathcal{C}_{\widehat{G}}$, the sequence (G_n^c) converges to $\mu_{\widehat{G}}^{\mathbf{K}}$ in the sense of Benjamini-Schramm.

Proof. In this proof the hypothesis that $\lim\limits_{n\to\infty} |V(G_n)| = \infty$ is necessary.

Take $\alpha \in \mathbf{G}_{r,s}^{\mathbf{K}}$, we denote by $\beta \in \mathbf{G}_r$ the underlying colorless graph. With this notation, we have $\mu_{\widehat{G}}(N_\alpha) = \frac{A_\beta^\alpha}{2^{s.|V(\alpha)|}}\mu_{\widehat{G}}(N_\beta)$ by the definition of the Bernoullization.

Our goal is thus to show that $\forall^* c \in \mathcal{C}_{\widehat{G}} \lim\limits_{n\to\infty} \mu_{G_n^c}(N_\alpha) = \frac{A_\beta^\alpha}{2^{s.|V(\alpha)|}}\mu_{\widehat{G}}(N_\beta)$.

For $u \in V(G_n)$ we let $\mathcal{C}_n^u(\alpha) = \{c \in \mathcal{C}_{\widehat{G}} : B_r^{G_n^c}(u) \sim \alpha\}$, then $\forall u \in V(G_n)$, $\nu(\mathcal{C}_n^u(\alpha)) = \frac{A_\beta^\alpha}{2^{s.|V(\alpha)|}}$. However, we can't directly apply the law of large numbers, since the $\mathcal{C}_n^u(\alpha)$ may not be independent for u ranging over $V(G_n)$.

Claim 2.18.1. Let $m \in \mathbb{N}$, then there is $l \in \mathbb{N}$ such that for any graph G there is a coloring of G by l colors such that two different vertices having same color are at distance at least m one of the other.

Proof. Set $l = d^{m+1} + 1$. It is a well known fact that any simple graph of degree bound k can be colored properly by $k+1$ colors. Define $G_{\leqslant m}$ to be the simple graph on $V(G)$ whose edges are the pairs $\{x, y\}$ such that $d_G(x, y) \leqslant m$. Note that since G is of degree bound d (every graph is supposed so in this section) $G_{\leqslant m}$ is a simple graph of degree bound less than d^{m+1}, so it can be colored properly with $l = d^{m+1} + 1$ colors. Now this induces a suitable coloring of G. ■

Recall that $V_\beta(G_n) = \{v \in V(G_n) : B_r^{G_n}(v) \sim \beta\}$ and $V_\alpha(G_n^c) = \{v \in V(G_n) : B_r^{G_n^c}(v) \sim \alpha\}$. By the claim, there is $l \in \mathbb{N}$ such that for any n, there is a partition $\mathcal{P}_n$ of $V_\beta(G_n)$ into l pieces such that $\forall P \in \mathcal{P}_n\ \forall u, v \in P\ d_{G_n}(u, v) \leqslant 2r \Rightarrow u = v$. For $q \geqslant 2$ let $\mathcal{P}'_{n,q}$ be the subset of $\mathcal{P}_n$ consisting of the elements P such that $\frac{|P|}{|V(G_n)|} > \frac{2^{-q}}{l}$.

Let $\varepsilon > 0$ be small enough. By construction of $\mathcal{P}_n$, for $P \in \mathcal{P}_n$ and any distinct elements $u_1, \ldots, u_k \in P$, the sets $\mathcal{C}_n^{u_j}(\alpha)$ are independent for $j = 1, \ldots, k$. As $\lim\limits_{n\to\infty} |V(G_n)| = \infty$, we have $\lim_{n\to\infty} \min_{P\in\mathcal{P}'_{n,q}} |P| = \infty$ and so by the law of large numbers, $\forall^* c \in \mathcal{C}_{\widehat{G}}$, for n big enough and $P \in \mathcal{P}'_{n,q}$ we have

$$\left|\frac{|V_\alpha(G_n^c) \cap P|}{|P|} - \frac{A^\alpha_\beta}{2^{s.|V(\alpha)|}}\right| < \varepsilon$$

Moreover, the elements of $\mathcal{P}'_{n,q}$ are disjoint so we get, for n big enough and $P \in \mathcal{P}'_{n,q}$

$$\left|\frac{|V_\alpha(G_n^c) \cap \bigcup \mathcal{P}'_{n,q}|}{|\bigcup \mathcal{P}'_{n,q}|} - \frac{A^\alpha_\beta}{2^{s.|V(\alpha)|}}\right| < \varepsilon$$

Furthermore, by definition of $\mathcal{P}'_{n,q}$, we have $\frac{|V_\beta(G_n)\setminus \bigcup \mathcal{P}'_{n,q}|}{|V(G_n)|} \leqslant \sum_{P\in\mathcal{P}_n} \frac{2^{-q}}{l} = 2^{-q}$ so for n and q big enough, $\left|\frac{|\bigcup \mathcal{P}'_{n,q}|}{|V(G_n)|} - \mu_{\widehat{G}}(N_\beta)\right| < \varepsilon$ and $\left|\frac{|V_\alpha(G_n^c)\ \cap\ \bigcup \mathcal{P}'_{n,q}|}{|V(G_n)|} - \mu_{G_n^c}(N_\alpha)\right| < \varepsilon$.

Combining these inequalities gives us

$$\left(\frac{A^\alpha_\beta}{2^{s.|V(\alpha)|}} - \varepsilon\right)(\mu_{\widehat{G}}(N_\beta) - \varepsilon) \leqslant \frac{|V_\alpha(G_n^c) \cap \bigcup \mathcal{P}'_{n,q}|}{|\bigcup \mathcal{P}'_{n,q}|}\frac{|\bigcup \mathcal{P}'_{n,q}|}{|V(G_n)|} \leqslant \left(\frac{A^\alpha_\beta}{2^{s.|V(\alpha)|}} + \varepsilon\right)(\mu_{\widehat{G}}(N_\beta) + \varepsilon)$$

which implies, for n and q big enough,

$$\frac{A^\alpha_\beta}{2^{s.|V(\alpha)|}}\mu_{\widehat{G}}(N_\beta) - 2\varepsilon \leqslant \frac{|V_\alpha(G_n^c) \cap \bigcup \mathcal{P}'_{n,q}|}{|V(G_n)|} \leqslant \frac{A^\alpha_\beta}{2^{s.|V(\alpha)|}}\mu_{\widehat{G}}(N_\beta) + 3\varepsilon$$

hence the final inequality,

$$\frac{A^\alpha_\beta}{2^{s.|V(\alpha)|}}\mu_{\widehat{G}}(N_\beta) - 3\varepsilon \leqslant \mu_{G_n^c}(N_\alpha) \leqslant \frac{A^\alpha_\beta}{2^{s.|V(\alpha)|}}\mu_{\widehat{G}}(N_\beta) + 4\varepsilon$$

□

Theorem 2.19. *Take a convergent sequence of finite graphs $\widehat{G} = (G_n)$ and let $\mathcal{G}_{\widehat{G}}$ be the graph $\mathscr{G}^{\mathbf{K}}_*$ endowed with the measure $\mu^{\mathbf{K}}_{\widehat{G}}$. Then $\mathcal{G}_{\widehat{G}}$ is a graphing and $\widehat{G}$ converges to $\mathcal{G}_{\widehat{G}}$. Moreover, $\forall^* x \in \mathscr{G}^{\mathbf{K}}_*$, $x \sim (\mathscr{G}^{\mathbf{K}}_*, x)$. We call $\mathcal{G}_{\widehat{G}}$ the canonical limit of $\widehat{G}$.*

Proof. First, by definition of the Bernoullization, the measure $\mu^{\mathbf{K}}_{\widehat{G}}$ is concentrated on graphs without automorphism. The following are consequences :

- By Lemma 2.13, $\forall^* x \in \mathscr{G}^{\mathbf{K}}_*$, $x \sim (\mathscr{G}^{\mathbf{K}}_*, x)$.
- By Lemma 2.18 $\mu^{\mathbf{K}}_{\widehat{G}}$ is a weak limit of linear combinations of Dirac measures and thus is unimodular by Lemma 2.16. As remarked before, in the case where the measure is concentrated on graphs without automorphism, this shows that $\mu^{\mathbf{K}}_{\widehat{G}}$ is $R_{\mathscr{G}^{\mathbf{K}}_*}$-invariant. In other words, $\mathcal{G}_{\widehat{G}}$ is a graphing.
- Finally, again by Lemma 2.13, for $\alpha \in \mathbf{G}_r$

$$\mu_{\mathcal{G}_{\widehat{G}}}(N_\alpha) = \mu^{\mathbf{K}}_{\widehat{G}}(C_\alpha^{\mathscr{G}^{\mathbf{K}}_*}) = \mu^{\mathbf{K}}_{\widehat{G}}(N_\alpha) = \mu_{\widehat{G}}(N_\alpha)$$

  so $\mathcal{G}_{\widehat{G}}$ is indeed a limit of $\widehat{G}$.

□

3 Hyperfiniteness

In this section every graph considered is still supposed to be of degree bound at most d, with the exception of the last corollary.

Definition 3.1. Let $\mathcal{G}(X,\mu)$ be a graphing. $\mathcal{G}$ is called **hyperfinite** if
$\forall \varepsilon > 0\ \exists M \in \mathbb{N}\ \exists Z \subseteq E(\mathcal{G})$ Borel such that $\mu_E(Z) < \varepsilon$ and the subgraphing $\mathcal{H} = \mathcal{G}\backslash Z$ has components of size at most M.

This section contains the proofs for two important properties of hyperfiniteness :

1. Every hyperfinite graphing $\mathcal{G}$ is the limit of a sequence of finite graphs $\widehat{G}$.
2. Hyperfiniteness is an invariant of statistical equivalence. That is, if $\mathcal{G}$ and $\mathcal{H}$ are statistically equivalent (recall that it means that $\mu_{\mathcal{G}} = \mu_{\mathcal{H}}$), then $\mathcal{G}$ is hyperfinite if and only if $\mathcal{H}$ is.

Before anything else, let's note that we can also define hyperfiniteness according to the next proposition. We will use both definitions indifferently in the rest of the paper.

Proposition 3.2. *Let $\mathcal{G}(X,\mu)$ be a graphing, then the following are equivalent :*

1. *$\mathcal{G}(X,\mu)$ is hyperfinite.*
2. *$\forall \varepsilon > 0\ \exists Z \subseteq E(\mathcal{G})$ Borel of edge measure $\mu_E(Z) < \varepsilon$ such that the components of the graphing $\mathcal{G}\backslash Z$ are finite.*

Proof. Only one direction is of interest. Let $\varepsilon > 0$ and $Z \subseteq E(\mathcal{G})$ Borel of edge measure $\mu_E(Z) < \varepsilon$ such that the components of the graphing $\mathcal{H} = \mathcal{G}\backslash Z$ are finite. For $n \in \mathbb{N}$ let $X_{\leqslant n}$ be the Borel subset of X consisting of the components of $\mathcal{H}$ of size at most n. It is clear that $X = \bigcup_{n\in\mathbb{N}} X_{\leqslant n}$ and μ is a probability measure so $\lim_{n\to\infty} \mu(X\backslash X_{\leqslant n}) = 0$. Take $M \in \mathbb{N}$ such that $\mu(X\backslash X_{\leqslant M}) < \frac{\varepsilon-\mu_E(Z)}{d}$ and set $Z' = Z \cup E^{\mathcal{H}}_{inc}(X\backslash X_{\leqslant M})$. Then $\mu_E(Z') < \varepsilon$ and moreover, the graphing $\mathcal{G}\backslash Z'$ has components of size at most M by definition. ☐

3.1 Hyperfinite graphings are limits of finite graphs

We begin with a very useful lemma.

Lemma 3.3. *If $\mathcal{G}(Z,\eta)$ is a graphing, $\mathcal{A}$ a subgraphing and $(\mathcal{A}_n)$ a sequence of subgraphings of $\mathcal{G}$ such that $\lim_{n\to\infty} \eta_E(E(\mathcal{A}) \triangle E(\mathcal{A}_n)) = 0$, then $\mathcal{A}_n$ converges to $\mathcal{A}$ in the sense of Benjamini and Schramm.*

Proof. We write D_n for $E(\mathcal{A}) \triangle E(\mathcal{A}_n)$. Let $r \in \mathbb{N}$ and $\alpha \in \mathbf{G}_r$, we set
$D^r_n = \{z \in Z : B^{\mathcal{G}}_r(z) \cap V_{inc}(D_n) \neq \varnothing\}$. By measure preservation, we get
$\eta(D^r_n) \leqslant d^r.\eta(V_{inc}(D_n)) \leqslant d^r.\eta_E(D_n)$, so $\lim_{n\to\infty} \eta(D^r_n) = 0$. Furthermore, for $z \notin D^r_n$, we have of course $B^{\mathcal{A}}_r(z) \sim B^{\mathcal{A}_n}_r(z)$, hence

$$\begin{aligned}
|\mu_{\mathcal{A}}(N_\alpha) - \mu_{\mathcal{A}_n}(N_\alpha)| &= |\eta(C^{\mathcal{A}}_\alpha) - \eta(C^{\mathcal{A}_n}_\alpha)| \\
&= |\eta(\{z \in Z : B^{\mathcal{A}}_r(z) \sim \alpha\}) - \eta(\{z \in Z : B^{\mathcal{A}_n}_r(z) \sim \alpha\})| \\
&= |\eta(\{z \in D^r_n : B^{\mathcal{A}}_r(z) \sim \alpha\}) - \eta(\{z \in D^r_n : B^{\mathcal{A}_n}_r(z) \sim \alpha\})| \\
&\leqslant 2\eta(D^r_n)
\end{aligned}$$

Thus $(\mathcal{A}_n)$ converges to $\mathcal{A}$. □

Definition 3.4. For $M \in \mathbb{N}$ let $\mathscr{G}_M$ be the set of unrooted connected graphs of size at most M. If G is a finite graph then for $S \in \mathscr{G}_M$ we let $C_S^G = \{v \in V(G) : [v]_G \sim S\}$. Moreover, we let $c_S^G = \frac{|C_S^G|}{|V(G)|}$. If $\mathcal{G}(X, \mu)$ is a graphing, then for $S \in \mathscr{G}_M$ we let $C_S^{\mathcal{G}} = \{x \in X : [x]_{\mathcal{G}} \sim S\}$ and we let $c_S^{\mathcal{G}} = \mu(C_S^{\mathcal{G}})$.

Theorem 3.5. *Let $\mathcal{G}$ be a hyperfinite graphing, then there is a sequence $\widehat{G}$ converging to $\mathcal{G}$.*

Proof. By Lemma 3.3, every hyperfinite graphing is a limit of graphings whose sizes of components are bounded. The conclusion then follows from

> **Claim 3.5.1.** Let $\mathcal{G}$ be a graphing whose components are of size at most M, then there is a sequence of finite graphs $\widehat{G}$ converging to $\mathcal{G}$.
>
> *Proof.* It is clear by measure preservation that
>
> $$\forall r \in \mathbb{N}\ \forall \alpha \in \mathbf{G}_r\ \mu_{\mathcal{G}}(N_\alpha) = \sum_{S \in \mathscr{G}_M} c_S^{\mathcal{G}}.\mu_S(N_\alpha)$$
>
> so choose a sequence $((k_n(S) : S \in \mathscr{G}_M))_{n \in \mathbb{N}}$ of elements of $\mathbb{N}^{\mathscr{G}_M}$ such that $\forall S \in \mathscr{G}_M \lim_{n\to\infty} \frac{k_n(S)}{\sum_{S \in \mathscr{G}_M} k_n(S)} = c_S^{\mathcal{G}}$. Such a sequence exists because $\sum_{S \in \mathscr{G}_M} c_S^{\mathcal{G}} = 1$. We then define the graphs G_n to be the disconnected union of $n_k(S)$ disconnected copies of S for each $S \in \mathscr{G}_M$. Obviously, $\mu_{G_n}(N_\alpha) = \sum_{S \in \mathscr{G}_M} \frac{k_n(S)}{\sum_{S \in \mathscr{G}_M} k_n(S)}.\mu_S(N_\alpha)$, thus $(\mu_{G_n}) \Rightarrow \mu_{\mathcal{G}}$. ■

□

3.2 Hyperfiniteness is an invariant of statistical equivalence

All the results of this subsection are due to Gábor Elek [Ele12].

Definition 3.6. Let G be a graph. For $A \subseteq V(G)$ finite, we define **the boundary of A in G**, denoted by $\partial_G(A)$ to be the set of edges incident to both A and $V(G) \backslash A$. We also define **the isoperimetric constant of A in G** by $i_G(A) = \frac{|\partial_G A|}{|A|}$.

We say that a graph G is **amenable** if $\forall \varepsilon > 0\ \exists A \subseteq V(G)$ finite such that $i_G(A) < \varepsilon$.

Lemma 3.7. *Let $\mathcal{G}(X, \mu)$ be a graphing, $\mathcal{H}$ be an induced subgraphing and $\varepsilon > 0$. Then there is a Borel $K \subseteq X$, which intersects every amenable component of $\mathcal{H}$ and such that the components of K are finite sets of isoperimetric constant less than ε in $\mathcal{H}$.*

Proof. We define K by induction. The idea is that for each $n \in \mathbb{N}$ we add to K finite sets of radius less than n and having isoperimetric constant less than ε, in a way that any set that we add stay disconnected from any other set in K.

For $n = 0$, let $K_0 = \varnothing$.

Suppose $K_{n-1} \subseteq X$ is defined. We then define K_n. First, by a result of Kechris, Solecki and Todorcevic, any Borel graph of degree bound d can be colored properly in a Borel way with $d + 1$ colors. Applying the proof of Claim 2.18.1, we see that there exists a Borel partition $X = \bigsqcup_{i \leqslant l_n} A_i$ such that two distinct elements of an A_i are at distance at least $2n+2$ in $\mathcal{H}$.

For $i = 0$ and $x \in A_0$, we let $R_x^{n,0}$ be the set of finite subsets $S \subseteq X$ such that :

- $x \in S$
- $S \subseteq B_n(x)$
- $i_{\mathcal{H}}(S) < \varepsilon$
- $E_{inc}^{\mathcal{H}}(S) \cap E_{inc}^{\mathcal{H}}(K_{n-1}) = \varnothing$

Take a Borel linear order on the finite subsets of X and let $K_n^0 = K_{n-1} \cup \bigcup_{x \in A_0} min\ R_x^{n,0}$, with the convention that $min\ \varnothing = \varnothing$.

Suppose K_n^{i-1} already defined and for $x \in A_i$ we let $R_x^{n,i}$ be the set of finite subsets $S \subseteq X$ such that :

- $x \in S$
- $S \subseteq B_n(x)$
- $i_{\mathcal{H}}(S) < \varepsilon$
- $E_{inc}^{\mathcal{H}}(S) \cap E_{inc}^{\mathcal{H}}(K_n^{i-1}) = \varnothing$

Then let $K_n^i = K_n^{i-1} \cup \bigcup_{x \in A_i} min\ R_x^{n,i}$. Finally let $K_n = K_n^{l_n}$ and $K = \bigcup_{n \in \mathbb{N}} K_n$.

By construction it is clear that the components of K are the sets of the form $min\ R_x^{n,i}$, which are finite sets of isoperimetric constant less than ε in $\mathcal{H}$. Let's prove that K intersects every amenable component of $\mathcal{H}$:

Let C be an amenable component of $\mathcal{H}$ and suppose that $K \cap C = \varnothing$. Then obviously $\forall n \in \mathbb{N}\ \forall i \leqslant l_n\ K_n^i \cap C = \varnothing$, thus the construction of the $R_x^{n,i}$ shows that $R_x^{n,i}$ is the set of finite subsets of X such that :

- $x \in S$
- $S \subseteq B_n(x)$
- $i_{\mathcal{H}}(S) < \varepsilon$

Since C is amenable, let $S \subseteq C$ be a finite subset of isoperimetric constant $i_{\mathcal{H}}(S) < \varepsilon$. Take any $x \in S$ and a natural number n such that $S \subseteq B_n(x)$. Let A_i be the set containing x in the partition $X = \bigsqcup_{i \leqslant l_n} A_i$. By the remark above, $S \in R_x^{n,i}$, therefore $x \in K \cap C \neq \varnothing$, a contradiction. □

Theorem 3.8. *Let $\mathcal{G}(X, \mu)$ be a graphing, then the following are equivalent :*

1. *$\mathcal{G}$ is hyperfinite.*
2. *For every subgraphing $\mathcal{H} \subseteq \mathcal{G}$ of positive measure, almost all the components of $\mathcal{H}$ are amenable.*

Proof. *2.* $\Rightarrow$ *1.* :

Suppose *2.*. We let $\mathfrak{A}$ be the set of families $\mathscr{A}$ of Borel subsets of X such that :

- $\forall A \in \mathscr{A}\ \mu(A) > 0$.
- $\forall A \neq A' \in \mathscr{A}$ no vertex in A is adjacent to a vertex in A'.

- For any $A \in \mathscr{A}$, the components of $\mathcal{G}_{\upharpoonright X \setminus \bigcup\limits_{A' \neq A} V^{\mathcal{G}}_{adj}(A')}$ are finite sets of isoperimetric constant less than ε in $\mathcal{G}_{\upharpoonright X \setminus \bigcup\limits_{A' \neq A} V^{\mathcal{G}}_{adj}(A')}$.

Let us order $\mathfrak{A}$ by inclusion. Then $\varnothing \in \mathfrak{A}$ and every chain of $\mathfrak{A}$ has an upper bound in $\mathfrak{A}$, namely its union, so we can apply Zorn's Lemma.

Let A be maximal in $\mathfrak{A}$. Suppose that $\mu(\bigcup\limits_{A \in \mathscr{A}} V^{\mathcal{G}}_{adj}(A)) < 1$, then let $Y = X \setminus \bigcup\limits_{A \in \mathscr{A}} V^{\mathcal{G}}_{adj}(A)$ and consider the subgraphing $\mathcal{H}$ induced on Y. By Lemma 3.7 and the hypothesis *2.*, there is a Borel set $K \subseteq Y$ which interesects almost every component of $\mathcal{H}$ and such that the components of K are finite sets of isoperimetric constant less than ε in $\mathcal{H}$. Then we have :

- As K is a complete section for Y, by Feldman-Moore Theorem, Y can be covered by a countable union of sets of measure $\mu(K)$ and therefore $\mu(K) > 0$.
- $\forall A \in \mathscr{A}$ no vertex in A is adjacent to a vertex in K by definition of Y.
- Note that two elements of Y adjacent in $\mathcal{G}$ are adjacent in $\mathcal{H}$ by definition, so the components of K for $\mathcal{G}$ are the components of K for $\mathcal{H}$ and therefore are finite sets in $\mathcal{G}$. Furthermore, the boundary of a component of K in $\mathcal{G}_{\upharpoonright X \setminus \bigcup\limits_{A \in \mathscr{A}} V^{\mathcal{G}}_{adj}(A)} = \mathcal{G}_{\upharpoonright Y}$ is contained in Y as well, so any component of K has same boundary in $\mathcal{G}_{\upharpoonright Y}$ and $\mathcal{H}$. It follows that the components of K in $\mathcal{G}_{\upharpoonright Y}$ are finite sets of isoperimetric constant less than ε in $\mathcal{G}_{\upharpoonright Y}$.

But this means that $\mathscr{A} \cup \{K\} \in \mathfrak{A}$, contradicting the maximality of $\mathscr{A}$. That proves that $\mu(\bigcup\limits_{A \in \mathscr{A}} V^{\mathcal{G}}_{adj}(A)) = 1$.

Finally, let $Z = \bigcup\limits_{A \in \mathscr{A}} \partial_{(\mathcal{G}_{\upharpoonright X \setminus \bigcup\limits_{A' \neq A} V^{\mathcal{G}}_{adj}(A')})} A \subseteq E(\mathcal{G})$ and consider the graphing $\mathcal{G}_{\upharpoonright \bigcup\limits_{A \in \mathscr{A}} V^{\mathcal{G}}_{adj}(A)} \setminus Z$. Since the sets A are pairwise disconnected for $A \in \mathscr{A}$, by removing Z we remove the boundary of each finite component of A in $\mathcal{G}$, and so we end up with only finite components in $\mathcal{G}_{\upharpoonright \bigcup\limits_{A \in \mathscr{A}} V^{\mathcal{G}}_{adj}(A)} \setminus Z$. Moreover, for each finite component C of an $A \in \mathscr{A}$, we have $|\partial_{(\mathcal{G}_{\upharpoonright X \setminus \bigcup\limits_{A' \neq A} V^{\mathcal{G}}_{adj}(A')})} C| \leqslant \varepsilon |C|$ so by averaging we get $\mu_E(\partial_{(\mathcal{G}_{\upharpoonright X \setminus \bigcup\limits_{A' \neq A} V^{\mathcal{G}}_{adj}(A')})} A) \leqslant \varepsilon \mu(A)$ and so $\mu_E(Z) \leqslant \varepsilon$.

1. $\Rightarrow$ *2.* :

Suppose that $\mathcal{G}$ is hyperfinite and *2.* does not hold. There is a subgraphing $\mathcal{H} \subseteq \mathcal{G}$ of positive measure such that for each component C of $\mathcal{H}$, $\inf\limits_{A \subseteq C \text{ finite}} i_{\mathcal{H}}(A) > 0$ and without loss of generality we may suppose that $\exists n \in \mathbb{N}$ such that for each component C of $\mathcal{H}$, we have $\inf\limits_{A \subseteq C \text{ finite}} i_{\mathcal{H}}(A) \geqslant \frac{1}{n}$, by σ-additivity of μ.

By hyperfiniteness of $\mathcal{G}$, let $Z \subseteq E(\mathcal{G})$ be such that $\mu_E(Z) < \frac{\mu(V(\mathcal{H}))}{n}$ and all the components of $\mathcal{G} \setminus Z$ are finite.

Take D a component of $\mathcal{G} \setminus Z$. For C a component of $\mathcal{H}$, as $C \cap D$ is a finite subset of C, we have $|\partial_{\mathcal{H}}(C \cap D)| \geqslant \frac{|C \cap D|}{n}$. Moreover, if $C \neq C'$ are components of $\mathcal{H}$, $\partial_{\mathcal{H}}(C \cap D) \cap \partial_{\mathcal{H}}(C' \cap D) \subseteq E^{\mathcal{H}}_{inc}(C) \cap E^{\mathcal{H}}_{inc}(C') = \varnothing$. Combining these inequalities for the components $C_1, \ldots, C_n$ that intersect D, we get $|\partial_{\mathcal{H}}(D \cap V(\mathcal{H}))| \geqslant \frac{|D \cap V(\mathcal{H})|}{n}$. Furthermore, $\partial_{\mathcal{H}}(D \cap V(\mathcal{H})) \subseteq E^{\mathcal{G}}_{inc}(D) \cap Z$, indeed every edge in $\mathcal{G} \setminus Z$ witnesses that its two extremities are in the same component for $\mathcal{G} \setminus Z$ so such an edge cannot be in the boundary of D.

We just proved that for each component D of $\mathcal{G} \setminus Z$, $|E^{\mathcal{G}}_{inc}(D) \cap Z| \geqslant \frac{|D \cap V(\mathcal{H})|}{n}$ and thus on average $\mu_E(Z) \geqslant \frac{\mu(V(\mathcal{H}))}{n}$, contradicting the definition of Z. $\square$

Definition 3.9. Let $\mathcal{G}(X,\mu)$ and $\mathcal{H}(Y,\nu)$ be two graphings. A **graphing factor map** is a pmp almost surjective map $\pi : Y \to X$ such that $\forall^* y \in Y$ $\pi_{\restriction [y]_{\mathcal{H}}}$ is an isomorphism of graphs.

We say that $\mathcal{G}$ **is a factor of** $\mathcal{H}$ and we write $\mathcal{G} \sqsubseteq \mathcal{H}$ if there exists a graphing factor map $\pi : Y \to X$.

Proposition 3.10. *Let $\mathcal{G}(X,\mu) \sqsubseteq \mathcal{H}(Y,\nu)$ two graphings. Then $\mathcal{G}$ is hyperfinite if and only if $\mathcal{H}$ is.*

Proof. Let $\pi : Y \to X$ be a graphing factor map.

Suppose that $\mathcal{G}$ is hyperfinite, take $\varepsilon > 0$ and $Z \subseteq E(\mathcal{G})$ such that $\mu_E(Z) < \varepsilon$ and the components of $\mathcal{G} \backslash Z$ are finite. Then the set $\pi^{-1}(Z) \subseteq E(\mathcal{H})$ witnesses the hyperfiniteness of $\mathcal{H}$.

Conversely, suppose that $\mathcal{G}$ is not hyperfinite. Then by Theorem 3.8 there is a subgraphing $\mathcal{K} \subseteq \mathcal{G}$ of positive measure such that not almost all components of $\mathcal{K}$ are amenable. Then $\pi^{-1}(\mathcal{K}) \subseteq \mathcal{H}$ is a subgraphing of positive measure and not almost all components of $\pi^{-1}(\mathcal{K})$ are amenable, therefore, $\mathcal{H}$ is not hyperfinite. □

Theorem 3.11. *Let $\mathcal{G}$ and $\mathcal{H}$ be statistically equivalent graphings, then $\mathcal{G}$ is hyperfinite if and only if $\mathcal{H}$ is.*

Proof. Let $\mathcal{H}(X,\mu)$ be a graphing. Consider the map $\pi_{R_{\mathcal{H}}} : R_{\mathcal{H}} \to \mathscr{G}_{**}$ that sends (x,y) to their component for $\mathcal{H}$ birooted in (x,y). It is easy to check that ${\pi_{R_{\mathcal{H}}}}_* \mu_l = \mu_{\mathcal{H}L}$ and ${\pi_{R_{\mathcal{H}}}}_* \mu_r = \mu_{\mathcal{H}R}$. But $\mathcal{H}$ is a graphing, so $\mu_l = \mu_r$ and it follows that $\mu_{\mathcal{H}}$ is unimodular. Therefore by taking its Bernoullization we get a graphing $\mathcal{G}_{\mu_{\mathcal{H}}}(\mathscr{G}_*^{\mathbf{K}}, \mu_{\mathcal{H}}^{\mathbf{K}})$.

The goal in this proof is to show that $\mathcal{H}$ is hyperfinite if and only if $\mathcal{G}_{\mu_{\mathcal{H}}}$ is. The conclusion will then follow from the fact that $\mathcal{G}_{\mu_{\mathcal{H}}}$ only depends on $\mu_{\mathcal{H}}$.

Claim 3.11.1. There is a common extension to $\mathcal{H}$ and $\mathcal{G}_{\mu_{\mathcal{H}}}$. In other words there is a graphing $\mathcal{K}$ such that both $\mathcal{H}$ and $\mathcal{G}_{\mu_{\mathcal{H}}}$ are factors of $\mathcal{K}$.

Proof. The chosen graphing will be the Bernoullization of $\mathcal{H}$. For any graphing $\mathcal{H}(X,\mu)$ (not necessarily on $\mathscr{G}_*$), the construction of its Bernoullization $\mathcal{H}^{\mathbf{K}}$ presented below is classical :

Definition 3.12. By Lusin-Novikov theorem, there exists a family $(f_i : i \in \mathbb{N})$ of Borel partial maps $X \to X$ such that $\mathcal{H} = \bigsqcup_{i\in\mathbb{N}} \Gamma(f_i)$, where $\Gamma(f_i)$ is the graph of f_i. Thus we can embed $\mathcal{H}$ in $X \times \mathbb{N}$ in a Borel way with the application $\Phi : (x,y) \to (x, \varphi_x(y))$ where $\varphi_x(y)$ is the only integer i such that $f_i(x) = y$.

The space on which the Bernoullization is defined is $X \times \mathbf{K}^{\mathbb{N}}$, we will denote it by $X_{\mathbf{K}}$. It is endowed with its σ-algebra of Borel sets for the product topology on $X_{\mathbf{K}}$, which is a Polish topology. Therefore, $X_{\mathbf{K}}$ is a standard Borel.

For the measure $\mu^{\mathbf{K}}$, we begin by defining a measure on $\mathbf{K}^{\mathbb{N}}$ for each $x \in X$ with the formula : $\mu_x = \bigotimes_{i\in\Phi(\mathcal{H})_x} \lambda \otimes \bigotimes_{i\notin\Phi(\mathcal{H})_x} \delta_0$. Then let $\mu^{\mathbf{K}} = \int_X \delta_x \times \mu_x \, d\mu(x) \in \mathfrak{P}(X_{\mathbf{K}})$.

Finally the Bernoullization of $\mathcal{H}$ is the graph $\mathcal{H}^{\mathbf{K}}$ on $X_{\mathbf{K}}$ defined by :

$$((x,f),(y,g)) \in E(\mathcal{H}^{\mathbf{K}}) \Longleftrightarrow (x,y) \in E(\mathcal{H}) \text{ and } f \circ \varphi_x = g \circ \varphi_y$$

The idea is to see the space $X_{\mathbf{K}}$ as the space of colorings by $\mathbf{K}$ on X, with the Lebesgue measure on colorings. We take the convention that on points of the space $X_{\mathbf{K}}$ which do not represent any element of $\mathcal{H}$, the colorings must have value 0.

To see that $\mathcal{H}^{\mathbf{K}}$ is indeed a graphing, let us define, for $(x,y) \in R_{\mathcal{H}}$ and $f \in \mathbf{K}^{\mathbb{N}}$, $f_{x\to y}$ to be the only element of $\mathbf{K}^{\mathbb{N}}$ such that $((x,f),(y,f_{x\to y})) \in R_{\mathcal{H}^{\mathbf{K}}}$.
Take any Borel non-negative $\varphi : R_{\mathcal{H}^{\mathbf{K}}} \to \infty$, we let $\Phi : R_{\mathcal{H}} \to [0,\infty]$ be defined by $\Phi(x,y) = \int_{\mathbf{K}^{\mathbb{N}}} \varphi((x,f),(y,f_{x\to y}))\, d\mu_z(f)$.
We have

$$\begin{aligned}
\int_{R_{\mathcal{H}^{\mathbf{K}}}} \varphi\, d(\mu^{\mathbf{K}})_l &= \int_{X_{\mathbf{K}}} \sum_{(y,g)\in[x,f]_{\mathcal{H}^{\mathbf{K}}}} \varphi((x,f),(y,g))\, d\mu^{\mathbf{K}}(x,f) \\
&= \int_X \int_{X_{\mathbf{K}}} \sum_{(y,g)\in[x,f]_{\mathcal{H}^{\mathbf{K}}}} \varphi((x,f),(y,g))\, d(\delta_z \times \mu_z)(x,f)\, d\mu(z) \\
&= \int_X \int_{\mathbf{K}^{\mathbb{N}}} \sum_{(y,g)\in[z,f]_{\mathcal{H}^{\mathbf{K}}}} \varphi((z,f),(y,g))\, d\mu_z(f)\, d\mu(z) \\
&= \int_X \int_{\mathbf{K}^{\mathbb{N}}} \sum_{y\in[z]_{\mathcal{H}}} \varphi((z,f),(y,f_{z\to y}))\, d\mu_z(f)\, d\mu(z) \\
&= \int_X \sum_{y\in[z]_{\mathcal{H}}} \int_{\mathbf{K}^{\mathbb{N}}} \varphi((z,f),(y,f_{z\to y}))\, d\mu_z(f)\, d\mu(z) \\
&= \int_X \sum_{y\in[z]_{\mathcal{H}}} \Phi(z,y)\, d\mu(z) \\
&= \int_X \sum_{y\in[z]_{\mathcal{H}}} \Phi(y,z)\, d\mu(z) \\
&= \int_X \sum_{y\in[z]_{\mathcal{H}}} \int_{\mathbf{K}^{\mathbb{N}}} \varphi((y,f),(z,f_{y\to z}))\, d\mu_y(f)\, d\mu(z) \\
&= \int_X \sum_{y\in[z]_{\mathcal{H}}} \int_{\mathbf{K}^{\mathbb{N}}} \varphi((y,f_{z\to y}),(z,f))\, d\mu_z(f)\, d\mu(z) \\
&= \int_X \int_{\mathbf{K}^{\mathbb{N}}} \sum_{y\in[z]_{\mathcal{H}}} \varphi((y,f_{z\to y}),(z,f))\, d\mu_z(f)\, d\mu(z) \\
&= \int_X \int_{\mathbf{K}^{\mathbb{N}}} \sum_{(y,g)\in[z,f]_{\mathcal{H}^{\mathbf{K}}}} \varphi((y,g),(z,f))\, d\mu_z(f)\, d\mu(z) \\
&= \int_X \int_{X_{\mathbf{K}}} \sum_{(y,g)\in[x,f]_{\mathcal{H}^{\mathbf{K}}}} \varphi((y,g)(x,f))\, d(\delta_z \times \mu_z)(x,f)\, d\mu(z) \\
&= \int_{X_{\mathbf{K}}} \sum_{(y,g)\in[x,f]_{\mathcal{H}^{\mathbf{K}}}} \varphi((y,g),(x,f))\, d\mu^{\mathbf{K}}(x,f) \\
&= \int_{R_{\mathcal{H}^{\mathbf{K}}}} \varphi\, d(\mu^{\mathbf{K}})_r
\end{aligned}$$

Now there are two natural candidates for graphing factor maps $X_{\mathbf{K}} \to X$ and $X_{\mathbf{K}} \to \mathscr{G}_*^{\mathbf{K}}$. Let $\pi : X_{\mathbf{K}} \to X$ be the projection on the first coordinate, and $\rho : X_{\mathbf{K}} \to \mathscr{G}_*^{\mathbf{K}}$ be the map sending (x, f) to the class of the rooted graph $([x]_{\mathcal{H}}, x)$, colored according to f.

It is quite obvious that π is a graph factor map from its definition. The case of ρ is a little bit more interesting. Note that for almost all $(x, f) \in X_{\mathbf{K}}$, f is injective so by Lemma 2.13, $\rho(x, f) \sim ([\rho(x, f)]_{\mathscr{G}_*^{\mathbf{K}}}, \rho(x, f))$, which exactly means that $\forall^*(x, f) \in X_{\mathbf{K}}$, $\rho_{\upharpoonright [x,f]_{\mathcal{H}^{\mathbf{K}}}}$ is a graph isomorphism.

Finally, let $r, s \in \mathbb{N}$, $\alpha \in \mathbf{G}_{r,s}^{\mathbf{K}}$ and $\beta \in \mathbf{G}_r$ the underlying colorless graph.

Note that the Lebesgue probability of a coloring of β to give a colored graph isomorphic to α is again $\frac{A_\beta^\alpha}{2^{s.|V(\alpha)|}}$, thus we have

$$\begin{aligned}
\rho_* \mu^{\mathbf{K}}(N_\alpha) &= \mu^{\mathbf{K}}(\{(x, f) \in X_{\mathbf{K}} : \rho(x, f) \in N_\alpha\}) \\
&= \mu^{\mathbf{K}}(\{(x, f) \in X_{\mathbf{K}} : x \in V_\beta(\mathcal{H}) \text{ and } f \circ \varphi_x \text{ colors } B_r^{\mathcal{H}}(x) \text{ according to } \alpha\}) \\
&= \int_{V_\beta(\mathcal{H})} \frac{A_\beta^\alpha}{2^{s.|V(\alpha)|}}\, d\mu \\
&= \mu_{\mathcal{H}}(N_\beta). \frac{A_\beta^\alpha}{2^{s.|V(\alpha)|}}\, d\mu \\
&= \mu_{\mathcal{H}}^{\mathbf{K}}(N_\alpha)
\end{aligned}$$

therefore ρ is pmp and as a consequence, it is a graphing factor map.

■

We conclude by using the Claim and Lemma 3.10 : $\mathcal{H}$ is hyperfinite if and only if $\mathcal{H}^K$ is hyperfinite, if and only if $\mathcal{G}_{\mu_{\mathcal{H}}}$ is. □

4 The Rokhlin Lemma

Recall that from now on, there is no more bound on degrees of graphs.

4.1 Measure preserving actions and Graphings

4.1.1 Classical Rokhlin Lemma

Rokhlin Lemma states that if τ is an aperiodic measure preserving transformation of a standard probability space (X, μ), that is a bijection $X \to X$ that preserves μ and such that $\mathrm{Supp}(\tau) = \{x \in X : \tau(x) = x\}$ is null, then $\forall n \in \mathbb{N}\ \forall \varepsilon > 0\ \exists A \subseteq X$ Borel such that the sets $A, \tau A, \ldots, \tau^{n-1} A$ are pairwise disjoint and $\mu(\bigsqcup_{i=0}^{n-1} \tau^i A) > 1 - \varepsilon$.

What we present in this paper is not a generalization of Rokhlin Lemma itself but rather of one of its important consequences :

Corollary 4.1 ("Rokhlin Lemma"). *Any two aperiodic measure preserving transformations τ_1 and τ_2 on standard probability spaces (X, μ) and (Y, ν) are strongly equivalent, meaning that $\forall \varepsilon > 0$ there is a measure preserving bijection $\rho : X \to Y$ such that $\mu(\{x \in X : \rho \circ \tau_1(x) = \tau_2 \circ \rho(x)\}) > 1 - \varepsilon$.*

Before presenting the proof, we have to introduce cycles.

Definition 4.2. Let (X, μ) be a standard probability space, a cycle c of period $n \in \mathbb{N}$ is a measure preserving Borel bijection $X \to X$ such that $\exists A \subseteq X$ Borel such that, up to a null set, $X = \bigsqcup_{i=0}^{n-1} c^i A$. We call such a A a base for the cycle c.

Lemma 4.3. *Let (X, μ) and (Y, ν) be standard probability spaces and c_X, c_Y cycles of period n respectively of X and Y. Then c_X and c_Y are conjugated, that is $\exists \rho : X \to Y$ a measure preserving bijection such that $\forall^* x \in X$ $c_Y \circ \rho(x) = \rho \circ c_X(x)$.*

Proof. Let A and B be respective bases for c_X and c_Y. By the definitions of cycles, it is clear that $\mu(A) = \frac{1}{n} = \nu(B)$ so by uniqueness of the standard Borel space, there exists a measure preserving bijection $\tau : A \to B$.

Now we extend τ into $\rho : X \to Y$ by letting $\rho_{\restriction c_X^i A} = c_Y^i \circ \tau \circ c_X^{-i}$. It is easy to check that ρ is a measure preserving bijection that conjugates c_X and c_Y. □

We are ready for the proof of the Corollary of Rokhlin Lemma :

Proof of "Rokhlin Lemma". Let $\tau_X : X \to X$ and $\tau_Y : Y \to Y$ be aperiodic measure preserving bijections.

First, by the usual Rokhlin Lemma, for $n \in \mathbb{N}^*$, there is a Borel $A \subseteq X$ (resp. $B \subseteq Y$) such that $\mu(\bigsqcup_{i=0}^{n} \tau_X^i A) \geqslant 1 - \frac{1}{n+1}$ (resp. $\nu(\bigsqcup_{i=0}^{n} \tau_Y^i B) \geqslant 1 - \frac{1}{n+1}$). Without loss of generality, we may furthermore suppose that $\mu(\bigsqcup_{i=0}^{n} \tau_X^i A) = \nu(\bigsqcup_{i=0}^{n} \tau_Y^i B) = 1 - \frac{1}{n+1}$.

Then we can define a cycle c_X (resp. c_Y) of period $n+1$ on $\bigsqcup_{i=0}^{n} \tau_X^i A$ (resp. $\bigsqcup_{i=0}^{n} \tau_Y^i B$) by setting

$$c_X : \left| \begin{array}{lcl} x \in \bigsqcup_{i=0}^{n-1} \tau_X^i A & \longmapsto & \tau(x) \\ x \in \tau_X^n A & \longmapsto & \tau^{-n}(x) \end{array} \right. \quad \text{and} \quad c_Y : \left| \begin{array}{lcl} y \in \bigsqcup_{i=0}^{n-1} \tau_Y^i B & \longmapsto & \tau(y) \\ y \in \tau_Y^n B & \longmapsto & \tau^{-n}(y) \end{array} \right.$$

Now, using Lemma 4.3, we can find a measure preserving bijection $\rho : \bigsqcup_{i=0}^{n} \tau_X^i A \to \bigsqcup_{i=0}^{n} \tau_Y^i B$ that conjugates c_X and c_Y and then extend it arbitrarily into a measure preserving bijection $\rho' : X \to Y$. It follows that

$$\begin{aligned} & \mu(\{x \in X : \tau_Y \circ \rho'(x) \neq \rho' \circ \tau_X(x)\}) \\ \leqslant \; & \mu(\{x \in \bigsqcup_{i=0}^{n} \tau_X^i A : \tau_Y \circ \rho(x) \neq \rho \circ \tau_X(x)\} + \frac{2}{n+1} \\ \leqslant \; & \{x \in \bigsqcup_{i=0}^{n} \tau_X^i A : \tau_X(x) \neq c_X(x)\} + \{y \in \bigsqcup_{i=0}^{n} \tau_Y^i B : \tau_Y(y) \neq c_Y(y)\} + \frac{2}{n+1} \\ \leqslant \; & \frac{4}{n+1} \end{aligned}$$

□

An aperiodic measure preserving transformation can be viewed as a free action of $\mathbb{Z}$. The goal of this section is to generalize the latter Corollary to hyperfinite actions of the free group having a given IRS (i.e. Invariant Random Subgroup, defined in subsection 4.1.3).

4.1.2 Hyperfiniteness for measure preserving actions

An equivalence relation is called finite if its classes are finite.

Definition 4.4. Let R be a Borel equivalence relation on a standard probability space (X,μ), we say that R is **hyperfinite** if there is a R-invariant connull subset of X on which R is a countable union of finite Borel equivalence relations.

We say that a measure preserving action α of a countable group Γ on (X,μ) is **hyperfinite** if the induced equivalence relation R_α is hyperfinite.

It is obvious that any subequivalence relation of a hyperfinite equivalence relation is hyperfinite. Conversely, we have the following :

Proposition 4.5 (Admitted,[Kec10])**.** *The union of an increasing sequence of hyperfinite equivalence relations on (X,μ) is a hyperfinite equivalence relation on (X,μ).*

Note that we defined the notion of hyperfiniteness relative to a standard probability space and thus to a measure. If we ask that R is the union of an increasing sequence of finite equivalence relations on the whole set X and not just on a connull subset, we obtain the classic definition of Borel hyperfiniteness. For this notion of hyperfiniteness, it is not known whether the union of an increasing sequence of hyperfinite equivalence relations is always finite.

Now any countable group Γ is the increasing union of a sequence of finitely generated subgroups, and so by Proposition 4.5 we get that a pmp action α of Γ is hyperfinite if and only if for every finitely generated subgroup Λ the restriction of α to Λ, which is a pmp action of Λ, is hyperfinite.

From now on, we focus our attention on pmp actions of finitely generated groups. We write finitely generated groups as couples (Γ,S) where Γ is a countable group and S is a finite symmetric generating subset.

Definition 4.6. Let F be a finite set. A **F-colored graphing** on a standard probability space (X,μ) is a graphing $\mathcal{G}(X,\mu)$ endowed with a Borel map $\varphi_{\mathcal{G}} : E(\mathcal{G}) \to F$. For $(x,y) \in E(\mathcal{G})$, we call $\varphi_{\mathcal{G}}(x,y)$ the color of (x,y).

We will simply write $\mathcal{G}$ and consider the color implicitely when dealing with colored graphings.

Fix $d \in \mathbb{N}$. Let $\mathscr{G}_*^F$ be the standard Borel space of rooted connected F-colored graphs of degree bound at most d. For a F-colored graphing $\mathcal{G}(X,\mu)$, we define a Borel probability measure $\mu_{\mathcal{G}}^F$ on $\mathscr{G}_*^F$ by letting $\pi : X \to \mathscr{G}_*^F$ be the map that sends x to its F-colored component rooted in x and setting $\mu_{\mathcal{G}}^F = \pi_*\mu$.

Definition 4.7. Let $\mathcal{G}(X,\mu)$ and $\mathcal{G}'(Y,\nu)$ be two F-colored graphings. A **colored graphing factor map** $\pi : Y \to X$ is a pmp almost surjective map such that $\forall^* y \in Y$, $\pi_{\restriction [y]_{\mathcal{H}}}$ is an isomorphism of F- colored graphs.

We say that $\mathcal{G}$ is a colored factor of $\mathcal{G}'$ and we write $\mathcal{G} \underset{c}{\sqsubseteq} \mathcal{G}'$ if there is a colored factor map $\pi : Y \to X$.

Let (Γ,S) be a finitely generated group. Let us consider a measure preserving action $\alpha : \Gamma \curvearrowright (X,\mu)$. We define a $\mathscr{P}(S)$-colored graphing $\mathcal{G}_\alpha$ on (X,μ) by $(x,y) \in E(\mathcal{G}_\alpha)$ if and only if $\exists s \in S\ y = sx$ and we color the edges of $\mathcal{G}_\alpha$ by letting the color of an edge (x, y) be $\{s \in S : y = sx\}$.

Lemma 4.8. *Let (Γ, S) be a finitely generated group and let $\alpha : \Gamma \curvearrowright (X, \mu)$ be a pmp action. Then α is hyperfinite if and only if $\mathcal{G}_\alpha$ is hyperfinite.*

Proof. Suppose α is hyperfinite and let $\varepsilon > 0$. Let (R_n) be an increasing sequence of finite Borel equivalence relations on X such that, up to a null set, $R_\alpha = \bigcup\limits_{n\in\mathbb{N}} R_n$. Since by definition $E(\mathcal{G}_\alpha) \subseteq R_\alpha$, we have, up to a null set, $E(\mathcal{G}_\alpha) = \bigcup\limits_{n\in\mathbb{N}} E(\mathcal{G}_\alpha) \cap R_n$.

Now the measure μ_E is finite (bounded by $|S|$) so $\lim\limits_{k\to\infty} \mu_E(E(\mathcal{G}_\alpha)\backslash E(\mathcal{G}_\alpha) \cap R_k) = 0$.

Thus for k big enough, the set $E(\mathcal{G}_\alpha)\backslash E(\mathcal{G}_\alpha) \cap R_k$ has edge measure less than ε and by removing this set from $E(\mathcal{G}_\alpha)$ we get a subgraphing $\mathcal{H}$ whose edges are contained in R_k, which implies of course that $\mathcal{H}$ has finite components.

Conversely, suppose $\mathcal{G}_\alpha$ is a hyperfinite graphing. For $n \in \mathbb{N}$, let $Z_n \subseteq E(\mathcal{G}_\alpha)$ be a Borel subset of edge measure less than $\frac{1}{2^n}$ such that the graphing $\mathcal{G}_\alpha\backslash Z_n$ has finite components.

For $n \in \mathbb{N}$, let $Z'_n = \bigcup\limits_{k\geqslant n} Z_k$. Then we have $\forall n \in \mathbb{N},\ \mu_E(Z'_n) \geqslant \frac{1}{2^{n-1}}$ and $\mathcal{G}_\alpha\backslash Z'_n \subseteq \mathcal{G}_\alpha\backslash Z_n$ has finite components.

Therefore, setting $\mathcal{H}_n := \mathcal{G}_\alpha\backslash Z'_n$, $(R_{\mathcal{H}_n})$ is an increasing sequence of finite subequivalence relations of $R_{\mathcal{G}_\alpha}$ and by definition of $\mathcal{H}_n$, $R_{\mathcal{G}_\alpha}$ and $\bigcup\limits_{n\in\mathbb{N}} R_{\mathcal{H}_n}$ differ only on a null set. ☐

4.1.3 Invariant random subgroups

Let $\alpha : \Gamma \curvearrowright (X, \mu)$ be a measure preserving action of the countable group Γ. To this action we can associate a probability measure on the Polish space of subgroups of Γ.

For Γ a countable group, $\{0, 1\}^\Gamma$ is a Polish space homeomorphic to the Cantor space. We let $Sg(\Gamma)$ be the closed subset of $\{0, 1\}^\Gamma$ consisting of the subgroups of Γ. Then $Sg(\Gamma)$ is of course a Polish space.

We have a natural map $\mathrm{Stab}^\alpha : X \to Sg(\Gamma)$ defined by $x \mapsto \mathrm{Stab}^\alpha(x) = \{g \in \Gamma : g^\alpha(x) = x\}$ and that gives us a probability measure $\mathrm{Stab}^\alpha_*\mu \in \mathfrak{P}(Sg(\Gamma))$ that we call the Invariant Random Subgroup (IRS in short) of α and denote by θ_α. Moreover, Γ acts on $Sg(\Gamma)$ by conjugation and the well known formula $\mathrm{Stab}^\alpha(gx) = g\mathrm{Stab}^\alpha(x)g^{-1}$ implies that the map Stab^α is equivariant. Therefore, θ_α is a Γ-invariant measure on $Sg(\Gamma)$. In general, we define an IRS of Γ to be a probability measure on $Sg(\Gamma)$ invariant for conjugacy.

For a finitely generated group, an IRS and a random colored graph are the same things in the way precised below :

Lemma 4.9. *If (Γ, S) is a finitely generated group, then there is a injective Borel map $\Psi : Sg(\Gamma) \to \mathscr{G}_*^{\mathscr{P}(S)}$ such that for all pmp action $\alpha : \Gamma \curvearrowright (X, \mu)$, Ψ is a measure preserving bijection $(Sg(\Gamma), \theta_\alpha) \to (\mathscr{G}_*^{\mathscr{P}(S)}, \mu_{\mathcal{G}_\alpha}^{\mathscr{P}(S)})$.*

Proof. Let $\Psi : Sg(\Gamma) \to \mathscr{G}_*^{\mathscr{P}(S)}$ that associates to $\Lambda \leqslant \Gamma$ the Cayley graph of Γ/Λ with generators $\{\bar{s}^\Lambda : s \in S\}$ (these are classes in the quotient), colored on edges by the map $(x, y) \mapsto \{s \in S : \bar{s}^\Lambda x = y\}$.

Let $\alpha : \Gamma \curvearrowright (X, \mu)$ be a pmp action. By definition of $\mathcal{G}_\alpha$, the measure $\mu_{\mathcal{G}_\alpha}^{\mathscr{P}(S)}$ is concentrated on the image of Ψ, thus we only need to prove that Ψ is injective to conclude that Ψ is the desired injection.

We construct a left inverse for Ψ. For $(G, o) \in \mathscr{G}_*^{\mathscr{P}(S)}$, define G' as follows. For $v \in V(G)$, add a loop at v if $\exists s \in S$ such that no edge starting at v contains s in its color, and color the loop by the set of such s. Now in (G', o), for any $s \in S$ and $v \in V(G')$, there is a unique edge starting at v that contains s in its color, so there is a bijection f between the set of walks in G' and the free group with $|S|$ generators F_S. Consider the unique morphism $\varphi : F_S \to \Gamma$ extending $c_S \to s$ where F_S is generated by the c_S for $s \in S$. Then let CW be the set of closed walks in G'. Note that the map $G \mapsto \varphi \circ f(CW)$ is a left inverse for Ψ. $\square$

If follows from the latter Lemma that two actions of (Γ, S) have the same IRS if and only if their respective graphings have same random $\mathscr{P}(S)$-colored graph.

An immediate consequence is that for α, β pmp actions of (Γ, S), if $\theta_\alpha = \theta_\beta$, then $\mathcal{G}_\alpha$ and $\mathcal{G}_\beta$ have same random graphs, seen as uncolored graphings. Indeed, if π is the map $\mathscr{G}_*^{\mathscr{P}(S)} \to \mathscr{G}_*$ associating to a $\mathscr{P}(S)$-colored graph its underlying uncolored graph, then the local statistics of a colored graphing $\mathcal{G}$ can be obtained from the random colored graph associated to $\mathcal{G}$ simply by the equality $\mu_\mathcal{G} = \pi_* \mu_\mathcal{G}^{\mathscr{P}(S)}$. It follows that for an IRS θ, by Theorem 3.11, either every pmp action of Γ with IRS θ are hyperfinite or none is.

Moreover, it is proved in [TD15] that every IRS is the IRS associated to a pmp action.

These properties motivate us to call an IRS on a finitely generated group **hyperfinite** if actions having this IRS are hyperfinite.

Furthermore, we can extend this definition to any countable group Γ and any θ IRS on Γ : For Γ' a finitely generated subgroup, let $\pi_{\Gamma'} : Sg(\Gamma) \to Sg(\Gamma')$ defined by $\pi_{\Gamma'}(\Lambda) = \Lambda \cap \Gamma'$ and let $\theta_{\Gamma'} = \pi_{\Gamma'*}\theta$. It is then clear that $\theta_{\Gamma'}$ is an IRS on Γ' such that the restriction to Γ' of any pmp action α of Γ which has IRS θ has IRS $\theta_{\Gamma'}$. Therefore either $\theta_{\Gamma'}$ is hyperfinite for every $\Gamma' \leqslant \Gamma$ finitely generated, and then by Proposition 4.5 every Γ-pmp action is hyperfinite, or there is a finitely generated $\Gamma' \leqslant \Gamma$ such that $\theta_{\Gamma'}$ is not hyperfinite and in this case for any pmp $\alpha : \Gamma \curvearrowright (X, \mu)$, $R_{\alpha_{\restriction \Gamma'}} \subseteq R_\alpha$ is a non-hyperfinite subequivalence relation, witnessing the non-hyperfiniteness of R_α and therefore of α.

Definition 4.10. Let Γ be a countable group. An IRS θ on Γ is called **hyperfinite** if one of the two equivalent following statements is satisfied :

1. There exists a hyperfinite pmp action which has IRS θ.
2. Every pmp action which has IRS θ is hyperfinite.

Definition 4.11. Let $\alpha : \Gamma \curvearrowright (X, \mu)$ and $\beta : \Gamma \curvearrowright (Y, \nu)$. An **action factor map** $\pi : Y \to X$ is a measure preserving almost surjective map such that $\forall^* y \in Y\ \forall \gamma \in \Gamma,\ \pi(\gamma^\beta y) = \gamma^\alpha \pi(y)$.

We say that α **is a factor of** β and we write $\alpha \sqsubseteq \beta$ if there exists an action factor map $\pi : Y \to X$.

Lemma 4.12. *Let α, β be hyperfinite actions of a finitely generated group (Γ, S) on standard probability spaces (X, μ) and (Y, ν) such that $\alpha \sqsubseteq \beta$ and $\theta_\alpha = \theta_\beta$. Then we have $\mathscr{G}_\alpha \underset{c}{\sqsubseteq} \mathscr{G}_\beta$*

as $\mathscr{P}(S)$-colored graphings. Precisely, every action factor map is a colored graphing factor map.

Proof. Let π be an action factor map $Y \to X$, as π is Γ-invariant, we have $\forall^* y\ \mathrm{Stab}^{\beta}(y) \subseteq \mathrm{Stab}^{\alpha}(\pi(y))$. Suppose now that $\exists^* y\ \mathrm{Stab}^{\beta}(y) \subsetneq \mathrm{Stab}^{\alpha}(\pi(y))$. By countability of Γ, $\exists \gamma \in \Gamma\ \exists^* y$, $\gamma \in \mathrm{Stab}^{\alpha}(\pi(y)) \backslash \mathrm{Stab}^{\beta}(y)$, thus

$$\begin{aligned} \theta_{\beta}(N_{\gamma}) &= \mathrm{Stab}^{\beta}_{*}\nu(N_{\gamma}) \\ &< (\mathrm{Stab}^{\alpha} \circ \pi)_{*}\nu(N_{\gamma}) \\ &= \mathrm{Stab}^{\alpha}_{*}(\pi_{*}\nu)(N_{\gamma}) \\ &= \mathrm{Stab}^{\alpha}_{*}\mu(N_{\gamma}) \\ &= \theta_{\alpha}(N_{\gamma}) \end{aligned}$$

A contradiction. This proves that $\forall^* y\ \mathrm{Stab}^{\beta}(y) = \mathrm{Stab}^{\alpha}(\pi(y))$. Applying the application Ψ from Lemma 4.9 to $\mathrm{Stab}^{\beta}(y) = \mathrm{Stab}^{\alpha}(\pi(y))$ we see that the components of y and $\pi(y)$ respectively in $\mathcal{G}_{\beta}$ and $\mathcal{G}_{\alpha}$ are colored isomorphic. ☐

Now we can state the generalization of Rokhlin Lemma that we prove in this paper :

Theorem 4.13 (Approximate conjugacy for pmp actions with a given hyperfinite IRS)**.** *Let Γ be a countable group and θ a hyperfinite IRS on Γ. Two actions $\alpha : \Gamma \curvearrowright (X, \mu)$ and $\beta : \Gamma \curvearrowright (Y, \nu)$ of Γ such that $\theta_{\alpha} = \theta_{\beta} = \theta$ are approximately conjugated, meaning that $\forall \gamma_1, \ldots, \gamma_n \in \Gamma\ \forall \varepsilon > 0$ there is a measure preserving bijection $\rho : X \to Y$ such that*

$$\mu(\{x \in X : \forall i \leqslant n\ \rho \circ \gamma_i^{\alpha}(x) = \gamma_i^{\beta} \circ \rho(x)\}) > 1 - \varepsilon$$

Ornstein and Weiss showed that any pmp action of an amenable group is hyperfinite. As being a free action means having IRS equal to $\delta_{\{e\}}$, the result we present here is indeed a generalization of Rokhlin Lemma.

We can reformulate the latter theorem as follows : The uniform metric $d_u(f, g) := \mu(\{x \in X : fx \neq gx\})$ on $\mathrm{Aut}(X, \mu)$ makes it a Polish space, and therefore $\mathrm{Aut}(X, \mu)^{\Gamma}$ is a Polish space for the product topology. For any enumeration $\Gamma = \{\gamma_n : n \in \mathbb{N}\}$, we get a complete metric $\delta_u(\alpha, \beta) = \sum_{n \in \mathbb{N}} 2^{-n} d_u(\gamma_n^{\alpha}, \gamma_n^{\beta})$ compatible with the product topology on $\mathrm{Aut}(X, \mu)^{\Gamma}$. Now we can see the space of pmp actions of Γ on (X, μ) with IRS θ as a subspace of $\mathrm{Aut}(X, \mu)^{\Gamma}$ and we call the induced topology on this space the uniform topology. Then it becomes clear that the latter theorem is equivalent to

Theorem 4.14 (Approximate conjugacy for pmp actions with a given hyperfinite IRS reformulated)**.** *If θ is hyperfinite, then every orbit of the conjugacy relation on the space of pmp actions of Γ on (X, μ) with IRS θ is dense for the uniform topology.*

4.2 Approximate conjugacy for pmp actions with a given hyperfinite IRS

We begin with the case where one of the actions is a factor of the other. In fact we prove a stronger version involving the stability of Borel parameters.

Definition 4.15. Let F_1, F_2 be two finite sets. A **(F_1, F_2)-bicolored graphing** on a standard probability space (X, μ) is a graphing $\mathcal{G}(X, \mu)$ endowed with two Borel maps $\varphi_{\mathcal{G}} : E(\mathcal{G}) \to F_1$ and $\psi_{\mathcal{G}} : X \to F_2$. We call $\psi_{\mathcal{G}}(x)$ the vertex-color of x and $\varphi_{\mathcal{G}}(x, y)$ the edge-color of (x, y).

Definition 4.16. Let $\mathcal{G}(X, \mu)$ and $\mathcal{G}'(Y, \nu)$ be two (F_1, F_2)-bicolored graphings. A **bicolored graphing factor map** $\pi : Y \to X$ is a F_1-colored graphing factor map such that $\psi_{\mathcal{G}} \circ \pi = \psi_{\mathcal{G}'}$.

We say that $\mathcal{G}$ is a bicolored factor of $\mathcal{G}'$ and we write $\mathcal{G} \underset{2c}{\sqsubseteq} \mathcal{G}'$ if there is a bicolored factor map $\pi : Y \to X$.

Theorem 4.17 (Approximate parameterized conjugacy for factor actions)**.** *Let (X, μ) and (Y, ν) be standard probability spaces and $A_1, \ldots, A_k \subseteq X$, $B_1, \ldots, B_k \subseteq Y$ be Borel subsets. Let Γ be a countable group, θ be a hyperfinite IRS on Γ and $\alpha : \Gamma \curvearrowright (X, \mu)$, $\beta : \Gamma \curvearrowright (Y, \nu)$ be pmp actions of Γ with IRS θ and such that $\alpha \sqsubseteq \beta$ for an action factor map $\pi : Y \to X$ such that $\forall i \leqslant k$, $\pi^{-1}(A_i) = B_i$. Then for $\varepsilon > 0$ and $\gamma_1, \ldots, \gamma_n \in \Gamma$, there exists a pmp bijection $\rho : X \to Y$ such that $\forall i \leqslant k$, $\rho(A_i) = B_i$ and*

$$\mu(\{x \in X : \forall i \leqslant n \ \rho \circ \gamma_i^{\alpha}(x) = \gamma_i^{\beta} \circ \rho(x)\}) > 1 - \varepsilon$$

Proof. We begin the proof with a claim about graphings.

Claim 4.17.1. Let $\mathcal{G}(X, \mu)$ and $\mathcal{G}'(Y, \nu)$ be hyperfinite (F_1, F_2)-bicolored graphings such that $\mathcal{G}(X, \mu) \underset{2c}{\sqsubseteq} \mathcal{G}'(Y, \nu)$. Then for any $\varepsilon > 0$ there exists a pmp bijection $\rho : X \to Y$ such that $\psi_{\mathcal{G}} = \psi_{\mathcal{G}'} \circ \rho$ and

$$\mu_E\left(\bigcup_{c \in F_1} \rho^{-1}\left(E(\mathcal{G}') \cap \varphi_{\mathcal{G}'}^{-1}(c)\right) \triangle \left(E(\mathcal{G}) \cap \varphi_{\mathcal{G}}^{-1}(c)\right)\right) < \varepsilon$$

Proof. Let π be a bicolored graphing factor map $Y \to X$. First take a Borel set $Z \subseteq E(\mathcal{G})$ of measure less than $\frac{\varepsilon}{d}$ and $M \in \mathbb{N}$ such that the graphing $\mathcal{H} = \mathcal{G} \backslash Z$ has components of size at most M. Let $Z' = \pi^{-1}(Z)$ and $\mathcal{H}' = \mathcal{G}' \backslash Z'$, by definition of π we know that $\mathcal{H}'$ has components of size at most M.

Consider the set $\mathscr{G}_M$ be the set of connected F_1-colored graphs of size at most M. We consider the two partitions $X = \bigsqcup_{S \in \mathscr{G}_M} C_S^{\mathcal{H}}$ and $Y = \bigsqcup_{S \in \mathscr{G}_M} C_S^{\mathcal{H}'}$, where $C_S^{\mathcal{H}}$ is defined like in Section 3. to be the set of vertices of $\mathcal{H}$ whose component are (F_1-colored) isomorphic to S. Since π induces F_1- colored graph isomorphisms, we have $C_S^{\mathcal{H}'} = \pi^{-1}(C_S^{\mathcal{H}})$.

In order to define ρ, it suffices to define a measure preserving bijection $\rho_S : C_S^{\mathcal{H}} \to C_S^{\mathcal{H}'}$ preserving colored graph structures for each $S \in \mathscr{G}_M$.

Indeed, the union of all these bijections would yield a measure preserving bijection $\rho : X \to Y$ preserving colors such that
$\forall x \in X \backslash V_{inc}(Z)$, $B_1^{\mathcal{G}}(x) = B_1^{\mathcal{H}}(x) \sim B_1^{\mathcal{H}'}(\rho(x)) = B_1^{\mathcal{G}'}(\rho(x))$, hence

$V_{inc}(\bigcup_{c\in F_1}\left(\rho^{-1}E(\mathcal{G}')\cap\psi_{\mathcal{G}'}^{-1}\right)\triangle(E(\mathcal{G})))\subseteq V_{inc}(Z)$, so

$$\mu_E\left(\bigcup_{c\in F_1}\left(\rho^{-1}E(\mathcal{G}')\cap\psi_{\mathcal{G}'}^{-1}\right)\triangle(E(\mathcal{G}))\right)\leqslant d\mu(V_{inc}(Z))\leqslant d\mu_E(Z)<\varepsilon$$

Take $S\in\mathscr{G}_M$ and let us define ρ_S. First we define a partition of $C_S^{\mathcal{H}}$ into Borel transversals $(T_\omega)_{\omega\in V(S)}$ such that the elements of T_ω occupy the same place in their component for $\mathcal{H}$ that ω in S. Precisely, choose $\omega_0\in S$. We define the Borel transversals T_ω by induction.

Suppose that the $T_{\omega'}$ are already defined for $\omega'\in R$ where R is a proper subset of $V(S)$. Take $\omega\in V(S)\backslash R$ incident to R and let $\widetilde{T_\omega}=\{x\in C_S^{\mathcal{H}}:([x]_{\mathcal{H}},x)\sim_R(S,\omega)\}$. Here $\sim_R$ means isomorphic over R, that is there exists an isomorphism $f:([x]_{\mathcal{H}},x)\to(S,\omega)$ of colored rooted graphs such that $\forall\omega'\in R,\ f([x]_{\mathcal{H}}\cap T_{\omega'})=\{\omega'\}$. Then let $n_\omega=|\{\omega'\in S:(S,\omega')\sim_R(S,\omega)\}|$. Since all $T_{\omega'}$ must have same measure, we must choose T_ω to be a subset of $\widetilde{T_\omega}$ of measure $\frac{\mu(\widetilde{T_\omega})}{n_\omega}$. In fact it suffices to take any such set as T_ω. Then we let $R'=R\cup\{\omega\}$ and we iterate the construction.

Again by definition of π, the family $(\pi^{-1}(T_\omega))_{\omega\in V(S)}$ is a partition of $C_S^{\mathcal{H}'}$ into Borel transversals such that the elements of $\pi^{-1}(T_\omega)$ occupy the same place in their component for $\mathcal{H}'$ that ω in S. We may now define ρ_S :

- We start by taking a measure preserving bijection $\rho_S^{\omega_0}:T_{\omega_0}\to\pi^{-1}(T_{\omega_0})$.
- Then for every $\omega\in S$, there is a unique way of extending $\rho_S^{\omega_0}$ to T_ω while respecting the graph structure of S. Indeed, take $x\in T_\omega$, there is a unique $x_0\in[x]_{\mathcal{H}}\cap T_{\omega_0}$ and we want to define $\rho_S^{\omega}(x)\in[\rho_S^{\omega_0}(x_0)]_{\mathcal{H}'}\cap\pi^{-1}(T_\omega)$ but again this intersection is a singleton. Define $\rho_S:C_S^{\mathcal{H}}\to C_S^{\mathcal{H}'}$ to be this unique extension of $\rho_S^{\omega_0}$ satisfying the condition above.

  As π is a colored graphing factor map, it is clear that ρ_S is a measure preserving bijection such that $\forall x\in C_S^{\mathcal{H}}$ ρ_S induces an isomorphism of colored graphs between $[x]_{\mathcal{H}}$ and $[\rho_S(x)]_{\mathcal{H}'}$.

■

We now want to apply the Claim to suitable graphings to conclude. Let $S=\{\gamma_1,\dots,\gamma_n,\gamma_1^{-1},\dots,\gamma_n^{-1}\}$ and $\Gamma':=\langle S\rangle$ be the subgroup of Γ generated by S. Let us denote the respective restrictions of α and β to (Γ',S) by α' and β' and finally consider the graphings $\mathcal{G}_{\alpha'}$ and $\mathcal{G}_{\beta'}$.

For the spaces of colors, we choose $F_1=\mathscr{P}(S)$ and $F_2=\mathscr{P}(\{1,\dots,k\})$. The way we color edges has already been explained, for vertices, simply color a vertex $x\in X$ by $\psi_{\mathcal{G}_\alpha}(x)=\{i\leqslant k:x\in A_i\}$ and $y\in Y$ by $\psi_{\mathcal{G}_\beta}(y)=\{i\leqslant k:y\in B_i\}$.

First, $\mathcal{G}_{\alpha'}$ and $\mathcal{G}_{\beta'}$ are indeed $(\mathscr{P}(S),\mathscr{P}(\{1,\dots,k\}))$-bicolored graphings, and are hyperfinite since α' and β' are hyperfinite actions of a finitely generated group.

The next step is to prove that π considered in the statement of the theorem is a bicolored factor map for the $(\mathscr{P}(S),\mathscr{P}(\{1,\dots,k\}))$-bicolored graphings $\mathcal{G}_{\alpha'}$ and $\mathcal{G}_{\beta'}$.

- First, π is indeed a pmp almost surjective map $Y\to X$.

- Then for $y \in Y$, we have

$$\psi_{\mathcal{G}_{\alpha'}}(\pi(y)) = \{i \leqslant k : \pi(y) \in A_i\} = \{i \leqslant k : y \in B_i\} = \psi_{\mathcal{G}_{\beta'}}(y)$$

- Finally, by Lemma 4.12, π is furthermore a colored graphing factor map between the $\mathscr{P}(S)$-colored graphings $\mathcal{G}_{\alpha'}$ and $\mathcal{G}_{\beta'}$.

Applying the Claim gives us a pmp bijection $\rho : X \to Y$ such that $\psi_{\mathcal{G}_{\alpha'}} = \psi_{\mathcal{G}_{\beta'}} \circ \rho$ and $\mu_E(\bigcup_{c\in\mathscr{P}(S)} \left(E(\mathcal{G}_{\alpha'} \cap \psi_{\mathcal{G}_{\alpha'}}) \triangle \left(\rho^{-1}E(\mathcal{G}_{\beta'}) \cap \psi_{\mathcal{G}_{\beta'}}\right)\right) < \varepsilon$. But then for $1 \leqslant i \leqslant k$, $\rho(A_i) = B_i$, and by definitions of $\mathcal{G}_{\alpha'}$ and $\mathcal{G}_{\beta'}$ the set $\{x \in X : \exists \gamma \in S\ \rho \circ \gamma^\alpha(x) \neq \gamma^\beta \circ \rho(x)\}$ is contained in $V_{inc}(\bigcup_{c\in\mathscr{P}(S)} \left(E(\mathcal{G}_{\alpha'} \cap \psi_{\mathcal{G}_{\alpha'}}) \triangle \left(\rho^{-1}E(\mathcal{G}_{\beta'}) \cap \psi_{\mathcal{G}_{\beta'}}\right)\right)$ so its measure is less than ε. ☐

To conclude the proof of Theorem 4.13, we will use the transitivity of the approximate conjugacy relation and show that for any two pmp actions $\alpha : \Gamma \curvearrowright (X, \mu)$ and $\beta : \Gamma \curvearrowright (Y, \nu)$ of Γ such that $\theta_\alpha = \theta_\beta$, there is a third pmp action $\zeta : \Gamma \curvearrowright (Z, \eta)$ of IRS θ such that both α and β are factors of ζ.

Proposition 4.18 (Disintegration theorem, Admitted). *Let X, Y be Radon spaces, $\mu \in \mathfrak{P}(Y)$ and $\pi : Y \to X$ a Borel map. We let $\nu = \pi_*\mu$. Then there is a ν-a.e. uniquely determined family of Borel probability measures $(\mu_x)_{x\in X} \in \mathfrak{P}(Y)^X$ such that*

1. *For each Borel $B \subseteq Y$, the map $x \mapsto \mu_x(B)$ is Borel measurable.*
2. *For ν-a.e. $x \in X$, μ_x is concentrated on the fiber $\pi^{-1}(x)$.*
3. *For every Borel map $f : Y \to [0, \infty]$, $\int_Y f(y)\ d\mu(y) = \int_X \int_Y f(y)\ d\mu_x(y)\ d\nu(x)$.*

We then write $\mu = \int_X \mu_x\ d\nu$.

With the help of disintegration, we define the relative independent joining of two ergodic systems over a common factor.

Definition 4.19. Let $\alpha : \Gamma \curvearrowright (X, \mu)$ and $\beta : \Gamma \curvearrowright (X', \mu')$ be pmp actions, and let $\xi : \Gamma \curvearrowright (Y, \nu)$ be a common factor of α and β for respective action factor maps $\pi : X \to Y$ and $\pi' : X' \to Y$.

Since standard Borel spaces are in particular Radon spaces, we can disintegrate μ and μ' with respect to ν using the Borel maps π and π' to get $\mu = \int_Y \mu_y\ d\nu$ and $\mu' = \int_Y \mu'_y\ d\nu$.

Consider $Z := X \times Y$ and $\eta \in \mathfrak{P}(Z)$ defined by $\eta = \int_Y \mu_y \times \mu'_y\ d\nu$.

We call the **independent joining of** α **and** β **over** ξ and we write $\alpha \underset{\xi}{\times} \beta$ the pmp action $\alpha \times \beta : \Gamma \curvearrowright (Z, \eta)$.

The independent joining of α and β over ξ is a factor of both α and β respectively for the projection on the first and second coordinates p_1 and p_2, which moreover makes the following diagram commute, up to a null set :

$$\begin{tikzcd}
 & \alpha \underset{\xi}{\times} \beta \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
\alpha \arrow[dr, "\pi"'] & & \beta \arrow[dl, "\pi'"] \\
 & \xi &
\end{tikzcd}$$

Lemma 4.20. *Let Γ be a countable group and θ be an IRS on Γ. Let $\alpha : \Gamma \curvearrowright (X, \mu)$, $\beta : \Gamma \curvearrowright (Y, \nu)$ be pmp actions of IRS θ. Then there is a standard probability space (Z, η) and a pmp $\zeta : \Gamma \curvearrowright (Z, \eta)$ of IRS θ such that both α and β are factors of ζ.*

Proof. Let θ be an IRS on Γ, we write $\boldsymbol{\theta}$ for the measure preserving action $\Gamma \curvearrowright (Sg(\Gamma), \theta)$. Let $\alpha : \Gamma \curvearrowright (X, \mu)$, $\beta : \Gamma \curvearrowright (Y, \nu)$ be pmp actions of IRS θ, then the maps $\mathrm{Stab}^\alpha : X \to Sg(\Gamma)$ and $\mathrm{Stab}^\beta : Y \to Sg(\Gamma)$ are action factor maps.

Consider $\alpha \underset{\boldsymbol{\theta}}{\times} \beta$ be the independent joining of α and β over $\boldsymbol{\theta}$. It only remains to prove that its IRS is θ.

Claim 4.20.1. Let $\zeta : \Gamma \curvearrowright (Z, \eta)$ be a joining of α and β, that is a pmp action such that both α and β are factors of ζ. We have $\theta_\zeta = \theta$ if and only if, up to a null set, the following diagram commutes :

$$\begin{tikzcd} & \zeta \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \alpha \arrow[dr, "\mathrm{Stab}^\alpha"'] & & \beta \arrow[dl, "\mathrm{Stab}^\beta"] \\ & \boldsymbol{\theta} & \end{tikzcd}$$

Proof. All equalities in this proof are up to a null set.

Suppose the diagram commutes. For $\gamma \in \Gamma$, we have
$\forall^*(x, y),\ \gamma x = x \Leftrightarrow \gamma y = y \Leftrightarrow \gamma(x, y) = (x, y)$. It follows that
$\forall^*(x, y),\ \mathrm{Stab}^\zeta(x, y) = \mathrm{Stab}^\alpha(x)$ or in other words, $\mathrm{Stab}^\zeta = \mathrm{Stab}^\alpha \circ p_1$. Therefore
$\theta_\zeta = \mathrm{Stab}^\zeta_* \eta = \mathrm{Stab}^\alpha_* (p_{1*}\eta) = \mathrm{Stab}^\alpha_* \mu = \theta_\alpha = \theta$.

Conversely, suppose $\theta_\zeta = \theta$. Then p_1 and p_2 are action factor maps between two actions with same IRS, thus repeating the proof of Lemma 4.12, we get
$\mathrm{Stab}^\alpha \circ p_1 = \mathrm{Stab}^\zeta = \mathrm{Stab}^\beta \circ p_2$. ■

We conclude simply by definition of the independent joining over a common factor that $\alpha \underset{\boldsymbol{\theta}}{\times} \beta$ is a suitable joining for the Lemma. □

The proof of the big theorem easily follows :

Proof : Approximate conjugacy for pmp actions with a given hyperfinite IRS.
Let $\alpha : \Gamma \curvearrowright (X, \mu)$ and $\beta : \Gamma \curvearrowright (Y, \nu)$ two actions of Γ having IRS θ and consider the joining $\zeta : \Gamma \curvearrowright (Z, \eta)$ from Lemma 4.20.

Applying twice Theorem 4.17 with no Borel parameters we get two pmp bijections $\rho : X \to Z$ and $\rho' : Y \to Z$ such that

$$\mu(\{x \in X : \forall i \leqslant n,\ \rho \circ \gamma_i^\alpha(x) = \gamma_i^\beta \circ \rho(x)\}) > 1 - \frac{\varepsilon}{2}$$

and

$$\nu(\{y \in Y : \forall i \leqslant n,\ \rho \circ \gamma_i^\beta(y) = \gamma_i^\beta \circ \rho(y)\}) > 1 - \frac{\varepsilon}{2}$$

Thus, $\rho'^{-1} \circ \rho : X \to Y$ witnesses the ε-approximate conjugacy of α and β. □

5 Model theory for hyperfinite actions

5.1 Probability algebras

The reference for countinuous model theory is [YBHU08]. We assume everything that is in this article and we will use the same notations.

Definition 5.1. A **probability algebra** is a Boolean algebra $(\mathcal{A}, \cup, \cap, ^c, 0, 1, \subseteq, \triangle)$ endowed with an application $\mu : \mathcal{A} \to [0,1]$ satisfying the following :

1. $\mu(1) = 1$.
2. $\forall A, B \in \mathcal{A},\ \mu(A \cap B) = 0 \Rightarrow \mu(A \cup B) = \mu(A) + \mu(B)$.
3. The application $d_\mu(A, B) := \mu(A \cap B)$ is a complete metric.

Definition 5.2. An element $A \in \mathcal{A}$ is an **atom** if $\forall B \in \mathcal{A},\ B \subseteq A \Rightarrow B \in \{0, A\}$. A probability algebra is **atomless** if it has no atom.

Proposition 5.3 (Admitted)**.** *If a probability algebra $\mathcal{A}$ is atomless, then*
$\forall A \in \mathcal{A}\ \forall r \in [0, \mu(A)]\ \exists B \subseteq A,\ \mu(B) = r.$

Proposition 5.4 (Admitted,[Fre02])**.** *Let $\mathfrak{A}$ be any probability algebra. Then there exists a probability space (X, μ) such that $\mathfrak{A}$ is isomorphic to* $\mathrm{MAlg}(X, \mu)$*. Moreover if $\mathfrak{A}$ is separable then (X, μ) can be taken to be a standard probability space.*

Take $f : (X, \mu) \to (Y, \nu)$ a measure preserving map then the map $\widetilde{f} : \mathrm{MAlg}(Y, \nu) \to \mathrm{MAlg}(X, \nu)$ sending $[A]_\nu$ to $[f^{-1}(A)]_\mu$ is a probability algebra morphism. Moreover, if f is a bijection, then $\widetilde{f}$ is an isomorphism.

5.2 Model theory of atomless probability algebras

We can axiomatize the theory APA of atomless probability algebras in the signature
$\mathcal{L} = \{\cup, \cap, ^c, 0, 1\}$ ($\subseteq$ and $\triangle$ are defined as usual) by :

- The axioms of Boolean algebras :
  - $\sup_{x,y}\ d(x \cup y, y \cup x) = 0$
  - $\sup_{x,y}\ d(x \cap y, y \cap x) = 0$
  - $\sup_x d(x \cup 0, x) = 0$
  - $\sup_x d(x \cap 1, x) = 0$
  - $\sup_x d(x \cup x^c, 1) = 0$
  - $\sup_x d(x \cap x^c, 0) = 0$
  - $\sup_{x,y,z}\ d(x \cup (y \cap z), (x \cup y) \cap (x \cup z)) = 0$
  - $\sup_{x,y,z}\ d(x \cap (y \cup z), (x \cap y) \cup (x \cap z)) = 0$
- The axioms for the measure :
  - $\mu(1) = 1$
  - $\sup_{x,y}\ \mu(x \cap y) \dot{-} \mu(x) = 0$
  - $\sup_{x,y}\ \mu(x) \dot{-} \mu(x \cup y) = 0$
  - $\sup_{x,y}\ |(\mu(x) - \mu(x \cap y)) - (\mu(x \cup y) - \mu(y))| = 0$

- The link between the metric d and the measure : $\sup_{x,y} \ |d(x,y) - \mu(x \triangle y)| = 0$
- The lack of atoms : $\sup_x \ \inf_y \ |\mu(x \cap y) - \mu(x \cap y^c)| = 0$

For (X, μ) a probability space, we let $\mathrm{MAlg}(X, \mu)$ be the boolean algebra of Borel subsets of X quotiented by the σ-ideal of null sets. For $A \subseteq X$ Borel we denote its class in $\mathrm{MAlg}(X, \mu)$ by $[A]_\mu$. We define an application $\mu : \mathrm{MAlg}(X, \mu) \to [0, 1]$ by $\mu([A]_\mu) = \mu(A)$. It is classic that $\mathrm{MAlg}(X, \mu)$ endowed with μ is a probability algebra.

Moreover, in the case where (X, μ) is a standard probability space then $\mathrm{MAlg}(X, \mu)$ is atomless and separable for the topology induced by d_μ.

Proposition 5.5. *The theory APA is separably categorical and therefore complete.*

Proof. Take $\mathcal{A}, \mathcal{B}$ separable probability algebras. We may suppose that there are standard probability spaces (X, μ) and (Y, ν) such that $\mathcal{A} = \mathrm{MAlg}(X, \mu)$ and $\mathcal{B} = \mathrm{MAlg}(Y, \nu)$. Now by uniqueness of the standard probability space there exists a measure preserving Borel bijection $f : X \to Y$, and that induces an isomorphism $\widetilde{f} : \mathcal{B} \to \mathcal{A}$. □

Finally we give a characterization of types in the theory APA :

For $\mathcal{A}$ a probability algebra, we can define a Hilbert space $L^2(\mathcal{A})$ in which $\mathcal{A}$ embeds. This construction can be done in many different ways (see [Fre02]) and in the end if $\mathcal{A} = \mathrm{MAlg}(X, \mu)$, then the linear map $L^2(\mathcal{A}) \to L^2(X, \mu)$ sending A to $\mathbb{1}_A$ is an isometry.

Definition 5.6. Let $\mathcal{A}$ be a probability algebra and $\mathcal{B}$ a measure subalgebra of $\mathcal{A}$. Then the space $L^2(\mathcal{B})$ is a closed vector subspace of the Hilbert space $L^2(\mathcal{A})$, we denote by $\mathbb{P}_\mathcal{B}$ the orthogonal projection on $L^2(\mathcal{B})$ and we call it the **conditional expectancy** with respect to $\mathcal{B}$. Particularly, for $A \in \mathcal{A}$, A can be seen as an element of $L^2(\mathcal{A})$ and we call $\mathbb{P}_\mathcal{B}(A)$ the **conditional probability** of A with respect to $\mathcal{B}$.

By definition, the conditional probability of A with respect to $\mathcal{B}$ is the only $\mathcal{B}$-measurable function such that for any $\mathcal{B}$-measurable function f, we have $\int \mathbb{P}_\mathcal{B}(A).f = \int \mathbb{1}_A.f$.

Proposition 5.7 (Admitted, [BH04])**.** *Let $M \models APA$, $\bar{a}, \bar{b}$ be n-uples of elements of M and*

$C \subseteq M$. Then $\mathrm{tp}(\bar{a}/C) = \mathrm{tp}(\bar{b}/C)$ if and only if for every map $\sigma : \{1, \ldots, n\} \to \{1, c\}$ we have $\mathbb{P}_{\mathrm{dcl}^M(C)}\left(\bigcap_{1 \leqslant i \leqslant n} a_i^{\sigma(i)}\right) = \mathbb{P}_{\mathrm{dcl}^M(C)}\left(\bigcap_{1 \leqslant i \leqslant n} b_i^{\sigma(i)}\right)$, where x^1 means x.

5.3 Morphisms and liftings

In general, given a morphism $\varphi : \mathrm{MAlg}(Y, \nu) \to \mathrm{MAlg}(X, \mu)$ there is no way to get a point to point map $f : X \to Y$ such that $\widetilde{f} = \varphi$. However, with standard probability spaces, we can do such constructions :

Proposition 5.8 (Admitted,[Fre02])**.** *Let (X, μ) and (Y, ν) be standard probability spaces.*

1. *Let φ be a morphism of probability algebras $\mathrm{MAlg}(Y, \nu) \to \mathrm{MAlg}(X, \mu)$. Then there is a lifting of φ, that is a measure preserving map $f : X \to Y$ such that $\varphi = \widetilde{f}$.*
2. *Let φ be an isomorphism of probability algebras $\mathrm{MAlg}(Y, \nu) \to \mathrm{MAlg}(X, \mu)$. Then there is a lifting of φ which is a bijection $f : X \to Y$.*

3. *Let Γ be a countable group acting by automorphisms on* $\mathrm{MAlg}(X,\mu)$ *by an action α. Then there is a lifting of α, that is an action $\beta : \Gamma \curvearrowright X$ acting by measure preserving transformations such that $\forall \gamma \in \Gamma\ \gamma^\alpha = \widetilde{\gamma^\beta}$.*

5.4 The theory $\mathfrak{A}_\theta$

Until now, we studied actions of any countable group. For the sake of simplicity, we now restrict to F_∞ actions, where F_∞ denotes the countably generated free group. It is clear that any action of a countable group can be represented as a F_∞-action.

We now expand the signature $\mathcal{L}$ with a countable set of function symbols $\{\underline{\gamma} : \gamma \in F_\infty\}$. We call this new signature $\mathcal{L}_\infty$. We begin by considering the theory $\mathfrak{A}_{F_\infty}$ consisting of the following axioms :

- The axioms of APA.
- For $\gamma \in F_\infty$, the axioms expressing that $\underline{\gamma}$ is a morphism :
  - $\sup_{x,y}\ d(\underline{\gamma}(x \cup y), \underline{\gamma}x \cup \underline{\gamma}y) = 0$
  - $\sup_{x,y}\ d(\underline{\gamma}(x \cap y), \underline{\gamma}x \cap \underline{\gamma}y) = 0$
  - $\sup_x\ |\mu(\underline{\gamma}x) - \mu(x)| = 0$
  - $\sup_x\ \inf_y\ d(x, \underline{\gamma}y) = 0$
- The axioms expressing that F_∞ acts on the probability algebra :
  - $\sup_x\ d(\underline{e}x, x) = 0$
  - For $\gamma_1, \gamma_2 \in F_\infty$ the axiom $\sup_x\ d(\underline{\gamma_1}(\underline{\gamma_2}x), \underline{\gamma_1\gamma_2}x) = 0$

By Proposition 5.8 any separable model of $\mathfrak{A}_{F_\infty}$ can be seen as the probability algebra endowed with the action associated with a measure preserving action $\alpha : F_\infty \curvearrowright X$ where (X,μ) is a standard probability space. We denote the model of $\mathfrak{A}_{F_\infty}$ induced by such an action α by M_α. Without loss of generality, from now on, whenever taking a separable model of $\mathfrak{A}_{F_\infty}$, we will take an action α and suppose our model is M_α.

For f any measure preserving bijection $(X,\mu) \to (X,\mu)$, where (X,μ) is a standard probability space. We call the support of f and we denote by $\mathrm{Supp}\ f$ the set $\{x \in X : fx \neq x\}$.

Lemma 5.9. *There is a $A \in \mathrm{MAlg}(X,\mu)$ such that $[\mathrm{Supp}\ f]_\mu = f^{-1}A \cup A \cup fA$ and $A \cap fA = 0$.*

Proof. $\mathrm{MAlg}(X,\mu)$ is a probability algebra and therefore is complete as a Boolean algebra so it has a maximal element A disjoint from its image by f.

Consider $B = f^2A \backslash (f^{-1}A \cup A \cup fA)$. We have

$$\begin{aligned}(A \cup B) \cap f(A \cup B) &= (A \cap fA) \cup (A \cap fB) \cup (B \cap fA) \cup (B \cap fB)\\ &\subseteq 0 \cup (A \backslash A) \cup (fA \backslash fA) \cup (f^2A \backslash f^2A)\\ &= 0\end{aligned}$$

Thus $A \cup B$ is disjoint from its image. By maximality of A, we then have $B \subseteq A$, but by definition $B \cap A = 0$, so $B = 0$, or in other words, $f^2(A) \subseteq f^{-1}A \cup A \cup fA$. Therefore by a simple induction, $f(f^{-1}A \cup A \cup fA) = f^{-1}A \cup A \cup fA$, and thus the restriction of f to $[\text{Supp } f]_\mu \backslash (f^{-1}A \cup A \cup fA)$ is an endomorphism of $\text{MAlg}(X,\mu)_{\restriction [\text{Supp } f]_\mu \backslash (f^{-1}A \cup A \cup fA)}$.

Claim 5.9.1. Let g be a measure preserving transformation of a standard probability space (Y,ν) such that $\nu(\text{Supp } g) > 0$, then there exists a nonnull Borel $C \subseteq Y$ disjoint from its image by g.

Proof. Let $(C_n : n \in \mathbb{N})$ be a countable family of Borel subsets of Y separating the points. For $n \in \mathbb{N}$, let $C'_n = C_n \backslash g(C_n)$. For $y \in \text{Supp } g$, there is n such that $y \in C_n$ and $g^{-1}(y) \notin C_n$ so $y \in C'_n$ and therefore $\mu(\bigcup_{n\in\mathbb{N}} C'_n) \geqslant \mu(\text{Supp } g) > 0$. Take a C'_n of positive measure as the desired C. ■

By the Claim it is clear that if $\text{Supp } f \backslash (f^{-1}A \cup A \cup fA) \neq 0$ then $\exists C \in \text{MAlg}(X,\mu)$ disjoint from its image and such that $0 \neq C \subseteq [\text{Supp } f]_\mu \backslash (f^{-1}A \cup A \cup fA)$, contradicting the maximality of A. We conclude that $[\text{Supp } f]_\mu \subseteq f^{-1}A \cup A \cup fA$.

Conversely, if A is disjoint from its image by f, then $f^{-1}A$ and fA have the same property and thus $f^{-1}A \cup A \cup fA \subseteq [\text{Supp } f]_\mu$. ☐

This encourages the following definition.

Definition 5.10. Let $(\mathscr{A},\mu)$ be a probability algebra, we can generalize the notion of **support** to any pmp isomorphism f of $\mathscr{A}$ (and not only those coming from a pmp bijection of a probability space) by letting $\text{Supp } f = \sup\{f^{-1}A \cup A \cup fA : A \in \mathscr{A}, A \cap fA = 0\}$. The previous Lemma assures that if f is actually of the form $\tilde{\rho}$ for ρ a pmp bijection of a probability space, we have $\text{Supp } f = [\text{Supp } \rho]_\mu$.

Now we can prove that the IRS of a pmp action on a probability algebra is determined by the theory of this action seen as a model of $\mathfrak{A}_{F_\infty}$.

Definition 5.11. For $\gamma \in F_\infty$ we let $t_\gamma(x)$ denote the term $\underline{\gamma^{-1}}(x \backslash \underline{\gamma} x) \cup (x \backslash \underline{\gamma} x) \cup \underline{\gamma}(x \backslash \underline{\gamma} x)$.

It is clear from this definition that for $M \models \mathfrak{A}_{F_\infty}$, $\text{Supp } \gamma = \sup\{t_\gamma(A) : A \in M\}$.

Lemma 5.12. *Let $M \models \mathfrak{A}_{F_\infty}$ and $\gamma \in F_\infty$, then the support of γ is definable without parameters in the theory $\mathfrak{A}_{F_\infty}$.*

Proof. We need to prove that the distance to $\text{Supp } \gamma$ is a definable formula. By definition of the distance, we have $\forall x \in M,\ d(x, \text{Supp } \gamma) = \mu(x \backslash \text{Supp } \gamma) + \mu(\text{Supp } \gamma \backslash x)$.

On the one hand, $\mu(x \backslash \text{Supp } \gamma) = \inf_y\ \mu(x \backslash t_\gamma(y))$ so the first part is definable.

On the other hand, $\mu(\text{Supp } \gamma \backslash x) = \sup_y\ \mu(t_\gamma(y) \backslash x)$ and therefore the second part is definable as well. ☐

Theorem 5.13. *Let M_α, M_β be two elementary equivalent separable models of $\mathfrak{A}_{F_\infty}$, then $\theta_\alpha = \theta_\beta$.*

Proof. As θ_α and θ_β are measures on $Sg(F_\infty)$, they are determined by their values on the sets $N_{F,G} = \{\Lambda \leqslant F_\infty : F \subseteq \Lambda, G \cap \Lambda = \varnothing\}$ where F and G are finite.

Note that $\theta_\alpha(N_{F,\varnothing}) = \mu(\bigcap_{\gamma\in F} \mathrm{Supp}\,\gamma^{M_\alpha})$ and $\theta_\beta(N_{F,\varnothing}) = \mu(\bigcap_{\gamma\in F} \mathrm{Supp}\,\gamma^{M_\beta})$ and by Lemma 5.12, $\mu(\bigcap_{\gamma\in F} \mathrm{Supp}\,\gamma)$ is a definable sentence, thus by elementary equivalence, for every finite $F \subseteq F_\infty$, we have $\theta_\alpha(N_{F,\varnothing}) = \theta_\beta(N_{F,\varnothing})$.

Now for F, G finite subsets of F_∞, write $N_{F,G} = N_{F,\varnothing} \setminus \bigcup_{\gamma\in G} N_{F\cup\{\gamma\},\varnothing}$. By the inclusion exclusion principle, we then get

$$\begin{aligned}\theta_\alpha(N_{F,G}) &= \theta_\alpha(N_{F,\varnothing}) + \sum_{i=1}^{|G|}(-1)^i \sum_{\{J\subseteq G\,:\,|J|=i\}} \theta_\alpha(N_{F\cup J,\varnothing})\\ &= \theta_\beta(N_{F,\varnothing}) + \sum_{i=1}^{|G|}(-1)^i \sum_{\{J\subseteq G\,:\,|J|=i\}} \theta_\beta(N_{F\cup J,\varnothing})\\ &= \theta_\beta(N_{F,G})\end{aligned}$$

□

For θ an IRS, let $\mathfrak{A}_\theta$ be the $\mathcal{L}_\infty$-theory consisting of :

- The axioms of $\mathfrak{A}_{F_\infty}$.
- For $F \subseteq F_\infty$ finite, the axiom $\sup_{\{x_\gamma:\gamma\in F\}}\ \mu(\bigcap_{\gamma\in F} t_\gamma(x_\gamma)) = \theta(N_{F,\varnothing})$.

Then the theory $\mathfrak{A}_\theta$ represents measure preserving actions of F_∞ of IRS θ.

5.5 Completeness and model completeness

Theorem 5.14. *Let θ be a hyperfinite IRS on F_∞, then the theory $\mathfrak{A}_\theta$ is complete.*

Proof. It suffices to show that any two separable models are elementary equivalent by the Löwenheim-Skolem theorem.

Let M_α, M_β be separable models of $\mathfrak{A}_\theta$, where α acts on (X, μ) and β on (Y, ν). We will prove by induction on formulas that for any $\mathcal{L}_\infty$ formula $\varphi(\bar{x})$ and $\varepsilon > 0$, there is a pmp bijection $\rho : Y \to X$ such that $\forall \bar{a} \subseteq M_\alpha, \left|\varphi^{M_\alpha}(\bar{a}) - \varphi^{M_\beta}(\widetilde{\rho}\bar{a})\right| < \varepsilon$.

Step 1 : We start with atomic formulas. Since the distance can be expressed with the help of the measure symbol μ, μ is the only predicate in the language and therefore the atomic formulas are of the form $\varphi(\bar{x}) := \mu(t(\underline{\gamma_1}\bar{x}, \ldots, \underline{\gamma_n}\bar{x}))$, where $t(\bar{u})$ is a $\mathcal{L}$-term.

In continuous logic, all functions are uniformly continuous and thus so are the terms, so we can choose $\delta > 0$ such that in any two tuples $\bar{u}$ and $\bar{v}$ in M_β, if $d_\nu(\bar{u}, \bar{v}) < \delta$ then $d_\nu(t^{M_\beta}(\bar{u}), t^{M_\beta}(\bar{v})) < \varepsilon$.

We can now apply our big theorem, Theorem 4.13, to get a pmp bijection $\rho : Y \to X$ such that

$$\nu(\{y \in Y : \forall i \leqslant n\ \rho \circ \gamma_i^\alpha(y) = \gamma_i^\beta \circ \rho(y)\}) > 1 - \delta$$

In terms in probability algebra, we get $\forall i \in \{1, \ldots, n\}$ $\forall a \in M_\alpha,\ d_\nu(\widetilde{\rho}\gamma_i^\beta a, \gamma_i^\alpha \widetilde{\rho} a)$ so for any tuple $\bar{a} \subseteq M_\alpha$ we have

$$\begin{aligned}
& \left|\varphi(\bar{a})^{M_\alpha} - \varphi(\widetilde{\rho}\bar{a})^{M_\beta}\right| \\
= \;& \left|\mu(t^{M_\alpha}(\gamma_1^\alpha \bar{a}, \ldots, \gamma_n^\alpha \bar{a})) - \nu(t^{M_\beta}(\gamma_1^\beta \widetilde{\rho}\bar{a}, \ldots, \gamma_n^\beta \widetilde{\rho}\bar{a}))\right| \\
= \;& \left|\nu(\widetilde{\rho} t^{M_\alpha}(\gamma_1^\alpha \bar{a}, \ldots, \gamma_n^\alpha \bar{a})) - \nu(t^{M_\beta}(\gamma_1^\beta \widetilde{\rho}\bar{a}, \ldots, \gamma_n^\beta \widetilde{\rho}\bar{a}))\right| \\
= \;& \left|\nu(t^{M_\beta}(\widetilde{\rho}\gamma_1^\alpha \bar{a}, \ldots, \widetilde{\rho}\gamma_n^\alpha \bar{a})) - \nu(t^{M_\beta}(\gamma_1^\beta \widetilde{\rho}\bar{a}, \ldots, \gamma_n^\beta \widetilde{\rho}\bar{a}))\right| \\
\leqslant \;& d_\nu\left(t^{M_\beta}(\widetilde{\rho}\gamma_1^\alpha \bar{a}, \ldots, \widetilde{\rho}\gamma_n^\alpha \bar{a}), t^{M_\beta}(\gamma_1^\beta \widetilde{\rho}\bar{a}, \ldots, \gamma_n^\beta \widetilde{\rho}\bar{a})\right) \\
< \;& \varepsilon
\end{aligned}$$

Step 2 : The case of connectives is trivial : simply use uniform continuity.

Step 3 : For quantifiers, let us for example consider the case of a formula of the form $\inf_x \varphi(x, \bar{y})$, knowing the result for $\varphi(x, \bar{y})$.

Take a pmp bijection $\rho : Y \to X$ such that $\forall (a, \bar{b}) \subseteq M_\alpha, \left|\varphi^{M_\alpha}(a, \bar{b}) - \varphi^{M_\beta}(\widetilde{\rho}a, \widetilde{\rho}\bar{b})\right| < \varepsilon$. Since $\widetilde{\rho}$ is surjective, we also have $\forall \bar{b} \subseteq M_\alpha, \left|\inf_x \varphi(x, \bar{b})^{M_\alpha} - \inf_x \varphi(x, \widetilde{\rho}\bar{b})^{M_\beta}\right| < \varepsilon$. Hence the conclusion.

But now if φ is a $\mathcal{L}_\infty$-sentence, what we just proved shows that $\forall \varepsilon > 0,\ \left|\varphi^{M_\alpha} - \varphi^{M_\beta}\right| < \varepsilon$. It follows that $\varphi^{M_\alpha} = \varphi^{M_\beta}$ and therefore $\mathfrak{A}_\theta$ is complete. □

For model completeness, we need a version of the latter proof with parameters.

Theorem 5.15. *Let θ be a hyperfinite IRS on F_∞, then the theory $\mathfrak{A}_\theta$ is model complete.*

Proof. It suffices to show that any inclusion of two separable models is elementary. Indeed, suppose this result and take any $M \subseteq N \models \mathfrak{A}_\theta$, $\varphi(\bar{x})$ a $\mathcal{L}_\infty$-formula and $\bar{a} \in M$ finite. By Löwenheim-Skolem theorem, take $M' \preceq M$ separable containing A. Again by Löwenheim-Skolem theorem, take $N' \preceq N$ separable and containing the separable structure M'. Using the hypothesis, $M' \preceq N'$ so we finally get

$$\varphi(\bar{a})^M = \varphi(\bar{a})^{M'} = \varphi(\bar{a})^{N'} = \varphi(\bar{a})^N$$

From there, the proof is similar to the one of completeness, except that we use Theorem 4.17 to stabilize Borel parameters.

Claim 5.15.1. For M_α and M_β two separable models of $\mathfrak{A}_\theta$, $M_\alpha \subseteq M_\beta$ if and only if $\alpha \sqsubseteq \beta$.

Proof. Suppose $\alpha \sqsubseteq \beta$ and let $\pi : Y \to X$ be a corresponding action factor map. Then $\widetilde{\pi} : M_\alpha \hookrightarrow M_\beta$ is an embedding.

Conversely, suppose we have an embedding $i : M_\alpha \hookrightarrow M_\beta$. By Proposition 5.8 we can take a lifting π of i such that $\forall \gamma \in F_\infty,\ \gamma^\beta \circ \pi = \pi \circ \gamma^\alpha$ and this lifting is then an action factor map between α and β ■

To complete the proof, we consider two models $M_\alpha \subseteq M_\beta$ of $\mathfrak{A}_\theta$ and we proceed by induction on formulas, showing that for any $\mathcal{L}_\infty$ formula $\varphi(\bar x, \bar a)$, where $\bar a \subseteq M_\alpha$, and $\varepsilon > 0$, there is a pmp bijection $\rho : Y \to X$ such that $\forall \bar b \subseteq M_\alpha, \left|\varphi^{M_\alpha}(\bar b, \bar a) - \varphi^{M_\beta}(\widetilde{\rho}\bar b, \bar a)\right| < \varepsilon$. The cases of connectives and quantifiers are exactly the same so we focus on atomic formulas :

Let $\bar a \subseteq M_\alpha$ and $\varphi(\bar x, \bar a) := \mu(t(\underline{\gamma_1}\bar x, \dots, \underline{\gamma_n}\bar x, \underline{\gamma_1}\bar a, \dots, \underline{\gamma_n}\bar a)$ be an atomic formula, where $t(\bar x)$ is a $\mathcal{L}$-term. Let $\delta > 0$ be such that in any two tuples $\bar u$ and $\bar v$ in M_β, if $d_\nu(\bar u, \bar v) < \delta$ then $d_\nu(t^{M_\beta}(\bar u), t^{M_\beta}(\bar v)) < \varepsilon$. Finally, let $\bar A$ be a tuple of Borel representatives of elements of $\bar a$ in (X, μ) and consider the action factor map π given by the Claim. Let $\bar B = \pi^{-1}(\bar A)$, so that $\bar B$ is a tuple of Borel representatives of elements of $\bar a$ in (Y, ν).

Thanks to the Claim, we can now apply Theorem 4.17 to get a pmp bijection $\rho : Y \to X$ such that $\rho(\bar A) = \bar B$ and

$$\nu(\{y \in Y : \forall i \leqslant n \ \rho \circ \gamma_i^\beta(y) = \gamma_i^\alpha \circ \rho(y)\}) > 1 - \delta$$

In terms in probability algebra, we get $\widetilde{\rho}\bar a = \bar a$ and $\forall i \in \{1, \dots, n\}\ \forall b \in M_\alpha,\ d_\nu(\widetilde{\rho}\gamma_i^\alpha b, \gamma_i^\beta \widetilde{\rho} b)$ so for any tuple $\bar b \subseteq M_\alpha$ we have

$$\begin{aligned}
& \left|\varphi(\bar b, \bar a)^{M_\alpha} - \varphi(\widetilde{\rho}\bar b, \bar a)^{M_\beta}\right| \\
= & \left|\mu(t^{M_\alpha}(\gamma_1^\alpha \bar b, \dots, \gamma_n^\alpha \bar b, \gamma_1^\alpha \bar a, \dots, \gamma_n^\alpha \bar a)) - \nu(t^{M_\beta}(\gamma_1^\beta \widetilde{\rho}\bar b, \dots, \gamma_n^\beta \widetilde{\rho}\bar b, \gamma_1^\beta \bar a, \dots, \gamma_n^\beta \bar a))\right| \\
= & \left|\nu(\widetilde{\rho} t^{M_\alpha}(\gamma_1^\alpha \bar b, \dots, \gamma_n^\alpha \bar b, \gamma_1^\alpha \bar a, \dots, \gamma_n^\alpha \bar a)) - \nu(t^{M_\beta}(\gamma_1^\beta \widetilde{\rho}\bar b, \dots, \gamma_n^\beta \widetilde{\rho}\bar b, \gamma_1^\beta \bar a, \dots, \gamma_n^\beta \bar a))\right| \\
= & \left|\nu(t^{M_\beta}(\widetilde{\rho}\gamma_1^\alpha \bar b, \dots, \widetilde{\rho}\gamma_n^\alpha \bar b, \widetilde{\rho}\gamma_1^\alpha \bar a, \dots, \widetilde{\rho}\gamma_n^\alpha \bar a)) - \nu(t^{M_\beta}(\gamma_1^\beta \widetilde{\rho}\bar b, \dots, \gamma_n^\beta \widetilde{\rho}\bar b, \gamma_1^\beta \widetilde{\rho}\bar a, \dots, \gamma_n^\beta \widetilde{\rho}\bar a))\right| \\
\leqslant & \ d_\nu\left(t^{M_\beta}(\widetilde{\rho}\gamma_1^\alpha \bar b, \dots, \widetilde{\rho}\gamma_n^\alpha \bar b, \widetilde{\rho}\gamma_1^\alpha \bar a, \dots, \widetilde{\rho}\gamma_n^\alpha \bar a), t^{M_\beta}(\gamma_1^\beta \widetilde{\rho}\bar b, \dots, \gamma_n^\beta \widetilde{\rho}\bar b, \gamma_1^\beta \widetilde{\rho}\bar a, \dots, \gamma_n^\beta \widetilde{\rho}\bar a)\right) \\
< & \ \varepsilon
\end{aligned}$$

Now if $\varphi(\bar a)$ is a $\mathcal{L}_\infty$-sentence with parameters $\bar a$, what we just proved shows that
$\forall \varepsilon > 0,\ \left|\varphi^{M_\alpha}(\bar a) - \varphi^{M_\beta}(\bar a)\right| < \varepsilon$. It follows that $\varphi^{M_\alpha}(\bar a = \varphi^{M_\beta}(\bar a)$ and therefore $\mathfrak{A}_\theta$ is model complete. ☐

5.6 Elimination of quantifiers

Proposition 5.16 (Admitted,[YBHU08])**.** *Let T be a countable theory, then T admits quantifier*
elimination if and only if for any separable $M, N \models T$, any substructure $A \subseteq M$ and any embedding $f : A \hookrightarrow N$, there is an elementary extension N' of N and an embedding $\tilde f : M \hookrightarrow N'$ extending f.

Definition 5.17. We say that a theory T admits amalgamation if for any $M_1, M_2 \models T$ and any common substructure A, there is $N \models T$ and embeddings $M_i \hookrightarrow N$ $(i = 1, 2)$ such that the following diagram commutes :

$$\begin{tikzcd}
& N & \\
M_1 \arrow[ur] & & M_2 \arrow[ul] \\
& A \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Lemma 5.18. *Let T be a theory. Then T admits quantifier elimination if and only if it admits amalgamation and is model complete.*

Proof. Suppose that T admits quantifier elimination. Let $M_1, M_2 \models T$ with a common substructure A, applying Proposition 5.16 where f is the inclusion $A \hookrightarrow M_2$, we get N as required.

Now let $M \subseteq N$ be two models of T. By quantifier elimination, we only need to prove that $M \models \varphi(\bar{a}) \Leftrightarrow N \models \varphi(\bar{a})$ for atomic formulas φ and finite tuples $\bar{a}$ of parameters in M. But this is trivial by the definition of inclusion for models.

Conversely, suppose T admits amalgamation and is model complete and let $M, N \models T$, $A \subseteq M$ be a substructure, and $f : A \hookrightarrow N$. By considering a monster model, we may suppose that $A \subseteq N$ and f is the identity. Then by amalgamation there is a model $N' \models T$ and embeddings φ, ψ such that the following diagram commutes :

$$\begin{tikzcd} & N' & \\ M \arrow[ur, "\varphi"] & & N \arrow[ul, "\psi"'] \\ & A \arrow[ul, "\mathrm{Id}"] \arrow[ur, "\mathrm{Id}"'] & \end{tikzcd}$$

Again we may suppose that $N \subseteq N'$ and ψ is the identity, thus by model completeness we have $N \preceq N'$. Furthermore, the diagram now exactly states that φ extends the inclusion $A \hookrightarrow N$. $\square$

In order to prove that our theories eliminate quantifiers, it only remains to prove that they have amalgamation. However, the following example shows that it is not the case in general.

Proposition 5.19. *As we already saw, considering F_∞ actions allows us to study any Γ action as well, when Γ is countable. Fix a surjective morphism $F_\infty \to \Gamma$. For θ an IRS on Γ, denote by θ' the IRS on F_∞ such that actions of Γ of IRS θ are represented by actions of F_∞ of IRS θ'. We write $\mathfrak{A}_{\Gamma,\theta}$ for $\mathfrak{A}_{\theta'}$.*

Let $\theta = \frac{1}{2}\delta_{\{e\}} + \frac{1}{2}\delta_\Gamma$ be an IRS on Γ, then $\mathfrak{A}_{\Gamma,\theta}$ does not have quantifier elimination.

Proof. Take any $\gamma \in \Gamma \backslash \{e\}$. We already saw in Lemma 5.12 that the support of γ was definable in the theory $\mathfrak{A}_\theta$. However, it is not definable with a quantifier free formula. Indeed, suppose $\varphi(x)$ is a quantifier free formula equivalent to $\mu(x \cap \operatorname{Supp}\gamma) := \sup_y \mu(x \cap (y\backslash\underline{\gamma}y \cup \underline{\gamma}(y\backslash\underline{\gamma}y) \cup \underline{\gamma}^2(y\backslash\underline{\gamma}y)))$:

Let κ_1 be a free pmp action on $([0,1], \lambda)$ and κ_2 be the trivial action on $([0,1], \lambda)$. Define

- $\alpha : \Gamma \curvearrowright (X = [0,1] \times \{1,2,3,4\}, \mu = \frac{1}{4}\lambda \times \delta_1 + \frac{1}{4}\lambda \times \delta_2 + \frac{1}{4}\lambda \times \delta_3 + \frac{1}{4}\lambda \times \delta_4)$ that acts like κ_1 on $[0,1] \times \{1\}$ and $[0,1] \times \{2\}$ and acts like κ_2 on $[0,1] \times \{3\}$ and $[0,1] \times \{4\}$.
- $\beta : \Gamma \curvearrowright (X = [0,1] \times \{1,2,3,4\}, \mu = \frac{1}{4}\lambda \times \delta_1 + \frac{1}{4}\lambda \times \delta_2 + \frac{1}{4}\lambda \times \delta_3 + \frac{1}{4}\lambda \times \delta_4)$ that acts like κ_1 on $[0,1] \times \{1\}$ and $[0,1] \times \{3\}$ and acts like κ_2 on $[0,1] \times \{2\}$ and $[0,1] \times \{4\}$.

We have $\theta_\alpha = \theta_\beta = \theta$.

Let M be the common substructure $\{A, B\}$ of M_α and M_β, where $A = [[0,1] \times \{1,2\}]_\mu$, $B = [[0,1] \times \{3,4\}]_\mu$.

As $\varphi(x)$ is quantifier free, we have $\varphi^{M_\alpha}(A) = \varphi^M(A) = \varphi^{M_\beta}(A)$, but in M_α, $A \cap \text{Supp } \gamma = [[0,1] \times \{1,2\}]_\mu$ and thus $M_\alpha \models \mu(A \cap \text{Supp } \gamma) = \frac{1}{2}$ whereas in M_β, $A \cap \text{Supp } \gamma = [[0,1] \times \{1\}]_\mu$ and thus $M_\alpha \models \mu(A \cap \text{Supp } \gamma) = \frac{1}{4}$. A contradiction. □

In the case where Γ is amenable, this is an example of theory of the form $\mathfrak{A}_\theta$ which is model complete but does not eliminate quantifiers.

In the other hand, we still have amalgamation for free actions of an amenable group Γ :

Proposition 5.20. *If θ is the Dirac measure $\delta_{\{e\}}$ on Γ an amenable group, then $\mathfrak{A}_{\Gamma,\theta}$ admits amalgamation and thus has quantifier elimination.*

Proof. Let M_α, M_β be two separable models of $\mathfrak{A}_{\Gamma,\theta}$ and A a common substructure. Then α and β are actions of Γ of IRS θ and the substructure A can be interpreted as an action ξ of Γ on a probability space, which is a common factor of α and β.

Consider the relative independent joining ζ of α and β over the common factor ξ. Since $\alpha \sqsubseteq \zeta$, the stabilizers in ζ must be smaller than the ones in α, but these are always the trivial group, hence ζ is a free action of Γ, and thus M_ζ is a suitable model of $\mathfrak{A}_{\Gamma,\theta}$ for amalgamation over A. □

We just saw that there are theories of the form $\mathfrak{A}_\theta$ for θ hyperfinite which admit quantifier elimination and others that do not. The next question is thus :

For which θ does the theory $\mathfrak{A}_\theta$ admit quantifier elimination ? Is there a simple sufficient condition on θ ?

At the time, we do not have any satisfying answer, however, we propose a conjecture. Indeed our counterexample in Proposition 5.19 highly relies on non-ergodicity of the IRS. We recall the definition of ergodicity.

Definition 5.21. Let $\alpha : \Gamma \curvearrowright (X, \mu)$ be an action of a group on a probability space. We say that α is ergodic if every Γ-invariant (for α) measurable subset of X is either null or connull.

For a given Borel action $\Gamma \to X$ on a Polish space, we say that $\mu \in \mathfrak{P}(X)$ is ergodic if every Γ-invariant measurable subset of X is either null or connull (for μ).

For Invariant Random Subgroups, we consider the notion of ergodicity with respect to the action $\Gamma \curvearrowright Sg(\Gamma)$ by conjugation. Thus $\frac{1}{2}\delta_{\{e\}} + \frac{1}{2}\delta_\Gamma$ is one of the simplest examples of non-ergodic IRS. In the proof of Proposition 5.19 we choose actions which decompose in a very specific way, and that could not be done with an ergodic IRS. Moreover, the same proof can be adapted to the case of many non-ergodic IRS, namely to any IRS in the proper convex hull of two IRS concentrated on two disjoint Borel subsets of $Sg(\Gamma)$. All these remarks suggest that ergodicity play a role in the quantifier elimination of $\mathfrak{A}_\theta$. We therefore also ask :

Does the theory $\mathfrak{A}_\theta$ admit quantifier elimination if and only if θ is ergodic ?

5.7 Stability and Independence

5.7.1 Definable closure in $\mathfrak{A}_\theta$

Even though we do not know for which IRS θ the theory $\mathfrak{A}_\theta$ admits quantifier elimination, another interesting question is to ask what we can add to the signature $\mathcal{L}_\infty$ to expand it into a simple signature in which the theories $\mathfrak{A}_\theta$ always have quantifier elimination. One can notice that in all this section, the supports of elements of the group play a big role. An idea is then to add constants $\{S_\gamma : \gamma \in F_\infty\}$ to the signature $\mathcal{L}_\infty$ and to consider the theory $\mathfrak{A}'_\theta$ consisting of :

- The axioms of $\mathfrak{A}_\theta$.
- For $\gamma \in F_\infty$, the axioms :
  - $\sup_x \; d(S_\gamma \cap t_\gamma(x), t_\gamma(x)) = 0$.
  - $\mu(S_\gamma) = \theta(N_\gamma)$.

This theory expresses that for $\gamma \in F_\infty$, the constant S_γ must be interpreted as $[\mathrm{Supp}\ \gamma^\alpha]_\mu$ in the model M_α, as it contains the class of the support by the first axiom and has the same measure by the second one.

Theorem 5.22. *Let θ be a hyperfinite IRS, then the theory $\mathfrak{A}'_\theta$ eliminates quantifiers in the language $F_\infty \cup \{S_\gamma : \gamma \in F_\infty\}$.*

Proof. We use Lemma 5.18.

First, take $M_\alpha \subseteq M_\beta$ two separable models and let us prove that $M_\alpha \preceq M_\beta$. In order to repeat the proof from Theorem 5.15 we only need to include atomic formulas in which the constants S_γ appear. But for $\varepsilon > 0$ and $\rho : Y \to X$ a pmp bijection such that

$$\nu(\{y \in Y : \rho \circ \gamma^\alpha(y) = \gamma^\beta \circ \rho(y)\}) > 1 - \varepsilon$$

we have $d_\nu(\tilde{\rho} S_\gamma^{M_\alpha}, S_\gamma^{M_\beta}) < \varepsilon$ so by uniform continuity of formulas we conclude as we did in Theorem 5.15.

Now for amalgamation we will use the constants S_γ. Indeed, take M_α and M_β two separable models of $\mathfrak{A}'_\theta$ and A a common substructure. We interpret A as a F_∞ pmp action ξ on a probability space which is a common factor of α and β for respective factor action maps π_1 and π_2.

Consider $\zeta : F_\infty \rightharpoonup (X \times Y, \eta)$ the relative independent joining of α and β over ξ. By definition, $\forall^*(x, y) \in X \times Y\ \pi_1(x) = \pi_2(y)$, but moreover, since we added the constants S_γ, A being a common substructure implies in particular that $\pi_1^{-1}(S_\gamma^A) = S_\gamma^{M_\alpha}$ and $\pi_2^{-1}(S_\gamma^A) = S_\gamma^{M_\beta}$. It follows that

$$\forall \gamma \in F_\infty\ \forall^*(x, y) \in X \times Y,\ x \in \mathrm{Supp}\ \gamma^\alpha \Leftrightarrow \pi_1(x) \in S_\gamma^A \Leftrightarrow \pi_2(y) \in S_\gamma^A \Leftrightarrow y \in \mathrm{Supp}\ \gamma^\beta$$

and so for all $\gamma \in F_\infty$, up to a null set, $\mathrm{Supp}\ \gamma^\zeta = \pi_1^{-1}(\mathrm{Supp}\ \gamma^\alpha)$.

Hence for any $F \subseteq F_\infty$ finite, we get

$$\begin{aligned}
\theta_\zeta(N_F) &= \eta(\bigcap_{\gamma\in F}\text{Supp } \gamma^\zeta) \\
&= \eta(\pi_1^{-1}(\bigcap_{\gamma\in F}\text{Supp } \gamma^\alpha)) \\
&= \theta_\alpha(N_F) \\
&= \theta(N_F)
\end{aligned}$$

We conclude as in Theorem 5.13 using the inclusion exclusion principle that $\theta_\zeta = \theta$ and thus ζ is a witness that $\mathfrak{A}'_\theta$ admits amalgamation. ☐

Corollary 5.23. *Let $M \models \mathfrak{A}_\theta$ and $A \subseteq M$, then the definable closure of A in M is the σ-algebra generated by elements of the form $\gamma_1^M(a)$ or $\gamma_1^M(\text{Supp } \gamma_2^M)$ for $a \in A$ and $\gamma_1, \gamma_2 \in F_\infty$.*

Proof. First $A \subseteq \text{dcl}^M(A)$ and by Lemma 5.12, for $\gamma \in F_\infty$, Supp $\gamma^M \in \text{dcl}^M(A)$. Moreover, definable closure is stable by translates by elements of F_∞, complements and countable reunion, thus we get the first inclusion.

In the other way, since $\mathfrak{A}'_\theta$ expands $\mathfrak{A}_\theta$, the definable closure of A in the theory $\mathfrak{A}_\theta$ is contained in the definable closure of A in the theory $\mathfrak{A}'_\theta$, which is contained in the σ-algebra generated by elements of the form $\gamma_1^M(a)$ or $\gamma_1^M(\text{Supp } \gamma_2^M)$ for $a \in A$ and $\gamma_1, \gamma_2 \in F_\infty$ by quantifier elimination.

Hence the conclusion. ☐

5.7.2 The stable independence relation

Definition 5.24. Let κ be a cardinal. A κ-universal domain for a theory T is a κ-saturated and strongly κ-homogeneous model of T. If $\mathcal{U}$ is a κ-universal domain and $A \subseteq \mathcal{U}$, we say that A is small if $|A| < \kappa$.

Definition 5.25. Let $\mathcal{U}$ be a κ-universal domain for T. A stable independence relation on $\mathcal{U}$ is a relation $A \underset{C}{\perp} B$ on triples of small subsets of $\mathcal{U}$ satisfying the following properties, for all small $A, B, C, D \subseteq \mathcal{U}$, finite $\bar{u}, \bar{v} \subseteq \mathcal{U}$ and small $M \preceq \mathcal{U}$:

1. *Invariance under automorphisms of $\mathcal{U}$.*
2. *Symmetry :* $A \underset{C}{\perp} B \iff B \underset{C}{\perp} A$.
3. *Transitivity :* $A \underset{C}{\perp} BD \iff A \underset{C}{\perp} B \wedge A \underset{BC}{\perp} D$.
4. *Finite character :* $A \underset{C}{\perp} B$ if and only if $\bar{a} \underset{C}{\perp} B$ for every finite $\bar{a} \subseteq A$.
5. *Extension :* There exists A' such that $\text{tp}(A'/C) = \text{tp}(A/C)$ and $A' \underset{C}{\perp} B$.
6. *Local character :* There exists $B_0 \subseteq B$ such that $|B_0| \leqslant |T|$ and $\bar{u} \underset{B_0}{\perp} B$.
7. *Stationarity of types :* If $\text{tp}(A/M) = \text{tp}(B/M)$ and $A \underset{M}{\perp} C$ and $B \underset{M}{\perp} C$ then $\text{tp}(A/M \cup C) = \text{tp}(B/M \cup C)$.

Proposition 5.26 (Admitted, [YBHU08])**.** *Let $\kappa > |T|$ and let $\mathcal{U}$ be a κ-universal domain. Then the theory T is stable if and only if there exists a stable independence relation on $\mathcal{U}$, and in this case the stable independence relation is the independence relation given by non-dividing.*

Thus, in order to prove that our theories are stable, we only need to define a stable independence relation. The independence considered will be the classical independence of events in probability theory.

Definition 5.27. From now on, we write $\langle A\rangle$ for $\mathrm{dcl}^{\mathcal{U}}(A)$.

Let $A, B, C \subseteq \mathcal{U}$ be small, we say that A and B are independent over C and we write $A \underset{C}{\perp} B$ if $\forall a \in \langle A\rangle$, $\forall b \in \langle B\rangle$, $\mathbb{P}_{\langle C\rangle}(a).\mathbb{P}_{\langle C\rangle}(b) = \mathbb{P}_{\langle C\rangle}(a \cap b)$.

Lemma 5.28. *Let $A, B, C \subseteq \mathcal{U}$ be small, then we have $A \underset{C}{\perp} B$ if and only if $\forall a \in \langle A\rangle$, $\mathbb{P}_{\langle BC\rangle}(a) = \mathbb{P}_{\langle C\rangle}(a)$.*

Proof. Suppose that $\forall a \in \langle A\rangle$, $\mathbb{P}_{\langle BC\rangle}(a) = \mathbb{P}_{\langle C\rangle}(a)$. Let $a \in \langle A\rangle$, $b \in \langle B\rangle$ and $c \in \langle C\rangle$,

$$\begin{aligned}
\int_c \mathbb{P}_{\langle C\rangle}(a).\mathbb{P}_{\langle C\rangle}(b) &= \int \mathbb{P}_{\langle C\rangle}(a).\mathbb{P}_{\langle C\rangle}(b).\mathbb{1}_c \\
&= \int \mathbb{P}_{\langle C\rangle}(a).\mathbb{1}_b.\mathbb{1}_c \\
&= \int \mathbb{P}_{\langle BC\rangle}(a).\mathbb{1}_b.\mathbb{1}_c \\
&= \int \mathbb{1}_a.\mathbb{1}_b.\mathbb{1}_c \\
&= \int \mathbb{1}_{a\cap b}.\mathbb{1}_c \\
&= \int_c \mathbb{P}_{\langle C\rangle}(a \cap b)
\end{aligned}$$

which proves that $\mathbb{P}_{\langle C\rangle}(a).\mathbb{P}_{\langle C\rangle}(b) = \mathbb{P}_{\langle C\rangle}(a \cap b)$.

Conversely, suppose that $A \underset{C}{\perp} B$. Let $a \in \langle A\rangle$. The conditional probability $\mathbb{P}_{\langle C\rangle}$ is $\langle BC\rangle$-measurable and moreover, for $b \in \langle B\rangle$ and $c \in \langle C\rangle$, we have

$$\begin{aligned}
\int_{b\cap c} \mathbb{P}_{\langle C\rangle}(a) &= \int \mathbb{P}_{\langle C\rangle}(a).\mathbb{1}_{b\cap c} \\
&= \int \mathbb{P}_{\langle C\rangle}(a).\mathbb{1}_b.\mathbb{1}_c \\
&= \int \mathbb{P}_{\langle C\rangle}(a).\mathbb{P}_{\langle C\rangle}(b).\mathbb{1}_c \\
&= \int \mathbb{P}_{\langle C\rangle}(a \cap b).\mathbb{1}_c \\
&= \int \mathbb{1}_a.\mathbb{1}_b.\mathbb{1}_c \\
&= \int_{b\cap c} \mathbb{1}_a
\end{aligned}$$

And thus for any $\langle BC\rangle$-measurable function f, we have $\int \mathbb{P}_{\langle C\rangle}(a).f = \int \mathbb{1}_a.f$, therefore $\mathbb{P}_{\langle BC\rangle}(a) = \mathbb{P}_{\langle C\rangle}(a)$. □

Theorem 5.29. *If θ is a hyperfinite IRS, the relation of independence $\perp$ defined above is a stable independence relation. Consequently, the theory $\mathfrak{A}_\theta$ is stable and the relation $\perp$ agrees with non-dividing.*

Proof. 1. *Invariance under automorphisms of* $\mathcal{U}$ *:* If ρ is an automorphism of $\mathcal{U}$, by uniqueness of the orthogonal projection, we know that $\mathbb{P}_{\langle\rho(C)\rangle} = \rho\circ\mathbb{P}_{\langle C\rangle}\circ\rho^{-1}$ and therefore $\mathbb{P}_{\langle C\rangle}(a).\mathbb{P}_{\langle C\rangle}(b) = \mathbb{P}_{\langle C\rangle}(a\cap b)$ if and only if $\mathbb{P}_{\langle\rho(C)\rangle}(\rho a).\mathbb{P}_{\langle\rho(C)\rangle}(\rho b) = \mathbb{P}_{\langle\rho(C)\rangle}(\rho(a\cap b))$.

2. *Symmetry :* The definition is symmetric.

3. *Transitivity :* Let A, B, C, D be small. First if $A \underset{C}{\perp} B$ and $A \underset{BC}{\perp} D$ then by Lemma 5.28, for $a \in \langle A\rangle$, we have $\mathbb{P}_{\langle BCD\rangle}(a) = \mathbb{P}_{\langle BC\rangle}(a) = \mathbb{P}_{\langle C\rangle}(a)$ so $A \underset{C}{\perp} BD$.

   In the other way, if $A \underset{C}{\perp} BD$ then $\mathbb{P}_{\langle BCD\rangle}(a) = \mathbb{P}_{\langle C\rangle}(a)$, but that implies that $\mathbb{P}_{\langle C\rangle}(a)$ is a $\langle C\rangle$-measurable function such that for all $\langle BCD\rangle$-measurable function f we have $\int \mathbb{P}_{\langle C\rangle}(a).f = \int \mathbb{1}_a.f$. We conclude that $\mathbb{P}_{\langle BCD\rangle}(a) = \mathbb{P}_{\langle BC\rangle}(a) = \mathbb{P}_{\langle C\rangle}(a)$, and therefore that $A \underset{C}{\perp} C$ and $A \underset{BC}{\perp} D$.

4. *Finite character :* It is trivial by the definition.

5. *Extension :* Let A, B, C be small subsets of $\mathcal{U}$. Let $\mathcal{A} = \langle AC\rangle$, $\mathcal{B} = \langle BC\rangle$ and $\mathcal{C} = \langle C\rangle$. The three structures $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ can be seen as Boolean rings, and we can therefore define the free product of Boolean algebras $\mathcal{D} = \mathcal{A}\underset{\mathcal{C}}{\otimes}\mathcal{B}$ along with the product measure. Then $\mathcal{D}$ can be seen as a probability probability algebra.

   By universality of $\mathcal{U}$ and smallness of $\mathcal{D}$, there is an embedding of $\mathcal{D}$ in $\mathcal{U}$ sending $\mathcal{B}$ back to $\mathcal{B}$. We denote A' the image of A by this embedding. Of course $\mathcal{C}$ is sent back to $\mathcal{C}$ so we have $\mathrm{tp}(A'/C) = \mathrm{tp}(A/C)$.

   Moreover, it is shown in [] that another characterization of independence is given by $A \underset{C}{\perp} B \iff \mathcal{A}\wedge\mathcal{B} = \mathcal{A}\underset{\mathcal{C}}{\otimes}\mathcal{B}$, where $\mathcal{A}\wedge\mathcal{B}$ is the probability algebra generated by $\mathcal{A}$ and $\mathcal{B}$. Thus, we have by construction $A' \underset{C}{\perp} B$.

6. *Local character :* Let $\bar{u} = (u_1, \ldots, u_n) \subseteq \mathcal{U}$ be finite. Consider the conditional probabilities $\mathbb{P}_{\langle B\rangle}(u_i)$. These are $\langle B\rangle$-measurable functions with real values and so there is a countably

   generated σ-subalgebra of $\langle B\rangle$, say $\langle B_0\rangle$ where $B_0 \subseteq B$ is countable, for which they are all measurable. But then we have $\mathbb{P}_{\langle B\rangle}(u_i) = \mathbb{P}_{\langle B_0\rangle}(u_i)$, so by Lemma 5.28 $\bar{u} \underset{B_0}{\perp} B$.

7. *Stationarity of types :* We denote by $\mathrm{tp}_{\mathcal{L}}(\bar{x}/Y)$ the type of a tuple $\bar{x}$ over a set of parameters Y in the language $\mathcal{L}$. In other words, this is the type of $\bar{x}$ over Y in the underlying atomless probability algebra of $\mathcal{U}$.

   Let $A, B, C \subseteq \mathcal{U}$ be small and $M \preceq \mathcal{U}$ be small. Suppose that $\mathrm{tp}(A/M) = \mathrm{tp}(B/M)$, $A \underset{M}{\perp} C$ and $B \underset{M}{\perp} C$.

   We begin by proving that $\mathrm{tp}_{\mathcal{L}}(A/\langle M\cup C\rangle) = \mathrm{tp}_{\mathcal{L}}(B/\langle M\cup C\rangle)$. Indeed, for $a \in \langle A\rangle_{\mathcal{L}}$ and $b \in \langle B\rangle_{\mathcal{L}}$, we have $\mathbb{P}_{\langle M\cup C\rangle}(a) = \mathbb{P}_M(a)$ and $\mathbb{P}_{\langle M\cup C\rangle}(b) = \mathbb{P}_M(b)$, but by Proposition 5.7 types in APA can be fully described with conditional probabilities and we know that $\mathrm{tp}_{\mathcal{L}}(A/M) = \mathrm{tp}_{\mathcal{L}}(B/M)$ so we get $\mathrm{tp}_{\mathcal{L}}(A/M\cup C) = \mathrm{tp}_{\mathcal{L}}(B/M\cup C)$.

Now Corollary 5.23 implies that $\mathrm{tp}(A/M \cup C)$ (resp. $\mathrm{tp}(B/M \cup C)$) is determined by $\mathrm{tp}_{\mathcal{L}}\left(\bigcup_{\substack{\gamma_1 \in F_\infty \\ \gamma_2 \in F_\infty}} \gamma_1 A \cup \{\gamma_1(S_{\gamma_2})\} / \langle M \cup C \rangle\right)$ (resp. $\mathrm{tp}_{\mathcal{L}}\left(\bigcup_{\substack{\gamma_1 \in F_\infty \\ \gamma_2 \in F_\infty}} \gamma_1 B \cup \{\gamma_1(S_{\gamma_2})\} / \langle M \cup C \rangle\right)$).

Thus, let $A' = \bigcup_{\substack{\gamma_1 \in F_\infty \\ \gamma_2 \in F_\infty}} \gamma_1 A \cup \{\gamma_1(S_{\gamma_2})\}$ and $B' = \bigcup_{\substack{\gamma_1 \in F_\infty \\ \gamma_2 \in F_\infty}} \gamma_1 B \cup \{\gamma_1(S_{\gamma_2})\}$.

It is clear that $\mathrm{tp}(A'/M) = \mathrm{tp}(B'/M)$, $A' \underset{M}{\perp} C$ and $B' \underset{M}{\perp} C$ and we can apply what we proved just above to conclude that

$$\mathrm{tp}_{\mathcal{L}}\left(\bigcup_{\substack{\gamma_1 \in F_\infty \\ \gamma_2 \in F_\infty}} \gamma_1 A \cup \{\gamma_1(S_{\gamma_2})\} / \langle M \cup C \rangle\right) = \mathrm{tp}_{\mathcal{L}}\left(\bigcup_{\substack{\gamma_1 \in F_\infty \\ \gamma_2 \in F_\infty}} \gamma_1 B \cup \{\gamma_1(S_{\gamma_2})\} / \langle M \cup C \rangle\right)$$

Hence the conclusion.

□

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