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Classification of measure-preserving actions of Invariant random subgroups and model theory

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Titre : Classification d’actions ergodiques de sous-groupes aléatoires invariants et théorie des modèles

Résumé :
Cette thèse est centrée sur l'étude des actions préservant la mesure de groupes dénombrables sur des espaces de probabilité, et des sous-groupes aléatoires invariants (IRS) associés. Elle consiste en deux parties.
Dans la première, on y étudie les actions préservant la mesure dont l'IRS associé est hyperfini, cadre qui généralise celui des actions libres de groupes moyennables. On redémontre un théorème de G. Elek qui dit que deux actions préservant la mesure de même IRS hyperfini sont approximativement conjuguées. La preuve fournit en fait un résultat plus précis qu'on utilise ensuite pour étudier les actions préservant la mesure d'IRS hyperfini du point de vue de la théorie des modèles métrique.
La seconde partie se focalise sur les IRS non hyperfinis, et plus généralement sur les groupoïdes préservant la mesure non hyperfinis (un IRS donnant naturellement lieu à un groupoïde). La propriété (T) des groupoïdes préservant la mesure est caractérisée en termes d'actions ergodiques, étendant de manière naturelle un résultat de Connes et Weiss pour les groupes dénombrables. Ce résultat est utilisé dans un travail en commun avec Alessandro Carderi et François Le Maître, où il est montré qu'une relation d'équivalence préservant une mesure de probabilité a la propriété (T) si et seulement si toutes les actions ergodiques non libres de son groupe plein sont fortement ergodiques. On étend ensuite un résultat de Hjorth sur l'espace des actions aux IRS, et en déduit un résultat de rigidité pour une nouvelle relation sur l'espace des actions. La thèse se conclut par un panorama des différentes relations sur l'espace des actions préservant la mesure d'un groupe dénombrable.

Mots clefs : Théorie ergodique, Sous-groupes aléatoires invariants, Hyperfinitude, Moyennabilité, Théorie des modèles continue

Title : Classification of measure-preserving actions of Invariant Random Subgroups and model theory

Abstract :
This PhD thesis is centered on the study of measure-preserving actions of countable groups on probability spaces, and of the associated invariant random subgroups (IRS). It is divided in two parts.

The first part is focused on measure-preserving actions whose IRS is hyperfinite. This framework extends that of free measure-preserving actions of amenable groups. We describe a new proof of a theorem of G. Elek which states that any two measure-preserving actions which share the same hyperfinite IRS must be approximately conjugate. Our proof provides a more precise statement which we use for the study of measure- preserving actions with hyperfinite IRS from a model-theoretic perspective.

The second part is focused on non hyperfinite IRS, and more generally on non hyperfinite measure-preserving groupoids (every IRS naturally yields a measure-preserving groupoid). Property (T) for measure-preserving groupoids is characterized in terms of ergodic actions, providing a natural extension a result of Connes and Weiss for countable groups. This result is used in a joint work with Alessandro Carderi and François Le Maître, where it is shown that an ergodic measure-preserving equivalence relation has property (T) if and only if all the non-free ergodic measure-preserving actions of its full group are strongly ergodic. We then extend a result of Hjorth on the space of free actions to actions with a given IRS, and deduce a rigidity result for a new relation on the space of actions. The thesis ends with an overview of the different relations on the space of measure-preserving actions of a countable group.

Keywords : Ergodic theory, Invariant random subgroups, Hyperfiniteness, Amenability, Continuous model theory

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Chapter 1

Introduction (version française)

Un des principaux champs de la théorie ergodique est l'étude des transformations et des actions de groupe préservant la mesure de probabilité (pmp) sur des espaces de probabilité standard, ainsi que leur classification. Dans cette thèse nous nous intéressons particulièrement à la classification d'actions pmp de groupes moyennables sur des espaces de probabilité standard et à ses liens avec les propriétés algébriques du groupe agissant. Dans toute cette introduction, on se restreint au contexte des groupes moyennables discrets.

La façon la plus naturelle de classifier de telles actions est de se demander quand deux actions sont conjuguées l'une de l'autre. Pour exemple, le développement de la théorie de l'entropie pour les actions pmp de groupes va dans cette direction. Dans [Orn70], D. Ornstein a montré que deux décalages de Bernoulli de $\mathbb{Z}$ sont conjugués si et seulement s'ils ont la même entropie, et dans [OW87], Ornstein et Weiss ont étendu ce résultat aux décalages de Bernoulli au dessus de n'importe quel groupe Γ moyennable. En revanche, de nombreux résultats d'anti-classification ont mis à jour qu'il ne peut pas exister d'invariant complet simplement calculable pour la conjugaison d'actions ergodiques dans le cas général.

Plutôt que de se restreindre à une classe spécifique d'actions, comme les décalages de Bernoulli, une autre approche au problème de classification consiste à remplacer la conjugaison par une relation d'équivalence plus grossière. Voici quelques exemples de telles relations : Deux actions pmp sur (X, μ) sont dites orbites-équivalentes si les deux partitions de X en orbites qu'elles engendrent sont équivalentes, à mesure nulle près. La classification des actions pmp à orbites-équivalence près est très différente de celle à conjugaison près. En effet, deux actions pmp ergodiques de groupes moyennables sont toujours orbites-équivalentes ([Dye59] and [OW87]), alors que deux groupes libres n'admettent d'actions libres ergodiques orbites-équivalentes l'une à l'autre que si ils ont le même rang ([Gab00]). Plus récemment, A. Kechris a introduit la notion de contenance faible et d'équivalence faible pour les actions pmp, s'inspirant des notions correspondantes en théorie des représentations unitaires.

La première partie de cette thèse est un article de recherche soumis pour publication. La relation d'équivalence qui y est étudiée provient de la logique mathématique et plus précisément de la théorie des modèles continue. La théorie des modèles continue est une extension de la théorie des modèles classique à des structures métriques plutôt que discrètes. Différentes formalisations de cette théorie existent, mais dans le cadre de cette thèse nous nous référons à celle présentée par I. Ben Yaacov, A. Berenstein, C. W. Henson et A. Usvyatsov dans [BBHU08].

Etant donné un groupe dénombrable Γ, toute action pmp α de Γ sur un espace de probabilité (X, μ) induit une action de Γ par isomorphismes sur l'algèbre de mesure de (X, μ). La structure modèle-théorique $\mathcal{M}_\alpha$ que l'on étudie est cette algèbre de mesure $\mathrm{MAlg}(X, \mu)$ munie de l'action de Γ induite par α. Deux actions pmp de Γ sont dites élémentairement équivalentes lorsque leurs algèbres de mesure respectives associent la même valuation à chaque énoncé du premier ordre. Cette approche logique de la théorie ergodique soulève la question suivante, qui est la première que l'on se pose dans cette thèse :

Question 1.0.1. *Que peut-on dire sur une action pmp d'un groupe Γ à l'aide d'énoncés du premier ordre ?*

A. Berenstein et C. W. Henson ont donné une première réponse à cette question, dans le particulier d'actions libres de $\Gamma = \mathbb{Z}$, et ensuite, dans un papier non publié, dans le cas libres d'un groupe moyennable donné Γ : la théorie de telles actions est complète, ce qui veut dire que toutes les actions libres de Γ sont élémentairement équivalentes. En revanche, on prouve dans la Partie I que le degré de liberté d'une action est un invariant d'équivalence élémentaire. Par exemple, l'action triviale n'est élémentairement équivalente qu'à elle-même et une action élémentairement équivalente à une action libre est nécessairement libre.

Plus précisément, l'ensemble $\mathrm{Sub}(\Gamma)$ des sous-groupes de Γ a une structure naturelle de borélien standard, et pour une action pmp α de Γ sur (X, μ), l'application stabilisateur $\mathrm{stab}_\alpha \colon X \to \mathrm{Sub}(\Gamma)$ est borélienne. Le sous-groupe aléatoire invariant (Invariant Random Subgroup, abrégé en IRS, en anglais) associé à α est la mesure de probabilité "poussée-en-avant" $\mathrm{stab}_{\alpha *}\mu$ sur $\mathrm{Sub}(\Gamma)$. Alors le résultat annoncé plus haut dit que deux actions élémentairement équivalement ont le même IRS. A une action libre de Γ correspond la mesure de Dirac concentrée sur le groupe trivial $\delta_{\{e\}}$ tandis que l'IRS associé à l'action trivial est la mesure de Dirac concentrée sur Γ.

On peut raffiner en montrant que la théorie des modèles reconnait la contenance faible. En particulier, cela implique que l'équivalence élémentaire est une relation d'équivalence plus fine que l'équivalence faible. Puisque l'IRS est un invariant d'équivalence faible, il est a fortiori un invariant d'équivalence élémentaire.

On peut alors se demander si l'IRS est un invariant complet :

Question 1.0.2. *Existe-t-il deux actions pmp du groupe Γ qui ont même IRS mais ne sont pas élémentairement équivalentes ?*

Encore une fois, la réponse à cette question est liée à la moyennabilité. Cependant, il ne s'agit pas forcément de la moyennabilité du groupe Γ mais plutôt celle de l'IRS considéré. En effet, si les actions considérées ne sont pas supposées libres, elles pourraient en fait provenir d'un quotient de Γ comme d'une extension de Γ. Ainsi le choix de Γ est assez arbitraire. On peut même aller plus loin : puisque tout groupe dénombrable est un quotient du groupe libre de rang infini dénombrable $\mathbb{F}_\infty$, toute action d'un groupe dénombrable Γ peut être vue comme une action non libre de $\mathbb{F}_\infty$. Pour cette raison, à partir de maintenant, on fixe $\Gamma = \mathbb{F}_\infty$ et on se restreint aux actions ayant un certain IRS donné, plutôt qu'aux actions d'un groupe donné.

G. Elek a montré qu'étant donné un IRS θ, soit toutes les relations d'équivalence orbitales associées aux actions pmp d'IRS θ sont moyennables (de manière équivalence hyperfinies), ou aucune ne l'est. On dit qu'un IRS est moyennable s'il vérifie le premier cas de cette dichotomie. On donne plus tard dans cette introduction une caractérisation plus élégante des IRS moyennables. Le théorème suivant offre une réponse partielle à la Question 1.0.2 :

Théorème A (Theorem 4.4.4)**.** *Soit θ un IRS moyennable. Alors toutes les actions pmp d'IRS θ sont élémentairement équivalentes. Autrement dit, la théorie $\mathfrak{A}_\theta$ des actions pmp sur une algèbre de mesure donnée ayant pour IRS θ est une théorie complète.*

Ce théorème est une conséquence directe d'un théorème obtenu indépendamment par G. Elek et par P. Burton pour les IRS de groupes moyennables. On donne dans la Partie I une preuve plus courte du résultat suivant de G. Elek ([Ele12, Theorem 9]). Ce théorème affirme que si θ est un IRS moyennable, alors les actions pmp d'IRS θ sont toutes approximativement conjuguées deux à deux, c'est-à-dire que pour $\varepsilon > 0$ et toute partie finie F de $\mathbb{F}_\infty$, il existe une transformation pmp T de (X, μ) telle que $g \in F$, $\mu(\{x \in X : \alpha(g)(x) = T^{-1}\beta(g)T(x)\}) > 1 - \varepsilon$.

C'est en fait un phénomène couramment observé en théorie des modèles métriques. Même si la relation de conjugaison pour les actions pmp de groupes est en général beaucoup trop compliquée, autoriser une petite perturbation la rend beaucoup moins rigide dans le cas des IRS moyennables. On dit que deux actions pmp d'IRS θ sont isomorphes à une petite perturbation près ou encore que la théorie $\mathfrak{A}_\theta$ de ces actions est $\aleph_0$-catégorique à petite perturbation près. Puisque la $\aleph_0$-catégoricité à petite perturbation près implique la complétude de la théorie, on obtient notre Théorème 4.4.4.

De plus, quelques ajustements dans notre démonstration du résultat d'Elek permettent d'obtenir une généralisation avec paramètres, que l'on utilise pour prouver que la théorie $\mathfrak{A}_\theta$ est modèle complète. Alors, en rajoutant au langage une collection raisonnable de constantes, on obtient une théorie qui se comporte bien :

Théorème B (Theorems 4.5.8 and 4.6.8)**.** *Soit θ un IRS moyennable. On étend le langage de la théorie $\mathfrak{A}_\theta$ à l'aide de constantes S_g pour $g \in \mathbb{F}_\infty$ et on considère la théorie obtenue en ajoutant à $\mathfrak{A}_\theta$ que S_g doit correspondre au support de g pour tout $g \in \mathbb{F}_\infty$. Alors cette théorie élimine les quantificateurs. En particulier $\mathfrak{A}_\theta$ est stable.*

Cependant, pour quels IRS θ la théorie $\mathfrak{A}_\theta$ élimine les quantificateurs dans son langage original reste une question ouverte.

La seconde partie de cette thèse a été motivée par une éventuelle réciproque au Théorème 4.4.4, c'est-à-dire :

Question 1.0.3. *Soit θ un IRS non moyennable.*

- *Existe-t-il deux actions pmp d'IRS θ qui ne soient pas élémentairement équivalente ?*
- *Existe-t-il deux actions pmp d'IRS θ qui ne soient pas conjuguées à petite perturbation près (autrement dit approximativement conjuguées) ?*

Du point de vue de la théorie des modèles, nous n'avons aucun élément de réponse. Par conséquent, nous nous concentrons à présent sur la deuxième partie de la précédente question. On préférera alors la terminologie de conjugaison approximative à celle de conjugaison à une petite perturbation près. En reprenant les idées utilisées par G. Hjorth afin d'étudier les classes de conjugaison d'actions pmp libres ergodiques d'un groupe Γ, nous avons obtenu quelques réponses partielles à cette question en étudiant les groupoides préservant la mesure de probabilité ainsi que leurs actions. Nous présentons ces résultats dans la seconde partie de cette thèse.

Il se trouve que la conjugaison approximative peut être reformulée simplement en termes topologiques. Soit un groupe dénombrable Γ et un espace de probabilité standard (X, μ). L'ensemble $\mathrm{A}(\Gamma, X, \mu)$ des actions pmp de Γ sur (X, μ) peut être muni de deux topologies naturelles provenant de topologies sur l'espace $\mathrm{Aut}(X, \mu)$ des isomorphismes isométriques de l'algèbre de mesure $\mathrm{MAlg}(X, \mu)$.

- La topologie faible sur $\mathrm{Aut}(X, \mu)$ est la topologie faible, c'est-à-dire la topologie engendrée par les applications $T \mapsto TA$ pour A parcourant $\mathrm{MAlg}(X, \mu)$.
- La topologie uniforme est la topologie induite par la distance complète $\delta'_u(S, T) = \sup_{A \in \mathrm{MAlg}(X,\mu)} \mu(SA \bigtriangleup TA)$. Une distance équivalente est donnée par $\delta_u(S, T) = \mu(\{x \in X \colon S(x) \neq T(x)\})$ où on confond S et T avec des relèvements respectifs en transformations pmp de (X, μ).

Ces deux topologies peuvent être étendues à $\mathrm{A}(\Gamma, X, \mu)$, vu comme sous-ensemble fermé du produit $\mathrm{Aut}(X, \mu)^\Gamma$. Pour $E \subset \mathrm{A}(\Gamma, X, \mu)$ on note $\overline{E}^w$ son adhérence faible et $\overline{E}^u$ son adhérence uniforme. On rappelle que $\mathrm{Aut}(X, \mu)$ agit sur $\mathrm{A}(\Gamma, X, \mu)$ par conjugaison. Alors l'équivalence faible et la conjugaison approximative peuvent être reformulées ainsi : α et β sont faiblement équivalentes si et seulement si $\overline{\mathrm{Aut}(X, \mu) \cdot \alpha}^w = \overline{\mathrm{Aut}(X, \mu) \cdot \beta}^w$ tandis que α et β sont approximativement conjuguées si et seulement si $\overline{\mathrm{Aut}(X, \mu) \cdot \alpha}^u = \overline{\mathrm{Aut}(X, \mu) \cdot \beta}^u$.

Afin d'étudier les classes d'orbites-équivalence d'actions de groupes non moyennables, Hjorth utilise une dichotomie autour de la propriété (T). Tout d'abord, un théorème de Connes et Weiss stipule qu'un groupe Γ a la propriété (T) si et seulement si toutes ses actions ergodiques pmp sont en fait fortement ergodiques. Puisque la forte ergodicité est un invariant d'équivalence faible, il s'ensuit que tout groupe non moyennable n'ayant pas la propriété (T) doit avoir au moins deux actions pmp ergodiques non faiblement équivalentes.

Dans un autre temps, G. Hjorth étudie la topologie uniforme sur l'espace des actions pmp libres ergodiques d'un groupe Γ avec propriété (T) et prouve que les classes de conjugaison de telles actions sont uniformément ouvertes et fermées. Par conséquent, pour les groupes avec propriété (T), la conjugaison approximative et la conjugaison coïncident sur les actions libres ergodiques. Ainsi, les résultats mentionnés plus haut dans l'introduction sur la non-classifiabilité de la conjugaison d'actions pmp impliquent que la conjugaison approximative n'est pas non plus classifiable par structures dénombrables. En particulier, dans ce cas, il y a un continuum de classes de conjugaison approximative d'actions libres ergodiques.

Dans la présente thèse, nous adaptons les outils utilisés de part et d'autre de cette dichotomie à des actions pmp de groupoïdes pmp, lesquels généralisent à la fois la notion de groupe moyennable mais aussi de relation d'équivalence pmp. On obtient

Théorème C (Theorem 5.4.10)**.** *Soit $\mathcal{G}$ un groupoïde pmp ergodique sur un espace de probabilité standard (X, μ), ayant la propriété (T), et soit (Y, ν) un espace de probabilité* *standard. Alors les orbites de l'action* $\mathrm{Aut}(Y,\nu) \curvearrowright \mathrm{A}(\mathcal{G},Y,\nu)$ *sur l'espace des actions pmp ergodiques de* $\mathcal{G}$ *sur* (Y,ν) *par conjugaison sont ouvertes et fermées pour la topologie uniforme.*

Cependant, comme nous verrons par la suite, la conjugaison par $\mathrm{Aut}(Y,\nu)$ n'est pas la notion souhaitée de conjugaison pour des actions pmp de groupoïde. Il faudrait en réalité demander que les actions pleines (cf. Définition 5.2.7) induites sur $(X \times Y, \mu \times \nu)$ soient conjuguées par un élément de $\mathrm{Aut}(X \times Y, \mu \times \nu)$. En général, la conjugaison par un élément de $\mathrm{Aut}(Y,\nu)$ est une condition strictement plus forte que la conjugaison des actions pleines par un élément de $(X \times Y, \mu \times \nu)$. Malheureusment, nous ne savons pas en général quelle est la complexité de la relation de conjugaison par $\mathrm{Aut}(Y,\nu)$ dans une classe de conjugaison par $(X \times Y, \mu \times \nu)$ donnée.

On passe au deuxième point de la dichotomie de Hjorth.

Théorème D (Theorem 5.3.28)**.** *Soit pG une groupoïde pmp ergodique sur un espace de probabilité standard* (X,μ)*. Alors* $\mathcal{G}$ *a la propriété (T) si et seulement si toute actions pmp ergodique de pG est fortement ergodique.*

Cette généralisation du théorème de Connes-Weiss est également utilisée dans un article en préparation en commun avec A. Carderi et F. Le Maître, afin de caractériser la propriété (T) d'une relation d'équivalence en termes d'actions pmp de son groupe plein $[\mathcal{R}]$.

Théorème E (Carderi, Giraud, Le Maître, Theorems 7.4.7 and 7.4.8)**.** *Soit* $\mathcal{R}$ *une relation d'équivalence pmp ergodique. Alors* $\mathcal{R}$ *a la propriété (T) si et seulement si toute action booléenne ergodique non libre de son groupe plein* $[\mathcal{R}]$ *est fortement ergodique.*

Nous expliquons maintenant l'utilisation que nous faisons des groupoïdes. Soit θ un IRS sur un groupe dénombrable Γ. On construit un groupoïde pmp $\mathcal{G}_\theta$ associé à θ, dont les actions pmp libres correspondent aux actions de Γ ayant pour IRS θ. L'étude de ce groupoïde semble constituer une approche prometteuse pour les questions concernant cet IRS. Pour exemple, un IRS θ est moyennable, au sens décrit précedemment, si et seulement si $\mathcal{G}_\theta$ est un groupoïde moyennable. De la même manière, toute actions pmp d'IRS θ induit une relation d'équivalence ayant (T) si et seulement si le groupoïde $\mathcal{G}_\theta$ a (T). Bien entendu, les définitions utilisées pour la moyennabilité et la propriété (T) d'un groupoïde pmp coïncident avec les définitions classiques dans le cas où le groupoïde est un groupe ou une relation d'équivalence pmp.

A l'aide de la correspondance bijective entre actions pmp libres du groupoïde $\mathcal{G}_\theta$ et actions pmp de Γ ayant pour IRS θ et du théorème de Connes-Weiss pour les groupoïdes, on obtient :

Théorème F (Connes-Weiss Theorem for Invariant Random Subgroups, Theorem 6.2.5)**.** *Soit* θ *un IRS ergodique sur un groupe dénombrable* Γ*. Alors* θ *a la propriété (T) si et seulement si toute action pmp ergodique de* Γ *qui a pour IRS* θ *est fortement ergodique.*

On introduit donc une dernière relation d'équivalence sur l'espace des actions pmp de Γ : deux actions sont dites stab-équivalentes si elles sont approximativement conjuguées par des transformations pmp qui préservent le stabilisateur de ces actions (si ces transformations pmp préservent le stabilisateur d'une des deux actions, alors les deux actions doivent avoir même stabilisateur). Alors deux actions stab-équivalentes de Γ ayant IRS θ induisent deux actions de $\mathcal{G}_\theta$ sur des espaces isomorphes et on peut appliquer le Théorème C afin d'obtenir

Théorème G (Theorem 6.2.8). *Soit θ un IRS ergodique ayant la propriété (T) sur un groupe Γ. Alors deux actions ergodiques de Γ ayant pour IRS θ sont conjuguées si et seulement si elles sont stab-équivalentes.*

En général, nous ne savons pas à quelle point la stab-équivalence diffère de la conjugaison ou de la conjugaison approximative. On peut noter en revanche que pour des actions libres d'un quotient de Γ, c'est-à-dire dans le cas où θ est une mesure concentrée en un point, la stab-équivalence coïncide avec la conjugaison approximative. Inversement, pour les actions totalement non libres, c'est-à-dire telles que l'application stabilisateur est une bijection $(X, \mu) \to (\mathrm{Sub}(\Gamma), \theta_\alpha)$, la stab-équivalence coïncide simplement avec la conjugaison.

Chapter 2

Introduction (english version)

A central topic in ergodic theory consists of the study of probability measure-preserving (pmp) transformations and actions of groups on standard probability spaces, and their classification. In this thesis our interest is focused on the classification of pmp actions of countable groups on standard probability spaces and its link with the algebraic properties of the acting group. From now on, we restrict to discrete countable groups.

The most obvious way to classify such actions is to ask when two actions are equal up to conjugation. A telling example is the development of entropy theory for pmp actions of certain groups. In [Orn70], D. Ornstein proved that two Bernoulli shifts of $\mathbb{Z}$ are conjugate if and only if they have the same entropy, and in [OW87], Ornstein and Weiss extended this result to Bernoulli shifts over Γ for any countable amenable group Γ. On the other hand, several anti-classification results such as [FW04] and [FRW11] made it clear that there was no hope for a nicely computable complete invariant for conjugation on ergodic pmp transformations in general.

Rather than restricting to a specific class of actions such as the Bernoulli shifts, another approach to the problem of classification is to replace conjugation by a slightly coarser equivalence relation. We give a couple of examples of such equivalence relations. Two pmp actions on (X, μ) are orbit equivalent if the two partitions of X into orbits they induce are isomorphic, up to a null set. With regard to orbit equivalence, the classification of pmp actions of groups is radically different from the one up to conjugation: any two ergodic pmp actions of amenable groups are orbit equivalent ([Dye59] and [OW87]), whereas if two free groups admit free ergodic pmp actions that are orbit equivalent, then they must have the same rank ([Gab00]). More recently A. Kechris introduced the relations of weak containment and weak equivalence for pmp actions, exporting the corresponding notions from the theory of unitary representations to ergodic group theory.

The first part of the thesis consists in an article submitted for publication. The equivalence relation we focus on arises from mathematical logic and more precisely from metric model theory. Metric model theory is an extension of classical model theory to metric structures instead of discrete ones. Many different formalizations of continuous model theory exist, but we refer to the one presented in [BBHU08] and brought up to date by I. Ben Yaacov, A. Berenstein, C. W. Henson and A. Usvyatsov.

Fix a countable group Γ. Any pmp action α of Γ on a probability space (X, μ) induces an action of Γ by isometric isomorphisms on the probability measure algebra $\mathrm{MAlg}(X, \mu)$. The first order structure $\mathcal{M}_\alpha$ we study is the measure algebra $\mathrm{MAlg}(X, \mu)$ endowed with the action of Γ induced by α. We say that two pmp actions of Γ on (X, μ) are elementarily equivalent if their associated measure algebras give the same valuation to every first-order sentence. This logical approach to ergodic group theory raises the following question, which was the first motivation for this thesis:

Question 2.0.1. *What can be said about a pmp action of* Γ *through first-order sentences?*

This question was answered by A. Berenstein and C. W. Henson in [BH04] in the case of free actions of $\mathbb{Z}$, and in an unpublished paper for the case of free actions of a given amenable group Γ: the theory of such actions is complete, meaning that elementary equivalence does not distinguish two free actions. However, we prove in Part I that the degree of freeness of an action is an invariant of elementary equivalence. For example, triviality and freeness are both invariants of elementary equivalence.

Let us detail what we mean by degree of freeness: the set $\mathrm{Sub}(\Gamma)$ of subgroups of Γ has a natural standard Borel space structure. For α a pmp action of Γ on (X, μ), the map $\mathrm{stab}_\alpha \colon X \to \mathrm{Sub}(\Gamma)$ is Borel. The invariant random subgroup (IRS) associated to α is the push-forward Borel probability measure $\mathrm{stab}_{\alpha *}\mu$ on $\mathrm{Sub}(\Gamma)$. Then two elementarily equivalent actions have the same IRS. A free action of Γ corresponds to the Dirac IRS $\delta_{\{e\}}$ whereas the IRS of the trivial action is δ_Γ.

In fact, model theory even recognizes weak containment. In particular, elementary equivalence is finer than weak equivalence. Since the IRS is a weak equivalence invariant it is all the more an elementary equivalence invariant.

Once this invariant has been isolated, one can ask the following:

Question 2.0.2. *Are there two pmp actions of* Γ *which are not elementarily equivalent yet have the same IRS?*

It turns out that once again the answer to this question revolves around amenability. But this answer does not depend on amenability of the group Γ but rather on the IRS considered. Indeed, if the actions we consider are not supposed to be free, then they could very well arise from any quotient or extension of Γ, and the choice of Γ in order to study them seems quite arbitrary. We can go even further: since every countable group is a quotient of the free group of infinite rank $\mathbb{F}_\infty$, any action of a countable group arises from an action of $\mathbb{F}_\infty$. For this reason, from now on we can set $\Gamma = \mathbb{F}_\infty$ and restrict ourselves to actions with a given fixed IRS instead of actions of a given group.

G. Elek proved that given an IRS θ, either all orbit equivalence relations of actions with IRS θ are amenable (or equivalently hyperfinite), or none are. We call an IRS amenable if it corresponds to the first case of this dichotomy. We will give a nice characterization of amenable IRSs later on in this introduction. The following theorem gives a partial answer to Question 2.0.2:

Theorem A (Theorem 4.4.4)**.** *Let* θ *be an amenable IRS. Then any two pmp actions with IRS* θ *are elementarily equivalent. Equivalently, the theory* $\mathfrak{A}_\theta$ *of pmp actions on a given probability algebra which have IRS* θ *is complete.*

This theorem is a straightforward consequence of a theorem obtained independently by G. Elek and by P. Burton for an IRS on an amenable group Γ. We provide in Part I a shorter proof of G. Elek's result ([Ele12, Theorem 9]). This theorem states that if θ is amenable then any two actions α and β on (X, μ) with IRS θ are approximately conjugate, i.e. for any $\varepsilon > 0$ and any finite $F \subset \mathbb{F}_\infty$, there exists a pmp transformation T such that for all $g \in F$, $\mu(\{x \in X \colon \alpha(g)(x) = T^{-1}\beta(g)T(x)\}) > 1 - \varepsilon$.

This phenomenon is frequently observed in metric model theory. Even though the relation of conjugation for pmp group actions is very complicated in general, allowing a small perturbation in the case of an amenable IRS makes it much simpler. We say that any two pmp actions with IRS θ are isomorphic up to a small perturbation, or in other words, that the theory $\mathfrak{A}_\theta$ is $\aleph_0$-categorical up to a small perturbation. Since $\aleph_0$-categoricity up to a small perturbation implies completeness, we get Theorem 4.4.4.

Moreover, small adjustments in our proof of Elek's result lead to a generalization with parameters, that we use to prove that the theory $\mathfrak{A}_\theta$ is model complete. We also study elimination of quantifiers and stability of the theory $\mathfrak{A}_\theta$ in this thesis. We see that up to a small extension of the language used in our first-order sentences, $\mathfrak{A}_\theta$ behaves as expected:

Theorem B (Theorems 4.5.8 and 4.6.8)**.** *Let θ be an amenable IRS. Then $\mathfrak{A}_\theta$ eliminates quantifiers in the extension of the language with constants S_g for $g \in \mathbb{F}_\infty$, expressing that S_g corresponds to the support of the element g, and $\mathfrak{A}_\theta$ is stable.*

Still, the question of knowing for which IRSs θ the theory $\mathfrak{A}_\theta$ eliminates quantifiers in its original langage remains open.

The second part of this thesis was motivated by the converse of Theorem 4.4.4, that is:

Question 2.0.3. *Let θ be a non-amenable IRS.*

- *Are there two non-elementary equivalent pmp actions with IRS θ?*
- *Are there two pmp actions with IRS θ which are not conjugated up to a small perturbation (approximately conjugate)?*

The author could not say anything about this question from the point of view of model theory. For this reason, we focused on the second part of the latter question. From now on, moving from model theory to more classical ergodic group theory, we will prefer the terminology of approximate conjugacy rather than conjugation up to a small perturbation. Following the ideas used by G. Hjorth to study conjugacy classes of free ergodic pmp actions of a group Γ, we obtained partial answers to this question, that we present in the second part of this thesis, by studying measure-preserving groupoids and their pmp actions.

In fact, approximate conjugation has a nice topological reformulation. Consider a countable group Γ and a standard probability space (X, μ). The set $\mathrm{A}(\Gamma, X, \mu)$ can be endowed with two natural topologies arising from topologies on the space $\mathrm{Aut}(X, \mu)$ of isometric isomorphisms of $\mathrm{MAlg}(X, \mu)$.

- The weak topology on $\mathrm{Aut}(X, \mu)$ is the topology of pointwise convergence, i.e. the topology generated by the maps $T \mapsto TA$ for $A \in \mathrm{MAlg}(X, \mu)$.
- The uniform topology on $\mathrm{Aut}(X, \mu)$ is the topology induced by the complete metric $\delta'_u(S, T) = \sup_{A \in \mathrm{MAlg}(X,\mu)} \mu(SA \triangle TA)$. If S and T arise from pmp transformations of (X, μ), another equivalent metric is given by $\delta_u(S, T) = \mu(\{x \in X \colon S(x) \neq T(x)\})$.

Both these topologies extend to $\mathrm{A}(\Gamma, X, \mu)$. For $E \subset \mathrm{A}(\Gamma, X, \mu)$, we write $\overline{E}^w$ for the weak closure of E and $\overline{E}^u$ for its uniform closure of E. Moreover, $\mathrm{Aut}(X, \mu)$ acts on $\mathrm{A}(\Gamma, X, \mu)$ by conjugation. Then weak equivalence and approximate conjugacy can be reformulated as follows: on the one hand, α and β are weakly equivalent when $\overline{\mathrm{Aut}(X,\mu)\cdot\alpha}^w = \overline{\mathrm{Aut}(X,\mu)\cdot\beta}^w$ whereas on the other hand α and β are approximately conjugate when $\overline{\mathrm{Aut}(X,\mu)\cdot\alpha}^u = \overline{\mathrm{Aut}(X,\mu)\cdot\beta}^u$.

In order to study orbit equivalence classes of actions of non-amenable groups, Hjorth used a dichotomy based on property (T). First, a theorem of Connes-Weiss states that a group Γ has property (T) if and only if any ergodic pmp action of Γ is strongly ergodic. Since strong ergodicity is an invariant of weak equivalence, a consequence is that any non-amenable group without property (T) must have at least two ergodic pmp actions which are not weakly equivalent.

For the case of property (T) groups, G. Hjorth studied the uniform topology on the space of free ergodic pmp actions of a group Γ with property (T), and he proved that the conjugacy classes of such actions were uniformly clopen. It follows that for property (T) groups, approximate conjugation and conjugation are, in fact, the same relation. Therefore, results mentioned earlier in this introduction about the non-classifiability of conjugation for pmp actions imply that approximate conjugation itself is not classifiable by countable structures and all the more admits a continuum of classes.

In the second part of this thesis, we extend the tools used in both sides of this dichotomy to pmp actions of pmp groupoids, which generalize both countable groups and pmp equivalence relations. We get

Theorem C (Theorem 5.4.10)**.** *Let $\mathcal{G}$ be an ergodic pmp groupoid with property (T) on a standard probability space (X,μ) and let (Y,ν) be a standard probability space. Then the orbits of the action $\mathrm{Aut}(Y,\nu) \curvearrowright \mathrm{A}(\mathcal{G}, Y, \nu)$ on the space of ergodic pmp actions of $\mathcal{G}$ on (Y,ν) by conjugation are clopen for the uniform topology.*

However, as we will see, this is not the right notion of conjugation for pmp actions of groupoids. One should rather ask that the full actions (see Definition 5.2.7) induced on $(X \times Y, \mu \times \nu)$ are conjugate via en element of $\mathrm{Aut}(X \times Y, \mu \times \nu)$. In general, conjugation (resp. approximate conjugation) by elements of $\mathrm{Aut}(Y,\nu)$ is strictly stronger than conjugation of the full actions on $(X \times Y, \mu \times \nu)$. Unfortunately, we do not know the complexity of conjugation by elements of $\mathrm{Aut}(Y,\nu)$ in a general conjugacy class of full actions on $(X \times Y, \mu \times \nu)$.

Now for the other part of Hjorth's dichotomy.

Theorem D (Theorem 5.3.28)**.** *Let $\mathcal{G}$ be an ergodic pmp groupoid on a standard probability space (X,μ). Then $\mathcal{G}$ has property (T) if and only if every ergodic pmp action of $\mathcal{G}$ is strongly ergodic.*

This generalization of the Connes-Weiss Theorem is then also used in a common work in preparation with A. Carderi and F. Le Maître to characterize property (T) for a pmp equivalence relation $\mathcal{R}$ in terms of pmp actions of its full group $[\mathcal{R}]$.

Theorem E (Carderi, Giraud, Le Maître, Theorems 7.4.7 and 7.4.8)**.** *Let $\mathcal{R}$ be an ergodic pmp equivalence relation. Then $\mathcal{R}$ has property (T) if and only if every non-free ergodic boolean action of its full group $[\mathcal{R}]$ is strongly ergodic.*

We now explain our interest in pmp groupoids. Let θ be an IRS on a countable group Γ. We construct a pmp groupoid $\mathcal{G}_\theta$ associated to θ whose free pmp actions correspond to pmp actions of Γ which have IRS θ. The study of this groupoid seems to be a very promising approach to questions regarding the IRS. For example, an IRS θ is amenable in the sense described earlier, that is every pmp action with IRS θ induces an amenable equivalence relation, if and only if the pmp groupoid $\mathcal{G}_\theta$ is amenable. In the same fashion, every pmp action with IRS θ induces an orbit equivalence relation with property (T) if and only if θ has property (T), i.e. $\mathcal{G}_\theta$ has property (T). Note that of course these definitions of amenability and property (T) for pmp groupoids extend those known for countable groups as well as for pmp equivalence relations.

Using the 1-to-1 correspondence between pmp free actions of $\mathcal{G}_\theta$ and pmp actions of Γ with IRS θ alongside the Connes-Weiss Theorem for groupoids, we obtain:

Theorem F (Connes-Weiss Theorem for Invariant Random Subgroups, Theorem 6.2.5)**.** *Let θ be an ergodic IRS on a countable group Γ. Then θ has property (T) if and only if every ergodic pmp action of Γ with IRS θ is strongly ergodic.*

We thus introduce a last equivalence relation on the space of pmp actions of Γ: two actions are said to be stab-equivalent if they are approximately conjugate by pmp transformations which preserve their stabilizer (if these pmp transformations preserve the stabilizer of one action, then both stabilizers must be equal), up to conjugation. Thus two stab-equivalent actions of Γ with IRS θ induce two actions of $\mathcal{G}_\theta$ on isomorphic spaces and one can apply Theorem C to obtain

Theorem G (Theorem 6.2.8)**.** *Let θ be an ergodic IRS with property (T) on Γ. Then two pmp ergodic actions of Γ with IRS θ are conjugate if and only if they are stab-equivalent.*

In general, we do not know how stab-equivalence compares to conjugation or approximate conjugation. Note however that for free actions of a quotient of Γ, that is the case where θ is a Dirac measure, stab-equivalence is simply approximate conjugation. On the opposite side of the spectrum, for totally non-free actions on (X, μ), i.e. actions α whose stabilizer map is a pmp bijection $(X, \mu) \to (\mathrm{Sub}(\Gamma), \theta_\alpha)$, stab-equivalence always coincides with conjugation.

Part I

Hyperfinite measure-preserving actions and their model theory

Classical ergodic theory consists of the study of probability measure-preserving (pmp in short) transformations of a probability space. A **pmp transformation** T of a probability space (X, μ) is a bimeasurable permutation of X such that for all measurable subsets A of X, $\mu(T^{-1}A) = \mu(A)$. It is called **ergodic** if any T-invariant subset of X is either null or conull, and it is called **aperiodic** if almost every T-orbit is infinite. In the case of a single transformation T of an atomless probability space, it is well-known that ergodicity implies aperiodicity. For now, we restrict ourselves to **standard probability spaces**, that is probability spaces that are isomorphic to the interval $[0, 1]$ equipped with the Lebesgue measure.

Two pmp transformations T and T' are said to be **conjugate**, or sometimes **isomorphic**, if there is a third pmp transformation S such that up to a null set, $T' = STS^{-1}$. One of the main goals of ergodic theory is to understand the conjugacy relation on pmp transformations, particularly on the set of ergodic pmp transformations. Conjugacy is completely understood in some specific cases, for example, entropy is a complete invariant of conjugacy for Bernoulli shifts [Orn70] and spectrum is a complete invariant of conjugacy for compact transformations. However, in general, conjugacy is a very complicated relation as shown in [FW04] and [FRW11].

In this paper we study the simpler relation of approximate conjugacy. Two pmp transformations T and T' of (X, μ) are said to be **approximately conjugate** if for all $\varepsilon > 0$ there is a third pmp transformation S of (X, μ) such that $T' = STS^{-1}$ up to a set of measure at most ε. It is a well-known consequence of Rokhlin Lemma that any two aperiodic pmp transformations of standard probability spaces are approximately conjugate [Kec10, Thm. 2.4]. We thus focus on understanding the approximate conjugacy relation for general pmp actions of countable discrete groups rather than single pmp transformations, which correspond to $\mathbb{Z}$-actions.

A **pmp action** of a countable group Γ on a probability space (X, μ) is an action of Γ on X by pmp transformations. For a pmp action $\Gamma \overset{\alpha}{\curvearrowright} (X, \mu)$ and $\gamma \in \Gamma$, we let γ^α denote the pmp transformation associated to γ in the action α. Two pmp actions α and β of a countable group Γ are **conjugate** if there is a pmp transformation S such that $S^{-1}\gamma^\alpha S = \gamma^\beta$ for all $\gamma \in \Gamma$. We say that α is a **factor** of β, denoted by $\alpha \sqsubseteq \beta$ if there is a measure-preserving map $S\colon X \to X$ such that $\gamma^\alpha S = S\gamma^\beta$ for every $\gamma \in \Gamma$.

We say that α and β are **approximately conjugate** if for every finite $F \subseteq \Gamma$ and every $\varepsilon > 0$, there exists a pmp transformation S of X such that

$$\mu\left(\{x \in X : \exists \gamma \in F,\ \gamma^\beta x \neq S\gamma^\alpha S^{-1}x\}\right) < \varepsilon.$$

This notion of approximate conjugacy comes from the study of the spaces $\mathrm{Aut}(X, \mu)$ and $A(\Gamma, X, \mu)$ of pmp transformations of (X, μ) and of pmp actions of Γ on (X, μ), respectively.

The space $\mathrm{Aut}(X, \mu)$ can be equipped with two topologies: the weak and the uniform topology (see [Kec10] for definitions). Two pmp transformations T and S are called **weakly equivalent** if $\overline{[T]}^w = \overline{[S]}^w$, where $[T]$ is the conjugacy class of T, and $\overline{A}^w$ denotes the closure of A in the weak topology. Then, the space of actions can be seen as a closed subspace of $\mathrm{Aut}(X, \mu)^\Gamma$ equipped with either product topology, and this induces two topologies on $A(\Gamma, X, \mu)$, that we respectively call again the weak and the uniform topology. In the same fashion as for transformations, we say that two actions α and β are weakly equivalent if $\overline{[\alpha]}^w = \overline{[\beta]}^w$.

Now approximate conjugacy is the uniform counterpart of weak equivalence, that is, two pmp actions α and β are approximately conjugate if and only if $\overline{[\alpha]}^u = \overline{[\beta]}^u$, where $\overline{A}^u$ is the uniform closure of A. The study of approximate conjugacy in the present paper was mostly motivated by similar results obtained for weak equivalence by R. Tucker-Drob in [Tuc15].

The first obstacle to approximate conjugacy is freeness : a pmp action of Γ is **free** if the set of fixed points of any nontrivial element of Γ is null. For $\mathbb{Z}$-actions, freeness corresponds to aperiodicity. It is easy to see that approximate conjugacy preserves the freeness of the actions, and that the trivial action is only approximately conjugate with itself.

In fact, we have a better result. For a pmp action $\Gamma \overset{\alpha}{\curvearrowright} (X, \mu)$, the pushforward of the measure μ by the stabilizer application $x \in X \mapsto \mathrm{stab}_\alpha(x)$ gives a measure θ_α on the space of subgroups of Γ. We call this measure the Invariant Random Subgroup (IRS in short, see [AGV14]) of the action α. Then it is not hard to see that the IRS is an invariant of approximate conjugacy. Moreover, free actions correspond to the case where the IRS is the Dirac measure on the trivial subgroup $\delta_{\{e\}}$ and the trivial action corresponds to the case where the IRS is δ_Γ.

In this paper we work with hyperfinite actions, which are defined as follows:

Definition 2.0.4. A pmp action $\Gamma \curvearrowright (X, \mu)$ is said to be **hyperfinite** if for any finite subset S of Γ and any $\varepsilon > 0$, there exists a finite group G acting in a measure-preserving way on (X, μ) such that

$$\mu\left(\{x \in X : S \cdot x \subseteq G \cdot x\}\right) > 1 - \varepsilon.$$

It is a theorem of D. S. Ornstein and B. Weiss [OW80] that pmp actions of amenable groups are hyperfinite.

In general, we have the following implications:

$$\text{approximate conjugacy} \implies \text{weak equivalence} \implies \text{same IRS}.$$

In the most general context, the IRS of an action is not a complete invariant of approximate conjugacy. However, G. Elek proved that when restricted to hyperfinite actions, it is:

Theorem I (G.Elek, [Ele12, Thm. 9])**.** *Let α and β be two pmp hyperfinite actions of a group Γ on a standard probability space such that $\theta_\alpha = \theta_\beta$. Then α and β are approximately conjugate.*

This theorem thus generalizes the consequence [Kec10, Thm. 2.4] of Rokhlin Lemma, which can be obtained by taking $\Gamma = \mathbb{Z}$ and $\theta_\alpha = \theta_\beta = \delta_{\{e\}}$.

In this paper, we give a shorter proof of this theorem, first by considering the critical case of actions which are factors one of another and then using a confluence argument to conclude in the general case. Moreover, when one of the actions is a factor of the other, we add a slight improvement to the theorem by requiring that the pmp transformations witnessing approximate conjugacy stabilize some measurable sets. This stronger version of the theorem will be used for the model theoretic study of pmp actions, which is the main topic of the present paper.

The formalism of continuous model theory that we use was developed by I. Ben Yaacov and A. Usvyatsov.

While classical model theory is concerned with algebraic theories such as discrete groups, algebraically closed or real closed fields, its continuous counterpart allows the study of metric structures. In recent years, continuous model theory has been used to study theories such as metrics spaces, Banach spaces, Hilbert spaces and measure algebras. More precisely, a particular attention was given to the study of formulas involving automorphisms of the latter theories.

In the present paper we are interested in the model theory of a group action on a probability space, in other words, we look at formulas involving finite subsets of automorphisms of a probability space (X, μ) from a given subgroup of the group of automorphisms of (X, μ). However, probability spaces do not admit a model theoretic treatment as such, where the elements of a structure are the points in probability spaces.

In order to solve this issue, we consider as structures not the probability spaces themselves but their associated measure algebra. For a probability space (X, Σ, μ), its associated measure algebra $\mathrm{MAlg}(X, \mu)$ is the quotient set $\Sigma/\mathcal{N}$ where $\mathcal{N}$ denotes the σ-ideal of null sets. It inherits the boolean operations $\vee, \cap, \cdot^{-1}$ of Σ and is endowed with a natural metric $d_\mu(\pi(A), \pi(B)) := \mu(A \triangle B)$, where π is the quotient map.

Moreover, the correspondence between probability spaces and measure algebras is functorial, so that a pmp action on a probability space induces an action by automorphisms on its measure algebra.

Following the latter remarks, we study the model theory of atomless measure algebras with a countable group Γ acting by automorphisms. This work follows the one in [BBHU08, Section 18] about free actions of $\mathbb{Z}$ and the more general case of free actions of amenable groups treated by A. Berenstein and C. W. Henson in an unpublished paper.

Without loss of generality, we restrict our study to actions of the free group over an infinite countable subset, F_∞, as any action of a countable group can be seen as an action of F_∞. Then one can see that the equivalence relation of elementary equivalence is weaker than approximate conjugacy but stronger than weak equivalence. This result highlights the link between model theory and the equivalence relations usually studied in ergodic theory.

For any IRS θ on F_∞, we define a theory $\mathfrak{A}_\theta$ axiomatizing pmp actions with IRS θ. By a result of G. Elek ([Ele12, Thm. 2]), the hyperfiniteness of an action is determined by its IRS. We thus call an IRS θ amenable if actions with IRS θ are hyperfinite (or equivalently amenable).

By Theorem I, in the context of hyperfinite actions, having the same IRS is equivalent to being elementarily equivalent. We prove (Theorems 4.4.4 and 4.4.3):

Theorem II. *If θ is an amenable IRS, then the theory $\mathfrak{A}_\theta$ is complete and model complete.*

However, unlike in [BBHU08, Section 18] these theories do not admit quantifier elimination in general. We nevertherless prove in Theorem 4.5.8 that there is a reasonable expansion of the theory which eliminates quantifier, and we then use this to prove (Theorem 4.6.8):

Theorem III. *If θ is an amenable IRS, then the theory $\mathfrak{A}_\theta$ is stable and the stable independence relation given by non dividing admits a natural characterization in terms of the classical probabilistic independence of events (in a sense described in Definition 4.6.4).*

Chapter 3

The generalization of Rokhlin Lemma

3.1 Graphings

Definition 3.1.1. A **graph** G is a pair $(\mathrm{V}(G), \mathrm{E}(G))$ where $\mathrm{V}(G)$ is a set and $\mathrm{E}(G)$ is an irreflexive and symmetric binary relation on $\mathrm{V}(G)$. Elements of $\mathrm{V}(G)$ are called **vertices** of G and elements of $\mathrm{E}(G)$ are called **edges** of G.

For G a graph, for each $v \in \mathrm{V}(G)$ we let $\deg_G(v) = |\{u \in \mathrm{V}(G) : (v,u) \in \mathrm{E}(G)\}|$ and we call $\sup_{v \in \mathrm{V}(G)} \deg_G(v) \in \mathbb{N} \cup \{\infty\}$ the degree bound of G.

Definition 3.1.2. An **isomorphism between the graphs** G **and** H is a bijection $f\colon \mathrm{V}(G) \to \mathrm{V}(H)$ such that $\forall x, y \in \mathrm{V}(G)$, $(x,y) \in \mathrm{E}(G) \Leftrightarrow (f(x), f(y)) \in \mathrm{E}(H)$.

Definition 3.1.3. Let G be a graph, $A \subseteq \mathrm{V}(G)$ and $B \subseteq \mathrm{E}(G)$. Then we define:

- $\mathrm{V}^G_{\mathrm{inc}}(B) = \{v \in \mathrm{V}(G) : \exists u \in \mathrm{V}(G),\ (u,v) \in B \vee (v,u) \in B\}$ the set of **vertices incident to** B.
- $\mathrm{E}^G_{\mathrm{inc}}(A) = \{(a,v) \in \mathrm{E}(G) : a \in A\}$ the set of **edges incident to** A.

We will write $\mathrm{V}_{\mathrm{inc}}(B)$ and $\mathrm{E}_{\mathrm{inc}}(A)$ when the context makes clear which graph G is considered.

Definition 3.1.4. Let G be a graph. A **subgraph** of G is a graph H such that $\mathrm{V}(H) = \mathrm{V}(G)$ and $\mathrm{E}(H) \subseteq \mathrm{E}(G)$. In this case, we write $H \subseteq G$.

If $\mathrm{V} \subseteq \mathrm{V}(G)$, the **subgraph of** G **induced by** V is the graph $(\mathrm{V}(G), \mathrm{E}(G) \cap V \times V)$. Nevertheless, in many cases it will be convenient to identify the induced graph on V and the graph $(V, \mathrm{E}(G) \cap V \times V)$ and therefore see the induced graph on V as a graph on the set of vertices V.

In general, we write $G \simeq H$ to indicate that G and H are isomorphic.

Definition 3.1.5. A **standard Borel space** is a measurable space isomorphic to $[0,1]$ equipped with its Borel σ-algebra. We call Borel the maps between two standard Borel spaces which are measurable.

Let us give some notations regarding probability spaces:

- If X is a measurable space, we denote by $\mathfrak{P}(X)$ the set of probability measures on X.
- If (X, μ) is a probability space and P is a property, we write $\forall^* x \in X\ P(x)$ for $\mu(\{x \in X : P(x)\}) = 1$ and $\exists^* x \in X\ P(x)$ for $\mu(\{x \in X : P(x)\}) > 0$.
- If (X, μ) is a probability space, Y is a measurable space and $T \colon X \to Y$ is a measurable map, we write $T_*\mu$ for the pushforward of μ by T, that is the measure in $\mathfrak{P}(Y)$ defined by $T_*\mu(A) = \mu(T^{-1}(A))$ for any Borel subset $A \subseteq Y$.

Definition 3.1.6. Let X be a standard Borel space and $\mathcal{R}$ be a Borel (as a subset of the measurable space $X \times X$) equivalence relation on X. We let $[\mathcal{R}]$ be the group of Borel automorphisms of X whose graphs are contained in $\mathcal{R}$. We say that a Borel probability measure μ on X is **$\mathcal{R}$-invariant** if every element of $[\mathcal{R}]$ preserves the measure μ, namely, $\forall T \in [\mathcal{R}],\ T_*\mu = \mu$.

Proposition 3.1.7 ([KM04, Section 8])**.** *With the same notations as above, for any $\mu \in \mathfrak{P}(X)$, we can define two measures μ_l and μ_r on $\mathcal{R}$ by*

- *for all non-negative Borel $f \colon \mathcal{R} \to [0, \infty]$, $\int_{\mathcal{R}} f\ d\mu_l = \int_X \sum\limits_{y \in [x]_{\mathcal{R}}} f(x, y)\ d\mu(x)$,*
- *for all non-negative Borel $f \colon \mathcal{R} \to [0, \infty]$, $\int_{\mathcal{R}} f\ d\mu_r = \int_X \sum\limits_{y \in [x]_{\mathcal{R}}} f(y, x)\ d\mu(x)$,*

where $[x]_{\mathcal{R}}$ denotes the equivalence class of x for $\mathcal{R}$. Then $\mu_l = \mu_r$ if and only if μ is $\mathcal{R}$-invariant.

Definition 3.1.8. Let $\mathcal{G}$ be a Borel graph on a standard probability space (X, μ) which has countable connected components. Then the equivalence relation $\mathcal{R}_{\mathcal{G}}$ induced by $\mathcal{G}$ is the equivalence relation on (X, μ) whose classes are the connected components of $\mathcal{G}$. By the Lusin-Novikov theorem, $\mathcal{R}_{\mathcal{G}}$ is a Borel equivalence relation. We say that $\mathcal{G}$ is a **graphing** when μ is $\mathcal{R}_{\mathcal{G}}$-invariant.

We can define a measure on the set of edges of a graphing by:

Definition 3.1.9. Let $\mathcal{G}(X, \mu)$ be a graphing and $Z \subseteq \mathrm{E}(\mathcal{G})$ be a Borel set. The **edge measure** of the set Z is defined by $\mu_{\mathrm{E}}(Z) \coloneqq \mu_l(Z) = \mu_r(Z)$, where μ_l and μ_r are defined with respect to the Borel equivalence relation $\mathcal{R}_{\mathcal{G}}$.

For a graphing of degree bound d, the edge measure of a set of edges is bounded by the measure of the vertices incident to this set. Namely, for all Borel $Z \subseteq \mathrm{E}(\mathcal{G})$ we have

$$\frac{1}{2}\mu(\mathrm{V}_{\mathrm{inc}}(Z)) \leqslant \mu_{\mathrm{E}}(Z) \leqslant d\mu(\mathrm{V}_{\mathrm{inc}}(Z)).$$

3.2 Classical Rokhlin Lemma

A measure-preserving transformation is called **aperiodic** if almost all its orbits are infinite.

Rokhlin Lemma states that if T is an aperiodic measure-preserving transformation of a standard probability space (X, μ), then for every $n \in \mathbb{N}$ and every $\varepsilon > 0$, there is a Borel subset $A \subseteq X$ such that the sets $A, TA, \ldots, T^{n-1}A$ are pairwise disjoint and

$$\mu\left(\bigsqcup_{i=0}^{n-1} T^i A\right) > 1 - \varepsilon.$$

What we present in this paper is not a generalization of Rokhlin Lemma itself but rather of one of its important and well-known consequences:

Corollary 3.2.1 (Uniform Approximation Theorem, [Kec10, Theorem 2.2])**.** *Any two aperiodic measure-preserving transformations τ_1 and τ_2 on standard probability spaces (X, μ) and (Y, ν) are approximately conjugate.*

An aperiodic measure-preserving transformation can be seen as a free action of $\mathbb{Z}$. The goal of this section is to generalize the latter Corollary to hyperfinite actions of a countable group which have a given IRS (i.e. Invariant Random Subgroup, defined in subsection 3.4).

3.3 Hyperfiniteness

The key point on the proof of Uniform Approximation Theorem 3.2.1 is that the dynamics of an aperiodic automorphism are understood on arbitrary large sets. In the section we define the notion of hyperfiniteness of a pmp action, which allows one to make this idea work in a much more general context.

Definition 3.3.1 (See "approximately finite group" in [Dye59])**.** A pmp action $\Gamma \curvearrowright (X, \mu)$ is said to be **hyperfinite** if for every finite $S \subseteq \Gamma$ and every $\varepsilon > 0$, there exists a finite group G acting in a measure-preserving way on (X, μ) such that

$$\mu\left(\{x \in X : S \cdot x \subseteq G \cdot x\}\right) > 1 - \varepsilon.$$

What we are mostly interested in is the characterization of hyperfiniteness for graphings.

Definition 3.3.2. Let $\mathcal{G}(X, \mu)$ be a graphing of bounded degree. $\mathcal{G}$ is called **hyperfinite** if for any $\varepsilon > 0$ there exists $M \in \mathbb{N}$ and a Borel set $Z \subseteq \mathrm{E}(\mathcal{G})$ such that $\mu_{\mathrm{E}}(Z) < \varepsilon$ and the subgraphing $\mathcal{H} = \mathcal{G} \setminus Z$ has all its connected components of size at most M.

Definition 3.3.3. Let F be a finite set. An **F-colored graphing** on a standard probability space (X, μ) is a graphing $\mathcal{G}(X, \mu)$ endowed with a Borel map $\varphi_{\mathcal{G}} \colon \mathrm{E}(\mathcal{G}) \to F$. For $(x, y) \in \mathrm{E}(\mathcal{G})$, we call $\varphi_{\mathcal{G}}(x, y)$ the color of (x, y).

Additionally, for $c \in F$, we write $\mathrm{E}^c(\mathcal{G})$ for the set of edges colored by c, namely $\varphi_{\mathcal{G}}^{-1}(c)$.

We will simply write $\mathcal{G}$ and consider the color implicitly when dealing with colored graphings.

Definition 3.3.4. Let $\mathcal{G}(X,\mu)$ and $\mathcal{G}'(Y,\nu)$ be two F-colored graphings. A **colored graphing factor map** $\pi\colon Y \to X$ is a pmp map such that for almost all $y \in Y$, $\pi \restriction_{[y]_{\mathcal{G}'}}$ is an isomorphism of F- colored graphs.

We say that $\mathcal{G}$ is a colored factor of $\mathcal{G}'$ and we write $\mathcal{G} \underset{c}{\sqsubseteq} \mathcal{G}'$ if there is a colored factor map $\pi\colon Y \to X$.

Let Γ be a group and S be a finite subset of Γ. Let us consider a measure-preserving action $\Gamma \overset{\alpha}{\curvearrowright} (X,\mu)$. We define a $\mathcal{P}(S)$-colored graphing $\mathcal{G}_{\alpha,S}$ on (X,μ) by $(x,y) \in \mathrm{E}(\mathcal{G}_{\alpha,S})$ if and only if there is a $s \in S$ such that $y = sx$ and we color the edges of $\mathcal{G}_{\alpha,S}$ by letting the color of an edge (x,y) be $\{s \in S : y = sx\}$. We call it the **Schreier graph** of the action α relative to S.

Lemma 3.3.5. *Let Γ be a countable group and let $\Gamma \overset{\alpha}{\curvearrowright} (X,\mu)$ be a pmp action. Then α is hyperfinite if and only if for every finite $S \subseteq \Gamma$, $\mathcal{G}_{\alpha,S}$ is hyperfinite.*

Proof. Suppose α is hyperfinite and let $S \subseteq \Gamma$ be finite and $\varepsilon > 0$.

By hyperfiniteness, there exists a finite group G along with a pmp action $G \curvearrowright (X,\mu)$ such that $\mu(\{x \in X : S \cdot x \subseteq G \cdot x\}) > 1 - \varepsilon$. In particular, when restricted to the set $\{x \in X : S \cdot x \subseteq G \cdot x\}$, the Schreier graph $\mathcal{G}_{\alpha,S}$ has finite components of size less than $|G|$.

For the converse, suppose that for any $S \subseteq \Gamma$ finite, the graphing $\mathcal{G}_{\alpha,S}$ is hyperfinite.

Let $S \subseteq \Gamma$ be finite and let $\varepsilon > 0$. Then there exist $Z \subseteq \mathrm{E}(\mathcal{G}_{\alpha,S})$ Borel and $M \in \mathbb{N}$ such that $\mu_{\mathrm{E}}(Z) < \frac{\varepsilon}{2}$ and $\mathcal{G}_{\alpha,S} \setminus Z$ has components of size at most M.

We define a pmp action of $\prod_{n \leqslant M} \mathbb{Z}/n\mathbb{Z}$ on (X,μ) as follows:

Since (X,μ) is a standard probability space, there is a Borel linear ordering $<$ of X. This induces, for $n \leqslant M$, an action of $\mathbb{Z}/n\mathbb{Z}$ on the set of elements of $\mathcal{G}_{\alpha,S} \setminus Z$ whose component is of size n by shifting any component according to the order $<$.

It follows that $\prod_{n \leqslant M} \mathbb{Z}/n\mathbb{Z}$ acts as a product on $X \setminus Z$ in a pmp way, and we extend this action to the whole X by letting $\prod_{n \leqslant M} \mathbb{Z}/n\mathbb{Z}$ act trivially on Z.

One can easily check that for $x \notin \mathrm{V}_{\mathrm{inc}}(Z)$, $S \cdot x$ is exactly the set of neighbors of x in $\mathcal{G}_{\alpha,S} \setminus Z$ and thus it is contained in $[x]_{\mathcal{G}_{\alpha,S} \setminus Z} = \left(\prod_{n \leqslant M} \mathbb{Z}/n\mathbb{Z}\right) \cdot x$. Moreover, $\mu(\mathrm{V}_{\mathrm{inc}}(Z)) \leqslant 2\mu_{\mathrm{E}}(Z) < \varepsilon$ so we conclude that α is hyperfinite. □

3.4 Invariant Random Subgroups

Let $\Gamma \overset{\alpha}{\curvearrowright} (X,\mu)$ be a measure-preserving action of the countable group Γ. With this action we can associate a probability measure on the Polish space of subgroups of Γ as follows. Consider the compact Polish space $\{0,1\}^{\Gamma}$. We let $\mathrm{Sub}(\Gamma)$ be the closed subset of $\{0,1\}^{\Gamma}$ consisting of the subgroups of Γ. Then $\mathrm{Sub}(\Gamma)$ is a compact Polish space.

We have a natural Borel map $\mathrm{stab}_\alpha\colon X \to \mathrm{Sub}(\Gamma)$ defined by $x \mapsto \mathrm{stab}_\alpha(x) = \{g \in \Gamma : g^\alpha(x) = x\}$ and that gives us a probability measure $\mathrm{stab}_{\alpha*}\mu \in \mathfrak{P}(\mathrm{Sub}(\Gamma))$ that we call the Invariant Random Subgroup (IRS in short) of α and denote by θ_α. Moreover, Γ acts on $\mathrm{Sub}(\Gamma)$ by conjugacy and the well known formula $\mathrm{stab}_\alpha(gx) = g\mathrm{stab}_\alpha(x)g^{-1}$ implies that the map stab_α is equivariant. Therefore, θ_α is a Γ-invariant measure on $\mathrm{Sub}(\Gamma)$. We thus define the general notion of an **IRS** on Γ to be a probability measure on $\mathrm{Sub}(\Gamma)$ invariant for the action $\Gamma \curvearrowright \mathrm{Sub}(\Gamma)$ by conjugacy.

G. Elek proved in [Ele12, Thm. 2] that two pmp actions of a countable group Γ with the same IRS are either both hyperfinite or both non-hyperfinite.

Moreover, Abert, Glasner and Virag proved in [AGV14, Prop. 13] that any IRS can be obtained as the IRS associated to a pmp action.

We can thus express hyperfiniteness as a property of the IRS itself:

Definition 3.4.1. Let Γ be a countable group. An IRS θ on Γ is called **amenable** if one of the following two equivalent statements is satisfied:

1. There exists a hyperfinite (or equivalently amenable) pmp action which has IRS θ.
2. Every pmp action which has IRS θ is hyperfinite (or equivalently amenable).

Definition 3.4.2. Let $\Gamma \overset{\alpha}{\curvearrowright} (X, \mu)$ and $\Gamma \overset{\beta}{\curvearrowright} (Y, \nu)$. An **action factor map** $\pi \colon Y \to X$ is a measure-preserving map such that $\forall^* y \in Y\ \forall \gamma \in \Gamma,\ \pi(\gamma^\beta y) = \gamma^\alpha \pi(y)$.

We say that α **is a factor of** β and we write $\alpha \sqsubseteq \beta$ if there exists an action factor map $\pi \colon Y \to X$.

Lemma 3.4.3. *Let α, β be two actions of a countable group Γ on standard probability spaces (X, μ) and (Y, ν). Suppose that there is an action factor map $\pi \colon Y \to X$ for α and β and that $\theta_\alpha = \theta_\beta$. Then $\forall^* y \in Y,\ \mathrm{stab}_\alpha(\pi(y)) = \mathrm{stab}_\beta(y)$.*

Proof. For $\gamma \in \Gamma$, let $\mathrm{N}_\gamma = \{\Lambda \in \mathrm{Sub}(\Gamma) : \gamma \in \Lambda\}$. Then $(\mathrm{N}_\gamma)_{\gamma \in \Gamma}$ is a subbasis of the topology of $\mathrm{Sub}(\Gamma)$ consisting of clopen sets and any measure on $\mathrm{Sub}(\Gamma)$ is determined by the values it takes on finite intersections of elements of this subbasis.

By the definition of action factor map, we have $\forall^* y\ \mathrm{stab}_\beta(y) \subseteq \mathrm{stab}_\alpha(\pi(y))$. Suppose now that $\exists^* y\ \mathrm{stab}_\beta(y) \subsetneq \mathrm{stab}_\alpha(\pi(y))$.

By countability of Γ, $\exists \gamma \in \Gamma\ \exists^* y,\ \gamma \in \mathrm{stab}_\alpha(\pi(y)) \setminus \mathrm{stab}_\beta(y)$, thus

$$\begin{aligned} \theta_\beta(\mathrm{N}_\gamma) &= \mathrm{stab}_{\beta_*}\nu(\mathrm{N}_\gamma) \\ &< (\mathrm{stab}_\alpha \circ \pi)_* \nu(\mathrm{N}_\gamma) \\ &= \mathrm{stab}_{\alpha *}(\pi_* \nu)(\mathrm{N}_\gamma) \\ &= \mathrm{stab}_{\alpha *}\mu(\mathrm{N}_\gamma) \\ &= \theta_\alpha(\mathrm{N}_\gamma), \end{aligned}$$

a contradiction. □

Corollary 3.4.4. *Let α, β be actions of a countable group Γ on standard probability spaces (X, μ) and (Y, ν) such that $\alpha \sqsubseteq \beta$ and $\theta_\alpha = \theta_\beta$, and let $S \subseteq \Gamma$ be finite . Then we have $\mathcal{G}_{\alpha,S} \underset{c}{\sqsubseteq} \mathcal{G}_{\beta,S}$ as $\mathcal{P}(S)$-colored graphings.*

Proof. Applying Lemma 3.4.3 to an action factor map $\pi \colon Y \to X$ gives us that for almost every $y \in Y$, $\pi \restriction_{\Gamma \cdot y}$ is a Γ-equivariant bijection $\Gamma \cdot y \to \Gamma \cdot \pi(y)$ and so it is an isomorphism of Schreier graphs. It follows that π is a graphing factor map. □

3.5 The proof of Theorem I

3.5.1 The preliminary case of factors

We begin with the case where one of the actions is a factor of the other. In fact we prove a stronger version involving the stability of Borel sets.

Definition 3.5.1. Let F_1, F_2 be two finite sets. An (F_1, F_2)**-bicolored graphing** on a standard probability space (X, μ) is a graphing $\mathcal{G}(X, \mu)$ endowed with two Borel maps $\varphi_{\mathcal{G}} \colon \mathrm{E}(\mathcal{G}) \to F_1$ and $\psi_{\mathcal{G}} \colon X \to F_2$. We call $\psi_{\mathcal{G}}(x)$ the vertex-color of x and $\varphi_{\mathcal{G}}(x, y)$ the edge-color of (x, y).

Definition 3.5.2. Let $\mathcal{G}(X, \mu)$ and $\mathcal{G}'(Y, \nu)$ be two (F_1, F_2)-bicolored graphings. A **bicolored graphing factor map** $\pi \colon Y \to X$ is an F_1-colored graphing factor map such that $\psi_{\mathcal{G}} \circ \pi = \psi_{\mathcal{G}'}$.

We say that $\mathcal{G}$ is a bicolored factor of $\mathcal{G}'$ and we write $\mathcal{G} \underset{bic}{\sqsubseteq} \mathcal{G}'$ if there is a bicolored factor map $\pi \colon Y \to X$.

Theorem 3.5.3 (Approximate parametrized conjugacy for factor actions)**.** *Let (X, μ) and (Y, ν) be standard probability spaces and $A_1, \ldots, A_k \subseteq X$, $B_1, \ldots, B_k \subseteq Y$ be Borel subsets. Let Γ be a countable group, θ be an amenable IRS on Γ and $\Gamma \overset{\alpha}{\curvearrowright} (X, \mu)$,$\Gamma \overset{\beta}{\curvearrowright} (Y, \nu)$ be pmp actions of Γ with IRS θ and such that $\alpha \sqsubseteq \beta$ for an action factor map $\pi \colon Y \to X$ such that $\forall i \leqslant k,\ \pi^{-1}(A_i) = B_i$. Then for $\varepsilon > 0$ and $\gamma_1, \ldots, \gamma_n \in \Gamma$, there exists a pmp bijection $\rho \colon X \to Y$ such that $\forall i \leqslant k,\ \rho(A_i) = B_i$ and*

$$\mu(\{x \in X : \forall i \leqslant n,\ \rho \circ \gamma_i^{\alpha}(x) = \gamma_i^{\beta} \circ \rho(x)\}) > 1 - \varepsilon.$$

Proof. We begin the proof with a claim about graphings.

Claim 3.5.3.1. *Let $\mathcal{G}(X, \mu)$ and $\mathcal{G}'(Y, \nu)$ be hyperfinite (F_1, F_2)-bicolored graphings of degree bound at most d such that $\mathcal{G}(X, \mu) \underset{bic}{\sqsubseteq} \mathcal{G}'(Y, \nu)$. Then for any $\varepsilon > 0$ there exists a pmp bijection $\rho \colon X \to Y$ such that $\psi_{\mathcal{G}} = \psi_{\mathcal{G}'} \circ \rho$ and*

$$\mu_{\mathrm{E}} \left(\bigcup_{c \in F_1} \rho^{-1} \left(\mathrm{E}^c(\mathcal{G}') \right) \triangle \mathrm{E}^c(\mathcal{G}) \right) < \varepsilon.$$

Proof. Let π be a bicolored graphing factor map $Y \to X$. First take a Borel set $Z \subseteq \mathrm{E}(\mathcal{G})$ of measure less than $\frac{\varepsilon}{2d}$ and $M \in \mathbb{N}$ such that the graphing $\mathcal{H} = \mathcal{G} \setminus Z$ has components of size at most M. Let $Z' = \pi^{-1}(Z)$ and $\mathcal{H}' = \mathcal{G}' \setminus Z'$. Since π is a graphing factor map, we know that $\mathcal{H}'$ has components of size at most M. Then $\mathcal{H}$ and $\mathcal{H}'$ have a (F_1, F_2)-bicolored graphing structure respectively for the maps $\varphi_{\mathcal{G}} \restriction_{\mathrm{E}(\mathcal{H})}$, $\psi_{\mathcal{G}}$ and $\varphi_{\mathcal{G}'} \restriction_{\mathrm{E}(\mathcal{H}')}$, $\psi_{\mathcal{G}'}$.

Consider the set $\mathbf{G}_M$ of connected (F_1, F_2)-colored graphs of size at most M. We consider the two partitions $X = \bigsqcup_{S \in \mathbf{G}_M} C_S^{\mathcal{H}}$ and $Y = \bigsqcup_{S \in \mathbf{G}_M} C_S^{\mathcal{H}'}$, where $C_S^{\mathcal{H}}$ is defined to be the set of vertices of $\mathcal{H}$ whose component is (F_1, F_2)-colored isomorphic to S. Since π induces (F_1, F_2)- colored graph isomorphisms, we have $C_S^{\mathcal{H}'} = \pi^{-1}(C_S^{\mathcal{H}})$.

In order to define ρ, it suffices to define a measure-preserving bijection $\rho_S \colon C_S^{\mathcal{H}} \to C_S^{\mathcal{H}'}$ preserving bicolored graph structures for each $S \in \mathbf{G}_M$.

Indeed, the union of all these bijections would yield a measure-preserving bijection $\rho \colon X \to Y$ preserving vertex-colors such that $\forall x \in X \setminus \mathrm{V}_{\mathrm{inc}}(Z)$, $\mathcal{B}^{\mathcal{G}}(x, 1) = \mathcal{B}^{\mathcal{H}}(x, 1) \simeq \mathcal{B}^{\mathcal{H}'}(\rho(x), 1) = \mathcal{B}^{\mathcal{G}'}(\rho(x), 1)$, where $\mathcal{B}^{G}(v, n)$ denotes the ball of size n centered at v in the graph G. Hence we would have $\mathrm{V}_{\mathrm{inc}} \left(\bigcup_{c \in F_1} \rho^{-1} \left(\mathrm{E}^c(\mathcal{G}') \right) \triangle \mathrm{E}^c(\mathcal{G}) \right) \subseteq \mathrm{V}_{\mathrm{inc}}(Z)$, and so

$$\mu_{\mathrm{E}} \left(\bigcup_{c \in F_1} \rho^{-1} \left(\mathrm{E}^c(\mathcal{G}') \right) \triangle \mathrm{E}^c(\mathcal{G}) \right) \leqslant d\mu(\mathrm{V}_{\mathrm{inc}}(Z)) \leqslant 2d\mu_{\mathrm{E}}(Z) < \varepsilon.$$

Take $S \in \mathbf{G}_M$ and let us define ρ_S. First we define a partition of $C_S^{\mathcal{H}}$ into Borel transversals $(T_v)_{v\in \mathrm{V}(S)}$ (for $\mathcal{H}$) by induction, such that the elements of T_v occupy the same place in their component for $\mathcal{H}$ as v in S.

Suppose that the $T_{v'}$ are already defined for $v' \in R$ where R is a proper subset of $\mathrm{V}(S)$. Take $v \in \mathrm{V}(S) \setminus R$ incident to R and let $\widetilde{T_v} = \{x \in C_S^{\mathcal{H}} : ([x]_{\mathcal{H}}, x) \simeq_R (S, v)\}$. Here $\simeq_R$ means isomorphic over R, that is there exists an isomorphism $f\colon ([x]_{\mathcal{H}}, x) \to (S, v)$ of colored rooted graphs such that $\forall v' \in R,\ f([x]_{\mathcal{H}} \cap T_{v'}) = \{v'\}$. Now since $\mathcal{H}$ has finite components, chose for T_v any Borel transversal of $\widetilde{T_v}$. Then we let $R' = R \cup \{v\}$ and we iterate the construction.

Again since π is a bicolored graphing factor map, the family $(\pi^{-1}(T_v))_{v\in \mathrm{V}(S)}$ is a partition of $C_S^{\mathcal{H}'}$ into Borel transversals (for $\mathcal{H}'$) such that the elements of $\pi^{-1}(T_v)$ occupy the same place in their component for $\mathcal{H}'$ as v in S. We may now define ρ_S:

- We start by chosing $v_0 \in S$ and taking a measure-preserving bijection $\rho_S^{v_0}\colon T_{v_0} \to \pi^{-1}(T_{v_0})$.
- Then for every $v \in S$, there is a unique way of extending $\rho_S^{v_0}$ to T_v while respecting the graph structure of S. Indeed, take $x \in T_v$, there is a unique $x_0 \in [x]_{\mathcal{H}} \cap T_{v_0}$ and we want to define $\rho_S^{v}(x) \in [\rho_S^{v_0}(x_0)]_{\mathcal{H}'} \cap \pi^{-1}(T_v)$ but again this intersection is a singleton. Define $\rho_S\colon C_S^{\mathcal{H}} \to C_S^{\mathcal{H}'}$ to be this unique extension of $\rho_S^{v_0}$ satisfying the condition above.

  As π is a colored graphing factor map, it is clear that ρ_S is a measure-preserving bijection and that for every $x \in C_S^{\mathcal{H}}$, ρ_S induces an isomorphism of colored graphs between $[x]_{\mathcal{H}}$ and $[\rho_S(x)]_{\mathcal{H}'}$.

■

We now want to apply the Claim to suitable graphings to conclude. Let S be the set $\{\gamma_1, \dots, \gamma_n, \gamma_1^{-1}, \dots, \gamma_n^{-1}\}$ and consider the graphings $\mathcal{G}_{\alpha,S}$ and $\mathcal{G}_{\beta,S}$.

For the spaces of colors, we choose $F_1 = \mathcal{P}(S)$ and $F_2 = \mathcal{P}(\{1, \dots, k\})$. The way we color edges has already been explained; for vertices, simply color a vertex $x \in X$ by $\psi_{\mathcal{G}_{\alpha,S}}(x) = \{i \leqslant k : x \in A_i\}$ and $y \in Y$ by $\psi_{\mathcal{G}_{\beta,S}}(y) = \{i \leqslant k : y \in B_i\}$.

First, $\mathcal{G}_{\alpha,S}$ and $\mathcal{G}_{\beta,S}$ are indeed $(\mathcal{P}(S), \mathcal{P}(\{1, \dots, k\}))$-bicolored graphings, and are hyperfinite since α and β are hyperfinite actions.

The next step is to prove that π considered in the statement of the theorem is a bicolored factor map for the $(\mathcal{P}(S), \mathcal{P}(\{1, \dots, k\}))$-bicolored graphings $\mathcal{G}_{\alpha,S}$ and $\mathcal{G}_{\beta,S}$.

- First, π is indeed a pmp map $Y \to X$.
- Then for $y \in Y$, we have

$$\psi_{\mathcal{G}_{\alpha,S}}(\pi(y)) = \{i \leqslant k : \pi(y) \in A_i\} = \{i \leqslant k : y \in B_i\} = \psi_{\mathcal{G}_{\beta,S}}(y).$$

- Finally, by Corollary 3.4.4, π is furthermore a colored graphing factor map between the $\mathcal{P}(S)$-colored graphings $\mathcal{G}_{\alpha,S}$ and $\mathcal{G}_{\beta,S}$.

Applying the Claim gives us a pmp bijection $\rho\colon X \to Y$ such that $\psi_{\mathcal{G}_{\alpha,S}} = \psi_{\mathcal{G}_{\beta,S}} \circ \rho$ and

$$\mu_{\mathrm{E}}\left(\bigcup_{c\in\mathcal{P}(S)} \mathrm{E}^c(\mathcal{G}_{\alpha,S}) \bigtriangleup \rho^{-1}\left(\mathrm{E}^c(\mathcal{G}_{\beta,S})\right) \right) < \frac{\varepsilon}{2}.$$

But then for $1 \leqslant i \leqslant k$, $\rho(A_i) = B_i$, and by definitions of $\mathcal{G}_{\alpha,S}$ and $\mathcal{G}_{\beta,S}$ we get

$$\{x \in X : \exists \gamma \in S,\ \rho \circ \gamma^\alpha(x) \neq \gamma^\beta \circ \rho(x)\} \subseteq \mathrm{V}_{\mathrm{inc}} \left(\bigcup_{c \in \mathcal{P}(S)} \mathrm{E}^c(\mathcal{G}_{\alpha,S}) \triangle \rho^{-1}\left(\mathrm{E}^c(\mathcal{G}_{\beta,S})\right) \right),$$

so its measure is less than $2 \cdot \frac{\varepsilon}{2} = \varepsilon$. □

3.5.2 Amalgamation of measure-preserving actions

To conclude the proof of Theorem I, we will use the transitivity of the approximate conjugacy relation and show that for any two pmp actions $\Gamma \overset{\alpha}{\curvearrowright} (X,\mu)$ and $\Gamma \overset{\beta}{\curvearrowright} (Y,\nu)$ of Γ such that $\theta_\alpha = \theta_\beta$, there is a third pmp action $\Gamma \overset{\zeta}{\curvearrowright} (Z,\eta)$ of IRS θ such that both α and β are factors of ζ.

We recall the definition of the relative independent joining following the presentation in [Gla03].

Proposition 3.5.4 (Disintegration theorem,[Gla03, A.7])**.** *Let X, Y be standard probability spaces, $\mu \in \mathfrak{P}(Y)$ and $\pi\colon Y \to X$ be a measurable map. We let $\nu = \pi_*\mu$. Then there is a ν-a.e. uniquely determined family of probability measures $(\mu_x)_{x\in X} \in \mathfrak{P}(Y)^X$ such that:*

1. *For each Borel $B \subseteq Y$, the map $x \mapsto \mu_x(B)$ is measurable.*
2. *For ν-a.e. $x \in X$, μ_x is concentrated on the fiber $\pi^{-1}(x)$.*
3. *For every Borel map $f\colon Y \to [0,\infty]$, $\int_Y f(y)\ d\mu(y) = \int_X \int_Y f(y)\ d\mu_x(y)\ d\nu(x)$.*

We then write $\mu = \int_X \mu_x\ d\nu$.

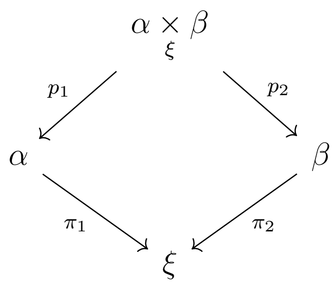
Definition 3.5.5 ([Gla03, Section 6.1])**.** Let $\Gamma \overset{\alpha}{\curvearrowright} (X,\mu)$ and $\Gamma \overset{\beta}{\curvearrowright} (X',\mu')$ be pmp actions on standard probability spaces, and let $\Gamma \overset{\xi}{\curvearrowright} (Y,\nu)$ be an action on a standard probability space common factor of α and β for respective action factor maps $\pi\colon X \to Y$ and $\pi'\colon X' \to Y$.

We can disintegrate μ and μ' with respect to ν using the Borel maps π and π' to get $\mu = \int_Y \mu_y\ d\nu$ and $\mu' = \int_Y \mu'_y\ d\nu$.

Consider $Z \coloneqq X \times X'$ and $\eta \in \mathfrak{P}(Z)$ defined by $\eta = \int_Y \mu_y \times \mu'_y\ d\nu$.

The pmp action $\Gamma \overset{\alpha\times\beta}{\curvearrowright} (Z,\eta)$ is called the **independent joining of α and β over ξ** and is denoted by $\alpha \underset{\xi}{\times} \beta$.

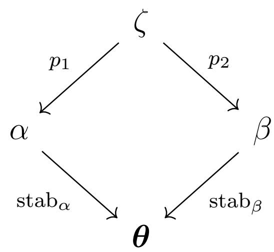
The action $\alpha \underset{\xi}{\times} \beta$ is indeed a **joining** of α and β over ξ, meaning that both α and β are factors of their independent joining over ξ, respectively for the projections on the first and second coordinates p_1 and p_2, and moreover the following diagram commutes, up to a null set:



Let θ be an IRS on Γ, we write $\boldsymbol{\theta}$ for the measure-preserving conjugation action $\Gamma \overset{\boldsymbol{\theta}}{\curvearrowright} (\mathrm{Sub}(\Gamma), \theta)$. For every pmp action $\Gamma \overset{\alpha}{\curvearrowright} (X, \mu)$, the map $\mathrm{stab}_\alpha \colon (X, \mu) \to (\mathrm{Sub}(\Gamma), \theta)$ is an action factor map.

Lemma 3.5.6. *Let Γ be a countable group and θ be an IRS on Γ. Let $\Gamma \overset{\alpha}{\curvearrowright} (X, \mu)$, $\Gamma \overset{\beta}{\curvearrowright} (Y, \nu)$ be pmp actions of IRS θ. Then $\alpha \underset{\boldsymbol{\theta}}{\times} \beta$ has IRS θ.*

Proof. Let ζ denote $\alpha \underset{\boldsymbol{\theta}}{\times} \beta$. We know that the following diagram commutes.



Therefore, for $\gamma \in \Gamma$, we have $\forall^*(x, y),\ \gamma x = x \Leftrightarrow \gamma y = y \Leftrightarrow \gamma(x, y) = (x, y)$. It follows that $\forall^*(x, y),\ \mathrm{stab}_\zeta(x, y) = \mathrm{stab}_\alpha(x)$ or in other words, $\mathrm{stab}_\zeta = \mathrm{stab}_\alpha \circ p_1$. We conclude that

$$\theta_\zeta = \mathrm{stab}_{\zeta *}\eta = \mathrm{stab}_{\alpha *}(p_{1*}\eta) = \mathrm{stab}_{\alpha *}\mu = \theta_\alpha = \theta.$$

□

Theorem I states that if α and β are two pmp hyperfinite actions of a group Γ on a standard probability space such that $\theta_\alpha = \theta_\beta$, then α and β are approximately conjugate. We can now prove this theorem:

Proof. Let $\Gamma \overset{\alpha}{\curvearrowright} (X, \mu)$ and $\Gamma \overset{\beta}{\curvearrowright} (Y, \nu)$ be two hyperfinite actions of Γ having IRS θ and consider the joining $\Gamma \overset{\zeta}{\curvearrowright} (Z, \eta)$ from Lemma 3.5.6.

Applying twice Theorem 3.5.3 with no Borel parameters we get two pmp bijections $\rho \colon X \to Z$ and $\rho' \colon Y \to Z$ such that:

$$\mu(\{x \in X : \forall i \leqslant n,\ \rho \circ \gamma_i^\alpha(x) = \gamma_i^\zeta \circ \rho(x)\}) > 1 - \frac{\varepsilon}{2}$$

and

$$\nu(\{y \in Y : \forall i \leqslant n,\ \rho' \circ \gamma_i^\beta(y) = \gamma_i^\zeta \circ \rho'(y)\}) > 1 - \frac{\varepsilon}{2}.$$

Thus, $\rho'^{-1} \circ \rho \colon X \to Y$ witnesses the ε-approximate conjugacy of α and β. □

Chapter 4

Model theory of hyperfinite actions

4.1 Measure algebras

The reader unfamiliar with continuous model theory is referred to [BBHU08]. We will use the same notations as theirs.

Definition 4.1.1. A **measure algebra** is a boolean algebra $(\mathcal{A}, \vee, \wedge, \neg, 0, 1, \subseteq, \triangle)$ endowed with a function $\mu\colon \mathcal{A} \to [0,1]$ satisfying the following:

1. $\mu(1) = 1$.
2. $\forall a, b \in \mathcal{A},\ \mu(a \wedge b) = 0 \Rightarrow \mu(a \vee b) = \mu(a) + \mu(b)$.
3. The function $d_\mu(a,b) \coloneqq \mu(a \triangle b)$ is a complete metric on $\mathcal{A}$.

Proposition 4.1.2 ([Fre02, 323G c)]). *Any measure algebra $\mathcal{A}$ is Dedekind complete, meaning that any subset $S \subseteq \mathcal{A}$ admits a supremum and an infimum, that we respectively denote by $\bigvee S$ and $\bigwedge S$.*

Definition 4.1.3. An element $a \in \mathcal{A}$ is an **atom** if $\forall b \in \mathcal{A},\ b \subseteq a \Rightarrow b \in \{0, a\}$. A measure algebra is **atomless** if it has no atoms.

Proposition 4.1.4 ([Fre02, 331C]). *If a measure algebra $\mathcal{A}$ is atomless, then*

$$\forall a \in \mathcal{A}\ \forall r \in [0, \mu(a)]\ \exists b \subseteq a,\ \mu(b) = r.$$

We introduce the classical example of a measure algebra: For (X, μ) a probability space, we let $\mathrm{MAlg}(X, \mu)$ be the quotient of the boolean algebra of measurable subsets of X by the σ-ideal of null sets. For $A \subseteq X$ Borel we denote its class in $\mathrm{MAlg}(X, \mu)$ by $[A]_\mu$. The measure μ descends to the quotient $\mathrm{MAlg}(X, \mu)$ and then $\mathrm{MAlg}(X, \mu)$ endowed with μ is a measure algebra. When (X, μ) is a standard probability space, $\mathrm{MAlg}(X, \mu)$ is atomless and separable for the topology induced by d_μ.

Conversely, we have:

Proposition 4.1.5 ([Fre02, 331L]). *Let $\mathcal{A}$ be a separable atomless measure algebra. Then there exists a standard probability space (X, μ) such that $\mathcal{A}$ is isomorphic to* $\mathrm{MAlg}(X, \mu)$.

Let $f\colon (X,\mu) \to (Y,\nu)$ be a measure-preserving map. Then the map $\widetilde{f}\colon \mathrm{MAlg}(Y,\nu) \to \mathrm{MAlg}(X,\nu)$ sending $[A]_\nu$ to $[f^{-1}(A)]_\mu$ is a measure algebra morphism. Moreover, if f is a bimeasurable bijection, then $\widetilde{f}$ is an isomorphism.

However, in general, given a morphism $\varphi\colon \mathrm{MAlg}(X,\nu) \to \mathrm{MAlg}(Y,\mu)$ there is no way to get a lifting of φ, that is a point to point measure-preserving map $\boldsymbol{\varphi}\colon Y \to X$ such that $\widetilde{\boldsymbol{\varphi}} = \varphi$. However, in the case of standard probability spaces, such a construction exists:

Proposition 4.1.6 ([Fre13, 425D])**.** *Let (X,μ) and (Y,ν) be standard probability spaces. For every morphism of measure algebras $\varphi\colon \mathrm{MAlg}(X,\mu) \to \mathrm{MAlg}(Y,\nu)$ there is a lifting $\boldsymbol{\varphi}\colon Y \to X$ of φ. Moreover, for Γ a countable group acting by automorphisms on $\mathrm{MAlg}(X,\mu)$ by an action α, there is a lifting of α, that is an action $\Gamma \overset{\boldsymbol{\alpha}}{\curvearrowright} X$ acting by measure-preserving transformations such that $\forall \gamma \in \Gamma$, $\widetilde{\gamma^{\boldsymbol{\alpha}}} = (\gamma^{-1})^{\alpha}$.*

4.2 Model theory of atomless measure algebras

We axiomatize the theory AMA of atomless measure algebras in the signature $\mathcal{L} = \{\vee, \wedge, \neg, 0, 1\}$ ($\triangle$ is defined as usual) as in [BBHU08, Section 16].

Proposition 4.2.1 ([BBHU08, 16.2])**.** *The theory* AMA *is separably categorical and therefore complete.*

We also have:

Proposition 4.2.2 ([BBHU08, 16.6 and 16.7])**.** *The theory* AMA *admits quantifier elimination. Moreover, the definable closure $\mathrm{dcl}^{\mathcal{M}}(C)$ of a subset C in a model $\mathcal{M}$ of* AMA *is the substructure $\langle C \rangle$ of $\mathcal{M}$ generated by C.*

We will now give a characterization of the types in the theory AMA. For that we need a little bit of terminology.

To any measure algebra $\mathcal{A}$ we can associate a natural Hilbert space $\mathrm{L}^2(\mathcal{A})$ called the L^2 **space of $\mathcal{A}$**. This construction is consistent in the sense that if $\mathcal{A}$ is the measure algebra of a probability space (X,μ), then there is a natural linear isometry between $\mathrm{L}^2(\mathcal{A})$ and $\mathrm{L}^2(X,\mu)$.

Definition 4.2.3. Let $\mathcal{A}$ be a measure algebra and $\mathcal{B}$ a measure subalgebra of $\mathcal{A}$. Then the space $\mathrm{L}^2(\mathcal{B})$ is a closed vector subspace of the Hilbert space $\mathrm{L}^2(\mathcal{A})$, we denote by $\mathbb{P}_{\mathcal{B}}$ the orthogonal projection on $\mathrm{L}^2(\mathcal{B})$ and we call it the **conditional expectation** with respect to $\mathcal{B}$. Particularly, for $a \in \mathcal{A}$, a can be seen as the element 1_a of $\mathrm{L}^2(\mathcal{A})$ and we call $\mathbb{P}_{\mathcal{B}}(1_a)$ the **conditional probability** of a with respect to $\mathcal{B}$. For simplicity, we will denote it by $\mathbb{P}_{\mathcal{B}}(a)$.

By definition, the conditional probability of a with respect to $\mathcal{B}$ is the only $\mathcal{B}$-measurable function such that for any $\mathcal{B}$-measurable function f, we have $\int \mathbb{P}_{\mathcal{B}}(a) f = \int 1_a f$.

Proposition 4.2.4 ([BBHU08, 16.5])**.** *Let $\mathcal{M} \models$ AMA, $\bar{a}, \bar{b}$ be n-tuples of elements of $\mathcal{M}$ and $C \subseteq \mathcal{M}$. Then $\mathrm{tp}(\bar{a}/C) = \mathrm{tp}(\bar{b}/C)$ if and only if for every map $\sigma\colon \{1,\dots,n\} \to \{-1,1\}$ we have*

$$\mathbb{P}_{\langle C \rangle}\left(\bigwedge_{1 \leqslant i \leqslant n} a_i^{\sigma(i)}\right) = \mathbb{P}_{\langle C \rangle}\left(\bigwedge_{1 \leqslant i \leqslant n} b_i^{\sigma(i)}\right),$$

where a^1 denotes a and a^{-1} denotes its complement $\neg\, a$ in $\mathcal{M}$.

4.3 The theory $\mathfrak{A}_\theta$

Until now, we studied actions of any countable group. Since any action of a countable group can be represented as an F_∞-action, for the sake of simplicity, we now restrict to F_∞-actions, where F_∞ denotes the countably generated free group.

We now expand the signature $\mathcal{L}$ with a countable set of function symbols indexed by F_∞, that we identify with F_∞ itself. We call this new signature $\mathcal{L}_\infty$. We begin by considering the theory $\mathfrak{A}_{F_\infty}$ consisting of the following axioms:

- The axioms of AMA.
- For $\gamma \in F_\infty$, the axioms expressing that γ is a measure algebra isomorphism:
  - $\sup_{a,b}\ d(\gamma(a \vee b), \gamma a \vee \gamma b) = 0$
  - $\sup_{a,b}\ d(\gamma(a \wedge b), \gamma a \wedge \gamma b) = 0$
  - $\sup_a\ |\mu(\gamma a) - \mu(a)| = 0$
  - $\sup_a\ \inf_b\ d(a, \gamma b) = 0$
- The axioms expressing that F_∞ acts on the measure algebra:
  - $\sup_a\ d(1_{F_\infty} a, a) = 0$
  - For $\gamma_1, \gamma_2 \in F_\infty$, the axiom $\sup_a\ d(\gamma_1(\gamma_2 a), (\gamma_1\gamma_2)a) = 0$

By Propositions 4.1.5 and 4.1.6 any separable model of $\mathfrak{A}_{F_\infty}$ can be seen as the action on a measure algebra associated with a measure-preserving action $F_\infty \curvearrowright (X, \mu)$ on a standard probability space. If α is a pmp action on a probability space, we write $\mathcal{M}_\alpha$ for the model of $\mathfrak{A}_{F_\infty}$ induced by α. Without loss of generality, from now on, separable models we consider are always of the form $\mathcal{M}_\alpha$ for α a pmp action on a standard probability space.

Definition 4.3.1. For f any measure-preserving transformation $(X, \mu) \to (X, \mu)$, where (X, μ) is a probability space, we call the set $\{x \in X : fx \neq x\}$ the **support** of f and we denote it by $\operatorname{Supp} f$.

Definition 4.3.2. Let $(\mathcal{A}, \mu)$ be a measure algebra, the **support** of an automorphism φ of $\mathcal{A}$ is defined by $\operatorname{supp} \varphi = \bigwedge\{a \in \mathcal{A} : \forall b \subseteq \neg a,\ \varphi b = b\}$.

It is classic that if f is a measure-preserving transformation of a standard probability space (X, μ), then $[\operatorname{Supp} f]_\mu = \operatorname{supp} \widetilde{f}$.

Our goal is now to give a first order description for the support of an automorphism of the measure algebra:

Lemma 4.3.3. *1. Let φ be an automorphism of a measure algebra $\mathcal{A}$ such that* $\operatorname{supp} \varphi \neq 0$. *Then there exists $b \neq 0 \in \mathcal{A}$ such that $\varphi b \wedge b = 0$.*

2. Let $\mathcal{A}$ be a measure algebra. Let φ be an automorphism of $\mathcal{A}$.

Then there is $a_0 \in \mathcal{A}$ such that $\operatorname{supp} \varphi = \varphi^{-1} a_0 \vee a_0 \vee \varphi a_0$ *and* $a_0 \wedge \varphi a_0 = 0$. *Furthermore, we have* $\operatorname{supp} \varphi = \bigvee\{\varphi^{-1} a \vee a \vee \varphi a : a \in \mathcal{A}, a \wedge \varphi a = 0\}$.

Proof. 1. Since $\operatorname{supp} \varphi \neq 0$, there is $a \in \mathcal{A}$ such that $\varphi a \neq a$. Then $\varphi^{-1} a \wedge \neg a \neq 0$, otherwise we would have $a \leqslant \varphi a$ and by measure preservation of φ, it would follow that $a = \varphi a$. Letting $b \coloneqq \varphi^{-1} a \wedge \neg a$, we get the conclusion.

2. First $\mathcal{A}$ is a measure algebra and therefore is complete as a boolean algebra so it has a maximal element a_0 disjoint from its image by φ.

   Consider $b = \varphi^2 a_0 \setminus (\varphi^{-1}a_0 \vee a_0 \vee \varphi a_0)$. We have

$$\begin{aligned}(a_0 \vee b) \wedge \varphi(a_0 \vee b) &= (a_0 \wedge \varphi a_0) \vee (a_0 \wedge \varphi b) \vee (b \wedge \varphi a_0) \vee (b \wedge \varphi b) \\ &\subseteq 0 \vee (a_0 \setminus a_0) \vee (\varphi a_0 \setminus \varphi a_0 \vee (\varphi^2 a_0 \setminus \varphi^2 a_0) \\ &= 0.\end{aligned}$$

   Thus $a_0 \vee b$ is disjoint from its image. By maximality of a_0, we then have $b \subseteq a_0$, but by definition $b \wedge a_0 = 0$, so $b = 0$, or in other words, $\varphi^2 a_0 \subseteq \varphi^{-1}a_0 \vee a_0 \vee \varphi a_0$.

   It follows that $\varphi(\varphi^{-1}a_0 \vee a_0 \vee \varphi a_0) \subseteq \varphi^{-1}a_0 \vee a_0 \vee \varphi a_0$ and since φ preserves the measure, the set $\varphi^{-1}a_0 \vee a_0 \vee \varphi a_0$ is invariant by φ.

   Furthermore, a_0 is disjoint from its image by φ, and so $\varphi^{-1}a_0$ and φa_0 are also disjoint from their respective image, so we have

$$\varphi^{-1}a_0 \vee a_0 \vee \varphi a_0 \subseteq \text{supp } \varphi.$$

   Conversely, let $c = \text{supp } \varphi \setminus (\varphi^{-1}a_0 \vee a_0 \vee \varphi a_0)$ and suppose that $c \neq 0$. Since c is invariant by φ, we can consider the automorphism $\varphi_{\upharpoonright c}$ of the measure algebra lying under c. Applying the first point of this lemma to this automorphism, we get a non trivial $b \subseteq c$ disjoint from its image by φ.

   But then, $a_0 \vee b$ contradicts the maximality of a_0. We conclude that

$$\varphi^{-1}a_0 \vee a_0 \vee \varphi a_0 = \text{supp } \varphi.$$

   Finally, as we already noticed, any set of the form $\varphi^{-1}a \vee a \vee \varphi a$ for $a \wedge \varphi a = 0$ is a subset of supp φ, so we have

$$\text{supp } \varphi = \bigvee \{\varphi^{-1}a \vee a \vee \varphi a : a \in \mathcal{A}, a \wedge \varphi a = 0\}.$$

□

Now we can prove that the IRS of a pmp action on a measure algebra is determined by the theory of this action seen as a model of $\mathfrak{A}_{F_\infty}$.

Definition 4.3.4. For $\gamma \in F_\infty$ we let $t_\gamma(a)$ denote the term $\gamma^{-1}(a \setminus \gamma a) \vee (a \setminus \gamma a) \vee \gamma(a \setminus \gamma a)$.

It follows from Lemma 4.3.3 that for $\mathcal{M} \models \mathfrak{A}_{F_\infty}$, supp $\gamma = \bigvee \{t_\gamma(a) : a \in \mathcal{M}\}$.

Lemma 4.3.5. *Let $\gamma \in F_\infty$. Then the support of γ is definable without parameters in the theory $\mathfrak{A}_{F_\infty}$.*

Proof. We need to prove that the distance to supp γ is definable. By definition of the distance, we have $\forall a \in \mathcal{M},\ d(a, \text{supp } \gamma) = \mu(a \setminus \text{supp } \gamma) + \mu(\text{supp } \gamma \setminus a)$.

On the one hand, $\mu(a \setminus \text{supp } \gamma) = \inf_b\ \mu(a \setminus t_\gamma(b))$ so the first part is definable.

On the other hand, $\mu(\text{supp } \gamma \setminus a) = \sup_b\ \mu(t_\gamma(b) \setminus a)$ and therefore the second part is definable as well. □

Theorem 4.3.6. *Let $\mathcal{M}_\alpha, \mathcal{M}_\beta$ be two elementarily equivalent models of $\mathfrak{A}_{F_\infty}$. Then $\theta_\alpha = \theta_\beta$.*

Proof. As θ_α and θ_β are measures on $\mathrm{Sub}(F_\infty)$, they are determined by their values on the sets $\mathrm{N}_{F,G} = \{\Lambda \leqslant F_\infty : F \cap \Lambda = \varnothing, G \subseteq \Lambda\}$ where F and G are finite.

Note that $\theta_\alpha(\mathrm{N}_{F,\varnothing}) = \mu(\bigcap_{\gamma \in F} \mathrm{Supp}\ \gamma^\alpha)$ and $\theta_\beta(\mathrm{N}_{F,\varnothing}) = \mu(\bigcap_{\gamma \in F} \mathrm{Supp}\ \gamma^\beta)$, but by Lemma 4.3.3 these supports are the same as those defined in the measure algebra. Furthermore, by Lemma 4.3.5, for each $\gamma \in F_\infty$, supp γ is definable over $\varnothing$ in the theory F_∞, and since the definable closure is a substructure, then $\bigwedge_{\gamma \in F} \mathrm{supp}\ \gamma$ must be definable over $\varnothing$ as well. Thus by elementary equivalence, for every finite $F \subseteq F_\infty$, we have $\theta_\alpha(\mathrm{N}_{F,\varnothing}) = \theta_\beta(\mathrm{N}_{F,\varnothing})$.

Now for F, G finite subsets of F_∞, write $\mathrm{N}_{F,G} = \mathrm{N}_{F,\varnothing} \setminus \bigcup_{\gamma \in G} \mathrm{N}_{F \cup \{\gamma\},\varnothing}$. By the inclusion-exclusion principle, we then get

$$
\begin{aligned}
\theta_\alpha(\mathrm{N}_{F,G}) &= \theta_\alpha(\mathrm{N}_{F,\varnothing}) + \sum_{i=1}^{|G|} (-1)^i \sum_{\{J \subseteq G \,:\, |J|=i\}} \theta_\alpha(\mathrm{N}_{F \cup J,\varnothing}) \\
&= \theta_\beta(\mathrm{N}_{F,\varnothing}) + \sum_{i=1}^{|G|} (-1)^i \sum_{\{J \subseteq G \,:\, |J|=i\}} \theta_\beta(\mathrm{N}_{F \cup J,\varnothing}) \\
&= \theta_\beta(\mathrm{N}_{F,G}).
\end{aligned}
$$

□

A version of this proof with existential quantifiers shows that actually, two elementary equivalent models of $\mathfrak{A}_{F_\infty}$ are weakly equivalent (see [BK20] for a general overview of weak equivalence).

For θ an IRS, let $\mathfrak{A}_\theta$ be the $\mathcal{L}_\infty$-theory consisting of:

- The axioms of $\mathfrak{A}_{F_\infty}$.
- For $F \subseteq F_\infty$ finite, the axiom $\sup_{\{a_\gamma : \gamma \in F\}} \mu(\bigwedge_{\gamma \in F} t_\gamma(a_\gamma)) = \theta(\mathrm{N}_{F,\varnothing})$.

Then the models of $\mathfrak{A}_\theta$ are exactly the measure-preserving actions of F_∞ which have IRS θ.

4.4 Completeness and Model Completeness

Definition 4.4.1. Let (X, μ) be a standard probability space and Γ be a countable group.

First, let $\mathrm{Aut}(X, \mu)$ be the space of automorphisms of $\mathrm{MAlg}(X, \mu)$. We equip it with a complete metric d_u called the **uniform metric** and defined by the formula $d_u(\varphi, \psi) := \sup_{a \in \mathrm{MAlg}(X,\mu)} d_\mu(\varphi a, \psi a)$. We call the topology induced the **uniform topology**.

Then we define the space $A(\Gamma, X, \mu)$ of pmp actions of Γ on (X, μ) naturally as a subspace of $\mathrm{Aut}(X, \mu)^\Gamma$. The uniform topology on $\mathrm{Aut}(X, \mu)$ gives rise to a product topology on $\mathrm{Aut}(X, \mu)^\Gamma$ which is completely metrizable and for which $A(\Gamma, X, \mu)$ is closed. Again, we call this topology the **uniform topology** on $A(\Gamma, X, \mu)$.

From now on, fix a complete metric d_u compatible with the uniform topology on $A(F_\infty, X, \mu)$.

Theorem 4.4.2. *Let $\varphi(\bar{x}, \bar{y})$ be an $\mathcal{L}_\infty$-formula, where $|\bar{x}| = n$, $|\bar{y}| = m$, let (X, μ) be a standard probability space and let $\bar{p} \in \mathrm{MAlg}(X, \mu)^m$.*

Then the map $\begin{array}{ccc}(A(F_\infty, X, \mu), d_u) & \longrightarrow & (l^\infty(\mathrm{MAlg}(X,\mu)^n), \| \ \|_\infty) \\ \alpha & \longmapsto & \left(\varphi^{\mathcal{M}_\alpha}(\bar{a},\bar{p})\right)_{\bar{a}\in \mathrm{MAlg}(X,\mu)^n}\end{array}$ *is uniformly continuous.*

Proof. We prove this result by induction on formulas. For now assume that the theorem holds for atomic formulas. First remark that if the theorem holds for certain formulas, then it holds for any combination of these formulas constructed with the help of connectives, by using their uniform continuity. Then it suffices to treat the case of quantifiers to conclude. But it is immediate, since we use the norm $\| \ \|_\infty$.

Let us now prove the theorem for atomic formulas. If $\varphi(\bar{x},\bar{y})$ is an atomic formula, then it is equivalent to a formula of the form $\varphi(\bar{x},\bar{y}) \coloneqq \mu(t(\gamma_1\bar{x},\ldots,\gamma_l\bar{x},\gamma_1\bar{y},\ldots,\gamma_l\bar{y})$ for an $\mathcal{L}$-term t and some $\gamma_1,\ldots,\gamma_l \in F_\infty$. Let $\varepsilon > 0$.

By definition of the terms, they are uniformly continuous and so there is $\delta > 0$ such that for $\bar{z}$ and $\bar{z'} \in \mathrm{MAlg}(X,\mu)^{(n+m)l}$, if $d_\mu(\bar{z},\bar{z'}) < \delta$ then $d_\mu(t(\bar{z}),t(\bar{z'})) < \varepsilon$.

Now if $\alpha,\beta \in A(F_\infty,X,\mu)$ are sufficiently d_u-close, then for every $a \in \mathrm{MAlg}(X,\mu)$ and $1 \leqslant i \leqslant l$, $d_\mu(\gamma_i^\alpha a, \gamma_i^\beta a) < \delta$. It follows that for all $\bar{a} \in \mathrm{MAlg}(X,\mu)^n$,

$$\left|\varphi^{\mathcal{M}_\alpha}(\bar{a},\bar{p}) - \varphi^{\mathcal{M}_\beta}(\bar{a},\bar{p})\right| \leqslant d_\mu\left(t(\gamma_1^\alpha\bar{a},\ldots,\gamma_l^\alpha\bar{a},\gamma_1^\alpha\bar{p},\ldots,\gamma_l^\alpha\bar{p}), t(\gamma_1^\beta\bar{a},\ldots,\gamma_l^\beta\bar{a},\gamma_1^\beta\bar{p},\ldots,\gamma_l^\beta\bar{p})\right) < \varepsilon,$$

which finishes the proof. □

Theorem 4.4.3. *Let θ be an amenable IRS on F_∞. Then the theory $\mathfrak{A}_\theta$ is model complete.*

Proof. It suffices to show that any inclusion of two separable models is elementary. Indeed, suppose this result and take any $\mathcal{M} \subseteq \mathcal{N} \models \mathfrak{A}_\theta$, $\varphi(\bar{x})$ a $\mathcal{L}_\infty$-formula and $\bar{p} \in \mathcal{M}$ finite. By the Löwenheim-Skolem theorem, find a separable $\mathcal{M}' \preceq \mathcal{M}$ containing $\bar{p}$. Again by the Löwenheim-Skolem theorem, find a separable $\mathcal{N}' \preceq \mathcal{N}$ containing the separable structure $\mathcal{M}'$. Using the hypothesis, $\mathcal{M}' \preceq \mathcal{N}'$ so we finally get

$$\varphi(\bar{p})^{\mathcal{M}} = \varphi(\bar{p})^{\mathcal{M}'} = \varphi(\bar{p})^{\mathcal{N}'} = \varphi(\bar{p})^{\mathcal{N}}.$$

Let $\mathcal{M} \subseteq \mathcal{N}$ be two separable models of $\mathfrak{A}_\theta$. Consider a $\mathcal{L}_\infty$-formula $\varphi(\bar{x})$ with k variables and $\bar{p} \in \mathrm{MAlg}(X,\mu)^k$.

A classical argument derived from Proposition 4.1.6 allows us to chose two pmp actions $F_\infty \overset{\alpha}{\curvearrowright} (X,\mu)$ and $F_\infty \overset{\beta}{\curvearrowright} (Y,\nu)$ on standard probability spaces along with a pmp map $\pi\colon Y \to X$, such that $\mathcal{M} \simeq \mathcal{M}_\alpha$, $\mathcal{N} \simeq \mathcal{M}_\beta$, and π is a lifting of the inclusion $\mathrm{MAlg}(X,\mu) \hookrightarrow \mathrm{MAlg}(Y,\nu)$, which is equivariant respectively to the actions α and β. For $1 \leqslant i \leqslant k$, let $A_i \subseteq X$ be a Borel representative of p_i and let $B_i = \pi^{-1}(A_i)$, which is also a Borel representative of p_i, in Y.

Then by Theorem 3.5.3, α is in the uniform closure of the set

$$\mathcal{C}(\beta) \coloneqq \{\rho^{-1}\beta\rho : \rho \text{ is a pmp bijection } X \to Y \text{ such that } \forall i \leqslant k, \rho^{-1}(A_i) = B_i\}.$$

But then Theorem 4.4.2 implies that $\varphi^{\mathcal{M}_\alpha}(\bar{p}) \in \overline{\{\varphi^{\mathcal{M}_{\beta'}}(\bar{p}) : \beta' \in \mathcal{C}(\beta)\}}$. Furthermore, for any $\beta' \in \mathcal{C}(\beta)$, we have $(\beta', \bar{A}) \simeq (\beta, \bar{B})$, so that $(\mathcal{M}_{\beta'},\bar{p}) \equiv (\mathcal{M}_\beta,\bar{p})$ and consequently $\varphi^{\mathcal{M}_{\beta'}}(\bar{p}) = \varphi^{\mathcal{M}_\beta}(\bar{p})$. This establishes that $\varphi^{\mathcal{M}_\alpha}(\bar{p}) = \varphi^{\mathcal{M}_\beta}(\bar{p})$.

Hence $\mathcal{M}_\alpha \preceq \mathcal{M}_\beta$ and therefore $\mathfrak{A}_\theta$ is model complete. □

Now for completeness we combine model completeness with the argument of amalgamation already seen in Section 3.5.2.

Theorem 4.4.4. *Let θ be an amenable IRS on F_∞. Then the theory $\mathfrak{A}_\theta$ is complete.*

Proof. As usual, it is sufficient to prove that two separable models of $\mathfrak{A}_\theta$ are elementarily equivalent.

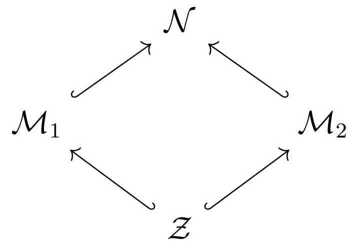
Let $\mathcal{M}_\alpha, \mathcal{M}_\beta \models \mathfrak{A}_\theta$ be two separable models and consider the action $\zeta := \alpha \underset{\boldsymbol{\theta}}{\times} \beta$. By Lemma 3.5.6, we have $\mathcal{M}_\zeta \models \mathfrak{A}_\theta$ and moreover, both $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$ are substructures of $\mathcal{M}_\zeta$.

Now since $\mathfrak{A}_\theta$ is model complete, we have $\mathcal{M}_\alpha \preceq \mathcal{M}_\zeta$ and $\mathcal{M}_\beta \preceq \mathcal{M}_\zeta$, so $\mathcal{M}_\alpha \equiv \mathcal{M}_\zeta \equiv \mathcal{M}_\beta$. □

4.5 Elimination of quantifiers

Proposition 4.5.1 ([BBHU08, Prop. 13.6]). *Let T be a countable theory. Then T admits quantifier elimination if and only if for any $\mathcal{M}, \mathcal{N} \models T$, any substructure $\mathcal{Z} \subseteq \mathcal{M}$ and any embedding $f\colon \mathcal{Z} \hookrightarrow \mathcal{N}$, there is an elementary extension $\mathcal{N}'$ of $\mathcal{N}$ and an embedding $\tilde{f}\colon \mathcal{M} \hookrightarrow \mathcal{N}'$ extending f.*

Definition 4.5.2. We say that a theory T admits **amalgamation** if for any $\mathcal{M}_1, \mathcal{M}_2 \models T$ and any common substructure $\mathcal{Z}$, there is a joining of $\mathcal{M}_1$ and $\mathcal{M}_2$ over $\mathcal{Z}$, that is a structure $\mathcal{N} \models T$ and embeddings $\mathcal{M}_i \hookrightarrow \mathcal{N}$ ($i = 1, 2$) such that the following diagram commutes:



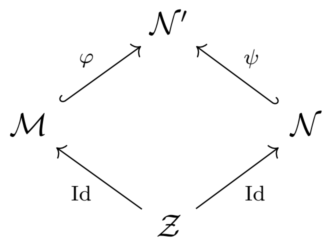
The next lemma is a classical result in discrete model theory and it easily extends to continuous model theory.

Lemma 4.5.3. *Let T be a theory. Then T admits quantifier elimination if and only if it admits amalgamation and is model complete.*

Proof. Suppose that T admits quantifier elimination. Let $\mathcal{M}_1, \mathcal{M}_2 \models T$ with a common substructure $\mathcal{Z}$, applying Proposition 4.5.1 where f is the inclusion $\mathcal{Z} \hookrightarrow \mathcal{M}_2$, we get $\mathcal{N}$ as required.

Now let $\mathcal{M} \subseteq \mathcal{N}$ be two models of T. By quantifier elimination, we only need to prove that $\mathcal{M} \models \varphi(\bar{a}) \Leftrightarrow \mathcal{N} \models \varphi(\bar{a})$ for atomic formulas φ and finite tuples $\bar{a}$ of parameters in $\mathcal{M}$. But this is trivial by the definition of inclusion for models.

Conversely, suppose T admits amalgamation and is model complete and let $\mathcal{M}, \mathcal{N} \models T$, $\mathcal{Z} \subseteq \mathcal{M}$ be a substructure, and $f\colon \mathcal{Z} \hookrightarrow \mathcal{N}$. By considering a monster model, we may suppose that $\mathcal{Z} \subseteq \mathcal{N}$ and f is the identity. Then by amalgamation there is a model $\mathcal{N}' \models T$ and embeddings φ, ψ such that the following diagram commutes:



Again we may suppose that $\mathcal{N} \subseteq \mathcal{N}'$ and ψ is the identity, thus by model completeness we have $\mathcal{N} \preceq \mathcal{N}'$. Furthermore, the diagram now exactly states that φ extends the inclusion $\mathcal{Z} \hookrightarrow \mathcal{N}$. $\square$

In order to prove that our theories eliminate quantifiers, it only remains to prove that they have amalgamation. However, the following example shows that this is not the case in general.

Definition 4.5.4. Let $\Gamma \overset{\alpha}{\curvearrowright} X$ be an action of a group on a standard Borel space. We say that an invariant probability measure $\mu \in \mathfrak{P}(X)$ is **ergodic** if every Γ-invariant for α measurable subset of X is either null or connull for μ.

It can be shown that ergodic measures are the extreme points of the convex space of invariant probability measures in $\mathfrak{P}(X)$.

For Invariant Random Subgroups, we consider the notion of ergodicity with respect to the action $\Gamma \curvearrowright \mathrm{Sub}(\Gamma)$ by conjugation.

Proposition 4.5.5. *Let θ be a non-ergodic IRS on F_∞. Then $\mathfrak{A}_\theta$ does not have quantifier elimination.*

Proof. Take any finite subset $F \subseteq F_\infty$. Then $\mu\left(x \wedge \bigwedge_{\gamma \in F} \operatorname{supp} \gamma\right) := \sup_{\{a_\gamma : \gamma \in F\}} \mu\left(x \wedge \bigwedge_{\gamma \in F} t_\gamma(a_\gamma)\right)$ is a definable predicate in the signature $\mathcal{L}_\infty$. However, as we shall see, not all predicates of this form are definable without quantifiers.

Indeed, suppose that for every finite subset $F \subseteq F_\infty$, there is a quantifier free formula $\varphi_F(x)$ equivalent to $\mu\left(x \wedge \bigwedge_{\gamma \in F} \operatorname{supp} \gamma\right)$.

Write $\theta = t\theta_1 + (1-t)\theta_2$ for a $t \in (0, \frac{1}{2}]$ and $\theta_1 \neq \theta_2$ two IRSs on F_∞. Let κ_1 be a pmp action on $([0,1], \lambda)$ with IRS θ_1 and κ_2 be a pmp action on $([0,1], \lambda)$ with IRS θ_2. Define

- $F_\infty \overset{\alpha}{\curvearrowright} (X = [0,1] \times \{1,2,3\}, \mu = t\lambda \times \delta_1 + t\lambda \times \delta_2 + (1-2t)\lambda \times \delta_3)$ that acts like κ_1 on $[0,1] \times \{1\}$ and acts like κ_2 both on $[0,1] \times \{2\}$ and on $[0,1] \times \{3\}$.

- $F_\infty \overset{\beta}{\curvearrowright} (X = [0,1] \times \{1,2,3\}, \mu = t\lambda \times \delta_1 + t\lambda \times \delta_2 + (1-2t)\lambda \times \delta_3)$ that acts like κ_1 on $[0,1] \times \{2\}$ and acts like κ_2 both on $[0,1] \times \{1\}$ and on $[0,1] \times \{3\}$.

We have $\theta_\alpha = \theta_\beta = \theta$.

Let $\mathcal{M}$ be the finite measure algebra generated by three atoms $\{a, b, c\}$ of respective measure t, t and $1-2t$. By sending a to $[0,1] \times \{1\}$, b to $[0,1] \times \{2\}$ and c to $[0,1] \times \{3\}$, one can embed $\mathcal{M}$ in both $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$. Then $\mathcal{M}$ endowed with the trivial action is a common substructure of $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$.

As $\varphi_F(x)$ is quantifier free, we have $\varphi_F^{\mathcal{M}_\alpha}(a) = \varphi_F^{\mathcal{M}}(a) = \varphi_F^{\mathcal{M}_\beta}(a)$, but

$$\mathcal{M}_\alpha \models \mu(a \wedge \bigwedge_{\gamma \in F} \operatorname{supp} \gamma) = t\theta_1(\mathrm{N}_{F,\varnothing}) \text{ whereas } \mathcal{M}_\beta \models \mu(a \wedge \bigwedge_{\gamma \in F} \operatorname{supp} \gamma) = t\theta_2(\mathrm{N}_{F,\varnothing}).$$

Since an IRS is determined by its values on the sets of the form $\mathrm{N}_{F,\varnothing}$, we get $\theta_1 = \theta_2$, a contradiction. □

Thus, non-ergodicity of the IRS is an obstacle to quantifier elimination. A natural question is to ask about a converse:

For which θ does the theory $\mathfrak{A}_\theta$ admit quantifier elimination? Is it the case for any ergodic IRS?

The author does not have any satisfying answer.

However, we answer another interesting question. One can ask what we can reasonably add to the theory $\mathfrak{A}_\theta$ to expand it into a theory $\mathfrak{A}'_\theta$ in a signature $\mathcal{L}'_\infty \supseteq \mathcal{L}_\infty$ which has quantifier elimination.

The issue encountered in Proposition 4.5.5 is that formulas involving the supports of the elements of F_∞ may not be equivalent to quantifiers free formulas in $\mathfrak{A}_\theta$. This motivates us to look at expansions that allow us to talk about the supports of elements of F_∞ in the language. For that we add constants $\{S_\gamma : \gamma \in F_\infty\}$ to the signature $\mathcal{L}_\infty$ to get a new signature $\mathcal{L}'_\infty$ and we consider the theory $\mathfrak{A}'_\theta$ consisting of:

- The axioms of $\mathfrak{A}_\theta$.
- For $\gamma \in F_\infty$, the axioms:
  - $\sup_a \; d(S_\gamma \wedge t_\gamma(a), t_\gamma(a)) = 0$.
  - $\mu(S_\gamma) = \theta(\mathrm{N}_\gamma)$.

This theory expresses that for $\gamma \in F_\infty$, the constant S_γ must be interpreted as supp $\gamma^{\mathcal{M}}$ in the model $\mathcal{M}$, as it contains the support by the first axiom and has the same measure by the second one.

We need a last definition in order to prove that the theories $\mathfrak{A}_\theta$ admit amalgamation for θ amenable:

Definition 4.5.6. Let $\mathcal{M} \models \mathfrak{A}_\theta$, we denote by $\mathcal{I}_\mathcal{M}$ and we call the IRS of $\mathcal{M}$ the substructure of $\mathcal{M}$ generated by the elements supp γ for $\gamma \in \Gamma$.

Note that this naming is consistent: let $\mathcal{M} = \mathcal{M}_\alpha$ for a pmp action $\Gamma \overset{\alpha}{\curvearrowright} (X,\mu)$ of IRS θ. Then $\mathcal{I}_\mathcal{M}$ is isomorphic to the measure algebra $\mathcal{I}_\theta$ associated to the action $\Gamma \overset{\theta}{\curvearrowright} (\mathrm{Sub}(\Gamma), \theta)$ and moreover, the map $\mathrm{stab}_\alpha \colon X \to \mathrm{Sub}(\Gamma)$ is a lifting of the inclusion $\mathcal{I}_\mathcal{M} \subseteq \mathcal{M}$.

Theorem 4.5.7. *Let θ be an IRS, then the theory $\mathfrak{A}'_\theta$ admits amalgamation in the signature $\mathcal{L}'_\infty$.*

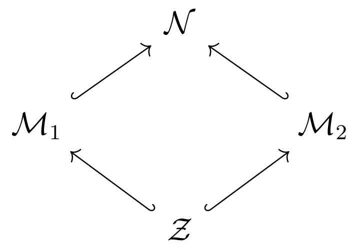
Proof. Let $\mathcal{M}_1, \mathcal{M}_2 \models \mathfrak{A}'_\theta$ and let $\mathcal{Z}$ be a common substructure of $\mathcal{M}_1$ and $\mathcal{M}_2$. Then by definition of the theory $\mathfrak{A}'_\theta$, $\mathcal{I}_\theta$ is a substructure of $\mathcal{Z}$ and the inclusions $\mathcal{Z} \hookrightarrow \mathcal{M}_1$ and $\mathcal{Z} \hookrightarrow \mathcal{M}_2$ send $\mathcal{I}_\theta$ on $\mathcal{I}_{\mathcal{M}_1}$ and $\mathcal{I}_{\mathcal{M}_2}$ respectively. For the sake of simplicity, we identify $\mathcal{Z}$ with its images in $\mathcal{M}_1$ and $\mathcal{M}_2$, which implies that $\mathcal{I}_\theta$, $\mathcal{I}_{\mathcal{M}_1}$ and $\mathcal{I}_{\mathcal{M}_2}$ are all identified.

Let X_1, X_2 and Z be the respective Stone spaces of $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{Z}$ (see [Fre02, 321J]) and let μ_1, μ_2 be the respective inner regular Borel probability measures on X_1 and X_2. We define an inner regular Borel probability measure ν on $X_1 \times X_2$ as in [Ben06, Construction 2.3] as the continuous extension of the map defined on cylinders by the formula:

$$\nu(a_1 \times a_2) = \int_Z \mu_1(a_1|\mathcal{Z})\mu_2(a_2|\mathcal{Z})\ dz \ \text{ for all } a_1 \in \mathcal{M}_1, a_2 \in \mathcal{M}_2.$$

The pmp action $F_\infty \curvearrowright (X_1 \times X_2, \nu)$ then induces a structure $\mathcal{N} \models \mathfrak{A}_{F_\infty}$ that we call the **relative independent joining of $\mathcal{M}_1$ and $\mathcal{M}_2$ over $\mathcal{Z}$.**

The following diagram is indeed commutative:



It remains to prove that $\mathcal{N} \models \mathfrak{A}_\theta$. For that note that

$$\begin{aligned} \neg \text{ supp } \gamma^{\mathcal{N}} &= \bigvee\{a : \forall b \subseteq a, \gamma b = b\} \\ &= \bigvee\{a_1 \times a_2 : \forall b \subseteq a_1 \times a_2, \gamma b = b\} \\ &= \bigvee\{a_1 \times a_2 : \forall b_1 \subseteq a_1\ \forall b_2 \subseteq a_2, \gamma b_1 = b_1 \text{ and } \gamma b_2 = b_2\} \\ &= \neg \text{ supp } \gamma^{\mathcal{M}_1} \times \neg \text{ supp } \gamma^{\mathcal{M}_2} \\ &= \neg\ S_\gamma^{\mathcal{Z}} \times \neg\ S_\gamma^{\mathcal{Z}} \end{aligned}$$

but the definition of ν implies that $\nu\left(\neg\ S_\gamma^{\mathcal{Z}} \times 1_{\mathcal{M}_2}\right) = \nu\left(\neg\ S_\gamma^{\mathcal{Z}} \times \neg\ S_\gamma^{\mathcal{Z}}\right)$, so that these two elements of $\mathcal{N}$ are equal. Letting i_1 denote the embedding $\mathcal{M}_1 \hookrightarrow \mathcal{N}$, we get the equalities $\neg$ supp $\gamma^{\mathcal{N}} = \neg\ S_\gamma^{\mathcal{Z}}$ and therefore supp $\gamma^{\mathcal{N}} = i_1\left(S_\gamma^{\mathcal{Z}}\right) = i_1\left(\text{supp } \gamma^{\mathcal{M}_1}\right)$. This being true for any $\gamma \in F_\infty$, it follows that i_1 maps any finite intersection of supports in $\mathcal{M}_1$ to the corresponding intersection of supports in $\mathcal{N}$, and since i_1 also preserves the measure, we can conclude that $\mathcal{N} \models \mathfrak{A}_\theta$. □

Theorem 4.5.8. *Let θ be an amenable IRS. Then the theory $\mathfrak{A}'_\theta$ eliminates quantifiers in the signature $\mathcal{L}'_\infty$.*

Proof. We use Lemma 4.5.3.

We just saw that $\mathfrak{A}'_\theta$ admits amalgamation.

For model completeness, take $\mathcal{M} \subseteq \mathcal{N}$ be two models of $\mathfrak{A}'_\theta$ and let us prove that $\mathcal{M} \preceq \mathcal{N}$. Let $\varphi(\bar{x})$ be an $\mathcal{L}'_\infty$-formula and $\bar{p} \in \mathcal{M}^n$. Then $\varphi(\bar{x})$ is equivalent to a formula of the form $\psi(\bar{x}, S_{\bar{\gamma}})$ where ψ is a $\mathcal{L}_\infty$-formula, and the constants of the form S_γ are preserved under the inclusion $\mathcal{M} \subseteq \mathcal{N}$. Therefore, it suffices to apply Theorem 4.4.3 to ψ and to consider the elements $S_{\bar{\gamma}}$ as parameters added to $\bar{p}$ to conclude. □

As a corollary, we get a class of IRSs θ for which the theory $\mathfrak{A}_\theta$ admits quantifier elimination.

Corollary 4.5.9. *The theory of free actions of an amenable group admits amalgamation. Namely, if θ is the Dirac measure δ_N for a co-amenable normal subgroup $N \leqslant F_\infty$, then $\mathfrak{A}_\theta$ has quantifier elimination.*

Proof. Simply note that the support of an element $\gamma \in F_\infty$ in a model of $\mathfrak{A}_\theta$ is either 0 (if $\gamma \in N$) or 1 (if $\gamma \notin N$). It follows that the theories $\mathfrak{A}_\theta$ and $\mathfrak{A}'_\theta$ completely coincide, hence the result. $\square$

For $\mathcal{M} \models \mathfrak{A}_\infty$ and $A \subseteq \mathcal{M}$, we write $\langle A \rangle$ for the closed subalgebra of $\mathcal{M}$ (that is, the substructure of $\mathcal{M}$ as a model of AMA) generated by A.

Theorem 4.5.10. *Let $\mathcal{M} \models \mathfrak{A}_\theta$ and $A \subseteq \mathcal{M}$. Then the definable closure of A in $\mathcal{M}$ is $\langle F_\infty A \cup \mathcal{I}_\mathcal{M} \rangle$.*

Proof. On the one hand, $A \subseteq \mathrm{dcl}^\mathcal{M}(A)$ and by Lemma 4.3.5, for $\gamma \in F_\infty$, $\mathrm{supp}\, \gamma^\mathcal{M} \in \mathrm{dcl}^\mathcal{M}(A)$. Thus we get the first inclusion.

On the other hand, since $\mathfrak{A}'_\theta$ expands $\mathfrak{A}_\theta$, the definable closure of A in the theory $\mathfrak{A}_\theta$ is contained in the definable closure of A in the theory $\mathfrak{A}'_\theta$. Let us compute this definable closure D.

First, we notice that the function symbols γ are interpreted by automorphisms and thus any atomic $\mathcal{L}_\infty$-formula with parameters in A is equivalent to an atomic $\mathcal{L}$-formula with parameters in $F_\infty A$. This remark then extends to quantifier free formulas.

Then, by Theorem 4.5.8, any $\mathcal{L}'_\infty$-formula with parameters in A is equivalent to a quantifier free $\mathcal{L}'_\infty$-formula with parameters in A and since we only added constants in $\mathcal{L}_\infty$, it is moreover equivalent to a quantifier free $\mathcal{L}_\infty$-formula with parameters in $A \cup \mathcal{I}_\mathcal{M}$.

Combining the two latter properties and the fact that $\mathrm{dcl}(A) = \langle A \rangle$ in the theory AMA, we get that $D = \langle F_\infty(A \cup \mathcal{I}_\mathcal{M}) \rangle$. Furthermore, $\mathcal{I}_\mathcal{M}$ is a substructure and so $\langle F_\infty(A \cup \mathcal{I}_\mathcal{M}) \rangle = \langle F_\infty A \cup \mathcal{I}_\mathcal{M} \rangle$.

Hence the conclusion. $\square$

4.6 Stability and Independence

We recall some definitions from [BBHU08].

Definition 4.6.1. Let κ be a cardinal. A **κ-universal domain** for a theory T is a κ-saturated and strongly κ-homogeneous model of T. If $\mathcal{U}$ is a κ-universal domain and $A \subseteq \mathcal{U}$, we say that A is **small** if $|A| < \kappa$.

Definition 4.6.2. Let $\mathcal{U}$ be a κ-universal domain for T. A **stable independence relation** on $\mathcal{U}$ is a relation $A \mathop{\smile\!\!\!\!\!|}_C B$ on triples of small subsets of $\mathcal{U}$ satisfying the following properties, for all small $A, B, C, D \subseteq \mathcal{U}$, finite $\bar{u}, \bar{v} \subseteq \mathcal{U}$ and small $\mathcal{M} \preceq \mathcal{U}$:

1. *Invariance under automorphisms:* If ρ is an automorphism of $\mathcal{U}$, then $A \mathop{\smile\!\!\!\!\!|}_C B \iff \rho(A) \mathop{\smile\!\!\!\!\!|}_{\rho(C)} \rho(B)$.
2. *Symmetry:* $A \mathop{\smile\!\!\!\!\!|}_C B \iff B \mathop{\smile\!\!\!\!\!|}_C A$.
3. *Transitivity:* $A \mathop{\smile\!\!\!\!\!|}_C BD \iff A \mathop{\smile\!\!\!\!\!|}_C B \wedge A \mathop{\smile\!\!\!\!\!|}_{BC} D$.
4. *Finite character:* $A \mathop{\smile\!\!\!\!\!|}_C B$ if and only if $\bar{a} \mathop{\smile\!\!\!\!\!|}_C B$ for every finite $\bar{a} \subseteq A$.

5. *Existence:* There exists A' such that $\mathrm{tp}(A'/C) = \mathrm{tp}(A/C)$ and $A' \underset{C}{\perp} B$.

6. *Local character:* There exists $B_0 \subseteq B$ such that $|B_0| \leqslant |T|$ and $\bar{u} \underset{B_0}{\perp} B$.

7. *Stationarity of types:* If $\mathrm{tp}(A/\mathcal{M}) = \mathrm{tp}(B/\mathcal{M})$ and $A \underset{\mathcal{M}}{\perp} C$ and $B \underset{\mathcal{M}}{\perp} C$, then

   $\mathrm{tp}(A/\mathcal{M} \cup C) = \mathrm{tp}(B/\mathcal{M} \cup C)$.

Proposition 4.6.3 ([BBHU08]). *Let $\kappa > |T|$ and let $\mathcal{U}$ be a κ-universal domain. Then the theory T is stable if and only if there exists a stable independence relation on $\mathcal{U}$, and in this case the stable independence relation is the independence relation given by non-dividing.*

Thus, in order to prove that our theories are stable, we only need to define a stable independence relation. Ben Yaacov proved in [Ben06, Thm. 4.1] that the classical relation of independence of events was the required one in the case of measure algebras without group actions. Now that we described the definable closures in our theories, the proof of Ben Yaacov naturally adapts to this case.

Definition 4.6.4. From now on, we write $\langle\langle A \rangle\rangle$ for $\mathrm{dcl}^{\mathcal{U}}(A)$.

Let $A, B, C \subseteq \mathcal{U}$, we say that A and B are independent over C and we write $A \underset{C}{\perp} B$ if we have $\forall a \in \langle\langle A \rangle\rangle$, $\forall b \in \langle\langle B \rangle\rangle$, $\mathbb{P}_{\langle\langle C \rangle\rangle}(a)\mathbb{P}_{\langle\langle C \rangle\rangle}(b) = \mathbb{P}_{\langle\langle C \rangle\rangle}(a \wedge b)$.

We will need the following propositions:

Proposition 4.6.5. *Let $A, B, C \subseteq \mathcal{U} \models \mathfrak{A}_{F_\infty}$. Then we have $A \underset{C}{\perp} B$ if and only if $\forall a \in \langle\langle A \rangle\rangle$,*

$$\mathbb{P}_{\langle\langle BC \rangle\rangle}(a) = \mathbb{P}_{\langle\langle C \rangle\rangle}(a).$$

We have the following characterization of independence in the theory of atomless probability algebras.

Proposition 4.6.6 ([Ben06, Lemma 2.7]). *Let $\mathcal{U} \models$ AMA and let $\mathcal{M}_1, \mathcal{M}_2$ be small substructures of $\mathcal{U}$. Let $\mathcal{Z}$ be a common substructure of $\mathcal{M}_1$ and $\mathcal{M}_2$. Let $\mathcal{M}_1 \wedge \mathcal{M}_2$ be the substructure of $\mathcal{U}$ generated by $\mathcal{M}_1$ and $\mathcal{M}_2$ and define $\mathcal{N}$ the relative independent joining of $\mathcal{M}_1$ and $\mathcal{M}_2$ over $\mathcal{Z}$ as in Theorem 4.5.7.*

Then in the theory AMA*, we have $\mathcal{M}_1 \underset{\mathcal{Z}}{\perp} \mathcal{M}_2$ if and only if $\mathcal{M}_1 \wedge \mathcal{M}_2 \simeq \mathcal{N}$.*

Here we work in the theory $\mathfrak{A}'_\theta$ so that the supports interpreted in a structure are the same when interpreted in any substructure. This ensures that the relative independent joining of two models of $\mathfrak{A}'_\theta$ over an IRS remains an elementary extension of these two models. Therefore, we have:

Proposition 4.6.7. *Let θ be an amenable IRS on F_∞.*

Let $\mathcal{U} \models \mathfrak{A}'_\theta$ and let $\mathcal{M}_1, \mathcal{M}_2$ be small substructures of $\mathcal{U}$. Let $\mathcal{Z}$ be a common substructure of $\mathcal{M}_1$ and $\mathcal{M}_2$. Let $\mathcal{M}_1 \wedge \mathcal{M}_2$ be the substructure of $\mathcal{U}$ generated by $\mathcal{M}_1$ and $\mathcal{M}_2$ and define $\mathcal{N}$ the relative independent joining of $\mathcal{M}_1$ and $\mathcal{M}_2$ over $\mathcal{Z}$.

Then in $\mathfrak{A}'_\theta$, we have $\mathcal{M}_1 \underset{\mathcal{Z}}{\perp} \mathcal{M}_2$ if and only if $\mathcal{M}_1 \wedge \mathcal{M}_2 \simeq \mathcal{N}$.

Theorem 4.6.8. *If θ is an amenable IRS, the relation of independence $\perp$ defined above is a stable independence relation when restricted to triples of small subsets, relative to the theory $\mathfrak{A}_\theta$. Consequently, the theory $\mathfrak{A}_\theta$ is stable and the relation $\perp$ agrees with non-dividing on triples of small subsets.*

Proof. 1. *Invariance under automorphisms* of $\mathcal{U}$: If ρ is an automorphism of $\mathcal{U}$, by uniqueness of the orthogonal projection, we know that $\mathbb{P}_{\langle\langle \rho(C)\rangle\rangle} = \rho \circ \mathbb{P}_{\langle\langle C\rangle\rangle} \circ \rho^{-1}$ and therefore

$$\mathbb{P}_{\langle\langle C\rangle\rangle}(a)\mathbb{P}_{\langle\langle C\rangle\rangle}(b) = \mathbb{P}_{\langle\langle C\rangle\rangle}(a \wedge b) \Leftrightarrow \mathbb{P}_{\langle\langle \rho(C)\rangle\rangle}(\rho a)\mathbb{P}_{\langle\langle \rho(C)\rangle\rangle}(\rho b) = \mathbb{P}_{\langle\langle \rho(C)\rangle\rangle}(\rho(a \wedge b)).$$

2. *Symmetry:* The definition is symmetric.

3. *Transitivity:* Let A, B, C, D be small. First if $A \underset{C}{\perp} B$ and $A \underset{BC}{\perp} D$ then by Proposition 4.6.5, for $a \in \langle\langle A\rangle\rangle$, we have $\mathbb{P}_{\langle\langle BCD\rangle\rangle}(a) = \mathbb{P}_{\langle\langle BC\rangle\rangle}(a) = \mathbb{P}_{\langle\langle C\rangle\rangle}(a)$ so $A \underset{C}{\perp} BD$.

   Conversely, suppose that $A \underset{C}{\perp} BD$. Then $\mathbb{P}_{\langle\langle BCD\rangle\rangle}(a) = \mathbb{P}_{\langle\langle C\rangle\rangle}(a)$, but that implies that $\mathbb{P}_{\langle\langle C\rangle\rangle}(a)$ is a $\langle\langle C\rangle\rangle$-measurable function such that for all $\langle\langle BCD\rangle\rangle$-measurable function f we have $\int \mathbb{P}_{\langle\langle C\rangle\rangle}(a)f = \int 1_a f$. We conclude that $\mathbb{P}_{\langle\langle BCD\rangle\rangle}(a) = \mathbb{P}_{\langle\langle BC\rangle\rangle}(a) = \mathbb{P}_{\langle\langle C\rangle\rangle}(a)$, and therefore that $A \underset{C}{\perp} C$ and $A \underset{BC}{\perp} D$.

4. *Finite character:* It follows from the definition and the continuity of $\mathbb{P}$.

5. *Existence:* Let A, B, C be small subsets of $\mathcal{U}$. By Löwenheim-Skolem theorem, let $\mathcal{A}$ and $\mathcal{B}$ be small structures such that $\langle\langle AC\rangle\rangle \subseteq \mathcal{A} \preceq \mathcal{U}$ and $\langle\langle BC\rangle\rangle \subseteq \mathcal{B} \preceq \mathcal{U}$, and let $\mathcal{C} = \langle\langle C\rangle\rangle$. Then $\mathcal{A}$ and $\mathcal{B}$ are both elementary substructures of $\mathcal{U}$ containing $\mathcal{I}_\mathcal{U}$. It follows that $\mathcal{A}$ and $\mathcal{B} \models \mathfrak{A}'_\theta$ when the constants S_γ are interpreted by supp $\gamma^\mathcal{U}$ in either of these models, and $\mathcal{C}$ is an $\mathcal{L}'_\infty$-common substructure of $\mathcal{A}$ and $\mathcal{B}$, so using Theorem 4.5.7, we see that the relative independent joining $\mathcal{D}$ of $\mathcal{A}$ and $\mathcal{B}$ over $\mathcal{C}$ is a small model of $\mathfrak{A}_\theta$.

   By saturation and homogeneity of $\mathcal{U}$, we can embed $\mathcal{D}$ in $\mathcal{U}$ while sending $\mathcal{B}$ back to $\mathcal{B}$. Taking the image of $\mathcal{A}$ by this embedding gives us a new copy $\mathcal{A}'$ of $\mathcal{A}$ and a new copy A' of A. Finally, $\mathcal{A}' \wedge \mathcal{B} \simeq \mathcal{D}$ so by Proposition 4.6.6 we get that $\mathcal{A}' \underset{\mathcal{C}}{\perp} \mathcal{B}$, which in turn implies that $A' \underset{C}{\perp} B$.

6. *Local character:* Let $\bar{u} = (u_1, \dots, u_n) \subseteq \mathcal{U}$ be finite. Consider the conditional probabilities $\mathbb{P}_{\langle\langle B\rangle\rangle}(u_i)$. These are $\langle\langle B\rangle\rangle$-measurable functions with real values and so there is a countably generated σ-subalgebra of $\langle\langle B\rangle\rangle$, say $\langle\langle B_0\rangle\rangle$ where $B_0 \subseteq B$ is countable, for which they are all measurable. But then we have $\mathbb{P}_{\langle\langle B\rangle\rangle}(u_i) = \mathbb{P}_{\langle\langle B_0\rangle\rangle}(u_i)$, so by Proposition 4.6.5 $\bar{u} \underset{B_0}{\perp} B$.

7. *Stationarity of types:* We denote by $\mathrm{tp}_\mathcal{L}(\bar{x}/Y)$ the type of a tuple $\bar{x}$ over a set of parameters Y in the language $\mathcal{L}$. In other words, this is the type of $\bar{x}$ over Y in the underlying atomless measure algebra of $\mathcal{U}$.

   Let $A, B, C \subseteq \mathcal{U}$ be small and $\mathcal{M} \preceq \mathcal{U}$ be small. Suppose that $\mathrm{tp}(A/\mathcal{M}) = \mathrm{tp}(B/\mathcal{M})$, $A \underset{\mathcal{M}}{\perp} C$ and $B \underset{\mathcal{M}}{\perp} C$.

We begin by proving that $\mathrm{tp}_{\mathcal{L}}(A/\langle\langle\mathcal{M}\cup C\rangle\rangle) = \mathrm{tp}_{\mathcal{L}}(B/\langle\langle\mathcal{M}\cup C\rangle\rangle)$. Indeed, for $a \in \langle A\rangle$ and $b \in \langle B\rangle$, we have $\mathbb{P}_{\langle\langle\mathcal{M}\cup C\rangle\rangle}(a) = \mathbb{P}_{\mathcal{M}}(a)$ and $\mathbb{P}_{\langle\langle\mathcal{M}\cup C\rangle\rangle}(b) = \mathbb{P}_{\mathcal{M}}(b)$, but by Proposition 4.2.4 types in AMA can be fully described with conditional probabilities and we have $\mathrm{tp}_{\mathcal{L}}(A/\mathcal{M}) = \mathrm{tp}_{\mathcal{L}}(B/\mathcal{M})$ so we get $\mathrm{tp}_{\mathcal{L}}(A/\langle\langle\mathcal{M}\cup C\rangle\rangle) = \mathrm{tp}_{\mathcal{L}}(B/\langle\langle\mathcal{M}\cup C\rangle\rangle)$.

Now Theorem 4.5.8 implies that $\mathrm{tp}(A/\mathcal{M}\cup C)$ (resp. $\mathrm{tp}(B/\mathcal{M}\cup C)$) is determined by the $\mathcal{L}$-type $\mathrm{tp}_{\mathcal{L}}\left(\langle F_\infty A \cup \mathcal{I}_{\mathcal{U}}\rangle / \langle\langle\mathcal{M}\cup C\rangle\rangle\right)$ (resp. $\mathrm{tp}_{\mathcal{L}}\left(\langle F_\infty B \cup \mathcal{I}_{\mathcal{U}}\rangle / \langle\langle\mathcal{M}\cup C\rangle\rangle\right)$).

Thus, let $A' = F_\infty A \cup \mathcal{I}_{\mathcal{U}}$ and $B' = F_\infty B \cup \mathcal{I}_{\mathcal{U}}$.

It is clear that $\mathrm{tp}(A'/\mathcal{M}) = \mathrm{tp}(B'/\mathcal{M})$, $A' \underset{\mathcal{M}}{\perp} C$ and $B' \underset{\mathcal{M}}{\perp} C$ and we can apply what we proved just above to conclude that $\mathrm{tp}_{\mathcal{L}}\left(A'/\langle\langle\mathcal{M}\cup C\rangle\rangle\right) = \mathrm{tp}_{\mathcal{L}}\left(B'/\langle\langle\mathcal{M}\cup C\rangle\rangle\right)$, that is

$$\mathrm{tp}_{\mathcal{L}}\left(\langle F_\infty A \cup \mathcal{I}_{\mathcal{U}}\rangle / \langle\langle\mathcal{M}\cup C\rangle\rangle\right) = \mathrm{tp}_{\mathcal{L}}\left(\langle F_\infty B \cup \mathcal{I}_{\mathcal{U}}\rangle / \langle\langle\mathcal{M}\cup C\rangle\rangle\right),$$

hence the conclusion.

$\square$

Part II

Measure-preserving actions of groupoids and applications

This part deals with probability measure-preserving groupoids and their measure-preserving actions on standard probability spaces. Their study was motivated by two separate objectives:

Our first objective is to study (in Chapter 6) pmp actions of a group Γ having a given IRS θ of Γ. It turns out such actions arise as actions of a groupoid $\mathcal{G}_\Gamma^\theta$ associated to Γ and we use this groupoid to classify pmp actions with IRS θ in some particular cases.

The second objective is to establish (in Chapter 7) a characterization of property (T) for pmp equivalence relations on a standard probability space in terms of boolean actions of the full group of the equivalence relation. Measure-preserving equivalence relations are very important particular examples of pmp groupoids, they correspond to the case of a groupoid with trivial isotropy.

Chapter 5 consists of a theoretical study of measure-preserving actions of pmp groupoids. We extend several results already known for countable groups to the more general setting of pmp groupoids. Most of this work relies on the correspondence between pmp actions of a groupoid $\mathcal{G}$ and so-called boolean full actions of its full group $[\mathcal{G}]$, as well as the correspondence between unitary representations of $\mathcal{G}$ and so-called full unitary actions of its full group $[\mathcal{G}]$.

In particular, we generalize the notion of Kazhdan pair for a pmp groupoid with property (T) and we extend a theorem of A. Connes and B. Weiss which states that a groupoid has property (T) if and only if all of its ergodic pmp actions are strongly ergodic.

Chapter 5

On measure-preserving actions of groupoids

5.1 Definitions

5.1.1 Topology and measure theory

We briefly recall the definitions for our general setting of standard probability spaces.

Definition 5.1.1. A topological space (X, τ) is called **Polish** if it is separable and completely metrizable.

Definition 5.1.2. A measurable space (X, Σ) is called a **standard Borel space** if it is isomorphic to the Borel underlying structure of a Polish space.

A standard Borel space is entirely determined up to isomorphism by its cardinality $\kappa \in \mathbb{N} \cup \{\aleph_0\} \cup \{2^{\aleph_0}\}$.

Definition 5.1.3. Let (X, μ), (Y, ν) and (Z, η) be standard Borel spaces and let $p_X : X \to Z$ and $p_Y : Y \to Z$ be Borel maps, then we define the **fibered product** of X and Y over p_X and p_Y by $X \ {}_{p_X}{*}_{p_Y} \ Y := \{(x, y) \in X \times Y : p_X(x) = p_Y(y)\}$. It is itself a standard Borel space.

Definition 5.1.4. A **standard probability space** (X, Σ, μ) is a standard Borel space endowed with a probability measure on the σ-algebra Σ.

A standard probability space is entirely determined by its cardinality and the measure of its atoms. In particular, there is a unique up to isomorphism atomless standard probability space.

Definition 5.1.5. The **measure algebra** (or **probability algebra**) of (X, Σ, μ) is the quotient of the set Σ by the σ-ideal of null sets, endowed with the operations of union, intersection, complement and with the measure map μ. We denote it $\mathrm{MAlg}(X, \mu)$.

There is a purely axiomatic description of a measure algebra that coincides with the latter definition, in the sense that for a standard probability space (X, μ), $\mathrm{MAlg}(X, \mu)$ is a measure algebra, and conversely, every separable measure algebra is isomorphic to a measure algebra of the form $\mathrm{MAlg}(X, \mu)$ for a standard probability space (X, μ).

Conventions

Let (X, μ) be a standard probability space, then the measure algebra $\mathrm{MAlg}(X, \mu)$ is complete, which means that any family $(A_i)_{i\in I}$ of elements has a supremum and an infimum. We will denote these by $\bigvee_{i\in I} A_i$ and $\bigwedge_{i\in I} A_i$. Since they coincide with set-theoretic union and intersection for countable families, when I is countable we will also use the notation $\bigcup_{i\in I} A_i$ and $\bigcap_{i\in I} A_i$.

All the subsets of X that we will consider are seen as elements of $\mathrm{MAlg}(X, \mu)$; in particular they are measurable. We will neglect what happens on measure zero sets.

Definition 5.1.6. A **probability measure-preserving transformation (pmp transformation in short)** of a measure algebra $\mathrm{MAlg}(X, \mu)$ is an isomorphism of the structure $(\mathrm{MAlg}(X, \mu), \emptyset, \cup, \cap, \cdot^c, \mu)$. We call $\mathrm{Aut}(X, \mu)$ the group of pmp transformations of $\mathrm{MAlg}(X, \mu)$.

If f is a Borel bijection $X \to X$ which preserves the measure μ, i.e. for $A \in \Sigma$, $\mu(f^{-1}A) = \mu(A)$, then the map f^{-1} passes to the quotient and induces a pmp transformation $\mathrm{MAlg}(X, \mu) \to \mathrm{MAlg}(X, \mu)$.

Conversely, every element $\rho \in \mathrm{Aut}(X, \mu)$ arises through this construction from such a Borel bijection f which preserves the measure.

Furthermore, if Γ is a countable group and $\rho\colon \Gamma \to \mathrm{Aut}(X, \mu)$ is an action of Γ on $\mathrm{MAlg}(X, \mu)$, then there exists an action a of Γ on X by measure-preserving Borel bijections such that for every $\gamma \in \Gamma$, $\rho(\gamma)$ is the pmp transformation induced by the Borel map $a(\gamma)$. This is not true in general for Γ uncountable.

5.1.2 Groupoids

Definition 5.1.7 (Borel groupoid)**.** A **Borel groupoid** is a tuple $(\mathcal{G}, X, s, r, \star, {}^{-1})$ where $\mathcal{G}$ is a standard Borel space called the **base space**, X is a standard Borel subspace of $\mathcal{G}$ called **the space of units**, s and r are Borel surjective maps $\mathcal{G} \to X$ called the **source** and the **range** maps, $\star$ is a Borel map from $\mathcal{G}^{(2)} := \mathcal{G}\ {}_s{*}_r\ \mathcal{G}$ to $\mathcal{G}$ and ${}^{-1}$ is a Borel map from $\mathcal{G}$ to itself such that:

1. $\forall x \in X,\ s(x) = r(x) = x$,
2. $\forall g \in \mathcal{G},\ g \star s(g) = g = r(g) \star g$,
3. $\forall (g, g') \in \mathcal{G}^{(2)},\ s(g \star g') = s(g')$ and $r(g \star g') = r(g)$,
4. $\forall g, g', g'' \in \mathcal{G},\ (g \star g') \star g'' = g \star (g' \star g'')$ if this product is well defined,
5. $\forall g \in \mathcal{G}$ we have $s(g^{-1}) = r(g), r(g^{-1}) = s(g)$ and moreover $g^{-1} \star g = s(g)$ and $g \star g^{-1} = r(g)$.

In this thesis, we only consider discrete groupoids, that is groupoids such that both the maps s and r are countable-to-1.

Example 5.1.8. Groupoids generalize both the notions of a group and an equivalence relation.

- A countable discrete group Γ is simply a Borel discrete groupoid with space of units $\{1_\Gamma\}$.

- A Borel countable equivalence relation on X is exactly a Borel discrete groupoid such that the map $r \times s \colon \mathcal{G} \to X^2$ is injective. In this case we identify an element $g \in \mathcal{G}$ with the couple $(r(g), s(g))$. After this identification, the product on an equivalence relation is defined by $(z, y) \star (y, x) = (z, x)$.

We will use the following notations: For $A, B \subseteq X$, we write $\mathcal{G}_A$ for $\mathcal{G} \star A = s^{-1}(A)$, $\mathcal{G}^B$ for $B \star \mathcal{G} = r^{-1}(B)$ and $\mathcal{G}_A^B$ for $\mathcal{G}_A \cap \mathcal{G}^B$. Moreover, from now on, we embed $\mathcal{G}$ into $[[\mathcal{G}]]$ using the inclusion map $g \mapsto \{g\}$, so that for $x, y \in X$, $\mathcal{G}_x$ and $\mathcal{G}^y$ simply denote $s^{-1}(x)$ and $r^{-1}(y)$ respectively. Finally, for $x \in X$ we let $\mathcal{G}(x)$ denote the set $\mathcal{G}_x^x$. It is a group called the **isotropy group** of $\mathcal{G}$ at x. We let $\mathcal{IG} = \{g \in \mathcal{G} : s(g) = r(g)\}$ be the union of all isotropy groups.

Definition 5.1.9 (Bitransversal)**.** Let Y be a standard Borel space and $\varphi \colon Y \to X$ be a Borel map. A (partial) **transversal** of Y for φ is a Borel subset $t \subseteq Y$ such that the φ is injective when restricted to t.

A (partial) **bitransversal** of a groupoid $\mathcal{G}$ is a Borel subset $T \subseteq \mathcal{G}$ which is a transversal for both the source s and the range r maps.

A transversal t for φ is called **total** if $\varphi(t) = X$.

A bitransversal T is called **total** if it is total both as a transversal for s and as a transversal for r.

Proposition 5.1.10 (Feldman-Moore for groupoids)**.** *Let $\mathcal{G}$ be a discrete Borel groupoid. Then there exists a countable set $\mathcal{C}$ of total bitransversals such that $\mathcal{G} = \bigcup \mathcal{C}$.*

Proof. We assume that X is uncountable. The case where X is at most countable is easier. Since X is a standard Borel space, suppose without loss of generality that $X = [0, 1]$.

First apply the Lusin-Novikov theorem to the spaces $(\mathrm{id} \times s)(\mathcal{G})$ and $(\mathrm{id} \times r)(\mathcal{G})$, which have countable sections, to obtain two sequences $(t_n^s)_{n\in\mathbb{N}}$ and $(t_n^r)_{n\in\mathbb{N}}$ of Borel partial transversals (respectively for s and r) such that $\bigcup_{n\in\mathbb{N}} t_n^r = \bigcup_{n\in\mathbb{N}} t_n^s = \mathcal{G}$.

For $m, n \in \mathbb{N}$ and $q \in \mathbb{Q} \cap (0, 1)$, let $T_{m,n,q} = \mathcal{G}_{[0,q)}^{(q,1]} \cap t_m^s \cap t_n^r$, so that $T_{m,n,q}$ is a bitransversal. Since the source and the range of $T_{m,n,q}$ are disjoint, complete it into a total bitransversal

$$T'_{m,n,q} \coloneqq X \setminus (s(T_{m,n,q}) \cup r(T_{m,n,q})) \sqcup T_{m,n,q} \sqcup T_{m,n,q}.$$

Note that $\bigcup_{m,n\in\mathbb{N},q\in\mathbb{Q}\cap(0,1)} T'_{m,n,q} = \mathcal{G} \setminus \mathcal{IG}$. Therefore, in order to conclude, it suffices to add the isotropy groups to the sequence considered.

For that, again apply the Lusin-Novikov theorem to $\mathcal{IG}$ to obtain total bitransversals $(T_n)_{n\in\mathbb{N}}$ from X to $\mathcal{IG}$ such that $\bigcup_{n\in\mathbb{N}} T_n = \mathcal{IG}$. □

The product $\star$ on $\mathcal{G}$ extends into an operation on partial bitransversals: for two partial bitransversals T and T', we define their product point by point by the formula $T \star T' \coloneqq \{g \star g' : (g, g') \in T \times T' \cap \mathcal{G}^{(2)}\}$. Note that $T \star T'$ is a Borel subset of $\mathcal{G}$ since it is the image of $T \times T' \cap \mathcal{G}^{(2)}$ by the map $\star$, which is injective when restricted to $T \times T' \cap \mathcal{G}^{(2)}$.

The set of all total bitransversals then forms a group $[\mathcal{G}]_{\mathrm{b}}$ for this product which we call the **Borel full group** of $\mathcal{G}$. The neutral element for this group is the trivial bitransversal X. Moreover, we call the set of all partial bitransversals the **Borel pseudo-full group** of $\mathcal{G}$ and we denote it by $[[\mathcal{G}]]_{\mathrm{b}}$.

Remark 5.1.11. According to the latter definitions and notations, we have:

1. For $T \in [[\mathcal{G}]]_{\mathrm{b}}$ and $x \in s(T)$, $T \star x$ is the unique element of $T \cap \mathcal{G}_x$, and for $x \in r(T)$, $x \star T$ is the unique element of $T \cap \mathcal{G}^x$.

2. For $T \in [[\mathcal{G}]]_{\mathrm{b}}$ and $x \in s(T)$, $T \star x \star T^{-1}$ is the unique $y \in X$ such that there exists $g \in T$ with $s(g) = x$ and $r(g) = y$. Similarly, for $x \in r(T)$, $T^{-1} \star x \star T$ is the unique $y \in X$ such that there exists $g \in T$ with $s(g) = y$ and $r(g) = x$.

3. The group $[\mathcal{G}]_{\mathrm{b}}$ therefore acts by Borel transformations on X via left (respectively right) conjugation. We denote by $\mathfrak{l}$ (resp. $\mathfrak{r}$) this action.

We conclude this section with similar definitions in the context of measure theory:

Definition 5.1.12 (Probability measure preserving groupoid)**.** Let (X, μ) be a standard probability space. A Borel groupoid on X is called **probability measure preserving** (or **pmp** in short) on (X, μ) when the Borel measures μ_L and μ_R on $\mathcal{G}$ defined by the formulas $\mu_L(D) = \int_X |D \star x| \; d\mu(x)$ and $\mu_R(D) = \int_X |x \star D| \; d\mu(x)$ are equal.

For $\mathcal{G}$ a pmp groupoid, we write μ^1 for the measure $\mu_L = \mu_R$ on $\mathcal{G}$.

An alternative for the definition of a pmp groupoid is the following: Let (X, μ) be a standard probability space and $\mathcal{G}$ Borel groupoid on X, then $\mathcal{G}$ is pmp if and only if the group action $[\mathcal{G}]_{\mathrm{b}} \overset{\mathfrak{l}}{\curvearrowright} X$ preserves μ.

The equivalence of these two definitions is a classical result for equivalence relations and can be obtained in the general setting of discrete groupoids with a similar proof.

Definition 5.1.13. A pmp groupoid $\mathcal{G}$ on (X, μ) is called **ergodic** if any $\mathfrak{l}$-invariant set $A \subset X$ is either null or connull.

Definition 5.1.14 (Measured full group)**.** Let $\mathcal{G}$ be a pmp groupoid on (X, μ). Its **full group** $[\mathcal{G}]$ is the quotient of the Borel full group of $\mathcal{G}$, seen as a groupoid on the standard Borel space X, by the normal subgroup consisting of bitransversals which are equal to the diagonal up to a μ^1-null set. Explicitely, we identify T and S in $[\mathcal{G}]_{\mathrm{b}}$ when $\mu^1(T \triangle S) = 0$, or equivalently, when $\mu(\{x \in X : T \star x \neq S \star x\}) = 0$.

The action $\mathfrak{l}$ of $[\mathcal{G}]_{\mathrm{b}}$ on (X, μ) passes to the quotient and induces an action of $[\mathcal{G}]$ that we also denote by $\mathfrak{l}$.

We briefly mention the topology used on the full group of a pmp groupoid: the map $d \colon (T, S) \mapsto \mu(\{x \in X : T \star x \neq S \star x\})$ defines a complete metric on $[\mathcal{G}]$ which makes it separable. The induced topology thus turns $[\mathcal{G}]$ into a Polish space.

5.2 Actions and representations

We choose to use to point of view of cocycles in this paper. Note that therefore, the definitions of actions and representations we give agree with the classical ones only when the groupoid is ergodic. This will always be the case for the results we are interested in.

Definition 5.2.1. Let $\mathcal{G}$ and $\mathcal{G}'$ be Borel groupoids. A **homomorphism of groupoids** $\mathcal{G} \to \mathcal{G}'$ is a Borel map $\varphi \colon \mathcal{G} \to \mathcal{G}'$ which commutes with the source and range maps and such that $\forall (g_1, g_2) \in \mathcal{G}^{(2)}, \varphi(g_1 \star g_2) = \varphi(g_1) \star \varphi(g_2)$.

Definition 5.2.2. Let $\mathcal{G}$ be a groupoid on a standard probability space (X, μ).

- A **measure-preserving $\mathcal{G}$-action** is a homomorphism $\alpha\colon \mathcal{G} \to \mathrm{Aut}(Y, \nu)$ where (Y, ν) is a standard probability space, possibly with atoms.
- A **unitary representation** of $\mathcal{G}$ is a homomorphism $\pi\colon \mathcal{G} \to \mathcal{U}(\mathcal{H})$, where $\mathcal{U}(\mathcal{H})$ is the unitary group of a separable Hilbert space $\mathcal{H}$.
- An **affine isometric action** of $\mathcal{G}$ is a homomorphism $\rho\colon \mathcal{G} \to \mathrm{Iso}(\mathcal{H})$, where $\mathrm{Iso}(\mathcal{H})$ denotes the group of affine isometries of a separable Hilbert space $\mathcal{H}$.

Remark 5.2.3. One could get another notion of action by considering Borel actions on the set Y which preserve the measure ν, rather that automorphisms of the measure algebra of (Y, ν). A **pmp point action** of $\mathcal{G}$ on (Y, ν) is a homomorphism $\alpha\colon \mathcal{G} \to \mathfrak{S}(Y)$ (the symmetric set over Y) such that the induced map $\mathcal{G} \times Y \to Y$ is Borel and for all $g \in \mathcal{G}$ and $A \subseteq Y$, $\nu(\alpha(g)^{-1}(A)) = \nu(A)$.

As we already noted, the inverse image map of a pmp point action of $\mathcal{G}$ on (Y, ν) induces a measure-preserving $\mathcal{G}$-action on $\mathrm{MAlg}(Y, \nu)$ and in the context of countable groups, every action on a measure algebra arises from an action on points. We prove in Proposition 5.2.13 that this is also true in the setting of discrete pmp groupoids.

Example 5.2.4. Let $\mathcal{G}$ be an ergodic pmp groupoid on (X, μ) and (Y, ν) be a standard probability space. We describe the standard construction of the **Bernoulli shift** of $\mathcal{G}$ over (Y, ν). Even though in this thesis we chose to represent groupoid actions on a single standard probability space after identification of all the fibers, such actions arise naturally as actions on a bundle of standard probability spaces rather than on a probability space (see [AD05]). Of course those two settings are equivalent. We give a description of the Bernoulli shift on the bundle $\bigsqcup_{x \in X} Y^{\mathcal{G}_x}$. One could obtain a version of the Bernoulli shift on a single probability space after the choice of an identification of all the $Y^{\mathcal{G}_x}$, however there is no canonical such identification for a general groupoid.

By the Lusin-Novikov theorem, fix a sequence $(t_k)_{k \in \mathbb{N}}$ of Borel partial transversals for s such that $\bigcup_{k \in \mathbb{N}} t_k = \mathcal{G}$. We construct a partition of $\mathcal{G}$ into Borel total transversals (T_n) for s as follows:

Suppose we already defined $T_0, \dots, T_{n-1}$ pairwise disjoint Borel total transversals for s. We define a Borel transversal T_n for s by letting $T_n \star x := t_m \star x$ for the least $m \in \mathbb{N}$ such that $t_m \star x \notin \bigcup_{k=0}^{n-1} T_k$ if such a m exists, and $T_n \star x = \emptyset$ otherwise. Note that the mapping of x to this m is a Borel map, so that T_n is indeed Borel. It is clear from its definition that T_n is disjoint to $\bigcup_{k=0}^{n-1} T_k$ and T_n is a transversal for s.

Moreover, T_n is either total or empty. Indeed, since $\bigcup_{k \in N} t_k = \mathcal{G}$, if $T_n \star x = \emptyset$, then $\mathcal{G}_x = \{T_0 \star x, \dots, T_{n-1} \star x\}$ and therefore we have $|\mathcal{G}_x| = n$ if and only if $T_n \star x = \emptyset$. However, the set of units x such that $|\mathcal{G}_x| = n$ is $[\![\mathcal{G}]\!]$-invariant and $\mathcal{G}$ is ergodic so the measure of this set has to be 0 or 1, or in other words, T_n is either total or empty.

We repeat this process for all $n \in \mathbb{N}$. Let us prove that the nonempty sets of the form T_n constructed partition $\mathcal{G}$ into Borel total transversals for s. From all our previous remarks, it only remains to observe that for $m \in \mathbb{N}$, almost surely $t_m \star x$ must be added in some T_n for $n \leqslant m$, and thus $\bigcup_{k \leqslant n} t_k \subseteq \bigcup_{k \leqslant n} T_k$, leading to $\bigcup_{n \in \mathbb{N}} T_n = \mathcal{G}$, hence the conclusion.

According to the latter construction, $\bigsqcup_{x\in X} Y^{\mathcal{G}_x}$ inherits a standard Borel structure via the bijection $\Phi\colon \bigsqcup_{x\in X} Y^{\mathcal{G}_x} \to Y^N \times X$, where $N \in \mathbb{N} \cup \{\mathbb{N}\}$ is the size of the sequence (T_n) (or in other words, the cardinal of almost all fibers of $\mathcal{G}$), defined by

$$\Phi(f) = (n \mapsto f(T_n \star d(f)), x) \text{ where } d(f) \text{ is the only } x \in X \text{ such that } f \in Y^{\mathcal{G}_x}.$$

This structure does not depend on the choice of (T_n). Indeed, if (S_n) is another Feldman-Moore sequence and Φ_T, Φ_S represent the respective injections $\bigsqcup_{x\in X} Y^{\mathcal{G}_x} \to Y^N \times X$, we can construct a Borel bijection $\sigma\colon Y^N \times X \to Y^N \times X$ such that $\Phi_T = \sigma \circ \Phi_S$ by letting

$$\sigma(f, x) = (n \mapsto f(m_{n,x}), x) \text{ where } m_{n,x} \text{ is the only } m < N \text{ such that } T_n \star x = S_m \star x.$$

Now we define a Borel map $\alpha\colon \mathcal{G} {}_r{*}_d \bigsqcup_{x\in X} Y^{\mathcal{G}_x} \to \bigsqcup_{x\in X} Y^{\mathcal{G}_x}$. For $f \in \bigsqcup_{x\in X} Y^{\mathcal{G}_x}$ and $g_0 \in \mathcal{G}^x$, $g_0 \cdot f\colon g \mapsto g_0 \cdot f(g) = f(g \star g_0)$. Almost surely, for $g_0 \in \mathcal{G}^x$ given, the map $\alpha(g_0, \cdot)$ sends the pushforward of the measure $\nu^{\otimes N}$ to $Y^{\mathcal{G}_x}$ on the pushforward of the measure $\nu^{\otimes N}$ to $Y^{\mathcal{G}_{r(g_0)}}$. Such a map is what we call a measure preserving action on a standard probability bundle. We call this action **the Bernoulli shift** over (Y, ν).

5.2.1 Full actions of full groups

We now want to compare actions of a groupoid $\mathcal{G}$ with actions of its full group $[\mathcal{G}]$. As we will see, any $\mathcal{G}$-action induces a $[\mathcal{G}]$-action. However, not all $[\mathcal{G}]$-actions arise from this construction. In this section we give a necessary and sufficient condition for an action of $[\mathcal{G}]$ to arise in this manner.

Definition 5.2.5. Let G be a group. A boolean G-action is an action of G by isomorphisms on a probability algebra, or equivalently, a group morphism $G \to \mathrm{Aut}(Y, \nu)$ for some probability algebra $\mathrm{MAlg}(Y, \nu)$.

We identify $\mathrm{MAlg}(X \times Y, \mu \times \nu)$ to the space $\mathrm{L}^0(X, \mu, \mathrm{MAlg}(Y, \nu))$ through the Borel map $A \mapsto (x \mapsto A_x)$, where $A_x = \{y \in Y : (x, y) \in A\}$. Then every $\mathcal{G}$-action $\alpha\colon \mathcal{G} \to \mathrm{Aut}(Y, \nu)$ induces a boolean $[\mathcal{G}]$-action $[\alpha]$ on $\mathrm{MAlg}(X \times Y, \mu \times \nu)$ defined by: For $f \in \mathrm{L}^0(X, \mu, \mathrm{MAlg}(Y, \nu))$, $T \in [\mathcal{G}]$ and almost all $x \in X$,

$$[\alpha](T)f\,(x) = \alpha(T \star x) f(\mathfrak{l}(T)x)$$

Moreover, note that the natural inclusion $\mathrm{MAlg}(X, \mu) \hookrightarrow \mathrm{MAlg}(X \times Y, \mu \times \nu)$ is a factor from $[\alpha]$ to $\mathfrak{l}$. This is a necessary condition for an action of $[\mathcal{G}]$ to be of the form $[\alpha]$ for a $\mathcal{G}$-action α. In the next definition, we isolate a stronger necessary condition for a boolean action of the full group to arise from an action of the groupoid. We prove in Theorem 5.2.12 that this condition is in fact sufficient.

Remark 5.2.6. Note that when a $\mathcal{G}$-action α^{-1} arises from a pmp point action α on (Y, ν), then $[\alpha^{-1}]$ can be defined as a point action on $(X \times Y, \mu \times \nu)$ via the simpler formula

$$[\alpha^{-1}](T)(x, y) = (\mathfrak{l}(T)x, \alpha(T \star x)(y)).$$

Definition 5.2.7 (Full action)**.** Let $\mathcal{G}$ be a pmp groupoid on (X, μ), let $G \leqslant [\mathcal{G}]$, let (Y, ν) be a standard probability space and let $\varphi\colon \mathrm{MAlg}(X, \mu) \to \mathrm{MAlg}(Y, \nu)$ an embedding. A boolean action $G \overset{\rho}{\curvearrowright} \mathrm{MAlg}(Y, \nu)$ is called a **full action** for φ if φ is a factor map from ρ to $\mathfrak{l}$, meaning that $\varphi\mathfrak{l} = \rho\varphi$, and for any $T \in G$, $A \in \mathrm{MAlg}(X, \mu)$ such that $T \star A = A$, we have $\rho(T)_{\restriction \varphi(A)} = \mathrm{id}_{\varphi(A)}$.

Remark 5.2.8. The condition $T \star A = A$ means that T is trivial when restricted to $s^{-1}(A)$. Multiplying both sides of this equality by an element of G to the left shows that, equivalently, a boolean G-action ρ is full if and only if φ is a factor map from ρ to $\mathfrak{l}$ and for any $T, S \in G$, $A \in \mathrm{MAlg}(X, \mu)$ such that $T \star A = S \star A$, we have $\rho(T)_{\restriction \varphi(A)} = \rho(S)_{\restriction \varphi(A)}$.

This remark justifies the terminology used. A full action of a full group is an action respecting the structure of group as well of full group, meaning that if $T \in G$ can be obtained by "cutting and pasting" a sequence $(T_n)_{n\in\mathbb{N}}$ respectively to a partition $(A_n)_{n\in\mathbb{N}}$ of X, then T must act as T_n does when restricted to A_n.

Definition 5.2.9. Let $\mathcal{G}$ be a pmp groupoid on (X, μ). The **full closure** $[G]$ of a subgroup G of $[\mathcal{G}]$ is the subgroup of $[\mathcal{G}]$ consisting of elements obtained by "cutting and pasting" elements of G. Precisely, $[G]$ is the set of elements $T \in [\mathcal{G}]$ defined by $T \star x = T_n \star x$ (again this is the pointwise product) for all $x \in A_n$, whenever (A_n) is a partition of X and (T_n) is a sequence of elements of G such that $(\mathfrak{l}(T_n)(A_n))$ is a partition of X.

Note that the full closure of $[G]$ is simply the full closure of G, or in other words, $[G]$ is stable by the operation of "cutting and pasting". It follows that the full closure of G is also the smallest subgroup of $[\mathcal{G}]$ containing G and stable by "cutting and pasting".

Remark 5.2.10. Consider a countable Feldman-Moore sequence $(T_n)_{n\in\mathbb{N}}$ (see Proposition 5.1.10). Then the group Γ generated by (T_n) is a countable subgroup of $[\mathcal{G}]$ and we have $[\Gamma] = [\mathcal{G}]$.

Indeed, for $T \in [\mathcal{G}]$, we have $T \subseteq \bigcup_{n\in\mathbb{N}} T_n$, therefore, letting $B_n = \{x \in X : T \star x = T_n \star x\}$, we have $\bigcup_{n\in\mathbb{N}} B_n = X$. We define a Borel partition $(A_n)_{n\in\mathbb{N}}$ of X such that $A_n \subseteq B_n$ by letting $A_n := B_n \setminus \bigcup_{k<n} B_k$. Then T can be obtained by cutting and pasting" the sequence $(T_n)_{n\in\mathbb{N}}$ respectively to $(A_n)_{n\in\mathbb{N}}$ so that $T \in [\Gamma]$.

We call such a group a **countable full generator** of $[\mathcal{G}]$. We will use such subgroups recurrently in the following, in order to use already existing tools for countable groups in the context of pmp groupoids.

Proposition 5.2.11. *Let $G \leqslant [\mathcal{G}]$. Suppose $\rho\colon G \to \mathrm{Aut}(Y, \nu)$ is a measure-preserving boolean action which is full over $\varphi\colon \mathrm{MAlg}(X, \mu) \to \mathrm{MAlg}(Y, \nu)$. Then ρ has a unique extension to a boolean $[G]$-action $\rho^{[G]}$ full over φ.*

Proof. Let $T \in [G]$, by definition there is a partition $(A_n)_{n\in\mathbb{N}}$ of X and a sequence $(T_n)_{n\in\mathbb{N}}$ of elements of G such that for every $n \in \mathbb{N}$, $T \star A_n = T_n \star A_n$. By equivariance and the fact that φ is a measure algebra embedding, we have that $(\rho(T_n)\varphi(A_n))$ is a partition of (Y, ν). Since we want our extension to be full, the only reasonable candidate for $\rho^{[G]}(T)$ must be defined by:

$$\rho^{[G]}(T)(y) = \rho(T_n)(y) \text{ for all } y \in \varphi(A_n),$$

which also shows uniqueness of the extension. Note that $\rho^{[G]}(T)$ is well-defined as a consequence of the fullness of ρ. The fact that $\rho^{[G]}$ is indeed an action and φ is still a factor map for this extension then follows from standard arguments. □

We now prove that every full action comes from a measure-preserving action of the underlying pmp groupoid.

Theorem 5.2.12. *Let $\mathcal{G}$ be a measure-preserving ergodic groupoid on (X, μ). Let $\rho\colon [\mathcal{G}] \to \mathrm{Aut}(Y, \nu)$ be an ergodic full boolean action. Then ρ is conjugate to an action of the form $[\alpha]$ for some measure-preserving action α of $\mathcal{G}$ on a standard probability space (Z, η).*

Proof. Let $\pi^{-1}\colon \mathrm{MAlg}(X,\mu) \hookrightarrow \mathrm{MAlg}(Y,\nu)$ be a embedding for which ρ is full and let Γ be a countable full generator of $[\mathcal{G}]$. By [Gla03, Thm. 3.18], the dynamical system $\Gamma \overset{\rho}{\curvearrowright} (Y,\nu)$ is isomorphic via Ψ to a skew-product of the form $\Gamma \overset{\tilde\rho}{\curvearrowright} (X \underset{\tilde\alpha}{\times} Z, \mu\times\eta)$ for some standard probability space (Z,η) and a Borel cocycle $\tilde\alpha\colon \Gamma\times X \to \mathrm{Aut}(Z,\eta)$ with $\Psi(\pi^{-1}(x)) = \{x\}\times Z$ for all $x\in X$. Explicitly, $\tilde\alpha$ is a Borel map such that $\tilde\alpha(\gamma', \mathfrak{l}(\gamma)x)\cdot \tilde\alpha(\gamma,x) = \tilde\alpha(\gamma'\star\gamma, x)$ and Γ acts on $X\times Z$ via the formula

$$\tilde\rho(\gamma)(x,z) = (\mathfrak{l}(\gamma)x, \tilde\alpha(\gamma,x)z)$$

Moreover, for $\gamma,\gamma'\in\Gamma$ and $A\in\mathrm{MAlg}(X,\mu)$, if $\gamma\star A = \gamma'\star A$, then $\rho(\gamma)_{\restriction \pi^{-1}(A)} = \rho(\gamma')_{\restriction\pi^{-1}(A)}$. Equivalently, $\tilde\rho(\gamma)_{\restriction A\times Z} = \tilde\rho(\gamma')_{\restriction A\times Z}$ and thus $\tilde\alpha(\gamma,x) = \tilde\alpha(\gamma',x)$ for $x\in A$.

This remark allows us to get a well defined Borel map $\alpha\colon\mathcal{G}\to\mathrm{Aut}(Z,\eta)$ by letting $\alpha(\gamma\star x) = \tilde\alpha(\gamma,x)$. Moreover, for $x\in X$ and $\gamma,\gamma'\in\Gamma$, we have

$$\begin{aligned}\alpha(\gamma'\star\mathfrak{l}(\gamma)x)\cdot\alpha(\gamma\star x) &= \tilde\alpha(\gamma',\mathfrak{l}(\gamma)x)\tilde\alpha(\gamma,x)\\ &= \tilde\alpha(\gamma'\star\gamma,x)\\ &= \alpha(\gamma'\star\gamma\star x)\\ &= \alpha\left((\gamma'\star\mathfrak{l}(\gamma)x)\star(\gamma\star x)\right),\end{aligned}$$

which proves that α is a measure-preserving $\mathcal{G}$-action on $\mathrm{MAlg}(Z,\eta)$.

Finally, after identification between $\mathrm{MAlg}(X\times Z,\mu\times\eta$ and $\mathrm{L}^0(X,\mu,\mathrm{MAlg}(Z,\eta))$, for $\gamma\in\Gamma$, $x\in X$ and $f\in\mathrm{L}^0(X,\mu,\mathrm{MAlg}(Z,\eta))$, we have

$$\begin{aligned}\tilde\rho(\gamma)f\,(x) &= \tilde\alpha(\gamma,x)f(\mathfrak{l}(\gamma)x)\\ &= \alpha(\gamma\star x)f(\mathfrak{l}(\gamma)x)\\ &= [\alpha](\gamma)f\,(x).\end{aligned}$$

We conclude that ρ is conjugate to an action of the form $[\alpha]$ for a measure-preserving $\mathcal{G}$-action α. ☐

Finally, we describe an explicit construction to recover a point action inducing a given measure-preserving $\mathcal{G}$-action α from the associated full action $[\alpha]$:

Proposition 5.2.13. *Let α be a measure-preserving $\mathcal{G}$-action on* $\mathrm{MAlg}(Y,\nu)$. *Then there exists a pmp point action β of $\mathcal{G}$ on (Y,ν) such that for almost any $g\in\mathcal{G}$, $\forall A\subseteq Y,\ \alpha(g)\left(\overline{A}^{\nu}\right) = \overline{\beta(g)^{-1}(A)}^{\nu}$.*

Proof. Let $\Gamma = \{T_n : n\in\mathbb{N}\}$ be a countable full generator for $[\mathcal{G}]$. Let us consider $\Gamma \overset{[\alpha]_{\restriction\Gamma}}{\curvearrowright} \mathrm{MAlg}(Y,\nu)$.

Since Γ is countable, this action arises from a point action $\Gamma\overset{\rho}{\curvearrowright}(Y,\nu)$. We define a map $\beta\colon\mathcal{G}\times Y\to Y$ as follows:

$$\beta(g,y) \coloneqq \rho(T_n)(s(g),y)) \text{ for the least } n\in\mathbb{N} \text{ such that } T_n\star s(g) = g.$$

It follows from the fullness of $[\alpha]_{\restriction\Gamma}$ that, up to a null set, β is a pmp point action of $\mathcal{G}$ on (Y,ν) which induces a measure-preserving $\mathcal{G}$-action β^{-1} on $\mathrm{MAlg}(Y,\nu)$. Moreover, β was constructed so that $[\beta^{-1}]_{\restriction\Gamma} = [\alpha]_{\restriction\Gamma}$, therefore $[\beta^{-1}] = [\alpha]$.

We conclude that $\alpha = \beta^{-1}$, and thus that β is the wanted pmp point action, by proving that the map $\alpha\mapsto[\alpha]$ is an injection.

Let $\alpha_1 \neq \alpha_2$ be two $\mathcal{G}$-actions on (Y, ν), then $\mu(\{x \in X : \exists g \in \mathcal{G}_x, \alpha_1(g) \neq \alpha_2(g)\}) > 0$. Thus there must be some nonzero $B \in \mathrm{MAlg}(Y, \nu)$ and $T \in [\mathcal{G}]$ such that

$$\mu\left(\{x \in X : \alpha_1(T \star x)(B) \neq \alpha_2(T \star x)(B)\}\right) > 0.$$

Let $A = \{x \in X : \alpha_1(T \star x)(B) \neq \alpha_2(T \star x)(B)\}$, then it follows that

$$\mu \times \nu([\alpha_1](T)(A \times B) \triangle [\alpha_2](T)(A \times B)) \geqslant \int_A \nu(\alpha_1(T \star x)(B) \triangle \alpha_2(T \star x)(B)) d\mu(x) > 0,$$

therefore $[\alpha_1] \neq [\alpha_2]$ and thus $\alpha \mapsto [\alpha]$ is an injection. □

5.2.2 Full unitary representations of the full group

We will now obtain an analogous result in the setup of unitary representations.

By an $\mathrm{L}^\infty(X, \mu)$**-module**, we mean a Hilbert space $\mathcal{K}$ an a *faithful* normal $*$-representation of $\mathrm{L}^\infty(X, \mu)$ in $\mathcal{B}(\mathcal{K})$. Given such a module, we identify $\mathrm{L}^\infty(X, \mu)$ to its image in $\mathcal{B}(\mathcal{K})$. For all $A \subseteq X$, we denote by $\mathcal{K}_A$ the subspace $\chi_A \mathcal{K}$.

Our motivating example of $\mathrm{L}^\infty(X, \mu)$-modules is provided by unitary representations of pmp groupoids. Given unitary representation $\pi \colon \mathcal{G} \to \mathcal{U}(\mathcal{H})$ of a pmp groupoid $\mathcal{G}$ on (X, μ), a **section** of the representation is a Borel map $\xi \colon X \to \mathcal{H}$. It is called **square-integrable** when $\int_X \|\xi(x)\|^2 d\mu(x)$ is finite. The space of square-integrable sections is naturally a Hilbert space for the scalar product $\langle \xi, \eta \rangle = \int_X \langle \xi(x), \eta(x) \rangle \, d\mu(x)$, and we denote this Hilbert space by $\mathrm{L}^2(X, \mu, \mathcal{H})$. Observe that $\mathrm{L}^2(X, \mu, \mathcal{H})$ has a natural $\mathrm{L}^\infty(X, \mu)$-module structure with $\mathrm{L}^\infty(X, \mu)$ acting by multiplication on sections. Moreover, one can associate to π a unitary representation $[\pi]$ of the full group of $\mathcal{G}$ on $\mathrm{L}^2(X, \mu, \mathcal{H})$: for all $T \in [\mathcal{G}]$ and all $\xi \in \mathrm{L}^2(X, \mu, \mathcal{H})$, we let

$$([\pi](T)\xi)(\mathfrak{t}(T)x) = \pi(T \star x)\xi(x).$$

Here are the properties of $[\pi]$ which will allow us to reconstruct π.

Definition 5.2.14. Let $\mathcal{G}$ be a pmp groupoid on (X, μ), let $\rho \colon [\mathcal{G}] \to \mathcal{U}(\mathcal{K})$ be a unitary representation on a Hilbert space $\mathcal{K}$ equipped with an $\mathrm{L}^\infty(X, \mu)$-module structure. Say that ρ is **full** when it satisfies the following two conditions:

(i) For all $A \subseteq X$ Borel, $\rho(T)\chi_A = \chi_{\mathfrak{t}(T)A}\rho(T)$;

(ii) For all $A \subseteq X$, and all $T, S \in [\mathcal{G}]$ such that $T \star A = S \star A$, we have $\rho(T)_{\restriction \mathcal{K}_A} = \rho(S)_{\restriction \mathcal{K}_A}$.

Note that the first condition yields that $\rho(T)\mathcal{K}_A = \mathcal{K}_{\mathfrak{t}(T)A}$ for all $T \in [\mathcal{G}]$, while the second is equivalent to asking that $\rho(T)_{\restriction \mathcal{K}_A} = \mathrm{id}_{\mathcal{K}_A}$ for all $T \in [\mathcal{G}]$ and $A \in \mathrm{MAlg}$ such that $T \star A = A$.

Theorem 5.2.15. *Let $\mathcal{G}$ be an ergodic pmp groupoid on (X, μ). Every full unitary representation $\rho \colon [\mathcal{G}] \to \mathcal{U}(\mathcal{K})$ is unitarily equivalent to a unitary representation of the form $[\pi]$ for some unitary representation π of $\mathcal{G}$.*

Proof. By [KR91, Thm. 14.2.1], the $\mathrm{L}^\infty(X, \mu)$-module $\mathcal{K}$ can be decomposed as a direct integral of Hilbert spaces $(\mathcal{H}_x)_{x \in X}$. By condition (i) and ergodicity the dimension of $\mathcal{H}_x$ is almost surely constant. We can thus assume $\mathcal{H}_x$ is constant equal to a fixed Hilbert space $\mathcal{H}$ and so $\mathcal{K} = \mathrm{L}^2(X, \mu, \mathcal{H})$.

Denote by $\lambda\colon [\mathcal{G}] \to \mathrm{L}^2(X,\mu,\mathcal{H})$ the unitary representation defined by $(\lambda(T)\xi)(\mathfrak{l}(T)x) = \xi(x)$. Then for all $A \subseteq X$ Borel, we have $\lambda(T)\chi_A = \chi_{T(A)}\lambda(T)$. So by condition (i), for every $T \in [\mathcal{G}]$, the unitary $\lambda(T)^*\rho(T)$ commutes with $\mathrm{L}^\infty(X,\mu)$ and is thus a decomposable operator

$$\lambda(T)^*\rho(T) = \int_X c(T,x)\ d\mu(x),$$

where $c(T,x) \in \mathcal{U}(\mathcal{H})$ for almost all $x \in X$. Observe that by uniqueness of the decomposition of a decomposable operator, for any $S \in [\mathcal{G}]$ we have $\lambda(S) \int_X c(T,x)\ d\mu(x)\lambda(S)^* = \int_X c(T,\mathfrak{l}(S)^{-1}x)\ d\mu(x)$. When $T_1, T_2 \in [\mathcal{G}]$, we thus have

$$\begin{aligned}
\lambda(T_1T_2)^*\rho(T_1T_2) &= \lambda(T_2)^*\lambda(T_1)^*\rho(T_1)\rho(T_2) \\
&= \lambda(T_2)^* \int_X c(T_1,x)\ d\mu(x)\rho(T_2) \\
&= \int_X c(T_1,\mathfrak{l}(T_2)x)\ d\mu(x)\lambda(T_2)^*\rho(T_2) \\
&= \int_X c(T_1,\mathfrak{l}(T_2)x)c(T_2,x)\ d\mu(x),
\end{aligned}$$

so for almost all $x \in X$, $c(T_1,\mathfrak{l}(T_2)x)c(T_2,x) = c(T_1T_2,x)$. We now pick a countable full generator Γ of $[\mathcal{G}]$. By restricting to a full measure set, we may assume that the above cocycle relation holds for all $\gamma \in \Gamma$ and all $x \in X$.

Furthermore by condition (ii), for $A \subseteq X$ and $\gamma \in \Gamma$, if $\gamma \star A = A$ then $\rho(\gamma)$ is trivial when restricted to $\mathcal{K}_A$, so up to restricting again we have $c(\gamma,x) = \mathrm{id}_{\mathcal{H}}$ for all $\gamma \in \Gamma$ and all $x \in X$ such that $\gamma \star x = x$. It follows that we have a well defined unitary representation $\pi\colon \mathcal{G} \to \mathcal{U}(\mathcal{H})$ by letting $\pi(\gamma \star x) = c(\gamma,x)$. By construction we have $\rho(T) = \lambda(T) \int_X c(T,x)\ d\mu(x)$, so for all $\xi \in \mathrm{L}^2(X,\mu,\mathcal{H})$ and all $x_0 \in X$ we have

$$\begin{aligned}
\rho(T)(\xi)(\mathfrak{l}(T)x_0) &= \lambda(T)\left(\int_X c(T,x)\ d\mu(x)\right)\xi(\mathfrak{l}(T)x_0) \\
&= \left(\int_X c(T,x)\ d\mu(x)\right)\xi(x_0) \\
&= c(T,x_0)\xi(x_0).
\end{aligned}$$

Now if we pick $\gamma \in \Gamma$ such that $\gamma \star x_0 = T \star x_0$ we have by condition (ii)

$$c(T,x_0)\xi(x_0) = c(\gamma,x_0)\xi(x_0) = \pi(\gamma \star x_0)\xi(x_0) = \pi(T \star x_0)\xi(x_0).$$

We thus have $\rho = [\pi]$ as wanted. □

5.3 Property (T)

5.3.1 First definition and strong ergodicity

Let $\mathcal{G}$ be a measure-preserving groupoid on (X,μ). Given a unitary representation $\pi\colon \mathcal{G} \to \mathcal{U}(\mathcal{H})$, a **unit section** is a section $\xi\colon X \to \mathcal{H}$ satisfying that for all $x \in X$, $\|\xi(x)\| = 1$.

We say that a sequence (ξ_n) of unit sections is **almost invariant** when the sequence of functions $g \in \mathcal{G} \mapsto \|\pi(g)\xi_n(s(g)) - \xi_n(r(g))\|$ converges pointwise to zero, up to restricting to a full measure subset of $\mathcal{G}$.

Remark 5.3.1. Equivalently, one could take a weaker definition for almost invariant unit sections by asking that the sequence of functions $g \mapsto \|\pi(g)\xi_n(s(g)) - \xi_n(r(g)))\|$ converges to zero *in measure*. Since every sequence converging in measure to zero has a subsequence which converges pointwise to zero, this does not affect the definition of property (T) (see [AD05, Lem. 4.1]).

Definition 5.3.2. A groupoid $\mathcal{G}$ has **property (T)** if whenever a unitary representation π of $\mathcal{G}$ has almost invariant unit sections, then it has an invariant unit section, i.e. we can find a section ξ such that for almost all $g \in \mathcal{G}$, we have $\pi(g)\xi(s(g)) = \xi(r(g))$.

Remark 5.3.3. An ergodic pmp groupoid $\mathcal{G}$ has property (T) if and only if every unitary representation with almost invariant unit sections has a nonzero invariant section. Indeed such a section must have an invariant norm which is thus constant by ergodicity. This constant must be non zero, so by rescaling we obtain an invariant unit section.

We now connect property (T) to the fundamental notion of strong ergodicity.

For a pmp equivalence relation $\mathcal{R}$ on (X, μ), we identify the classical notion of full group of an equivalence relation $\{T \in \mathrm{Aut}(X, \mu) : \forall^* x \in X, (T(x), x) \in \mathcal{R}\}$ with the full group $[\mathcal{R}]$ of $\mathcal{R}$ seen as a groupoid, defined in Definition 5.1.14, via the map $T \mapsto \mathrm{Graph}(T^{-1})$.

A sequence (A_n) of subsets of X is called **asymptotically invariant** if for all $T \in [\mathcal{R}]$, $\mu(T(A_n) \triangle A_n) \to 0$, and **asymptotically trivial** if $\mu(A_n)(1 - \mu(A_n)) \to 0$.

Remark 5.3.4. Given a pmp groupoid $\mathcal{G}$ on a standard probability space (X, μ), the **induced equivalence relation** $\mathcal{R}_\mathcal{G}$ on (X, μ) is the orbit equivalence relation of the action $[\mathcal{G}] \overset{\wr}{\curvearrowright} (X, \mu)$. It is characterized by $(x, x') \in \mathcal{R}_\mathcal{G} \Leftrightarrow \exists g \in \mathcal{G}, s(g) = x \wedge r(g) = x'$.

Remark that the action $\wr$ can therefore be seen as a morphism $[\mathcal{G}] \to [\mathcal{R}_\mathcal{G}]$, and this morphism is surjective. Indeed, if $S \in [\mathcal{R}_\mathcal{G}]$, then for almost any $x \in X$, there exists $g \in \mathcal{G}_x$ such that $r(g) = S(x)$. Consider a Feldman-Moore sequence $(T_n)_{n \in \mathbb{N}}$ for $\mathcal{G}$ and then for $n \in \mathbb{N}$, let $A_n = \{x \in X : n \text{ is the least integer such that } (T_n \star x) = S(x)\}$. We define $T = \bigsqcup_{n \in \mathbb{N}} T_n \star A_n$, then we have $\wr(T) = S$.

Definition 5.3.5.

1. A pmp equivalence relation $\mathcal{R}$ on a standard probability space (X, μ) is **strongly ergodic** if every asymptotically invariant sequence of subsets of X has to be asymptotically trivial.
2. A pmp groupoid $\mathcal{G}$ is **strongly ergodic** if the induced equivalence relation $\mathcal{R}_\mathcal{G}$ is.
3. For a pmp groupoid $\mathcal{G}$ on a standard probability space (X, μ), a pmp $\mathcal{G}$-action α on a standard probability space (Y, ν) is **strongly ergodic** if any sequence in $(X \times Y, \mu \times \nu)$ asymptotically invariant by $[\alpha]$ is asymptotically in (X, μ) (according to the natural embedding $\mathrm{MAlg}(X, \mu) \hookrightarrow \mathrm{MAlg}(X \times Y, \mu \times \nu)$).

By considering, for a $\mathcal{R}$-invariant set A, the constant sequence equal to A, we see that strong ergodicity of a pmp groupoid implies ergodicity. We now note that for an ergodic pmp groupoid, property (T) implies strong ergodicity.

Proposition 5.3.6 (Folklore)**.** *Let $\mathcal{R}$ be an ergodic pmp equivalence relation with property (T). Then $\mathcal{R}$ is strongly ergodic.*

Proof. Suppose by contradiction that $\mathcal{R}$ has (T) but is not strongly ergodic. Denote by E_0 the cofinite equivalence relation on $\{0,1\}^{\mathbb{N}}$, defined by $(x_i)E_0(y_i)$ if and only if the $x_i = y_i$ for all but finitely many indices $i \in \mathbb{N}$. We equip $\{0,1\}^{\mathbb{N}}$ with the Bernoulli product measure $\nu := (\frac{1}{2}(\delta_0 + \delta_1))^{\otimes\mathbb{N}}$. By [JS87], there is a measure-preserving map $\pi\colon (X,\mu) \to (\{0,1\}^{\mathbb{N}}, \nu)$ such that $(\pi \times \pi)(\mathcal{R}) = E_0$.

For every $n \in \mathbb{N}$, let $\mathcal{S}_n$ be the equivalence relation on $\{0,1\}^{\mathbb{N}}$ defined by $(x_i)\mathcal{S}_n(y_i)$ if and only if $x_i = y_i$ for all $i > n$. Observe that $E_0 = \bigcup_{n\in\mathbb{N}} \mathcal{S}_n$, so if $\mathcal{R}_n := \pi \times \pi^{-1}(\mathcal{S}_n)$ then $\mathcal{R} = \bigcup_{n\in\mathbb{N}} \mathcal{R}_n$.

But since $\mathcal{R}$ has (T) it is not *approximable*: there is $n \in \mathbb{N}$ and a non-negligible Borel set $A \subseteq X$ such that $\mathcal{R} \cap (A \times A) = \mathcal{R}_n \cap (A \times A)$ (see [Pic07, Prop. 16]; see also [GT16] for more on approximability). Now observe that each $\mathcal{S}_n$ has diffuse ergodic decomposition because it has only finite classes, hence so does each $\mathcal{R}_n$. However $\mathcal{R}$ is ergodic, so cannot have $\mathcal{R} \cap (A \times A) = \mathcal{R}_n \cap (A \times A)$, a contradiction. ☐

Then we have

Proposition 5.3.7. *Let $\mathcal{G}$ be an ergodic pmp groupoid on (X,μ) with property (T), then $\mathcal{G}$ is strongly ergodic.*

Proof. Consider the equivalence relation $\mathcal{R}_{\mathcal{G}}$ induced by $\mathcal{G}$. By definition $\mathcal{G}$ is strongly ergodic if and only if $\mathcal{R}_{\mathcal{G}}$ is. Moreover, $\mathcal{G}$ has property (T) so $\mathcal{R}_{\mathcal{G}}$ has property (T) as well by [AD05, Theorem 5.18]. Since it is an ergodic pmp equivalence relation, it is strongly ergodic, hence the conclusion. ☐

Our aim is now to give a notion of Kazhdan pair (and triple) which provides us more quantitative versions of property (T) as in [BdlHV08, Prop. 1.1.9]. Such a characterization of property (T) will be given purely in terms of full unitary representations of the full group, using implicitely Theorem 5.2.15. This will rely crucially on the following spectral gap characterization of strong ergodicity due to Houdayer, Marrakchi and Verraedt [HMV17].

Theorem 5.3.8 (Houdayer, Marrakchi, Verraedt)**.** *Let $\mathcal{R}$ be a strongly ergodic pmp equivalence relation. Then there is a finite set $F \subseteq [\mathcal{R}]$ and $\kappa > 0$ such that for all $\eta \in \mathrm{L}^2(X,\mu)$,*

$$\left\|\eta - \int_X \eta(x)d\mu(x)\right\|^2 \leqslant \kappa \sum_{T\in F} \left\|\eta - \eta \circ T^{-1}\right\|^2.$$

When we have a finite set $F \subseteq [\mathcal{R}]$ and $\kappa > 0$ as above, we say that (F,κ) is a **spectral gap pair** for $\mathcal{R}$.

When $\mathcal{G}$ is a pmp groupoid on a standard proability space (X,μ), for $F \subseteq [\mathcal{G}]$ and $\kappa > 0$ we say that (F,κ) is a **spectral gap pair** for $\mathcal{G}$ when $(\mathfrak{l}(F),\kappa)$ is a spectral gap pair for $\mathcal{R}_{\mathcal{G}}$. Note that the latter theorem and the surjectivity of $\mathfrak{l}$ imply that if $\mathcal{G}$ is strongly ergodic, then it admits a spectral gap pair.

Kazhdan pairs and proximity of invariant vectors

As explained before, one can associate to every unitary representation π of a pmp groupoid $\mathcal{G}$ a unitary representation $[\pi]$ of its full group on the space of square integrable sections: for all $T \in [\mathcal{G}]$ and all square-integrable section ξ, we let

$$([\pi](T)\xi)(\mathfrak{l}(T)x)) = \pi(T \star x)\xi(x).$$

Observe that a square-integrable section is $[\pi]$-invariant if and only if it is a π-invariant section. Given a unitary representation $\pi\colon G \to \mathcal{U}(\mathcal{H})$ of a group G and a finite subset F of G, we say that a vector $\xi \in \mathcal{H}$ is **(F,ϵ)-invariant** if for all $g \in F$,

$$\|\pi(g)\xi - \xi\| < \epsilon \|\xi\| .$$

Definition 5.3.9. Let $\mathcal{G}$ be a pmp groupoid, let F be a finite subset of $[\mathcal{G}]$ and let $\epsilon > 0$. Then (F,ϵ) is a **Kazhdan pair** for the groupoid $\mathcal{G}$ if whenever $[\pi]$ is a full unitary representation of $[\mathcal{G}]$ admitting an (F,ϵ)-invariant vector, then $[\pi]$ has a nonzero invariant vector.

Proposition 5.3.10. *A pmp ergodic groupoid $\mathcal{G}$ has property (T) if an only if it admits a Kazhdan pair.*

Proof. Suppose (F,ϵ) is a Kazhdan pair for $\mathcal{G}$. By the dominated convergence theorem, if (ξ_n) is a sequence of almost invariant unit sections for a unitary representation π of $\mathcal{G}$, then for all $T \in F$ we have $\|[\pi](T)\xi_n - \xi_n\| \to 0$. In particular for n large enough, ξ_n is (F,ϵ)-invariant and thus we have a nonzero invariant section, which by ergodicity and rescaling yields an invariant unit section.

Conversely, assume that $\mathcal{G}$ has (T) but no Kazhdan pair. Then $\mathcal{G}$ is strongly ergodic, so by Theorem 5.3.8 and the following remark, $\mathcal{G}$ has a spectral gap pair (F,κ). By the Feldman-Moore theorem, there is an increasing sequence (F_n) of finite subsets of $[\mathcal{G}]$ so that $\mathcal{G} = \bigcup_{n\in\mathbb{N}} F_n$. For every $n \in \mathbb{N}$, $(F \cup F_n, \frac{1}{n})$ is not a Kazhdan pair so we find a unitary representation π_n of $\mathcal{G}$ on $\mathcal{H}_n$ so that $[\pi_n]$ has an $(F \cup F_n, \frac{1}{n})$-invariant unit vector ξ_n but no nonzero invariant vector.

For each $n \in \mathbb{N}$, define $\eta_n(x) = \|\xi_n(x)\|$. By the reversed triangle inequality, we have for all $T \in [\mathcal{G}]$:

$$\begin{aligned}
\|[\pi](T)\xi_n - \xi_n\|^2 &= \int_X \|\pi(T)\xi_n(x) - \xi_n(x)\|^2 \, d\mu(x) \\
&\geqslant \int_X (\|\pi(T)\xi_n(x)\| - \|\xi_n(x)\|)^2 \, d\mu(x) \\
&= \int_X \left|\eta_n(\mathfrak{l}(T)^{-1}x)) - \eta_n(x)\right|^2 d\mu(x).
\end{aligned}$$

Since $(\mathfrak{l}(F),\kappa)$ is a spectral gap pair for $\mathcal{R}_{\mathcal{G}}$, we then have

$$\left\|\eta_n - \int_X \eta_n(x) d\mu(x)\right\|^2 \leqslant \kappa \sum_{T\in F} \left\|\eta_n - \eta_n \circ \mathfrak{l}(T)^{-1}\right\|^2 \leqslant \kappa \sum_{T\in F} \|\pi(T)\xi_n - \xi_n\|^2 \leqslant \frac{\kappa\,|F|}{n^2}.$$

Moreover since $\|\eta_n\| = 1$ we deduce $\left\|1 - \int_X \eta_n(x) d\mu(x)\right\|^2 \leqslant \frac{\kappa|F|}{n^2}$, so

$$\|\eta_n - 1\| \leqslant \frac{2\sqrt{\kappa\,|F|}}{n}.$$

Let $v_n \in \mathcal{H}_n$ be a fixed unit vector. Define $\xi'_n\colon X \to \mathcal{H}_n$ by

$$\xi'_n(x) = \begin{cases} \frac{1}{\eta_n(x)}\xi_n(x) & \text{if } \eta_n(x) \neq 0 \\ v & \text{otherwise.} \end{cases}$$

Each ξ'_n is by construction a unit section for π_n. We have

$$\|\xi_n - \xi'_n\|^2 = \int_X |\eta_n(x) - 1|^2 \, d\mu(x) = \|\eta_n - 1\|^2 \leqslant \frac{4\kappa\,|F|}{n^2}.$$

Since ξ_n is $(F_n, \frac{1}{n})$-invariant, it follows from the triangle inequality that each ξ'_n is an $\left(F_n, \frac{1}{n} + \frac{4\sqrt{\kappa|F|}}{n}\right)$-invariant unit section.

Consider the infinite direct sum $\mathcal{H} \coloneqq \bigoplus_n \mathcal{H}_n$, we have a unitary representation $\bigoplus_n \pi_n$ of $\mathcal{R}$ on $\mathcal{H}$. Each ξ'_n defines a unit section of $\bigoplus_n \pi_n$ and since $\mathcal{G} = \bigcup_{n\in\mathbb{N}} F_n$, by [AD05, Lem. 4.1] after taking a subsequence the sequence (ξ'_n) is almost invariant. But since no π_n had an invariant section, neither does their direct sum, contradicting that $\mathcal{G}$ had (T) as wanted. □

We can now give an even more quantitative version of property (T) by controlling how far our invariant vector will be from the (F, ϵ)-invariant vector, obtaining the desired pmp groupoid version of [BdlHV08, Prop. 1.1.9] (see also [Pic07, Thm. 20] for a related sequential and pointwise version of what follows).

Proposition 5.3.11. *Let $\mathcal{G}$ be an ergodic pmp groupoid with a Kazhdan pair (F_1, ϵ) and a spectral gap pair (F_2, κ). Suppose that $\pi\colon \mathcal{G} \to \mathcal{U}(\mathcal{H})$ is a unitary representation and that we have a section $\xi \in \mathrm{L}^2(X, \mu, \mathcal{H})$ which is $(F_1 \cup F_2, \delta\epsilon)$-invariant for some $\delta > 0$. Then $[\pi]$ admits an invariant vector which is at distance at most $\delta(1 + \epsilon\sqrt{\kappa\,|F_2|})\,\|\xi\|$ from ξ.*

Proof. Let $\mathcal{K}_1$ be the subspace consisting of $[\pi]$-invariant vectors. Denote by p_1 the orthogonal projection onto $\mathcal{K}_1$. Let $\mathcal{K}$ be be the $\mathrm{L}^\infty(X)$-module spanned by $\mathcal{K}_1$. Observe that the restriction of $[\pi]$ to $\mathcal{K}$ is a multiple of $[\theta]$ where θ is the trivial unitary representation of $\mathcal{G}$ (on $\mathbb{C}$). Write $\xi = \xi' + \xi''$ with $\xi' \in \mathcal{K}$ and $\xi'' \in \mathcal{K}^\perp$, then by construction $p_1(\xi')$ is the invariant vector which is the closest to ξ.

Since $\mathcal{K}^\perp$ is a $[\pi]$-invariant $\mathrm{L}^\infty(X)$-module without nonzero invariant vectors and (F_1, ϵ) is a Kazhdan pair, there is some $T \in F_1$ such that $\|\xi' - [\pi](T)\xi'\| \geqslant \epsilon\,\|\xi'\|$. On the other hand, we have $\|\xi' - [\pi](T)\xi'\| \leqslant \|\xi - [\pi](T)\xi\| < \epsilon\delta\,\|\xi\|$, so $\|\xi'\| < \delta\,\|\xi\|$.

Finally, since the restriction of $[\pi]$ to $\mathcal{K}$ is a multiple of $[\theta]$, we have

$$\|\xi'' - p_1(\xi'')\|^2 \leqslant \kappa \sum_{T'\in F_2} \|\xi'' - [\pi](T')\xi''\|^2 \leqslant \kappa\delta^2\epsilon^2\,|F_2|\,\|\xi''\|^2 \leqslant \kappa\delta^2\epsilon^2\,|F_2|\,\|\xi\|^2\,.$$

Since $p_1(\xi) = p_1(\xi'')$, we finally have

$$\|\xi - p_1(\xi)\| \leqslant \|\xi'\| + \|\xi'' - p_1(\xi'')\| \leqslant \delta(1 + \epsilon\sqrt{\kappa\,|F_2|})\,\|\xi\|\,.$$ □

5.3.2 From strongly ergodic actions to property (T)

We begin this section with a result which is a direct consequence of Anantharaman-Delaroche's Theorem 5.15 from [AD05]. We provide a version of her proof in terms of full unitary representations and Kazhdan pairs for the convenience of the reader.

Proposition 5.3.12. *Let $\mathcal{G}$ be an ergodic pmp groupoid on (X, μ) with property (T), suppose that $\alpha\colon \mathcal{G} \to \mathrm{Aut}(Y, \nu)$ is a pmp action which induces an ergodic pmp equivalence relation $\mathcal{R}_\alpha$. Then $\mathcal{R}_\alpha$ has property (T).*

Proof. From now on, see $[\alpha]$ as a the continuous group homomorphism $[\mathcal{G}] \to [\mathcal{R}_\alpha]_{\mu\times\nu}$. Let (F, ε) be a Kazhdan pair for $\mathcal{G}$, we will actually show that $([\alpha](F), \varepsilon)$ is a Kazhdan pair for $\mathcal{R}_\alpha$. Let $[\pi] : [\mathcal{R}_\alpha] \to \mathcal{U}(\mathcal{H})$ be a full unitary representation of $\mathcal{R}_\alpha$ with a $([\alpha](K), \epsilon)$-invariant vector, we view its space of square-integrable sections as an $\mathrm{L}^\infty(X \times Y)$-module. In particular, it is an $\mathrm{L}^\infty(X)$-module and it follows from fullness of $[\alpha]$ that $[\pi] \circ [\alpha]$ is a full unitary representation of $[\mathcal{G}]$. Since $[\pi]$ has a $([\alpha](F), \epsilon)$-invariant vector and (F, ϵ) is a Kazhdan pair for $\mathcal{G}$, there is a nonzero $[\pi] \circ [\alpha]$-invariant vector $\xi \in \mathrm{L}^2(X \times Y, \mathcal{H})$. Now $[\alpha]$ is a surjection onto $[\mathcal{R}_\alpha]$ (see Remark 5.3.4 for the case $[\alpha] = \mathfrak{l}$; the general proof does not differ) we have $[\mathcal{R}_\alpha] = [\alpha]([\mathcal{G}])$ and therefore the vector ξ is actually π-invariant, which concludes the proof. □

Our next goal is to prove a version of the Connes-Weiss characterization of property (T) for groups, but for pmp groupoids.

Theorem 5.3.13. *Let $\mathcal{G}$ be an ergodic pmp groupoid. Then $\mathcal{G}$ has property (T) if and only if every ergodic pmp action of $\mathcal{G}$ is strongly ergodic.*

The direct implication is an immediate consequence of Proposition 5.3.12 and Proposition 5.3.6. This section will thus be devoted to the converse, which is proved by contraposition. Towards this, we need as in the group case a weaker version of property (T) in terms of finite dimensional subrepresentations (or rather finitely generated submodules in the present case) instead of invariant vectors [VB93]. This is where affine actions come in, as in the group case.

The Bekka-Valette characterization of property (T)

Definition 5.3.14. Let $\mathcal{G}$ be a pmp groupoid and π be a unitary representation of $\mathcal{G}$ on a Hilbert space $\mathcal{H}$. **A finitely generated submodule** of π is a nontrivial $[\pi]$-invariant subspace $\mathcal{K}$ of $\mathrm{L}^2(X, \mu, \mathcal{H})$ such that there exists a finite set $F \subseteq \mathcal{K}$ with $\mathrm{L}^\infty(X, \mu)F$ dense in $\mathcal{K}$.

This section is devoted to the proof of the following theorem, which is the pmp groupoid version of the Bekka-Valette theorem [VB93, Thm. 1].

Theorem 5.3.15. *Let $\mathcal{G}$ be an ergodic pmp groupoid. Then $\mathcal{G}$ has property (T) if and only if every unitary representation of $\mathcal{G}$ which has almost invariant unit sections contains a finitely generated submodule.*

Note that the latter condition is a weaker version of property (T). Thus only the right-to-left direction remains to be proven.

Proposition 5.3.16 ([DV89, Chapitre 4])**.** *Let $\mathcal{H}$ be a real affine Hilbert space. Then for any $t > 0$, there exists a unique complex Hilbert space $\mathcal{H}_t$ and a continuous mapping $\xi \mapsto \xi_t$ from $\mathcal{H}$ to the unit sphere of $\mathcal{H}_t$ such that for any $\xi, \eta \in \mathcal{H}$, $\langle \xi_t, \eta_t \rangle = \exp(-t\|\xi - \eta\|^2)$ and the image of $\{\xi_t \colon \xi \in \mathcal{H}\}$ is total in $\mathcal{H}_t$.*

For $t > 0$, we define a morphism $\phi \mapsto \phi_t$ from the group of affine isometries $\mathrm{Iso}(\mathcal{H})$ of $\mathcal{H}$ to the unitary group $\mathcal{U}(\mathcal{H}_t)$ of $\mathcal{H}_t$, where ϕ_t is the only element of $\mathcal{U}(\mathcal{H}_t)$ such that $\forall \xi \in \mathcal{H}, \phi_t(\xi_t) = (\phi(\xi))_t$. This morphism is continuous.

Indeed, let (ϕ_n) be a sequence in $\mathrm{Iso}(\mathcal{H})$ converging to ϕ. Then for $\xi, \eta \in \mathcal{H}$, $\langle (\phi_n)_t(\xi_t), \eta_t \rangle = \langle (\phi_n(\xi))_t, \eta_t \rangle = \exp(-t\|\phi_n(\xi) - \eta\|^2) \underset{n\to\infty}{\longrightarrow} \exp(-t\|\phi(\xi) - \eta\|^2)$.

Therefore, any affine isometric action ρ of a pmp groupoid $\mathcal{G}$ on $\mathcal{H}$ gives rise to a unitary representation ρ_t on $\mathcal{H}_t$. We now give a cocycle free proof of some results from [AD05].

Lemma 5.3.17 ([AD05, Lem. 3.20])**.** *Let $\mathcal{G}$ be an ergodic pmp groupoid. Let ρ be an affine isometric action of $\mathcal{G}$ on $\mathcal{H}$, suppose that there is $A \subseteq X$ of positive measure and a Borel section $\xi\colon X \to \mathcal{H}$ such that for all $x \in A$, we have*

$$\sup_{g \in \mathcal{G}_x^A} \|\rho(g)\xi(x) - \xi(r(g))\| < +\infty.$$

Then there exists a section $\xi'\colon X \to \mathcal{H}$ such that for almost all $x \in X$, we have

$$\sup_{g \in \mathcal{G}_x} \|\rho(g)\xi'(x) - \xi'(r(g))\| < +\infty.$$

Proof. Fix a sequence (T_n) in $[\mathcal{G}]$ such that the graphs of the T_n cover $\mathcal{G}$. We define a Borel map $f\colon X \to \mathcal{G}$ by letting $f\colon x \mapsto T_n \star x$ where n is the least integer such that $T_n \star x \in A$. Such an integer exists by ergodicity of $\mathcal{G}$. Note that for almost any $x \in X$, we have $s(f(x)) = x$ and $r(f(x)) \in A$.

Now let $\xi'\colon x \mapsto \rho(f(x)^{-1})\xi(r(f(x)))$. For $g \in \mathcal{G}_x$, we compute

$$\begin{aligned}
\|\rho(g)\xi'(x) - \xi'(r(g))\| &= \left\|\rho(g)\rho(f(x)^{-1})\xi(r(f(x))) - \rho(f(r(g))^{-1})\xi(r(f(r(g))))\right\| \\
&= \left\|\rho(g \star f(x)^{-1})\xi(r(f(x))) - \rho(f(r(g))^{-1})\xi(r(f(r(g))))\right\| \\
&= \left\|\rho(f(r(g)) \star g \star f(x)^{-1})\xi(r(f(x))) - \xi(r(f(r(g))))\right\| \\
&\leqslant \sup_{g' \in \mathcal{G}^A_{r(f(x))}} \|\rho(g')\xi(r(f(x))) - \xi(r(g'))\| \\
&< +\infty,
\end{aligned}$$

since $r(f(x)) \in A$. □

Lemma 5.3.18 ([AD05, Thm. 3.19])**.** *Let $\mathcal{G}$ be an ergodic pmp groupoid. Let ρ be an affine isometric action of $\mathcal{G}$ on $\mathcal{H}$ and let $\xi : X \to \mathcal{H}$ be a Borel section, suppose that for all $x \in X$, we have $\sup_{g \in \mathcal{G}_x} \|\rho(g)\xi(x) - \xi(r(g))\| < +\infty$. Then ρ admits an invariant section.*

Proof. For all $g \in \mathcal{G}$, since $\rho(g)^{-1}$ is an isometry which is the inverse of $\rho(g)$, we have $\sup_{g \in \mathcal{G}_x} \|\rho(g^{-1})\xi(r(g)) - \xi(x)\| < +\infty$. Using a Feldman-Moore group for $\mathcal{G}$, we see that the section η which takes $x \in X$ to the circumcenter of the closed convex hull of $\{\rho(g^{-1})\xi(r(g)) : g \in \mathcal{G}_x\}$ is Borel. Such a section is easily checked to be fixed by ρ since for $g \in \mathcal{G}_x^y$, $g' \mapsto g'g^{-1}$ is a bijection $\mathcal{G}_x \to \mathcal{G}_y$ and we have

$$\rho((g'g^{-1})^{-1})\xi(r(g'g^{-1})) = \rho(g)\rho(g'^{-1})\xi(r(g')),$$

so that $\rho(g)\eta(x) = \eta(y)$. □

Proposition 5.3.19. *Let $\mathcal{G}$ be an ergodic pmp groupoid. Let ρ be an affine isometric action of $\mathcal{G}$ on $\mathcal{H}$ and $t > 0$. Then ρ admits a fixed section if and only if ρ_t has an invariant unit section.*

Proof. If ξ is a fixed section for ρ then it is straightforward that $\xi_t\colon x \mapsto \xi(x)_t$ is an invariant unit section for ρ_t.

Conversely, if there exists an invariant unit section ζ for ρ_t, suppose by contradiction that ρ does not admit any fixed section. Fix a dense sequence $(h_n)_{n\geqslant 1} \in \mathcal{H}$ and for $m, n \in \mathbb{N}^*$, let $E_{m,n} = \{x \in X\colon |\langle (h_m)_t, \zeta(x)\rangle| \geqslant \frac{1}{n}\}$. Since the span of the $(h_m)_t$ for $m \geqslant 1$ is dense in $\mathcal{H}_t$ and ζ is a unit section, we have $\bigcup_{m,n\geqslant 1} E_{m,n} = X$ so there exist $m, n \in \mathbb{N}^*$ such that $\mu(E_{m,n}) > 0$.

Applying Lemmas 5.3.18 and 5.3.17 to the constant section $x \mapsto h_m$, we get that for almost any $x \in E_{m,n}$,

$$\sup_{g\in\mathcal{G}_x^{E_{m,n}}} \|\rho(g)h_m - h_m\| = +\infty.$$

Therefore, for almost any $x \in E_{m,n}$, let (g_k) be a sequence in $\mathcal{G}_x^{E_{m,n}}$ such that $(\|\rho(g_k)h_m - h_m\|)_k$ tends to $+\infty$ and equivalently, $(\|\rho(g_k^{-1})h_m - h_m\|)_k$ tends to $+\infty$. Then $(\rho(g_k^{-1})h_m)_k$ tends to ∞ in $\mathcal{H}$. On the one hand, for $k \in \mathbb{N}$ we have

$$|\langle (\rho(g_k^{-1})h_m)_t, \zeta(x)\rangle| = |\langle (h_m)_t, \rho_t(g_k)\zeta(x)\rangle| = |\langle (h_m)_t, \zeta(r(g_k))\rangle| \geqslant \frac{1}{n}.$$

On the other hand, for $\xi \in \mathcal{H}$, we have

$$|\langle (\rho(g_k^{-1})h_m)_t, \xi_t\rangle| = \exp(-t\left\|\rho(g_k^{-1})h_m - \xi\right\|^2) \underset{k\to\infty}{\longrightarrow} 0.$$

Since the image of $\mathcal{H}$ is total in $\mathcal{H}_t$, $\big((\rho(g_k^{-1})h_m)_t\big)_{k\in\mathbb{N}}$ weakly converges to 0 in $\mathcal{H}_t$, which contradicts the latter inequality. □

Lemma 5.3.20. *Let ρ be an affine isometric action of a pmp groupoid $\mathcal{G}$ on a real Hilbert space $\mathcal{H}$. Consider the family $(\mathcal{H}_t, \rho_t)_t$ associated to ρ in Proposition 5.3.16. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence of positive reals converging to 0. Then $\bigoplus_{n\in\mathbb{N}} \rho_{t_n}$ has almost invariant unit sections.*

Proof. Set $\pi := \bigoplus_{n\in\mathbb{N}} \rho_{t_n}$ and for $n \in \mathbb{N}$, let ξ_n be the constant section with value 0_{t_n}.

Then ξ_n is a unit section and moreover $\forall^* g \in \mathcal{G}$,

$$\begin{aligned}
&\|\rho_{t_n}(g)\xi_n(s(g)) - \xi_n(r(g))\|^2 \\
&= \|(\rho(g)(0))_{t_n} - 0_{t_n}\|^2 \\
&= \|(\rho(g)(0))_{t_n}\|^2 + \|0_{t_n}\|^2 - 2\langle (\rho(g)(0))_{t_n}, 0_{t_n}\rangle \\
&= 2\left(1 - \exp(-t_n\|\rho(g)(0)\|^2)\right)
\end{aligned}$$

It easily follows that (ξ_n), seen as a sequence in $\bigoplus_{n\in\mathbb{N}} \mathcal{H}_{t_n}$ is a sequence of almost invariant unit sections for π. □

Lemma 5.3.21. *Let ρ be an affine isometric action of a pmp groupoid $\mathcal{G}$ on a real Hilbert space $\mathcal{H}$. Let us denote by ρ^2 the diagonal action $\rho \times \rho$ on $\mathcal{H} \oplus \mathcal{H}$. Then for $t > 0$, $(\rho^2)_t = \rho_t \otimes \rho_t$.*

Proof. Fix $t > 0$. We define a map $\Psi_t\colon \mathcal{H}\times\mathcal{H} \to \mathcal{H}_t\otimes\mathcal{H}_t$ by the formula $\Psi_t(\xi, \eta) := \xi_t\otimes\eta_t$.

First, for $\xi, \xi', \eta, \eta' \in \mathcal{H}$ we have

$$\begin{aligned}
\langle \Psi_t(\xi,\eta), \Psi_t(\xi',\eta')\rangle &= \exp(-t\|\xi-\xi'\|^2)\exp(-t\|\eta-\eta'\|^2) \\
&= \exp(-t\|(\xi,\eta) - (\xi',\eta')\|^2)
\end{aligned}$$

and it is clear that the image of Ψ_t is total in $\mathcal{H}_t \otimes \mathcal{H}_t$. Moreover,

$$\begin{aligned}\Psi_t(\rho^2(g)(\xi,\eta)) &= \Psi_t(\rho(g)\xi, \rho(g)\eta)\\ &= (\rho(g)\xi)_t \otimes (\rho(g)\eta)_t\\ &= \rho_t(g)\xi_t \otimes \rho_t(g)\eta_t\\ &= (\rho_t \otimes \rho_t)(g)(\xi_t \otimes \eta_t)\\ &= (\rho_t \otimes \rho_t)(g)\Psi_t(\xi,\eta).\end{aligned}$$

By uniqueness of the construction in Proposition 5.3.16, it follows that $\Psi_t(\xi,\eta) = (\xi,\eta)_t$ and $(\rho^2)_t = \rho_t \otimes \rho_t$. □

Lemma 5.3.22 ([GL17], Lemma 3.16.(2))**.** *Let $\mathcal{G}$ be a pmp groupoid and let π be a unitary representation of $\mathcal{G}$. The following assertions are equivalent:*

1. *$\pi \otimes \overline{\pi}$ has an invariant unit section,*
2. *there exists a unitary representation σ of $\mathcal{G}$ such that $\pi \otimes \sigma$ has an invariant unit section,*
3. *π contains a finitely generated submodule.*

Finally, let us introduce an affine version of property (T):

Definition 5.3.23 (Property (FH), [AD05], Section 4.3)**.** A pmp groupoid $\mathcal{G}$ is said to have **property (FH)** if every affine isometric action of $\mathcal{G}$ on a separable real Hilbert space admits a fixed section.

Proposition 5.3.24 ([AD05, Theorems 4.8 and 4.12])**.** *Let $\mathcal{G}$ be a pmp groupoid on a standard probability space. Then $\mathcal{G}$ has property (T) if and only if $\mathcal{G}$ has property (FH).*

We are now ready to prove Theorem 5.3.15.

Proof. Recall that the left-to-right direction is trivial. For the other direction, suppose that every unitary representation of $\mathcal{G}$ which has almost invariant unit sections contains a finitely generated submodule and let us show that $\mathcal{G}$ has property (FH).

Let ρ be an affine isometric action of $\mathcal{G}$ on a separable real Hilbert space $\mathcal{H}$. Let us consider the representation $\pi := \bigoplus_{n\in\mathbb{N}^*} \rho_{1/n}$. Then π admits almost invariant unit sections by Lemma 5.3.20, and therefore π contains a finitely generated submodule, or in other words, $\pi \otimes \overline{\pi}$ has an invariant unit section ξ. But $\pi \otimes \overline{\pi} = \bigoplus_{n,m\in\mathbb{N}^*} \rho_{1/n} \otimes \overline{\rho_{1/m}}$ and $\mathcal{G}$ is ergodic so there must be $n_0, m_0 \in \mathbb{N}^*$ such that ξ is a section of $\rho_{1/n_0} \otimes \overline{\rho_{1/m_0}}$. Then ρ_{1/n_0} contains a finitely generated subrmodule and finally $\rho_{1/n_0} \otimes \overline{\rho_{1/n_0}}$ has an invariant unit section. Since ρ_t is always unitarily equivalent to $\overline{\rho_t}$, we conclude that $\rho_{1/n_0} \otimes \rho_{1/n_0}$ admits an invariant unit section.

Applying Lemma 5.3.21, we get that $\rho \times \rho$ has a fixed section. Therefore ρ has a fixed section and $\mathcal{G}$ has property (FH). □

A reminder on Gaussian actions

Let $\mathcal{G}$ be a pmp groupoid on a standard probability space (X, μ).

Definition 5.3.25. Let α be a measure-preserving $\mathcal{G}$-action on a standard probability space (Y,ν). Then α induces a unitary representation κ^α of $\mathcal{G}$ on $\mathrm{L}^2(Y,\nu)$ defined by the formula $\kappa^\alpha(g)f\colon y\mapsto f(\alpha(g)^{-1}y)$. Note that 1_Y is κ^α-invariant. Let κ_0^α be the restriction of κ^α to the orthogonal to the subspace of constant functions $\mathrm{L}^2(Y,\nu)\ominus\mathbb{C}1_Y$. We call this unitary representation κ_0^α the **Koopman representation** associated to α.

We now briefly present the classical construction of Gaussian actions associated to unitary representations, which given our definitions of unitary representations and of measure-preserving actions of pmp groupoids (Def. 5.2.2), work exactly as in the group case. For more details, see [KL17, Appendix E].

In this section we use orthogonal representations on real Hilbert spaces whereas the rest of this thesis deals with unitary representations of complex Hilbert spaces. This is dealt with thanks to the notions of realification and complexification of representations. For a unitary representation π on a complex Hilbert space $\mathcal{H}$, write $\pi_\mathbb{R}$ and $\mathcal{H}_\mathbb{R}$ the respective realifications of π and $\mathcal{H}$, and for an orthogonal representation π on a real Hilbert space $\mathcal{H}$, write $\pi_\mathbb{C}$ and $\mathcal{H}_\mathbb{C}$ the respective complexifications of π and $\mathcal{H}$. We will use the fact that $(\pi_\mathbb{R})_\mathbb{C}$ is unitarily equivalent to $\pi\oplus\bar{\pi}$ (see [KL17, Proposition E.1]).

First we define the symmetric Fock space of a Hilbert space. For a real (resp. complex) Hilbert space $\mathcal{H}$, $n\in\mathbb{N}$ and $\sigma\in\mathfrak{S}_n$, σ induces an orthogonal (resp. unitary) operator U_σ on $\mathcal{H}^{\otimes n}$ defined on simple tensors by $U_\sigma(h_1\otimes\cdots\otimes h_n)=h_{\sigma^{-1}(1)}\otimes\cdots\otimes h_{\sigma^{-1}(n)}$. We let the **n-th symmetric space of** $\mathcal{H}$ be the subspace of $\mathcal{H}^{\otimes n}$ of elements that are invariant by all U_σ for $\sigma\in\mathfrak{S}_n$, and we denote it by $\mathcal{H}^{\odot n}$. For $n=0$, we take the convention that $\mathcal{H}^{\odot 0}=\mathbb{R}$ (resp. $\mathbb{C}$). The **symmetric Fock space of** $\mathcal{H}$ is the sum $S(\mathcal{H})\coloneqq\bigoplus_{n\in\mathbb{N}}\mathcal{H}^{\odot n}$.

Any orthogonal (resp. unitary) representation π on $\mathcal{H}$ then induces orthogonal (resp. unitary) representations $\pi^{\odot n}$ on $\mathcal{H}^{\odot n}$ and an orthogonal (resp. unitary) representation $S(\pi)$ on $S(\mathcal{H})$ naturally.

Now let $\mathcal{G}$ be an ergodic pmp groupoid on (X,μ) and let π be an unitary representation of $\mathcal{G}$ on a complex Hilbert space $\mathcal{H}$. Then one can associate to π a pmp action α_π, called the **Gaussian action associated to** π, on a standard probability space (Y,ν) such that $\kappa_0^{\alpha_\pi}$ contains π as a direct summand, as follows.

One can construct a Gaussian real Hilbert space $\widetilde{\mathcal{H}}\subset\mathrm{L}^2(Y,\nu,\mathbb{R})$, that is a real Hilbert space such that all elements of $\widetilde{\mathcal{H}}$ are Gaussian centered random variables over (Y,ν), such that $\widetilde{\mathcal{H}}$ has dimension dim $\mathcal{H}_\mathbb{R}$, and the symmetric Fock space $S(\widetilde{\mathcal{H}})$ of $\widetilde{\mathcal{H}}$ is isometrically isomorphic to $\mathrm{L}^2(Y,\nu,\mathbb{R})$ for an isomorphism sending $\widetilde{\mathcal{H}}^{\odot 0}$ on $\mathbb{R}1_Y$. By the means of a fixed isometric isomorphism, let us identify $\widetilde{\mathcal{H}}$ with $\mathcal{H}_\mathbb{R}$ as well as $S(\widetilde{\mathcal{H}})$ with $\mathrm{L}^2(Y,\nu,\mathbb{R})$. We can then consider $S(\pi_\mathbb{R})$ as an orthogonal representation on $\mathrm{L}^2(Y,\nu,\mathbb{R})$. The construction of $S(\pi_\mathbb{R})$ then ensures it preserves the subset $\mathrm{MAlg}(Y,\nu)\subseteq\mathrm{L}^2(Y,\nu,\mathbb{R})$ and acts by pmp transformations on it. Call α_π the restriction of $S(\pi_\mathbb{R})$ to $\mathrm{Aut}(Y,\nu)$. Since $\mathrm{MAlg}(Y,\nu)$ is closed in $\mathrm{L}^2(Y,\nu,\mathbb{R})$, α_π is a Borel morphism and so it is a measure-preserving $\mathcal{G}$-action on (Y,ν).

Furthermore, by construction, κ^{α_π} is unitarily equivalent to $S(\pi_\mathbb{R})_\mathbb{C}$ and $\kappa_0^{\alpha_\pi}$ is unitarily equivalent to $\bigoplus_{n\in\mathbb{N}^*}\pi^{\odot n}$, so $\pi=\pi^{\odot 1}$ is a direct summand of $\kappa_0^{\alpha_\pi}$.

The Connes-Weiss characterization of property (T)

First, we introduce weak mixing for measure-preserving actions of a groupoid.

Definition 5.3.26 (Weakly mixing action)**.** Let $\mathcal{G}$ be a pmp groupoid on a standard probability space (X,μ) and let α be a measure-preserving $\mathcal{G}$-action on some standard probability space (Y, ν). Then α is called **weakly mixing** if the diagonal action $\alpha \times \alpha$ on $(Y \times Y, \nu \times \nu)$ is ergodic.

Both ergodicity and weak mixing for a measure-preserving $\mathcal{G}$-action α can be expressed in terms of the Koopman representation κ_0^α.

Proposition 5.3.27 ([GL17], Lemmas 3.16.(3), 3.38 and 3.42)**.** *Let $\mathcal{G}$ be a pmp groupoid and let α be a measure-preserving $\mathcal{G}$-action on a standard probability space (Y, ν). Then*

- *α is ergodic if and only if the Koopman representation π_0^α associated to α does not admit an invariant unit section.*
- *α is weakly mixing if and only if the Koopman representation π_0^α associated to α does not contain any nontrivial finitely generated submodule.*

In this section we use exclusively the latter characterization of weak mixing in terms of representations instead of the definition in terms of actions.

We can now prove the following theorem, which generalizes a theorem of Connes and Weiss from discrete countable groups to pmp groupoids:

Theorem 5.3.28 (Connes-Weiss for pmp groupoids)**.** *Let $\mathcal{G}$ be a pmp ergodic groupoid on a standard probability space. Then the following assertions are equivalent:*

(1) $\mathcal{G}$ has property (T).

(2) Every ergodic measure-preserving $\mathcal{G}$-action is strongly ergodic.

(3) Every weakly mixing measure-preserving $\mathcal{G}$-action is strongly ergodic.

Proof. The implication *(1) $\Rightarrow$ (2)* can be derived from Propositions 5.3.6 and 5.3.12. Moreover, it a consequence of Proposition 5.3.27 that weak mixing of a $\mathcal{G}$-action implies ergodicity of this action. It follows that *(2)* implies *(3)*. It only remains to prove *(3) $\Rightarrow$ (1)*.

Let us suppose that $\mathcal{G}$ does not have property (T) and construct a weakly mixing action which is not strongly ergodic.

By Theorem 5.3.15, there is a unitary representation π of $\mathcal{G}$ which has almost invariant unit sections but does not admit any finitely generated submodule. We construct the Gaussian action α_π associated to π (see 5.3.2) and prove that it corresponds to what we seek:

Since π does not admit any finitely generated submodule, then neither do the $\pi^{\otimes n}$ for $n \in \mathbb{N}^*$ by Lemma 5.3.22. In particular, for $n \in \mathbb{N}^*$, $\pi^{\odot n}$ does not admit any finitely generated submodule. It follows that for any $n, m \in \mathbb{N}^*$, $\pi^{\odot n} \otimes \overline{\pi^{\odot m}}$ does not have an invariant unit section. But then $\kappa_0^{\alpha_\pi} \otimes \overline{\kappa_0^{\alpha_\pi}} = \bigoplus_{n,m \in \mathbb{N}^*} \pi^{\odot n} \otimes \overline{\pi^{\odot m}}$ does not have any invariant unit section and thus does not admit any finitely generated submodule. Therefore, α_π is weakly mixing.

Then we construct a sequence of asymptotically $[\alpha_\pi]$-invariant Borel sets $A_n \subset X \times Y$ which is not asymptotically trivial:

Let (ξ_n) be a sequence of almost invariant unit sections for π. Recall that through the construction of the Gaussian action α_π, we identified $\mathcal{H}$ to a Gaussian Hilbert space, so for almost any $x \in X$, $\xi_n(x)$ is a centered Gaussian random variable of variance 1.

Write ξ_n^x for $\xi_n(x)$ and for $x \in X$, let $A_n^x \subset Y$ be the set $\{y \in Y : \xi_n^x(y) \geqslant 0\}$. Since ξ_n^x is centered, we have $\nu(A_n^x) = \frac{1}{2}$.

Fix $g \in \mathcal{G}$. We write $\pi(g)\xi_n^{s(g)} = \cos\theta_{n,g}\xi_n^{r(g)} + \sin\theta_{n,g}\eta_{n,g}$ in $\mathcal{H}$, for some $\theta_{n,g} \in [0,\pi]$ and $\eta_{n,g} \in \left(\xi_n^{r(g)}\right)^{\perp}$. Then $\xi_n^{r(g)}$ and $\eta_{n,g}$ are orthogonal gaussian random variables of variance 1 so they are independent and their joint distribution is a probability measure m on $\mathbb{R}^2$ which is rotation invariant. Last, (ξ_n) is a sequence of almost invariant sections so for n big enough, $\theta_{n,g} \in [0, \frac{\pi}{2})$. It follows that

$$\begin{aligned}
&\nu(\alpha_\pi(g)A_n^{s(g)} \triangle A_n^{r(g)}) \\
&= \nu\left(\{\pi(g)\xi_n^{s(g)} \geqslant 0 \wedge \xi_n^{r(g)} < 0\} \cup \{\pi(g)\xi_n^{s(g)} < 0 \wedge \xi_n^{r(g)} \geqslant 0\}\right) \\
&= m\left(\{(a,b) \in \mathbb{R}^2 \colon \cos\theta_{n,g}a + \sin\theta_{n,g}b \geqslant 0 \wedge a < 0\}\right) \\
&\quad + m\left(\{(a,b) \in \mathbb{R}^2 \colon \cos\theta_{n,g}a + \sin\theta_{n,g}b < 0 \wedge a \geqslant 0\}\right) \\
&= m\left(\{(a,b) \in \mathbb{R}^2 \colon y \geqslant -\frac{1}{\tan\theta_{n,g}}a \wedge a < 0\}\right) \\
&\quad + m\left(\{(a,b) \in \mathbb{R}^2 \colon y < -\frac{1}{\tan\theta_{n,g}}a \wedge a \geqslant 0\}\right) \\
&= \frac{\theta_{n,g}}{\pi}
\end{aligned}$$

Let now $A_n = \{(x,y) \in X \times Y : y \in A_n^x\}$. Then $\mu \otimes \nu(A_n) = \frac{1}{2}$ so that A_n is not asymptotically trivial. Let $T \in [\mathcal{G}]$, then

$$\begin{aligned}
\mu \otimes \nu([\alpha_\pi](T)A_n \triangle A_n) &= \int_X \nu(\alpha_\pi(x \star T)A_n^{\iota(T)^{-1}x} \triangle A_n^x)\ d\mu(x) \\
&= \frac{1}{\pi}\int_X \theta_{n,x\star T}\ d\mu(x)
\end{aligned}$$

which converges to 0 by dominated convergence theorem. Therefore (A_n) is asymptotically invariant.

We thus get that α_π is weakly mixing and not strongly ergodic, which concludes the proof. □

5.4 Properties of groupoids reflected by their actions

Given a pmp groupoid, a measure-preserving action of this groupoid can be seen itself as a pmp groupoid.

Definition 5.4.1 (Translation groupoid)**.** Let $\mathcal{G}$ be a pmp groupoid on (X,μ) and α be a measure-preserving $\mathcal{G}$-action on $\mathrm{MAlg}(Y,\nu)$. Since we want to fiber over (Y,ν), we use the equivalent point of view of pmp point actions and let α denote indifferently the action on the measure algebra or any point action which induces it. We define a pmp groupoid $\mathcal{G} \underset{\alpha}{\ltimes} Y$ as follows:

- The base space is $\mathcal{G} \times Y$.
- The space of units is $(X \times Y, \mu \times \nu)$.
- The source and range maps are defined by $s(g,y) = (s(g),y)$ and $r(g,y) = (r(g), \alpha(g)y)$.

- The product is defined by $(g', \alpha(g)y) \star (g, y) = (g'g, y)$ for $(g', g) \in \mathcal{G}^{(2)}$.
- Accordingly, we have $(g, y)^{-1} = (g^{-1}, \alpha(g)y)$.

It is easy to check that $\mathcal{G} \underset{\alpha}{\ltimes} Y$ is indeed pmp. We call it the **translation groupoid** of the action α.

5.4.1 Amenability

Definition 5.4.2 (Amenable pmp groupoid)**.** A pmp groupoid $\mathcal{G}$ on (X, μ) is called **amenable** if there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of elements of $\mathrm{L}^0(\mathcal{G}, \mu^1, [0, \infty))$ such that:

1. For all $n \in \mathbb{N}$ and for almost all $x \in X$, $\lambda_n^x := \lambda_{n \restriction \mathcal{G}_x} \in l^1(\mathcal{G}_x)$,
2. for all $n \in \mathbb{N}$ and for almost all $x \in X$, $\|\lambda_n^x\|_1 = 1$,
3. for almost all $g \in \mathcal{G}$, $\|g \cdot \lambda_n^{s(g)} - \lambda_n^{r(g)}\|_1 \underset{n\to\infty}{\longrightarrow} 0$, where for $f \in \mathbb{R}^{\mathcal{G}_{s(g)}}$, $g \cdot f \in \mathbb{R}^{\mathcal{G}_{r(g)}}$ denotes the map $g' \mapsto f(g' \star g)$.

Theorem 5.4.3. *Let $\mathcal{G}$ be an ergodic pmp groupoid on a standard probability space (X, μ) and α be a pmp $\mathcal{G}$-action on a standard probability space (Y, ν). Then $\mathcal{G}$ is amenable if and only if $\mathcal{G} \underset{\alpha}{\ltimes} Y$ is amenable.*

Note that if α is free, then $\mathcal{G} \underset{\alpha}{\ltimes} Y$ is isomorphic to the orbit equivalence relation of $[\alpha]$ on $(X \times Y, \mu \times \nu)$ via the map $r \times s$. Therefore, a consequence of this theorem is that a free pmp $\mathcal{G}$-action α on a standard probability space (Y, ν) induces an amenable action $[\alpha]$ of $[\mathcal{G}]$ if and only if $\mathcal{G}$ is amenable. This is well-known for countable groups, for which we have $[\alpha] = \alpha$.

Proof.
- First, suppose that $\mathcal{G}$ is amenable and choose a sequence $(\lambda_n)_{n\in\mathbb{N}}$ witnessing it. We construct a sequence $(\xi_n)_{n\in\mathbb{N}}$ witnessing the amenability of $\mathcal{G} \underset{\alpha}{\ltimes} Y$. For any $y \in Y$ and $g \in \mathcal{G}$, let $\xi_n(g, y) := \lambda_n(g)$.

  Then for all $n \in \mathbb{N}$, for almost all $(x, y) \in X \times Y$, $\xi_n^{(x,y)}(\cdot, y) = \lambda_n^x$ and moreover, the map $g \mapsto (g, y)$ defines a bijection between $\mathcal{G}_x$ and $\left(\mathcal{G} \underset{\alpha}{\ltimes} Y\right)_{(x,y)}$. The first two points follow.

  For 3., we compute $(g, y) \cdot \xi_n^{(s(g),y)}(g', \alpha(g)y)$ for $(g', g) \in \mathcal{G}^{(2)}$ and $y \in Y$:

$$
\begin{aligned}
(g, y) \cdot \xi_n^{(s(g),y)}(g', \alpha(g)y) &= \xi_n^{(s(g),y)}((g', \alpha(g)y) \cdot (g, y)) \\
&= \xi_n(g'g, y) \\
&= \lambda_n(g'g) \\
&= g \cdot \lambda_n(g').
\end{aligned}
$$

  Thus $\|(g, y) \cdot \xi_n^{(s(g),y)} - \xi_n^{(r(g),\alpha(g)y)}\|_1 = \|g \cdot \lambda_n^{s(g)} - \lambda_n^{r(g)}\|_1 \underset{n\to\infty}{\longrightarrow} 0$ for almost all $(g, y) \in \mathcal{G} \times Y$.

- Conversely, suppose that $\mathcal{G} \underset{\alpha}{\ltimes} Y$ is amenable, as witnessed by a sequence $(\xi_n)_{n\in\mathbb{N}}$. We construct a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of elements of $\mathrm{L}^0(\mathcal{G}, \mu^1, [0,\infty))$. For $g \in \mathcal{G}$, let $\lambda_n(g) := \int_Y \xi_n(g,y)\ d\nu(y)$.
  Then for $x \in X$,
  $$\|\lambda^x\|_1 = \sum_{g\in\mathcal{G}_x} \lambda_n^x(g) = \sum_{g\in\mathcal{G}_x} \int_Y \xi_n^{(x,y)}(g,y) d\nu(y) = \int_Y \|\xi_n^{(x,y)}\|_1 d\nu(y) = 1.$$
  Moreover, for $(g',g) \in \mathcal{G}^{(2)}$, since α is pmp,
  $$\begin{aligned} g\cdot\lambda_n^{s(g)}(g') &= \lambda_n^{s(g)}(g'g) \\ &= \int_Y \xi_n(g'g,y)\ d\nu(y) \\ &= \int_Y (g,y)\cdot\xi_n(g',\alpha(g)y)\ d\nu(y) \\ &= \int_Y (g,\alpha(g)^{-1}y)\cdot\xi_n(g',y)\ d\nu(y), \end{aligned}$$
  and so
  $$\begin{aligned} \|g\cdot\lambda_n^{s(g)} - \lambda_n^{r(g)}\|_1 &= \sum_{g'\in\mathcal{G}_{r(g)}} \int_Y (g,\alpha(g)^{-1}y)\cdot\xi_n(g',y) - \xi_n(g',y)\ d\nu(y) \\ &= \int_Y \sum_{g'\in\mathcal{G}_{r(g)}} (g,\alpha(g)^{-1}y)\cdot\xi_n^{(s(g),\alpha(g)^{-1}y)}(g',y) - \xi_n^{(r(g),y)}(g',y)\ d\nu(y) \\ &= \int_Y \|(g,\alpha(g)^{-1}y)\cdot\xi_n^{(s(g),\alpha(g)^{-1}y)} - \xi_n^{(r(g),y)}\|_1\ d\nu(y) \end{aligned}$$
  tends to 0 when n goes to infinity, by the dominated convergence theorem. It follows that $\mathcal{G}$ is amenable.

□

5.4.2 Property (T)

The same phenomenon can be observed for property (T).

Theorem 5.4.4 ([AD05] Theorem 5.15)**.** *Let $\mathcal{G}$ be a pmp groupoid on a standard probability space (X,μ) and let α be a pmp $\mathcal{G}$-action on a standard probability space (Y,ν). Then $\mathcal{G}$ has property (T) if and only if $\mathcal{G} \underset{\alpha}{\ltimes} Y$ has property (T).*

5.4.3 Strong ergodicity of the Bernoulli shift

Here we prove that the Bernoulli shift over of a nonamenable ergodic pmp groupoid is strongly ergodic. We use this result later on in Corollary 6.2.6.

Let $\mathcal{G}$ be an ergodic pmp groupoid on a standard probability space (X,μ).

Definition 5.4.5. Let V be a countable set. A $\mathcal{G}$-action on V is a groupoid morphism $\mathcal{G}\to\mathfrak{S}(V)$, the symmetric group of V. Such an action is called **amenable** if there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of elements of $\mathrm{L}^0(X,\mu,l^1(V))$ such that:

1. for all $n \in \mathbb{N}$ and for almost all $x \in X$, $\lambda_n(x)$ is non-negative and $\|\lambda_n(x)\|_1 = 1$,
2. for almost all $g \in \mathcal{G}$, $\|g \cdot \lambda_n(s(g)) - \lambda_n(r(g))\|_1 \underset{n\to\infty}{\longrightarrow} 0$, where for $f \in l^1(V)$, $g \cdot f \in \mathbb{R}^V$ denotes the map $v \mapsto f(g^{-1} \cdot v)$.

Lemma 5.4.6. *Let $\mathcal{G} \curvearrowright V$ and $\mathcal{G} \curvearrowright W$ be $\mathcal{G}$-actions on countable sets. Suppose that there exists a factor map $\pi\colon V \to W$ and that $\mathcal{G} \curvearrowright V$ is amenable.*

Then $\mathcal{G} \curvearrowright W$ is amenable.

Proof. For $(\lambda_n)_{n\in\mathbb{N}}$ a sequence witnessing the amenability of $\mathcal{G} \curvearrowright V$, the sequence $(\eta_n)_{n\in\mathbb{N}}$ defined by $\eta_n(x)(w) := \sum_{v\in\pi^{-1}(w)} \lambda_n(x)(v)$ witnesses the amenability of $\mathcal{G} \curvearrowright W$. □

Consider the fibered action $\mathcal{G} \curvearrowright \mathcal{G}$ by $g \cdot g' = g' \star g^{-1}$ for $g' \in \mathcal{G}_{s(g)}$. Let $(T_n)_{n<N}$ be a sequence of Borel total transversals for s which partitions $\mathcal{G}$ for some $N \in \mathbb{N} \cup \{\mathbb{N}\}$, so that $\Psi\colon (x, n) \mapsto T_n \star x$ is a Borel bijection $X \times N \to \mathcal{G}$ such that $s \circ \Psi$ is the projection on the first coordinate.

Then Ψ induces a $\mathcal{G}$-action σ on N, defined by $\sigma(g)n := \pi_N\left(\Psi^{-1}(r(g), g \cdot \Psi(s(g), n)\right)$. It is a cocycle associated to $\mathcal{G} \curvearrowright \mathcal{G}$, meaning that for $\mathfrak{l} \times \sigma$ is conjugate (by Ψ) to $\mathcal{G} \curvearrowright \mathcal{G}$.

Note that by construction, $\mathcal{G}$ is amenable if and only if σ is.

Consider the Bernoulli shift $\mathcal{B}_{\mathcal{G}}$ (see example 5.2.4) of $\mathcal{G}$ over $(\mathbb{Z}/2\mathbb{Z}, \frac{1}{2}(\delta_0 + \delta_1))$, on the standard probability bundle $\bigsqcup_{x\in X} \mathbb{Z}/2\mathbb{Z}^{\mathcal{G}_x}$. Then Ψ also induces a measure preserving $\mathcal{G}$-action $b_{\mathcal{G}}$ on $(\mathbb{Z}/2\mathbb{Z}^N, \left(\frac{1}{2}(\delta_0 + \delta_1)\right)^{\otimes N})$ defined by the formula

$$b_{\mathcal{G}}(g)((u_n)_{n<N}) := (u_{\sigma(g)^{-1}n})_{n<N},$$

which is a cocycle associated to $\mathcal{B}_{\mathcal{G}}$.

Proposition 5.4.7. *If $\mathcal{G}$ is not amenable, then $b_{\mathcal{G}}$ is strongly ergodic.*

Proof. We prove in fact the stronger proposition that if the Koopman representation $\kappa_0^{b_{\mathcal{G}}}$ has almost invariant unit sections, then $\mathcal{G}$ is amenable.

Recall that $\kappa_0^{b_{\mathcal{G}}}$ is a unitary representation on $\mathrm{L}^2\left(\mathbb{Z}/2\mathbb{Z}^N, \left(\frac{1}{2}(\delta_0 + \delta_1)\right)^{\otimes N}\right) \ominus \mathbb{C}1_{\mathbb{Z}/2\mathbb{Z}^N}$. By the Fourier transform, $\kappa_0^{\mathcal{B}_{\mathcal{G}}}$ is unitarily equivalent to the unitary representation of $\mathcal{G}$ on $l^2(\mathbb{Z}/2\mathbb{Z}^{(N)})$, where $\mathbb{Z}/2\mathbb{Z}^{(N)}$ denotes the set of finitely supported nonzero functions $N \to \mathbb{Z}/2\mathbb{Z}$, which is itself unitarily equivalent to the unitary representation of $\mathcal{G}$ on $l^2(\mathcal{P}_f^*(N))$ defined by $g \cdot \phi(F) = \phi(\sigma(g)^{-1}(F))$, where $\mathcal{P}_f^*(N)$ denotes the set of finite nonempty subsets of N.

We define an isometric embedding $\Xi\colon l^1(\mathcal{P}_f^*(N)) \hookrightarrow l^1(N^{(<\infty)})$, where $N^{(<\infty)}$ denotes the set of injective finite nonempty sequences of elements of N, with the formula:

$$\Xi(\phi)\colon s \mapsto \frac{1}{\mathrm{Card}(\mathrm{Im}\, s)!}\phi(\mathrm{Im}\, s).$$

Indeed, Ξ is linear and for $\psi \in l^1(\mathcal{P}_f^*(N))$,

$$\begin{aligned}
\|\Xi(\phi)\|_1 &= \sum_{s\in N^{(<\infty)}} \frac{1}{\operatorname{Card}(\operatorname{Im} s)!}|\phi(\operatorname{Im} s)| \\
&= \sum_{F\in\mathcal{P}_f^*(N)} \sum_{\substack{s\in N^{(<\infty)}\\ \operatorname{Im} s=F}} \frac{1}{\operatorname{Card}(F)!}|\phi(F)| \\
&= \sum_{F\in\mathcal{P}_f^*(N)} \operatorname{Card}(F)!\frac{1}{\operatorname{Card}(F)!}|\phi(F)| \\
&= \sum_{F\in\mathcal{P}_f^*(N)} |\phi(F)| \\
&= \|\phi\|_1 .
\end{aligned}$$

Furthermore, note that Ξ intertwines the actions of $\mathcal{G}$ on $l^1(\mathcal{P}_f^*(N))$ and $l^1(N^{(<\infty)})$ respectively defined by $g\cdot\phi(F) = \phi(\sigma(g)^{-1}(F))$ and $g\cdot\phi(s) = \phi(\sigma^{<\infty}(g)(s))$, where $\mathcal{G} \overset{\sigma^{<\infty}}{\curvearrowright} N^{<\infty}$ is the product action associated to σ.

Now suppose that $l^2(\mathcal{P}_f^*(N))$ has a sequence of almost invariant unit sections $(\xi_n)_{n\in\mathbb{N}}$. Then the sequence $(\Xi(|\xi_n(x)|^2))_{n\in\mathbb{N}}$ witnesses the amenability of $\mathcal{G} \overset{\sigma^{<\infty}}{\curvearrowright} N^{<\infty}$.

However, the projection $\pi\colon N^{<\infty} \to N$ is a factor map between $\mathcal{G} \overset{\sigma^{<infty}}{\curvearrowright} N^{<\infty}$ and $\mathcal{G} \overset{\sigma}{\curvearrowright} N$, so that σ is amenable and therefore $\mathcal{G}$ is amenable. □

5.4.4 Topologies on the space of $\mathcal{G}$-actions

In order to define a useful topology on the space of measure-preserving $\mathcal{G}$-actions on a given standard probability space, we use the correspondence between $\mathcal{G}$-actions and full boolean actions of $[\mathcal{G}]$.

Fix standard probability spaces (X,μ) and (Y,ν) and let $\mathcal{G}$ be a pmp groupoid on (X,μ). Let $\mathrm{G} \leqslant [\mathcal{G}]$ and let $\varphi\colon \operatorname{MAlg}(X,\mu) \hookrightarrow \operatorname{MAlg}(Y,\nu)$ be an embedding.

We call $[\mathrm{A}](\mathrm{G},Y,\nu,\varphi)$ the set of φ-full boolean measure-preserving actions of G on $\operatorname{MAlg}(Y,\nu)$, seen as a subset of $\operatorname{Aut}(Y,\nu)^{\mathrm{G}}$.

The weak and uniform topologies on $\operatorname{Aut}(Y,\nu)$ induce respective topologies τ_w and τ_u on $[\mathrm{A}](\mathrm{G},Y,\nu,\varphi)$, defined to be the restriction of the product topology on $\operatorname{Aut}(Y,\nu)^{\mathrm{G}}$ for the respective topologies on $\operatorname{Aut}(Y,\nu)$.

A basis of τ_w is given by

$$U^{\rho_0,T_1,\dots,T_n}_{A_1,\dots,A_p,\varepsilon}(\mathrm{G}) = \{\rho \in [\mathrm{A}](\mathrm{G},Y,\nu,\varphi)\colon \forall (i,k) \in [\![1,n]\!]\times[\![1,p]\!],\ \nu(\rho(T_i)(A_k)\triangle\rho_0(T_i(A_k)) < \varepsilon\}$$

Lemma 5.4.8. *Let $\mathcal{G}$ be a pmp groupoid on (X,μ) and $\mathrm{G} \leqslant [\mathcal{G}]$. Then*

- *$[\mathrm{A}](\mathrm{G},Y,\nu,\varphi)$ is closed in $\operatorname{Aut}(Y,\nu)^{\mathrm{G}}$ for both τ_w and τ_u.*
- *$([\mathrm{A}]([\mathcal{G}],Y,\nu,\varphi),\tau_w)$ is a Polish space.*

Proof. First, observe that for $\rho \in \operatorname{Aut}(Y,\nu)^{\mathrm{G}}$, $\rho \in [\mathrm{A}](\mathrm{G},Y,\nu,\varphi)$ if and only if for all $T,T' \in \mathrm{G}$, $\rho(T'T) = \rho(T')\rho(T)$ and for all $T \in \mathrm{G}$, for all $A \subseteq \varphi(\operatorname{Fix}(T))$, we have $\rho(T)(A) = A$. Consequently, $[\mathrm{A}](\mathrm{G},Y,\nu,\varphi)$ is an intersection of closed sets representing the latter conditions, and thus $[\mathrm{A}](\mathrm{G}, Y, \nu, \varphi)$ is closed in $\mathrm{Aut}(Y, \nu)^{\mathrm{G}}$.

If G is countable, since the weak topology on $\mathrm{Aut}(Y, \nu)$ is Polish, then $\mathrm{Aut}(Y, \nu)^{\mathrm{G}}$ is Polish for the weak topology . Then $([\mathrm{A}](\mathrm{G}, Y, \nu, \varphi), \tau_w)$ is closed in a Polish space and therefore Polish . Let Γ be a countable full generator of $[\mathcal{G}]$. We will prove that $([\mathrm{A}]([\mathcal{G}], Y, \nu, \varphi), \tau_w)$ is homeomorphic to $([\mathrm{A}](\Gamma, Y, \nu, \varphi), \tau_w)$ and therefore Polish.

As we already saw, any full boolean measure-preserving action α of Γ can be extended into a unique full boolean measure-preserving action $\rho^{[\mathcal{G}]}$ of $[\Gamma] = [\mathcal{G}]$, and conversely, taking the restriction $\rho_{\restriction\Gamma}$ of an action α of $[\mathcal{G}]$ yields an action of Γ. The maps $\rho \mapsto \rho^{[\mathcal{G}]}$ and $\rho \mapsto \rho_{\restriction\Gamma}$ are inverses of each other and form a 1-to-1 correspondence between $[\mathrm{A}]([\mathcal{G}], Y, \nu, \varphi)$ and $[\mathrm{A}](\Gamma, Y, \nu, \varphi)$. Moreover, the restriction to Γ is clearly weak-to-weak continuous. Let us show that $\rho \mapsto \rho^{[\mathcal{G}]}$ is weak-to-weak continuous as well.

Let $T_1, \ldots, T_n \in [\mathcal{G}]$ and let $(S_k)_{k\in\mathbb{N}}$ be an enumeration of Γ. Since Γ is a countable full generator for $[\mathcal{G}]$, for any $T \in [\mathcal{G}]$, we have $\mu(\bigcup_{m\in\mathbb{N}}\{x \in X \colon \exists j \leqslant m, T \star x = S_j \star x\}) = 1$. Therefore let $m \in \mathbb{N}$ be such that $\mu(\{x \in X : \exists j \leqslant m, T_i \star x = S_j \star x\}) > 1 - \frac{\varepsilon}{2}$ for all $i \in [\![1, n]\!]$ and let $E_{i,j}$ be the set $\{x \in X \colon T_i \star x = S_j \star x\}$. Let $\rho_0 \in [\mathrm{A}]([\mathcal{G}], Y, \nu, \varphi)$ and let $A_1, \ldots, A_p \in \mathrm{MAlg}(Y, \nu)$.

Now consider $\rho \in U^{S_1,\ldots,S_m}_{\rho_{0\restriction\Gamma}, A_1,\ldots,A_p,\frac{\varepsilon}{2m}}(\Gamma)$. For $i \in [\![1, n]\!]$ and $k \in [\![1, p]\!]$, we have

$$\begin{aligned}
\nu(\rho^{[\mathcal{G}]}(T_i)(A_k) \triangle \rho_0(T_i)(A_k)) &< \sum_{j=1}^{m} \nu\left(\varphi(E_{i,j}) \cap \rho^{[\mathcal{G}]}(T_i)(A_k) \triangle \rho_0(T_i)(A_k)\right) + \frac{\varepsilon}{2} \\
&= \sum_{j=1}^{m} \nu\left(\varphi(E_{i,j}) \cap \rho(S_j)(A_k) \triangle \rho_0(S_j)(A_k)\right) + \frac{\varepsilon}{2} \\
&< m \cdot \frac{\varepsilon}{2m} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

So that $\rho^{[\mathcal{G}]} \in U^{\rho_0, T_1,\ldots,T_n}_{A_1,\ldots,A_p,\varepsilon}([\mathcal{G}])$, which proves that $\rho \mapsto \rho^{[\mathcal{G}]}$ is indeed continuous. □

Remark 5.4.9. A similar proof allows to show that $([\mathrm{A}]([\mathcal{G}], Y, \nu, \varphi), \tau_u)$ and $([\mathrm{A}](\Gamma, Y, \nu, \varphi), \tau_u)$ are homeomorphic. Although $\mathrm{Aut}(Y, \nu)$ is not Polish for the uniform topology, we still get that τ_u is a completely metrizable topology on $[\mathrm{A}]([\mathcal{G}], Y, \nu, \varphi)$.

Let $\mathrm{Aut}_\varphi(Y, \nu)$ be the subgroup of $\mathrm{Aut}(Y, \nu)$ consisting of elements T such that $T\varphi = \varphi$. Then $\mathrm{Aut}_\varphi(Y, \nu)$ acts by conjugation on $[\mathrm{A}]([\mathcal{G}], Y, \nu, \varphi)$. We have

Theorem 5.4.10. *Let $\mathcal{G}$ be a pmp groupoid on (X, μ) with property (T). Let ϕ^{-1} be an embedding $\mathrm{MAlg}(X, \mu) \hookrightarrow \mathrm{MAlg}(Y, \nu)$, for some pmp map $\phi\colon Y \to X$. Then the $\mathrm{Aut}_{\phi^{-1}}(Y, \nu)$-orbit of any ergodic full boolean action of $[\mathcal{G}]$ on (Y, ν) is uniformly clopen in $[\mathrm{A}]([\mathcal{G}], Y, \nu, \phi^{-1})$.*

Proof. Let Γ be a countable full generator of $[\mathcal{G}]$. We want to apply [Kec10, Theorem 14.2] , however there is no reason why Γ should have property (T).

Taking a closer look at the proof of [Kec10, Theorem 14.2], one can see that property (T) is used to get an invariant vector from a sequence of almost invariant vectors, for a given unitary representation. Since $\mathcal{G}$ has property (T), it suffices to find a similar representation which is full (see Definition 5.2.14) and proceed with the same method using a Kazhdan pair (see Proposition 5.3.11).

The representation we consider is the following: let $\rho_1, \rho_2 \in [\mathrm{A}]([\mathcal{G}], Y, \nu, \phi^{-1})$, let $\mathcal{R} = \mathcal{R}_{\rho_1} \vee \mathcal{R}_{\rho_2}$ be the equivalence relation on Y generated by the orbit equivalence relations of ρ_1 and ρ_2 and let $\mathcal{R}_{\phi^{-1}} = \mathcal{R} \cap Y \underset{\phi}{*} Y$, where $Y \underset{\phi}{*} Y$ is the fibered product over ϕ, i.e. $\{(y, y') \in Y^2 : \phi(y) = \phi(y')\}$. Recall that for $i = 1, 2$, since $\mathcal{R}_{\rho_i}$ is in particular the orbit equivalence relation of the restriction of ρ_i to any countable full generator of $[\mathcal{G}]$, it is a countable equivalence relation, and therefore $\mathcal{R}_{\phi^{-1}}$ is also countable. Consider the σ-finite measure ν^1 defined in 5.1.12 on $\mathcal{R}$ and its restriction $\nu_{\phi^{-1}}$ to $Y \underset{\phi}{*} Y$, which is a set of measure at least 1. Note that the product action $\rho_1 \times \rho_2$ on Y^2 leaves $\mathcal{R}$ invariant and moreover, since ϕ^{-1} is a factor map from both ρ_1 and ρ_2 to $\mathfrak{l}$, $\rho_1 \times \rho_2$ also leaves $Y \underset{\phi}{*} Y$ invariant. Finally, the action $\rho_1 \times \rho_2$ preserves the measure $\nu_{\phi^{-1}}$. It follows that $\rho_1 \times \rho_2$ induces a unitary representation $\pi \colon [\mathcal{G}] \to \mathcal{U}(\mathrm{L}^2(\mathcal{R}_{\phi^{-1}}, \nu_{\phi^{-1}}, \mathbb{R}))$.

We define a faithful normal $*$-representation of $\mathrm{L}^\infty(X, \mu)$ on $\mathrm{L}^2(\mathcal{R}_{\phi^{-1}}, \nu_{\phi^{-1}}, \mathbb{R})$ by letting, for $A \in \mathrm{MAlg}(X, \mu)$, $\chi_A \cdot \xi(y, y') = \chi_{\phi^{-1}(A)}(y)\xi(y, y')$. Therefore, $\mathrm{L}^2(\mathcal{R}_{\phi^{-1}}, \nu_{\phi^{-1}}, \mathbb{R})$ is a $\mathrm{L}^\infty(X, \mu)$-module. Let us prove that π is full, as defined in Definition 5.2.14.

First, recall that ρ_1 and ρ_2 are full for ϕ^{-1}, so ϕ^{-1} is a factor map from both ρ_1 and ρ_2 to $\mathfrak{l}$, therefore, for $A \in \mathrm{MAlg}(X, \mu)$ and $T \in [\mathcal{G}]$, $\chi_{\phi^{-1}(A)} \circ \rho_1(T)^{-1} = \chi_{\phi^{-1}(A)} \circ \rho_2(T)^{-1} = \chi_{\phi^{-1}(\mathfrak{l}(T)A)}$. Now let $T \in [\mathcal{G}]$, $A \in \mathrm{MAlg}(X, \mu)$, $f \in \mathrm{L}^2(\mathcal{R}_{\phi^{-1}}, \nu_{\phi^{-1}})$ and $(y, y') \in \mathcal{R}_{\phi^{-1}}$. On the one hand, we have

$$\begin{aligned}
\pi(T)\chi_A f(y, y') &= \chi_A f(\rho_1(T)^{-1}y, \rho_2(T)^{-1}y') \\
&= \chi_{\phi^{-1}(A)}(\rho_1(T)^{-1}y) f(\rho_1(T)^{-1}y, \rho_2(T)^{-1}y') \\
&= \chi_{\phi^{-1}(\mathfrak{l}(T)A)}(y)\pi(T)f(y, y') \\
&= \chi_{\mathfrak{l}(T)A}\pi(T)f(y, y')
\end{aligned}$$

so $\pi(T)\chi_A = \chi_{\mathfrak{l}(T)A}\pi(T)$. On the other hand, if $T \star A = A$ we have

$$\begin{aligned}
&\pi(T)\chi_A f(y, y') - \chi_A f(y, y') \\
&= \chi_{\phi^{-1}(\mathfrak{l}(T)A)}(y) f(\rho_1(T)^{-1}y, \rho_2(T)^{-1}y') - \chi_{\phi^{-1}(A)}(y) f(y, y') \\
&= \chi_{\phi^{-1}(A)}(y)(f(\rho_1(T)^{-1}y, \rho_2(T)^{-1}y') - f(y, y'))
\end{aligned}$$

but by fullness of ρ_1 and ρ_2, we have

$$y \in \phi^{-1}(A) \Rightarrow y, y' \in \phi^{-1}(A) \Rightarrow f(\rho_1(T)^{-1}y, \rho_2(T)^{-1}y') - f(y, y') = 0.$$

We conclude $\pi(T)\chi_A = \chi_A$.

Thus π is full. We can now proceed as in [Kec10, Lemma 14.1] for the end of the proof:

By Proposition 5.3.11 let (F, κ) be a Kazhdan pair for $\mathcal{G}$ such that any unitary representation of $\mathcal{G}$ which has a (F, κ)-invariant nonzero section ξ admits an invariant section at distance less than $\frac{1}{4}\|\xi\|$ of ξ.

A straightforward computation shows that for $T \in [\mathcal{G}]$,

$$\left\|\pi(T)^{-1}(\Delta) - \Delta\right\|_2^2 = 2d_u(\rho_1(T), \rho_2(T)),$$

where $\Delta \in \mathrm{L}^2(\mathcal{R}_{\phi^{-1}}, \nu_{\phi^{-1}}, \mathbb{R})$ is the transversal of Y, that is $\Delta(y, y') = 1$ if $y = y'$, 0 otherwise. Moreover, $\|\Delta\|_2^2 = \int_Y 1 \, d\nu(y) = 1$.

Suppose now that ρ_1 is ergodic, and for all $T \in F$, $d_u(\rho_1(T), \rho_2(T)) < \frac{\kappa^2}{2}$. Then for all $T \in F$, $\|\pi(T)^{-1}(\Delta) - \Delta\|_2 < \kappa \|\Delta\|_2$, therefore, Δ is (F, κ)-invariant, so π admits an invariant vector $f \in \mathrm{L}^2(\mathcal{R}_{\phi^{-1}}, \nu_{\phi^{-1}}, \mathbb{R})$ such that $\|f - \Delta\|_2 < \frac{1}{4}$.

Let $\mathrm{E} = \mathcal{R}_{\phi^{-1}} \cap f^{-1}([\frac{1}{2}, \frac{3}{2}])$. Let us prove that E is the graph of a bijection when restricted to some set of positive measure.

Let $A' = \{y \in Y : \exists! y' \in Y, (y, y') \in \mathrm{E}\}$, $B = \{y \in Y : \exists! y' \in A', (y, y') \in \mathrm{E}\}$ and $A = \{y \in A' : \exists y' \in B, (y, y') \in \mathrm{E}\}$. Clearly, $\mathrm{E} \cap A \times B$ is the graph of a Borel bijection $\Psi : A \to B$ which intertwines ρ_1 and ρ_2.

Moreover, let $D_1 = \{y \in Y : \forall y' \in Y, (y', y) \notin \mathrm{E}\}$, $D_2 = \{y \in Y : \exists y' \neq y, (y, y') \in \mathrm{E}\}$, $D_3 = \{y \in Y : \exists y' \neq y, (y', y) \in \mathrm{E}\}$ and $D_4 = \{y \in Y : \exists y', y'' \in A', y' \neq y'' \wedge (y', y) \in \mathrm{E} \wedge (y'', y) \in \mathrm{E}\}$.

Remark that we have $Y \setminus B \subseteq \bigcup_{i=1}^4 D_i$ and compute:

$$\begin{aligned}
\frac{1}{16} &> \|f - \Delta\|_2^2 \\
&= \int_Y \sum_{y \in [y_0]_{\mathcal{R}_\phi^{-1}}} |f(y, y_0) - \Delta(y, y_0)|^2 \, d\nu(y_0) \\
&\geqslant \int_{D_1} |f(y_0, y_0) - 1|^2 \, d\nu(y_0) \\
&\geqslant \frac{1}{4} \nu(D_1),
\end{aligned}$$

so $\nu(D_1) < \frac{1}{4}$.

$$\begin{aligned}
\frac{1}{16} &> \|f - \Delta\|_2^2 \\
&= \int_Y \sum_{y \in [y_0]_{\mathcal{R}_\phi^{-1}}} |f(y_0, y) - \Delta(y_0, y)|^2 \, d\nu(y_0) \\
&\geqslant \int_{D_2} |\frac{1}{2} - 0|^2 \, d\nu(y_0) \\
&\geqslant \frac{1}{4} \nu(D_2),
\end{aligned}$$

so $\nu(D_2) < \frac{1}{4}$.

$$\begin{aligned}
\frac{1}{16} &> \|f - \Delta\|_2^2 \\
&= \int_Y \sum_{y \in [y_0]_{\mathcal{R}_\phi^{-1}}} |f(y, y_0) - \Delta(y, y_0)|^2 \, d\nu(y_0) \\
&\geqslant \int_{D_3} |\frac{1}{2} - 0|^2 \, d\nu(y_0) \\
&\geqslant \frac{1}{4} \nu(D_3),
\end{aligned}$$

so $\nu(D_3) < \frac{1}{4}$.

$$\begin{aligned}\frac{1}{16} &> \|f-\Delta\|_2^2 \\ &= \int_Y \sum_{y\in[y_0]_{\mathcal{R}_\varphi}} |f(y,y_0)-\Delta(y,y_0)|^2 \, d\nu(y_0) \\ &\geqslant \int_{D_4} |\frac{1}{2}-0|^2 \, d\nu(y_0) \\ &\geqslant \frac{1}{4}\nu(D_4),\end{aligned}$$

so $\nu(D_4) < \frac{1}{4}$.

Therefore $\nu(B) > 0$. However, $\operatorname{graph}(\Psi) \subseteq \mathcal{R}_{\phi^{-1}}$ so $\Psi \in [[\mathcal{R}_{\phi^{-1}}]]$ preserves the measure ν. It follows that $\nu(A) = \nu(B) > 0$.

Finally, the construction of A and B ensures that A is ρ_1-invariant, B is ρ_2-invariant. Since ρ_1 is ergodic we have $\nu(A) = \nu(B) = 1$, but then $\Psi \in \mathrm{Aut}_{\phi^{-1}}(Y,\nu)$ conjugates ρ_1 and ρ_2.

In conclusion, we proved that if $\rho_1 \in [\mathrm{A}]([\mathcal{G}], Y, \nu, \phi^{-1})$ is ergodic and $\mathcal{U}$ is a uniformly open neighborhood of ρ_1 such that for $\rho_2 \in [\mathrm{A}]([\mathcal{G}], Y, \nu, \phi^{-1})$, $\rho_2 \in \mathcal{U} \Rightarrow \forall T \in F$, $d_u(\rho_1(T), \rho_2(T)) < \frac{\kappa^2}{2}$, then $\mathcal{U}$ is contained in the conjugacy class (by elements of $\mathrm{Aut}_{\phi^{-1}}(Y,\nu)$) of ρ_1. Consequently, the $\mathrm{Aut}_{\phi^{-1}}(Y,\nu)$-orbit of any ergodic full boolean action of $[\mathcal{G}]$ is uniformly open in $[\mathrm{A}]([\mathcal{G}], Y, \nu, \phi^{-1})$. It follows that those orbits are uniformly clopen. □

Definition 5.4.11. Let G be a group and α, β be two pmp actions of G on $\mathrm{MAlg}(Y,\nu)$. We say that α and β are **approximately conjugate** and we write $\alpha \sim^u \beta$ if for any $\varepsilon > 0$ and $F \subset \mathrm{G}$ finite, there exists $T \in \mathrm{Aut}(Y,\mu)$ such that $d_u(\alpha(g), T\beta(g)T^{-1}) < \varepsilon$ for all $g \in F$.

Remark 5.4.12. If Γ is a countable group, let $\mathrm{Aut}(Y,\nu)$ act on the space $\mathrm{A}(\Gamma, Y, \nu)$ of pmp Γ-actions on (Y,ν) by conjugation. For $E \subset \mathrm{A}(\Gamma, Y, \nu)$, we write $\overline{E}^u$ (resp. $\overline{E}^w$) for the uniform (resp. weak) closure of E. Then by definition $\alpha \sim^u \beta \Leftrightarrow \alpha \in \overline{\mathrm{Aut}(Y,\nu)\cdot\beta}^u$.

Recall for comparison that α is said to be **weakly contained** in β, denoted by $\alpha \prec^w \beta$, if $\alpha \in \overline{\mathrm{Aut}(Y,\nu)\cdot\beta}^w$. Moreover, α and β are called **weakly equivalent**, denoted by $\alpha \sim^w \beta$ if both $\alpha \prec^w \beta$ and $\beta \prec^w \alpha$.

However, unlike the weak topology, the uniform topology admits a $\mathrm{Aut}(Y,\nu)$-invariant metric from where we get $\alpha \in \overline{\mathrm{Aut}(Y,\nu)\cdot\beta}^u \Leftrightarrow \beta \in \overline{\mathrm{Aut}(Y,\nu)\cdot\alpha}^u$ and therefore $\sim^u$ is an equivalence relation, stronger than $\sim^w$.

We will use the following refinement of approximate conjugation: If $\varphi\colon \mathrm{MAlg}(X,\mu) \hookrightarrow \mathrm{MAlg}(Y,\nu)$ is an embedding, we say that α and β are **φ-approximately conjugate**, which we write $\alpha \sim^u_\varphi \beta$ when $\alpha \in \overline{\mathrm{Aut}_\varphi(Y,\nu)\cdot\beta}^u$.

According to the latter terminology, Theorem 5.4.10 has the following consequence:

Let $\mathcal{G}$ be a pmp groupoid on (X,μ) with property (T). Let α and $\beta \in [\mathrm{A}]([\mathcal{G}], Y, \nu, \varphi)$ and suppose that β is ergodic. Then $\alpha \sim^u_\varphi \beta \Leftrightarrow \alpha \simeq \beta$.

5.4.5 Approximate conjugation for groupoids

In this section, we prove a groupoid counterpart to the main theorem of the second chapter of this thesis.

This main theorem (Theorem I) states that if θ is an amenable IRS of a countable group Γ, then any two ergodic actions with IRS θ are approximately conjugate. Applied to an IRS that is a Dirac measure, this theorem implies that any two ergodic free pmp actions of a given amenable group Γ are approximately conjugate. The pmp groupoid version of this theorem is the following:

Let $\mathcal{G}$ be an amenable pmp groupoid and let α and β be two ergodic free pmp actions of $\mathcal{G}$ on a standard probability space (Y,ν), are $[\alpha]$ and $[\beta]$ approximately conjugate?

The answer is positive and can be derived from Theorem I itself:

Theorem 5.4.13. *Let $\mathcal{G}$ be an amenable pmp groupoid on (X,μ). Then any two ergodic free pmp actions of $\mathcal{G}$ on standard probability spaces induce two approximately conjugate boolean actions of $[\mathcal{G}]$.*

Proof. Fix a countable full generator Γ of $\mathcal{G}$ and consider two ergodic free pmp $\mathcal{G}$-actions α and β on a standard probability space (Y,ν).

Let $(x,y)\in X\times Y$ and let $\gamma\in\Gamma$, then almost surely we have

$$\begin{aligned}\gamma\in \operatorname{stab}_{[\alpha]}(x,y) &\Leftrightarrow \mathfrak{l}(\gamma)x=x \ \wedge\ \alpha(\gamma x)y=y\\ &\Leftrightarrow \gamma x=x\\ &\Leftrightarrow \mathfrak{l}(\gamma)x=x \ \wedge\ \beta(\gamma x)y=y\\ &\Leftrightarrow \gamma\in \operatorname{stab}_{[\beta]}(x,y).\end{aligned}$$

Therefore, $[\alpha]_{\restriction\Gamma}$ and $[\beta]_{\restriction\Gamma}$ have the same IRS θ. Moreover, the orbit equivalence relation induced by α is the same as the one induced by $[\alpha]_{\restriction\Gamma}$ and the same goes for β. Since $\mathcal{G}$ is amenable, these equivalence relations are amenable, or in other words, the actions $[\alpha]_{\restriction\Gamma}$ and $[\beta]_{\restriction\Gamma}$ are hyperfinite.

By Theorem I, $[\alpha]_{\restriction\Gamma}$ and $[\beta]_{\restriction\Gamma}$ are approximately conjugate.

Finally, given $T_1,\dots,T_n\in[\mathcal{G}]$ and $\varepsilon>0$, there exist $\gamma_1,\dots,\gamma_m\in\Gamma$ such that $\mu(\{x\in X:\forall i\in[\![1,n]\!],\exists j\in[\![1,m]\!],T_ix=\gamma_jx\})>1-\frac{\varepsilon}{2n}$. Thus, for $\rho\in\operatorname{Aut}(Y,\nu)$ such that $\mu\times\nu(\{(x,y)\in X\times Y:\forall j\in[\![1,m]\!],\rho[\alpha]\rho^{-1}(\gamma_j)(x,y)=[\beta](\gamma_j)(x,y)\})>1-\frac{\varepsilon}{2}$, we have $\mu\times\nu(\{(x,y)\in X\times Y:\forall j\in[\![1,n]\!],\rho[\alpha]\rho^{-1}(T_i)(x,y)=[\beta](T_i)(x,y)\})>1-\varepsilon$, by fullness of $[\alpha]$ and $[\beta]$, hence the conclusion. ☐

Chapter 6

Application to Invariant Random Subgroups

In this chapter we study Invariant Random Subgroups (IRS in short) and in particular the classification of pmp actions of a countable group Γ having a given IRS of Γ.

We first associate to any IRS θ of Γ a pmp groupoid $\mathcal{G}_\Gamma^\theta$ such that there exists a correspondence between pmp actions of Γ having IRS θ and pmp free $\mathcal{G}_\Gamma^\theta$. We then use results obtained in Chapter 5 to study pmp actions having IRS θ.

6.1 Invariant Random Subgroups

Definition 6.1.1 (Space of subgroups)**.** Let Γ be a countable group. Consider the space $\{0,1\}^\Gamma$ equipped with the product topology. It is a Polish space and can be identified as the set of subsets of Γ. Through this identification, the subspace $\mathrm{Sub}(\Gamma)$ of subgroups of Γ is closed, hence Polish. From now on, we consider $\mathrm{Sub}(\Gamma)$ as a standard Borel space.

The group Γ naturally acts in a Borel way on $\mathrm{Sub}(\Gamma)$ by conjugation.

Definition 6.1.2 (Invariant random subgroup)**.** An **Invariant Random Subgroup (IRS)** on Γ is a Borel probability measure on $\mathrm{Sub}(\Gamma)$ invariant for the action of Γ by conjugation.

Example 6.1.3. Let $\Gamma \overset{\alpha}{\curvearrowright} (Y,\nu)$ be a pmp group action on a standard probability space. Consider the stabilizer map $\mathrm{stab}_\alpha \colon Y \to \mathrm{Sub}(\Gamma)$. Note that it is Borel and thus the pushforward $\mathrm{stab}_{\alpha*}\nu$ of ν defines a Borel probability measure on $\mathrm{Sub}(\Gamma)$. Using the well-known formula $\mathrm{stab}(\gamma y) = \gamma^{-1}\mathrm{stab}(y)\gamma$, one can see that this measure is invariant by conjugation. In other words, it is an IRS on Γ.

We call this measure the **IRS associated to the action** α, and we denote it θ_α.

In fact, any IRS on a group Γ arises from the latter example. As we will see later on, the proof given in [Tuc15, Prop. 5.9]) can be reformulated quite nicely with the following terminology:

Definition 6.1.4 (Coset groupoid)**.** Let Γ be a countable group. We define a Borel groupoid $\mathcal{G}_\Gamma$ as follows:

- The base space is $\mathrm{Cos}(\Gamma) := \bigsqcup_{\Lambda \in \mathrm{Sub}(\Gamma)} \Gamma/\Lambda$. Since $\mathrm{Cos}(\Gamma) = \bigcup_{\gamma \in \Gamma} \gamma \mathrm{Sub}(\Gamma)$, it is a Borel subset of $\{0,1\}^{\Gamma}$ and inherits its standard Borel space structure.
- The space of units is $\mathrm{Sub}(\Gamma)$.
- The source and range maps are defined by the formulas $s(\gamma\Lambda) = \Lambda$ and $r(\gamma\Lambda) = \gamma\Lambda\gamma^{-1}$.
- The product is then defined by $\gamma'(\gamma\Lambda\gamma^{-1}) \cdot \gamma\Lambda = \gamma'\gamma\Lambda$.
- Accordingly, the inverse of $\gamma\Lambda$ is $\gamma^{-1}(\gamma\Lambda\gamma^{-1})$.

We call this groupoid the **coset groupoid of** Γ and we denote it by $\mathcal{G}_\Gamma$.

Note that for each $\gamma \in \Gamma$, the set $\gamma\mathrm{Sub}(\Gamma) \subseteq \mathrm{Cos}(\Gamma)$ defines an element of $[\mathcal{G}_\Gamma]_{\mathrm{b}}$, so that Γ canonically embeds in $[\mathcal{G}_\Gamma]_{\mathrm{b}}$. Moreover, $\bigcup_{\gamma\in\Gamma} \gamma\mathrm{Sub}(\Gamma) = \mathrm{Cos}(\Gamma)$ and therefore Γ is a countable full generator for $[\mathcal{G}_\Gamma]_{\mathrm{b}}$.

Let θ be an IRS on a group Γ. We say that Γ is faithful to θ if the restriction to Γ of the quotient map from the full group of $\mathcal{G}$, seen as a Borel groupoid on $\mathrm{Sub}(\Gamma)$, to the full group of $\mathcal{G}$, seen as a groupoid on $(\mathrm{Sub}(\Gamma), \theta)$, is injective. In other words, if for any $\gamma \in \Gamma \setminus \{e\}$, $\theta(\{\Lambda : \gamma \in \Lambda\}) < 1$.

Remark 6.1.5. Let θ be an IRS on a group Γ. Then there is a quotient Γ' of Γ which is faithful to the quotient measure $\theta_{\Gamma'}$, such that the quotient map $(\mathrm{Sub}(\Gamma), \theta) \to (\mathrm{Sub}(\Gamma'), \theta_{\Gamma'})$ is an isomorphism of probability spaces and therefore induces an isomorphism of pmp groupoids $\mathcal{G}_\Gamma \to \mathcal{G}_{\Gamma'}$. Indeed, let $N = \{\gamma \in \Gamma : \theta(\gamma \in \Lambda) = 1\}$. Since θ is an IRS, N is a normal subgroup of Γ, and so we let $\Gamma' = \Gamma/N$.

From now on, without loss of generality, every IRS θ considered is supposed to be an IRS on a group Γ which is faithful to θ. Consequently, Γ embeds in $[\mathcal{G}_\Gamma]$.

Lemma 6.1.6. *Let θ be a Borel probability measure on* $\mathrm{Sub}(\Gamma)$. *Then $\mathcal{G}_\Gamma$ on* $(\mathrm{Sub}(\Gamma), \theta)$ *is pmp if and only if θ is an IRS.*

Proof. Since Γ is a full generator for $[\mathcal{G}_\Gamma]$, the action $[\mathcal{G}_\Gamma] \overset{\mathfrak{l}}{\curvearrowright} \mathrm{Sub}(\Gamma)$ preserves θ if and only if its restriction to Γ preserves θ. However, the restriction of $\mathfrak{l}$ to Γ is simply the action of $\Gamma \curvearrowright \mathrm{Sub}(\Gamma)$ by conjugation.

It follows that $\mathcal{G}_\Gamma$ is a pmp groupoid on $(\mathrm{Sub}(\Gamma), \theta)$ if and only if $[\mathcal{G}_\Gamma] \overset{\mathfrak{l}}{\curvearrowright} \mathrm{Sub}(\Gamma)$ preserves θ, if and only if θ is an IRS on Γ. □

From now on, we will always consider $\mathcal{G}_\Gamma$ as a pmp groupoid on the standard probability space $(\mathrm{Sub}(\Gamma), \theta)$ for some IRS θ. We use the notation $\mathcal{G}_\Gamma^\theta$ to indicate that we consider $\mathcal{G}_\Gamma$ as a pmp groupoid on $(\mathrm{Sub}(\Gamma), \theta)$.

Definition 6.1.7. An IRS θ on Γ is called **ergodic** if it is an ergodic measure for the action of Γ on $\mathrm{Sub}(\Gamma)$, or in other words, if the pmp groupoid $\mathcal{G}_\Gamma^\theta$ is ergodic.

6.2 Actions of the coset groupoid

In this section we define a very useful correspondence between faithful pmp actions of the group Γ and free (for a well chosen IRS) actions of the pmp groupoid $\mathcal{G}_\Gamma$.

Definition 6.2.1. Let $\mathcal{G}$ be a pmp groupoid on (X, μ) and let α be a pmp action of $\mathcal{G}$ on (Y, ν). We say that α is **free** if for almost all g in the isotropy $\mathcal{IG}$ of $\mathcal{G}$ (that is any g such that $s(g) = r(g)$) and almost all $y \in Y$, $\alpha(g)y = y \Rightarrow g \in X$.

Note that this definition extends the definition of free action for a countable group. However, this is not a useful notion in the context of pmp equivalences relations. Indeed, every action of a pmp equivalence relation is automatically free.

Let θ be an ergodic IRS on Γ. Let $\Gamma \overset{\rho}{\curvearrowright} (Y, \nu)$ be a pmp action such that $\theta_\rho = \theta$. Just as before, we identify Γ with a subgroup of $[\mathcal{G}_\Gamma]$. Let us prove that ρ is full for the map $\mathrm{stab}_\rho^{-1} \colon \mathrm{MAlg}(\mathrm{Sub}(\Gamma), \theta) \to \mathrm{MAlg}(Y, \nu)$.

First off, stab_ρ^{-1} is clearly a factor map from ρ to $\mathfrak{l}$ (recall that the latter is the action of Γ by conjugation).

Moreover, let $A \in \mathrm{MAlg}(\mathrm{Sub}(\Gamma), \theta)$ and suppose that $\gamma\Lambda = \Lambda$, or in other words, $\gamma \in \Lambda$. Then for $y \in \mathrm{stab}_\rho^{-1}(\Lambda)$, we have $\gamma \in \Lambda = \mathrm{stab}_\rho(y)$ and it follows that $\rho(\gamma)y = y$.

We can now apply Proposition 5.2.12 to the extensions to $[\mathcal{G}_\Gamma]$ of full actions of Γ, in order to get a standard probability space (Z_θ, η_θ) and an isomorphism $\Phi_\theta \colon \mathrm{Sub}(\Gamma) \times Z_\theta \to Y$ which intertwines the inclusion $\mathrm{MAlg}(\mathrm{Sub}(\Gamma), \theta) \hookrightarrow \mathrm{MAlg}(\mathrm{Sub}(\Gamma) \times Z_\theta, \theta \times \eta_\theta)$ and the stabilizer map $\mathrm{MAlg}(\mathrm{Sub}(\Gamma), \theta) \to \mathrm{MAlg}(Y, \nu)$, and such that for any action ρ of Γ on (Y, ν) with IRS θ, the action $\Phi_\theta^{-1} \rho \Phi_\theta$ is of the form $[\alpha]_{\restriction\Gamma}$ for a $\mathcal{G}_\Gamma$-action α on (Z_θ, η_θ). We call α_ρ this $\mathcal{G}_\Gamma$-action.

Furthermore, α_ρ is free. Indeed, fix a $\gamma\Lambda \in \mathcal{IG}_\Gamma$ and suppose that $\exists^* z \in Z_\theta$ such that $\alpha_\rho(\gamma\Lambda)z = z$. Then we have $[\alpha_\rho](\gamma)(\Lambda, z) = (\gamma\Lambda\gamma^{-1}, \alpha_\rho(\gamma\Lambda)z) = (\Lambda, z)$. It follows that $\gamma \in \mathrm{stab}_\rho(\Phi_\theta(\Lambda, z))$, but $\mathrm{stab}_\rho \circ \Phi_\theta$ is the projection onto the first coordinate so $\gamma \in \Lambda$. We conclude that $\gamma\Lambda = \Lambda \in \mathrm{Sub}(\Gamma)$ and therefore α_ρ is free.

Conversely, consider $\mathcal{G}_\Gamma$ as a pmp groupoid on $(\mathrm{Sub}(\Gamma), \theta)$. Let α be a free pmp $\mathcal{G}_\Gamma$-action on a standard probability space (Z, η) and let $\rho_\alpha = [\alpha]_{\restriction\Gamma}$. Then ρ_α is a pmp Γ-action. Let us prove that $\theta_{\rho_\alpha} = \theta$.

For almost all $(\Lambda, z) \in \mathrm{Sub}(\Gamma) \times Z$ and for any $\gamma \in \Gamma$,

$$\begin{aligned} \gamma \in \mathrm{stab}_{\rho_\alpha}(\Lambda, z) &\Leftrightarrow \gamma\Lambda\gamma^{-1} = \Lambda \wedge \alpha(\gamma\Lambda)z = z \\ &\Leftrightarrow \gamma\Lambda \in \mathcal{IG} \wedge \alpha(\gamma\Lambda)z = z \\ &\Leftrightarrow \gamma \in \Lambda \end{aligned}$$

It follows that $\theta_{\rho_\alpha}(\gamma \in \Lambda) = \theta \times \eta(\gamma \in \mathrm{stab}_{\rho_\alpha}(\Lambda, z)) = \theta(\gamma \in \Lambda)$, which proves that $\theta_{\rho_\alpha} = \theta$.

Corollary 6.2.2. *([AGV14, Prop. 14]) Let θ be an ergodic IRS on Γ. Then there exists a pmp Γ-action which has IRS θ.*

Proof. We consider the Bernoulli shift $\mathcal{B}_{(Y,\nu)}$ of $\mathcal{G}_\Gamma$ on an atomless standard probability space (Y, ν) and show that it is free.

Because Y is atomless, almost any element $f \in Y^n$ is an injection. For such a f and for almost any $\gamma\Lambda \in \mathcal{IG}_\Gamma$, we have

$$\begin{aligned}\mathcal{B}_{(Y,\nu)}(\gamma\Lambda)f = f &\Rightarrow \forall k < n, T_\Lambda^{-1}(T_\Lambda(k)\gamma\Lambda) = k \\ &\Rightarrow \forall g \in (\mathcal{G}_\Gamma)_\Lambda, g(\gamma\Lambda) = g \\ &\Rightarrow \gamma\Lambda \in \mathrm{Sub}(\Gamma).\end{aligned}$$

Therefore $\mathcal{B}_{(Y,\nu)}$ is free and it follows that $[\mathcal{B}_{(Y,\nu)}]_{|\Gamma}$ is a pmp Γ-action of IRS θ. □

Corollary 6.2.3. *Let θ be an ergodic IRS of a group Γ and $\Gamma \overset{\rho}{\curvearrowright} (Y,\nu)$ be a pmp action with IRS θ. Then the orbit equivalence relation $\mathcal{R}_\rho$ is amenable if and only if $\mathcal{G}_\Gamma$ is amenable on $(\mathrm{Sub}(\Gamma),\theta)$.*

Proof. Let $\mathcal{G}$ be a pmp groupoid and α be a pmp $\mathcal{G}$-action on (Z,η). Then the map $(g,z) \mapsto (gz,z)$ defines a pmp homomorphism between the groupoids $\mathcal{G} \underset{\alpha}{\ltimes} Z$ and $\mathcal{R}_\alpha$. It is an isomorphism if and only if the action α is free.

Applying the latter isomorphism to α_ρ and Theorem 5.4.3, we see that $\mathcal{R}_\rho = \mathcal{R}_{\alpha_\rho}$ is amenable if and only if $\mathcal{G}_\Gamma \underset{\alpha_\rho}{\ltimes} Z_\theta$ is amenable, if and only if $\mathcal{G}_\Gamma$ is amenable on $(\mathrm{Sub}(\Gamma),\theta)$. □

An IRS θ is called **amenable** if $\mathcal{G}_\Gamma$ is amenable on $(\mathrm{Sub}(\Gamma),\theta)$.

The latter corollary ensures that this definition is consistent with the definition of an amenable IRS given in the introduction, that is, an IRS is amenable when every pmp action having this IRS is amenable (or equivalently hyperfinite).

Corollary 6.2.4. *Let θ be an ergodic IRS of a group Γ and $\Gamma \overset{\rho}{\curvearrowright} (Y,\nu)$ be a pmp action with IRS θ. Then the orbit equivalence relation $\mathcal{R}_\rho$ has property (T) if and only if $\mathcal{G}_\Gamma$ has property (T) on $(\mathrm{Sub}(\Gamma),\theta)$.*

Proof. Since α_ρ is free, $\mathcal{R}_{\alpha_\rho}$ is isomorphic to $\mathcal{G}_\Gamma \underset{\alpha_\rho}{\ltimes} Z_\theta$. We use Theorem 5.4.4 to get see that $\mathcal{R}_\rho = \mathcal{R}_{\alpha_\rho}$ has property (T) if and only if $\mathcal{G}_\Gamma \underset{\alpha_\rho}{\ltimes} Z_\theta$ has property (T), if and only if $\mathcal{G}_\Gamma$ has property (T) on $(\mathrm{Sub}(\Gamma),\theta)$. □

An IRS θ is said to have **property (T)** if $\mathcal{G}_\Gamma$ has property (T) on $(\mathrm{Sub}(\Gamma),\theta)$.

Theorem 6.2.5. *Let Γ be a countable group and θ be an ergodic IRS on Γ. Then θ has property (T) if and only if every ergodic pmp action of Γ which has IRS θ is strongly ergodic.*

Proof. Any pmp action of Γ which has IRS θ corresponds to a free pmp $\mathcal{G}_\Gamma$-action where $\mathcal{G}$ is seen as a groupoid over $(\mathrm{Sub}(\Gamma),\theta)$. This groupoid, by definition, has property (T) if and only if θ has property (T). Therefore by Theorem 5.3.28, θ has (T) if and only if every ergodic pmp $\mathcal{G}_\Gamma$-action is strongly ergodic.

It follows that if θ has (T), then every ergodic pmp action with IRS θ is strongly ergodic.

Conversely, if θ does not have (T), then $\mathcal{G}$ admits an ergodic pmp action α which is not strongly ergodic. Without loss of generality, one may replace α with its product action with the atomless Bernoulli shift to ensure it is free, ergodic and not stronlgy ergodic. Therefore $[\alpha]_{|\Gamma}$ is an ergodic pmp action of Γ which has IRS θ and is not strongly ergodic. □

Corollary 6.2.6. *Let Γ be a countable group and θ be an ergodic IRS on Γ. Suppose that θ is not amenable and does not have property (T). Then Γ admits two ergodic pmp actions of IRS θ which are not weakly equivalent (in particular, they are not even approximately conjugate).*

Proof. On the one hand, θ does not have property (T) so there must be an ergodic action of Γ which has IRS θ but is not strongly ergodic. On the other hand, the Bernoulli shift of $\mathcal{G}_\Gamma^\theta$ over $(\mathbb{Z}/2\mathbb{Z}, \frac{1}{2}(\delta_0 + \delta_1))$ induces cocycles which are strongly ergodic.

Therefore, Γ admits a strongly ergodic pmp action of IRS θ, and an ergodic yet not strongly ergodic pmp action of IRS θ. Those two actions cannot be weakly equivalent since strong ergodicity is an invariant of weak equivalence. □

We end this section with an application of Theorem 5.4.10. Though it does not look very promising, we introduce the equivalence relation of stab-equivalence between pmp actions of a group Γ and prove that for actions which have an IRS with property (T), stab-equivalence coincides with conjugation.

We define the equivalence relation of **stab-equivalence** on pmp actions of a countable group Γ to be the equivalence relation generated by conjugation and approximate conjugation relatively that the stabilizers. Rigorously, for two pmp Γ-actions α and β, α is **stab-equivalent** to β, written $\alpha \underset{\text{stab}}{\sim} \beta$, when up to conjugation, $\alpha \in \overline{\mathrm{Aut}_{\mathrm{stab}_\beta{}^{-1}}(Y,\nu)\cdot\beta}^u$.

Remark 6.2.7. Note that stab-equivalence is implied by conjugation and in turn implies approximate conjugation. In the case where $\theta = \delta_{\{e\}}$ is the free IRS on Γ, the stab-equivalence relation is exactly the approximate conjugation, whereas if β is totally non-free, its stab-equivalence class is equal to its conjugation class.

Theorem 6.2.8. *Let Γ be a countable group. If α and β are pmp Γ-actions on (Y,ν) such that β is ergodic, $\alpha \underset{\text{stab}}{\sim} \beta$ and θ_β has property (T), then α and β are conjugate.*

Proof. Let α_0 be a conjugate of α on (Y,ν) such that $\alpha_0 \in \overline{\mathrm{Aut}_{\mathrm{stab}_\beta^{-1}}\cdot\beta}^u$. We have that $\mathrm{Aut}_{\mathrm{stab}_\beta^{-1}}\cdot\beta \subset [\mathrm{A}](\Gamma, Y, \nu, \mathrm{stab}_\beta^{-1})$ which is uniformly closed, so $\alpha_0 \in [\mathrm{A}](\Gamma, Y, \nu, \mathrm{stab}_\beta^{-1})$.

Furthermore, the action β is ergodic so its IRS must be ergodic as well. Therefore we can apply Theorem 5.4.10 to $\alpha_0^{[\mathcal{G}_\Gamma^\theta]}$ and $\beta^{[\mathcal{G}_\Gamma^\theta]} \in [\mathrm{A}]([\mathcal{G}_\Gamma], Y, \nu, \mathrm{stab}_\beta^{-1})$, which leads to $\alpha_0^{[\mathcal{G}_\Gamma^\theta]} \simeq \beta^{[\mathcal{G}_\Gamma^\theta]}$. Thus we have $\alpha_0 \simeq \beta$ and consequently $\alpha \simeq \beta$. □

Chapter 7

Application to the classification of boolean actions of a full group

The present chapter is the first version of a work in common with A. Carderi and F. Le Maître. The main theorem describes a classification of boolean non-free actions of measured full groups. We then use the results established in Chapter 5 and applied to the particular case of equivalence relations to give a characterization of Kazhdan's property (T) of an equivalence relation in terms of boolean actions of its full group.

In a groundbreaking work, Nicolás Matte Bon [MB18] was able to classify actions of any minimal topological full group over a Cantor space X on another Cantor space Y. He proved that either the action has some free behaviour, or that it has to come from the standard action of the full group on a symmetric power of X. The purpose of this work is to exhibit a similar classification for full groups in the sense of Dye, in the context of measure preserving boolean actions on standard probability spaces.

Let us start by fixing some notation. Given a standard probability space (X, μ), denote by $\mathrm{Aut}(X, \mu)$ its group of measure-preserving transformations, two such transformations being identified if they coincide up to a measure zero set. A subgroup $\mathbb{G} \leqslant \mathrm{Aut}(X, \mu)$ is **full** if for every measurable partition $(A_n)_{n\in\mathbb{N}}$ of X and every sequence (g_n) of elements of $\mathbb{G}$ such that $(g_n(A_n))$ is also a partition of X, the element $T \in \mathrm{Aut}(X, \mu)$ defined by

$$T(x) = g_n(x) \text{ for all } x \in A_n \text{ and all } n \in \mathbb{N}$$

actually belongs to $\mathbb{G}$. A full subgroup of $\mathrm{Aut}(X, \mu)$ is called **Polish** when it admits a Polish group topology which refines the topology induced by the weak topology of $\mathrm{Aut}(X, \mu)$. Examples are provided by full groups of countable pmp equivalence relations, and more generally by orbit full groups of measure-preserving actions of Polish groups (see [CLM16]).

Finally, a pmp **boolean action** of a full group $\mathbb{G}$ is simply a group homomorphism $\mathbb{G} \to \mathrm{Aut}(Y, \nu)$. Although full groups do not admit non-trivial measure-preserving actions on standard probability space because all their boolean actions are *whirly* (see [GW05] and [GP07, Sec. 5]), they do admit boolean actions such as the one provided by their inclusion into $\mathrm{Aut}(X, \mu)$. Other examples are provided by (symmetric) diagonal actions, which we now define along with some useful auxiliary notation.

Given a standard Borel space X and $n \in \mathbb{N}$, we have a natural action of the symmetric group $\mathfrak{S}_n$ on X^n by permuting coordinates and the quotient $X^n/\mathfrak{S}_n$ is still a standard Borel space which we denote by $X^{\odot n}$. Moreover, if (X, μ) is a standard probability space, we can endow the space $X^{\odot n}$ with the pushforward measure via the quotient map $\pi_n : (X^n, \mu^{\otimes n}) \to X^{\odot n}$, and we denote by $\mu^{\odot n} := \pi_{n*}\mu^{\otimes n}$ this measure. Given $A_1, \dots, A_n \subset X$ we will use the notation $[A_1, \dots, A_n] := \pi_n(A_1 \times \dots \times A_n) \subseteq X^{\otimes n}$. We have a group homomorphism $\iota^{\odot n} : \mathrm{Aut}(X, \mu) \to \mathrm{Aut}(X^{\odot n}, \mu^{\odot n})$ defined by $\iota^{\odot n}(T)[A_1, \dots, A_n] = [TA_1, \dots, TA_n]$, in particular every full group admits countably many boolean actions obtained by restricting $\iota^{\odot n}$ to the full group in question. It is not hard to see that these boolean actions are not conjugate as soon as the full group is not the trivial group.

We can now state our main theorem, which states roughly that every boolean action decomposes as a free part plus a countable union of boolean actions factoring onto the above defined $\iota^{\odot n}$ in a measure-preserving (and "support-preserving") manner.

Theorem I. *Let $\mathbb{G}$ be an ergodic full group over the standard probability space (X, μ) and let $\rho : \mathbb{G} \to \mathrm{Aut}(Y, \nu)$ be a boolean action on another standard probability space (Y, ν). Then there is a unique measurable $\rho(\mathbb{G})$-invariant partition $\{A_n\}_{n=0,\dots,\omega}$ of Y such that the boolean actions $\rho_n = \rho_{\restriction\left(A_n, \frac{\nu_{\restriction A_n}}{\nu(A_n)}\right)}$ are subject to the following conditions*

1. *the map ρ_0 maps every element to the identity on A_0;*
2. *for all $n \in \omega \setminus \{0\}$, there is a (unique) measure preserving map $\varphi_n : A_n \to X^{\odot n}$ satisfying the following two properties*
   (a) *for all $g \in \mathbb{G}$ and almost all $y \in A_n$ we have $\varphi_n(\rho_n(g)y) = \iota^{\odot n}(g)\varphi_n(y)$;*
   (b) *for all $g \in \mathbb{G}$ and almost all $y \in A_n$, if $\rho_n(g)(y) \neq y$ then $\iota^{\odot n}(g)\varphi_n(y) \neq \varphi_n(y)$.*
3. *for every $g \in \mathbb{G} \setminus \{\mathrm{id}\}$ we have that $\{y \in A_\omega : \ \rho_\omega(g)y = y\}$ is a null-set.*

Let us briefly describe our approach to the above result. An important distinction with the case of topological full groups is that full groups of measure-preserving actions are uncountable. This difficulty is counter-balanced by the presence of the so-called *uniform metric*, which turns them into SIN Raikov-complete groups, not separable in general. Although our main result does not mention the topology, it relies crucially on it, as our first observation towards it is an automatic continuity result which actually makes our life much easier than in the topological context (see Cor. 7.1.8). We also make a crucial use of a classification of the invariant random subgroups of the group of dyadic permutations due to Thomas and Tucker-Drob, while an important part of Matte Bon's proof is to first obtain an analogous classification in the topological context, and to somehow extend it to the full group using the topology. All in all, although our result is heavily inspired by Matte Bon's, the proof is different and much easier in our setup. We also use the following result which might be of independent interest.

Theorem II. *Let $\mathbb{G}$ be an ergodic full group, let τ be a SIN group topology on $\mathbb{G}$. Then τ is either the discrete or the uniform topology.*

Note that the SIN hypothesis above is important, otherwise one can always endow $\mathbb{G}$ with the weak topology induced by $\mathrm{Aut}(X, \mu)$.

Our main application of the classification theorem for boolean actions is about full groups which are *separable* for the uniform topology. Such full groups are better-known as full groups of *pmp equivalence relations* and are thus closely related to the well-developed field of orbit equivalence of pmp actions. This connection is particularly apparent in Dye's reconstruction theorem, which states that any abstract group isomorphism between full groups of pmp aperiodic equivalence relations must come from an isomorphism between the equivalence relations themselves. One thus expects group properties of full groups to reflect properties of the equivalence relation. To our knowledge, only three such examples were known so far for pmp equivalence relations:

- A non-trivial pmp equivalence relation is aperiodic if and only if its full group has no index two normal subgroup [LM16, Thm. 1.14].
- A non-trivial pmp equivalence relation is ergodic if and only if its full group is simple [Eig81].
- An ergodic pmp equivalence relation is amenable if and only if every action by homeomorphism of its full group on a compact metrizable space admits a fixed point [KT10, Cor. 1.4].

We should also mention that if one is rather concerned with topological properties of full groups, the two last properties have natural continuous counterparts which are crucial in establishing the above algebraic properties, using automatic continuity. Another topological property which reflects properties of the pmp equivalence relation is the topological rank [LM], but it has no natural purely algebraic counterpart.

Let us now explain how our main result allows us to obtain one more of the above kind. First note that every pmp action of an equivalence relation on a probability space (Y, ν) induces a boolean action of its associated full group. The main result of the present paper has the following consequence: *every non-free ergodic boolean action of the full group of a pmp equivalence relation comes from a measure-preserving action of the equivalence relation itself or one of its symmetric powers.*

We can then obtain a dynamical characterization of property (T) for pmp equivalence relations, purely in terms of their full group, thus adding a fourth item to the above list.

Theorem III. *Let $\mathcal{R}$ be a pmp ergodic equivalence relation. Then $\mathcal{R}$ has (T) if and only if all the non-free ergodic boolean actions of its full group on standard probability spaces are strongly ergodic.*

Let us finish this introduction by highlighting the key ingredients of the above result, apart from Theorem I. The direct implication (Theorem 7.4.7) relies on the fact that every ergodic pmp action of a pmp equivalence relation with property (T) is strongly ergodic, and that if $\mathcal{R}$ has (T) then so do all its symmetric powers. On the other hand, the converse relies on a natural generalization to pmp equivalence relations of the Connes-Weiss result that if all pmp ergodic actions of a countable group are strongly ergodic, then the group must have property (T). The following question remains open:

Let $\mathcal{R}$ be a pmp ergodic equivalence relation with property (T). Does its full group $[\mathcal{R}]$ have strong property (T) ?

Conventions

Let (X, μ) be a standard probability space, we then denote by $\mathrm{MAlg}(X, \mu)$ its measure algebra, which is the boolean algebra consisting of measurable subsets of X up to measure zero. This algebra is complete, which means that any family $(A_i)_{i \in I}$ of elements has a supremum and an infimum. We will denote these by $\bigvee_{i\in I} A_i$ and $\bigwedge_{i\in I} A_i$. Since they coincide with set-theoretic union and intersection for countable families, when I is countable we will also use the notation $\bigcup_{i\in I} A_i$ and $\bigcap_{i\in I} A_i$.

All the subsets of X that we will consider are seen as elements of $\mathrm{MAlg}(X,\mu)$; in particular they are measurable. We will neglect what happens on measure zero sets.

7.1 Ergodic full groups

7.1.1 Definition and automatic continuity

Let us start by recalling once more Dye's definition of full groups.

Definition 7.1.1. A **full group** is a subgroup $\mathbb{G}$ of $\mathrm{Aut}(X,\mu)$ which is stable under **cutting and pasting**, that is: whenever (A_n) is a partition of X and (T_n) is a sequence of elements of $\mathbb{G}$ such that $(T_n(A_n))$ is a partition of $\mathbb{G}$, then the element $T\in\mathrm{Aut}(X,\mu)$ defined by $T(x)=T_n(x)$ for all $x\in A_n$ actually belongs to $\mathbb{G}$.

Given $G\leqslant\mathrm{Aut}(X,\mu)$, the smallest full group containing G is denoted by $[G]$; it can be constructed by cutting and pasting the elements of G. A subgroup $G\leqslant\mathrm{Aut}(X,\mu)$ is **ergodic** if the only G-invariant elements of $\mathrm{MAlg}(X,\mu)$ are X and $\emptyset$. Here is an important property of ergodic full groups, proved by a maximality argument.

Proposition 7.1.2 ([Dye59, Lem. 3.2])**.** *Let $\mathbb{G}\leqslant\mathrm{Aut}(X,\mu)$ be an ergodic full group. Then for every $A,B\in\mathrm{MAlg}(X,\mu)$ such that $\mu(A)=\mu(B)$, there is an involution $T\in\mathbb{G}$ such that $T(A)=B$.*

Since every set can be written as the disjoint union of two sets of equal measure, we have the following immediate corollary.

Corollary 7.1.3. *Let $\mathbb{G}\leqslant\mathrm{Aut}(X,\mu)$ be an ergodic full group, let $A\in\mathrm{MAlg}(X,\mu)$. Then there is an involution $U\in\mathbb{G}$ whose support is equal to A.* □

We will often use the following other well-known consequence of the above proposition.

Proposition 7.1.4. *Let $\mathbb{G}\leqslant\mathrm{Aut}(X,\mu)$ be an ergodic full group. Then two involutions in $\mathbb{G}$ are conjugate iff their supports have the same measure.*

Proof. Since all the elements of $\mathbb{G}$ preserve the measure, if two involutions are conjugate then their supports must have the same measure.

Conversely, let $U,V\in\mathbb{G}$ be two involutions whose supports have the same measure. Let $<$ be a Borel linear order on X, let $A=\{x\in X: U(x)>x\}$ and $B=\{x\in X: V(x)>x\}$, then it is straightforward to check that $\operatorname{supp}U=A\sqcup U(A)$ and $\operatorname{supp}V=B\sqcup V(B)$. Since U and V preserve the measure and $\mu(\operatorname{supp}U)=\mu(\operatorname{supp}V)$, we deduce that $\mu(A)=\mu(B)$. Using the above proposition, we then find $T\in\mathbb{G}$ such that $T(A)=B$. Define a partial bijective measure-preserving map W by

$$W(x)=\begin{cases} T(x) & \text{if } x\in A\\ VTU(x) & \text{if } x\in U(A).\end{cases}$$

Using the previous proposition we may extend arbitrarily W to an element of $\mathbb{G}$. Then it is straightforward to check that $WUW^{-1}=V$. □

The following result is key to all automatic continuity results on full groups. It is due to Ryzhikov [Ryz85] (see also the very neat proof in Miller's thesis [Mil]).

Theorem 7.1.5 (Ryzhikov)**.** *Let $T \in \mathrm{Aut}(X,\mu)$. Then T can be written as the product of three involutions which belong to the full group generated by T.*

Recall that $\mathrm{Aut}(X,\mu)$ carries two important metrizable topologies. The first is the **weak topology**, defined by letting $T_n \to T$ if and only if for all $A \in \mathrm{MAlg}(X,\mu)$, $\mu(T_n(A) \bigtriangleup T(A)) \to 0$. It is a Polish group topology. The second is the **uniform topology**, which is the topology induced by the **uniform metric** d_u defined by $d_u(S,T) = \mu(\{x \in X : S(x) \neq T(x)\})$. We will use the fact that the uniform topology can be interpreted as a topology of uniform convergence:

Proposition 7.1.6 ([Dye59, Lem. 5.4])**.** *The metric δ_u on $\mathrm{Aut}(X,\mu)$ defined by*

$$\delta_u(T,U) = \sup_{A\in\mathrm{MAlg}(X,\mu)} \mu(T(A) \bigtriangleup U(A)) = 2 \sup_{A\in\mathrm{MAlg}(X,\mu)} \mu(T(A) \setminus U(A))$$

is equivalent to the uniform metric d_u.

Both metrics d_u and δ_u are complete. The uniform topology refines the weak topology and it is not separable. Moreover, full groups are always closed for the uniform topology, so the uniform metric restricts to a complete metric on them [Dye59, Lem. 5.4]. The full groups which are moreover separable for the uniform topology are exactly the full groups of *countable pmp equivalence relations* (see [CLM16, Prop. 3.8]). The following result was proved by Kittrell and Tsankov for such full groups [KT10], but their proof extends verbatim to the general case, as was already observed by Ben Yaacov, Berenstein and Melleray in the case $\mathbb{G} = \mathrm{Aut}(X,\mu)$ [BYBM13].

Theorem 7.1.7. *Let $\mathbb{G} \leqslant \mathrm{Aut}(X,\mu)$ be an ergodic full group. Then every group homomorphism $\mathbb{G} \to H$, where H is a Polish group, has to be continuous with respect to the uniform topology on $\mathbb{G}$.*

We will only need the following corollary.

Corollary 7.1.8. *Let $\mathbb{G}$ be an ergodic full group, suppose that we have a homomorphism $\rho : \mathbb{G} \to \mathrm{Aut}(Y,\nu)$. Then ρ is uniform to weak continuous.*

A full group is called **Polish** when it carries a Polish group topology. If the full group is ergodic, by [CLM16, Thm. 4.7] it carries at most one Polish group topology, and this topology refines the weak topology while being refined by the uniform topology. Examples of ergodic Polish full groups whose topology is neither the uniform nor the weak topology are provided by some orbit full groups of pmp actions of Polish groups, see [CLM16, Thm. 1].

7.1.2 Another automatic continuity result

The uniform metric on $\mathrm{Aut}(X,\mu)$ is biinvariant, which implies that the uniform topology is SIN (the identity element admits a basis of conjugacy invariant neighborhoods). We now prove that there is only one other SIN topology on any ergodic full group.

Theorem 7.1.9. *Let $\mathbb{G} \leqslant \mathrm{Aut}(X,\mu)$ be an ergodic full group, let τ be a SIN group topology on $\mathbb{G}$. Then τ is either the discrete or the uniform topology.*

Proof. First we recall that all the involutions with support of some fixed measure are conjugated in $\mathbb{G}$ (Prop. 7.1.4) We start by proving τ has to refine the uniform topology. For this let us fix $\varepsilon \in (0, \frac{1}{2})$ and an involution U whose support has measure 2ε. Since the topology is Hausdorff and SIN, we find a conjugacy-invariant τ-neighborhood of the identity W which does not contain U, and thus does not contain *any* involution of support of measure 2ε. Choose a conjugacy-invariant τ-neighborhood of the identity V such that $VV^{-1} \subset W$.

We claim that V is contained in $\{T \in \mathbb{G} : \ \mu(T(A) \setminus A) < \varepsilon, \forall A \subset X\}$. Since these subsets form a uniform basis of neighborhoods of the identity by Proposition 7.1.6, the claim implies that τ refines the uniform topology.

Let us prove the claim by contradiction. Assume that there exists $T \in V$ and a measurable subset $A \subset X$ such that $\mu(T(A) \setminus A) \geqslant \varepsilon$. By shrinking A if necessary, we may as well assume $\mu(T(A) \setminus A) = \varepsilon$. Let U an involution with support equal to $A' = T(A) \setminus A$. Then observe that $UTU \in V$ and hence $UTUT^{-1} \in W$. On the other hand, TUT^{-1} is an involution with support $T(A')$ and hence $UTUT^{-1}$ is an involution with support $T(A') \cup A'$ which has measure 2ε, contradicting our hypothesis on W.

Let us now show that if τ is not discrete, it has to be equal to the uniform topology. For this we will use the fact that for every $\varepsilon > 0$, every element in $\mathbb{G}$ which has support of measure less than ε is the product of at most 3 involutions whose support has measure less than ε (Thm. 7.1.5). So let W be a τ-neighborhood of the identity. We claim that there exists ε such that $W \supseteq \{T \in \mathbb{G} : \ d_u(T, \mathrm{id}) < \varepsilon\}$. Indeed, since τ is not discrete, there exists a conjugacy-invariant, symmetric τ-neighborhood of the identity $V \subset W$ such that $V \neq \{\mathrm{id}\}$ and $V^6 \subset W$. There exists $\varepsilon > 0$, $T \in V$ and $A \subset X$ such that $\mu(A) = \varepsilon$ and $T(A) \cap A = \emptyset$. Proceeding as in the second paragraph, we see that V^2 has to contain every involution of support less than 2ε. Hence V^6 contains every element of $\mathbb{G}$ whose support has measure less than 2ε, as claimed. □

We will only use the following corollary when the target group G is $\mathrm{Aut}(Y, \nu)$ equipped with the uniform topology.

Corollary 7.1.10. *Let $\mathbb{G}$ be an ergodic full group and let G be a group with a SIN group topology τ. Assume that we have a homomorphism $\rho : \mathbb{G} \to G$. If the image of $\mathbb{G}$ is not discrete, then ρ is (uniform to τ) continuous.*

Proof. We may as well assume there is some $g \in \mathbb{G}$ such that $\rho(g) \neq 1$ because otherwise ρ is clearly continuous. It follows that ρ is injective since $\mathbb{G}$ is simple (see [Fat78]; the proof there adapts verbatim to ergodic full groups). Consider the pullback topology $\rho^*(\tau)$ on $\mathbb{G}$, that is the topology generated by the pre-images of open sets in G. Since ρ is injective and τ is Hausdorff, $\rho^*(\tau)$ is Hausdorff. Moreover τ is SIN so $\rho^*(\tau)$ is a SIN group topology. So by the above theorem, if the image of $\mathbb{G}$ is not discrete, $\rho^*(\tau)$ has to be the uniform topology and hence ρ is continuous (actually an embedding) of topological groups. □

Our main theorem implies that whenever the morphism $\rho : \mathbb{G} \to \mathrm{Aut}(Y, \nu)$ induces an everywhere non-free action (that is the image of $\mathbb{G}$ is not discrete for the uniform topology), then ρ is actually weak to weak continuous. Corollary 7.1.10 moreover implies that ρ is uniform to uniform continuous. We wonder what happens when the image of ρ is discrete in the uniform topology. Is ρ in this case still weak to weak continuous?

7.2 Preliminaries on boolean actions

This section prepares the ground for the proof of Theorem I. We start by some useful definitions and an important observation about total non freeness (Lemma 7.2.3).

7.2.1 Support-preserving factor maps and total non freeness

Given $T \in \mathrm{Aut}(X, \mu)$, we put $\mathrm{Fix}(T) := \{x \in X : Tx = x\}$ and we denote by $\mathrm{supp}(T)$ its complement, called the **support** of T.

Given two quasi-actions $\rho_1 : G \to \mathrm{Aut}(Y_1, \nu_1)$ and $\rho_2 : G \to \mathrm{Aut}(Y_2, \nu_2)$, we say that a measure-preserving map $\varphi : (Y_1, \nu_1) \to (Y_2, \nu_2)$ is a **factor map** if for every $g \in G$, and almost all $y \in Y_1$, we have $\varphi(\rho_1(g)y) = \rho_2(g)\varphi(y)$. Equivalently, $\varphi^{-1} : \mathrm{MAlg}(Y_2, \nu_2) \to \mathrm{MAlg}(Y_1, \nu_1)$ is a factor map when for all $A \in \mathrm{MAlg}(Y_2, \nu_2)$, we have $\varphi^{-1}(\rho_2(g)A) = \rho_1(g)\varphi^{-1}(A)$ (see [Gla03, Chap. 2] for details on this).

Given a factor map φ as above, we always have $\varphi^{-1} \operatorname{supp} \rho_2(g) \subseteq \operatorname{supp} \rho_1(g)$ for all $g \in G$. When we have equality, we will say that φ is a **support-preserving** factor map. Equivalently, this means that for all $g \in G$, for almost all $y \in Y_1$, if $\rho_1(g)y \neq y$ then $\rho_2(g)\varphi(y) \neq \varphi y$.

Remark 7.2.1. Using this terminology, the two conditions from item 2 of Theorem I may simply be restated as: the maps φ_n are support-preserving factor maps.

Let $G \leqslant \mathrm{Aut}(X, \mu)$ be a subgroup. The associated **isotropy subalgebra** $\mathrm{Iso}_G(X, \mu)$ is the measure algebra generated by the support (or equivalently the set of fixed points) of all the elements of G, that is $\mathrm{Iso}_G(X, \mu) = \langle \operatorname{supp} \alpha(g) : g \in G \rangle$.

Definition 7.2.2. A subgroup $G \leqslant \mathrm{Aut}(X, \mu)$ is said to be **totally non free** (TNF) if the measure algebra $\mathrm{Iso}_G(X, \mu)$ coincides with the measure algebra of (X, μ).

The following lemma is a simple observation, but it shows that totally non free actions have some kind of rigidity with respect to support-preserving factor maps, and will yield the uniqueness part of our main theorem.

Lemma 7.2.3. *Let $\rho_1 : G \to \mathrm{Aut}(Y_1, \nu_1)$ and $\rho_2 : G \to \mathrm{Aut}(Y_2, \nu_2)$ be two boolean actions of a group G. Suppose $\rho_2(G)$ is totally non free, then there is at most one support-preserving factor map $\phi : (Y_1, \nu_1) \to (Y_2, \nu_2)$.*

Proof. Suppose φ is a support-preserving factor map. By total non freeness the measure algebra generated by $\{\operatorname{supp} \rho_2(g) : g \in G\}$ is equal to $\mathrm{MAlg}(Y_2, \nu_2)$, so φ^{-1} is completely determined by the values it takes on $\{\operatorname{supp} \rho_2(g) : g \in G\}$. Since φ is support-preserving, for all $g \in G$ we have $\varphi^{-1}(\operatorname{supp} \rho_2(g)) = \operatorname{supp} \rho_1(g)$, which proves uniqueness. □

7.2.2 High absolute non freeness

Let G be a group and assume we have fixed a boolean action $\rho : G \to \mathrm{Aut}(Y, \nu)$. We define the following three $\rho(G)$-invariant elements of $\mathrm{MAlg}(Y, \nu)$.

- The **free part** $A_\omega(\rho)$ is defined by $A_\omega(\rho) = \bigwedge_{T \in G} \operatorname{supp} \rho(g)$
- The **trivial part** $A_0(\rho)$ is defined by $A_0(\rho) = \bigwedge_{T \in G} \mathrm{Fix}(\rho(g))$
- The **non-free part** $A_{[0,\omega[}(\rho)$ is defined by $A_{[0,\omega[} = Y \setminus (A_\omega(\rho))$

Let us now assume $G \leqslant \mathrm{Aut}(X,\mu)$. Gluing together symmetric powers, we can build a natural class of boolean actions of G.

Definition 7.2.4. Let $G \leqslant \mathrm{Aut}(X,\mu)$ be a subgroup. Given a sequence of non-negative reals $(\alpha_i)_{i\in\omega}$ which sums to 1, we define a boolean action $\tilde{\iota}$ called the **symmetric diagonal sum** of parameter $(\alpha_i)_{i\in\omega}$ associated to the inclusion $G \leqslant \mathrm{Aut}(X,\mu)$ as follows.

We let (Z_0,ν_0) be the one-point probability space, and ι_0 be the trivial morphism $G \to \mathrm{Aut}(Z_0,\nu_0)$. For $i \geqslant 1$, let $(Z_i,\nu_i) := (X^{\odot i},\mu^{\odot i})$ and denote by ι_i the symmetric diagonal embedding of G in $\mathrm{Aut}(Y_i,\nu_i)$. Finally define $(Z,\nu) = \sqcup_i (Z_i,\alpha_i\nu_i)$ and let $\tilde{\iota} = \bigsqcup_i \iota_i$ be the embedding of G in $\mathrm{Aut}(Z,\nu)$ obtained by gluing together the ι_i's.

We now give a condition on the inclusion $G \leqslant \mathrm{Aut}(X,\mu)$ which will yield that all associated symmetric diagonal sums are totally non free, and have non discrete image for the uniform topology.

Definition 7.2.5. A subgroup $G \leqslant \mathrm{Aut}(X,\mu)$ is called *highly absolutely non free* if for every partition of X into n pieces $A_1,...,A_n$ and every $\epsilon > 0$, there are $T_1,...,T_n \in G$ with disjoint support such that for all $i \in \{1,...,n\}$ we have

$$\mu(A_i \triangle \operatorname{supp} T_i) < \epsilon.$$

The name is motivated by the fact that this is a strenghtening of absolute non freeness (see [DG18, Def. 11]). This condition is met by ergodic full groups as well as some natural countable subgroups of $\mathrm{Aut}(X,\mu)$ such as the group of dyadic permutations (cf. Section 7.2.3). Here are the boolean actions we want to consider.

Proposition 7.2.6. *Let $G \leqslant \mathrm{Aut}(X,\mu)$ be a highly absolutely non free subgroup. Let $(\alpha_i)_{i\in\omega}$ be a sequence of non-negative reals such that $\sum_{i\in\omega}\alpha_i = 1$, and let $\tilde{\iota}$ be the associated symmetric diagonal sum boolean action. Then $\tilde{\iota}(G) \leqslant \mathrm{Aut}(Z,\nu)$ is not discrete for the uniform topology and totally non free.*

Proof. As in Definition 7.2.4, we view $\tilde{\iota}$ as a boolean action on $(Z,\nu) = \sqcup_{i\in\omega}(Z_i,\alpha_i\nu_i)$ where $(Z_i,\nu_i) := (X^{\odot i},\mu^{\odot i})$.

Observe that by high absolute non freeness we have a sequence of non-trivial elements $g_n \in G$ such that $\mu(\operatorname{supp} g_n) \to 0$, and this implies $\mu^{\odot i}(\operatorname{supp} \iota^{\odot i}(g_n)) \to 0$ for all $i < \omega$, so that $\tilde{\iota}(G)$ is not discrete as claimed.

Let us now assume $\alpha_0 = 0$ so that the trivial part vanishes. Take $0 < i < \omega$ and consider $T \in G$. Then $\operatorname{supp}\iota_i(T) = [\operatorname{supp}(T), Z_i, \dots, Z_i]$. Hence if $T_1,\dots,T_k \in G$ have pairwise disjoint support, the set $\bigcap_j \operatorname{supp}\iota_i(T_j)$ is empty if $k \geqslant i$ and it is equal to $[\operatorname{supp} T_1,\dots,\operatorname{supp} T_i]$ for $k = i$. Since $G \subseteq \mathrm{Aut}(X,\mu)$ is highly absolutely non free, these measurable subsets of Z_i are generating its measure algebra.

Moreover, the union of these sets is equal to Z_i, so we can recover each Z_n (now seen as a subset of Z) from the measure algebra generated by the supports of the elements of G using the following recursive formula:

$$\bigvee_{i\geqslant n} Z_i = \bigvee_{\operatorname{supp} T_1,\dots,\operatorname{supp} T_n \text{ pairwise disjoint}} \bigwedge_{i=1}^{n} \operatorname{supp}\tilde{\iota}(T_i).$$

It follows that the measure algebra generated by supports of elements of the form $\tilde{\iota}(g)$ is equal to $\mathrm{MAlg}(Z,\nu)$, so the inclusion $\tilde{\iota}(G) \leqslant \mathrm{Aut}(Z,\nu)$ is totally non free as wanted. ☐

Our main theorem roughly states that every boolean action of an ergodic full group factors in a unique manner through a boolean action such as in the proposition, except for the free part. We will first show this is the case for the group of *dyadic permutations*, using as a black box a result of Thomas and Tucker-Drob which classifies its IRS.

7.2.3 Non-free actions and IRS of the dyadic permutation group

Fix the standard Cantor space $(X,\mu) := (\{0,1\}^{\mathbb{N}}, \mathcal{B}(1/2)^{\otimes\mathbb{N}})$, which we equip with the product of 1/2 Bernoulli measures $\mathcal{B}(1/2) := \frac{1}{2}(\delta_0+\delta_1)$. We view the symmetric groups $\mathfrak{S}_{2^n} := \mathfrak{S}(\{0,1\}^n)$ as a subgroup of $\mathrm{Aut}(X,\mu)$ as follows: for each $(x_k) \in X$ and each $\sigma \in \mathfrak{S}_{2^n}$, we let $\sigma \cdot (x_k)$ be the concatenation $\sigma((x_k)_{k<n}) \cdot (x_k)_{k\geqslant n}$. Note that with this identification in mind, for each $n \in \mathbb{N}$ we have $\mathfrak{S}_{2^n} \leqslant \mathfrak{S}_{2^{n+1}}$. We then define $\mathfrak{S}_{2^\infty}$ as the following subgroup of $\mathrm{Aut}(X,\mu)$:

$$\mathfrak{S}_{2^\infty} = \bigcup_{n\in\mathbb{N}} \mathfrak{S}_{2^n}.$$

One can easily show that this subgroup is highly absolutely non free. Indeed any of the cylinder subset of X is the fixed-point set of some element of $\mathfrak{S}_{2^\infty}$.

Finally, denote by E_0 the equivalence relation on $\{0,1\}^{\mathbb{N}}$ which relates any two sequences which only disagree on a finite set. We will use the following fact several times

Proposition 7.2.7 (see [Kec10, Prop. 3.8]))**.** *The full group of E_0 is equal to the closure of $\mathfrak{S}_{2^\infty}$ in the uniform topology.*

Our main goal in this section is the following result.

Proposition 7.2.8. *Assume that we have a measure-preserving action $\rho : \mathfrak{S}_{2^\infty} \to \mathrm{Aut}(Y,\nu)$. Then its restriction to its non-free part factors in a support-preserving manner onto a symmetric diagonal sum $\tilde{\iota}$ associated to the inclusion $\mathfrak{S}_{2^\infty} \leqslant \mathrm{Aut}(X,\mu)$.*

The above proposition will be deduced from a theorem of Thomas and Tucker-Drob [TT14]. This theorem is stated in the context of invariant random subgroups (IRS) and we will need to introduce a little bit of terminology before applying it.

For a countable group Γ, we denote by $\mathrm{Sub}(\Gamma) \subset \{0,1\}^\Gamma$ the Borel set of subgroups of Γ. The group Γ acts on $\mathrm{Sub}(\Gamma)$ by conjugation and an IRS of Γ is by definition a conjugacy invariant probability measure on $\mathrm{Sub}(\Gamma)$. Note that the stabilizer of any $\Lambda \in \mathrm{Sub}(\Gamma)$ for the Γ-action by conjugacy is equal to its normalizer.

An IRS $\zeta \in \mathrm{Prob}(\mathrm{Sub}(\Gamma))$ is *self-normalizing* if for ζ-almost every $\Lambda \in \mathrm{Sub}(\Gamma)$ we have that the normalizer of Λ equals Λ. A pmp action of Γ on (X,μ) is said *totally non free* if the image of Γ in $\mathrm{Aut}(X,\mu)$ is totally non free. Every IRS supported on self-normalizing subgroups is totally non free as a pmp action.

As noted by Vershik, the IRS associated to any measure-preserving totally non free action is both totally non free and self-normalizing since it is isomorphic as a dynamical system to the original action.

Let us fix a pmp action of the countable group Γ on (X,μ). We define the measurable stabilizer map $\mathrm{Stab}_\Gamma : X \to \mathrm{Sub}(\Gamma)$ which maps $x \in X$ to its stabilizer $\mathrm{Stab}_\Gamma(x) := \{\gamma \in \Gamma : \gamma x = x\}$. The pushforward measure $(\mathrm{Stab}_\Gamma)_*(\mu)$ is an IRS, which we call the IRS *of the action*. The action of Γ is totally non free if and only if Stab_Γ induces a measurable isomorphism [Ver12, Prop. 1].

Here is a useful observation.

Lemma 7.2.9. *Consider a pmp action $\alpha : \Gamma \to \mathrm{Aut}(X, \mu)$ of a countable group Γ, and denote by $\beta : \Gamma \to \mathrm{Aut}(\mathrm{Sub}(\Gamma), (\mathrm{Stab}_\Gamma)_*(\mu))$ the action associated to the IRS of α. Assume that the IRS $(\mathrm{Stab}_\Gamma)_*(\mu)$ is self-normalizing. Then the stabilizer map is support-preserving: for every $g \in \Gamma$*

$$\mathrm{Stab}_\Gamma^{-1}(\mathrm{supp}\,\beta(g)) = \mathrm{supp}\,\alpha(g).$$

Proof. Because β is a factor of α via the stabilizer map, we have $\mathrm{Stab}^{-1}(\mathrm{supp}\,\beta(g)) \subseteq \mathrm{supp}\,\alpha(g)$ for every $g \in \Gamma$. For the converse, since the IRS is self-normalizing we have

$$\mathrm{supp}\,\beta(g) = \{H \leqslant \Gamma : g \notin N_G(H)\} = \{H \leqslant \Gamma : \ g \notin H\}.$$

Using the definition of the stabilizer map, we finally have

$$\mu(\mathrm{Stab}^{-1}(\mathrm{supp}\,\beta(g))) = \mu(\{x \in X : \alpha(g)x \neq x\}) = \mu(\mathrm{supp}\,\alpha(g)))$$

and since $\mathrm{Stab}^{-1}(\mathrm{supp}\,\beta(g)) \subseteq \mathrm{supp}\,\alpha(g)$ we are done. □

Let us go back to the proof of Proposition 7.2.8. Assume that $(X, \mu) := (\{0,1\}^{\mathbb{N}}, \mathcal{B}(1/2)^{\otimes \mathbb{N}})$ and consider $\Gamma := \mathfrak{S}_{2^\infty} \subseteq \mathrm{Aut}(X, \mu)$. For every $i \geqslant 1$, we denote by ζ_i the IRS associated to the diagonal action of G on $(X^{\odot i}, \mu^{\odot i})$. Finally we denote by ζ_ω the trivial IRS, that is the Dirac measure on the identity and by ζ_0 the Dirac mass on G.

Theorem 7.2.10 ([TT14])**.** *Every IRS of $\mathfrak{S}_{2^\infty}$ can be written uniquely as an infinite convex combination of elements of the family $(\zeta_i)_{i=0,\dots,\omega}$.*

We are now ready to give the proof of Proposition 7.2.8.

Proof of Prop. 7.2.8. Consider a pmp action ρ of $\mathfrak{S}_{2^\infty}$ on (Y, ν), denote by $\zeta := (\mathrm{Stab}_\Gamma)_*(\nu)$ the IRS associated to the action. By Theorem 7.2.10 we have that $\zeta = \sum_i \alpha_i \zeta_i$. By restricting to the non-free part of the action, we may as well assume $\alpha_\omega = 0$. But then by definition of the ζ_i's, the IRS ζ is a pmp action which is a symmetric diagonal sum associated to the inclusion $\mathfrak{S}_{2^\infty} \leqslant \mathrm{Aut}(X, \mu)$. In particular by Prop. 7.2.6 it is totally non free, hence self-normalizing. Moreover the stabilizer map is support-preserving as a consequence of Lemma 7.2.9, which finishes the proof. □

7.3 Proof of the classification theorem

Let $\mathbb{G}$ be an ergodic full group over the standard probability space (X, μ). Since all the standard probability spaces are isomorphic, we can assume that $(X, \mu) = (\{0,1\}^{\mathbb{N}}, \mathcal{B}(1/2)^{\otimes \mathbb{N}})$. Moreover we claim that we can always assume that $\mathbb{G} \subset \mathrm{Aut}(X, \mu)$ always contains the standard copy of $\mathfrak{S}_{2^\infty}$ acting on (X, μ). Indeed by different results of Dye, every ergodic full group contains a copy of the full group of the hyperfinite equivalence relation, all these full groups are conjugated in $\mathrm{Aut}(X, \mu)$ and $\mathfrak{S}_{2^\infty}$ is contained in such a full group. Therefore from now on, we will always assume that $\mathfrak{S}_{2^\infty} \subset \mathbb{G}$.

Assume now that we have a morphism $\rho : \mathbb{G} \to \mathrm{Aut}(Y, \nu)$. We first need to show that the free part of ρ and the restriction of ρ to $\mathfrak{S}_{2^\infty}$ agree.

7.3.1 The free part is the same as the free part of the restriction to $\mathfrak{S}_{2^\infty}$

We first show two intermediate results.

Lemma 7.3.1. *The subset $A_\omega(\rho_{\restriction \mathfrak{S}_{2^\infty}})$ is $\rho(\mathbb{G})$-invariant.*

Proof. Let $A = A_\omega(\rho_{\restriction \mathfrak{S}_{2^\infty}})$, suppose by contradiction that A is not $\rho(\mathbb{G})$-invariant. Let $g' \in \mathbb{G}$ such that $\rho(g')A \neq A$. By automatic continuity (Cor. 7.1.8), we have that ρ is uniform to weak continuous. Therefore we can find a small cylinder subset $B \subseteq X$ and an element $g \in \mathbb{G}$ uniformly close to g' such that $\mathrm{supp}(g) \cap B = \emptyset$ and $\rho(g)A \neq A$. Fix ε such that $\nu(\rho(g)A \setminus A) > \varepsilon$.

Consider the character $\chi : \mathbb{G} \to \mathbb{R}$ defined by $\chi(g) := \nu(\mathrm{Fix}\,\rho(g))$. Proposition 7.2.8 implies that $\chi(\sigma) = \alpha_0 + \sum_{1 \leqslant i < \omega} \alpha_i \mu(\mathrm{Fix}\,\sigma)^i + \alpha_\omega \delta_e(\sigma)$ for every $\sigma \in \mathfrak{S}_{2^\infty}$, where $\alpha_\omega = \nu(A_\omega(\rho_{\restriction \mathfrak{S}_{2^\infty}}))$. In particular if σ_n tends uniformly to the identity, $\chi(\sigma_n) = \nu(\mathrm{Fix}\,\rho(\sigma_n))$ tends to $1 - \alpha_\omega$. So we can choose an element $\sigma \in \mathfrak{S}_{2^\infty}$ whose support is contained in B and such that $\nu(\mathrm{supp}(\rho(\sigma))) < \nu(A) + \varepsilon$. Clearly g and σ commute. On the other hand $\mathrm{supp}(\rho(g\sigma g^{-1})) = \rho(g)\,\mathrm{supp}(\rho(\sigma)) \supseteq \rho(g)A$. Therefore

$$\nu(\mathrm{supp}(\rho(\sigma))) \geqslant \nu(A) + \nu(\rho(g)A \setminus A)) > \nu(A) + \varepsilon,$$

a contradiction. □

Proposition 7.3.2. *For every $g \in \mathbb{G}$ which is not the identity, we have that* $\mathrm{supp}(\rho(g)) \supset A_\omega(\rho_{\restriction \mathfrak{S}_{2^\infty}})$.

Proof. By the previous lemma, we can restrict ρ and assume that $A_\omega(\rho_{\restriction \mathfrak{S}_{2^\infty}}) = Y$. Observe also that every involution $U \in \mathbb{G}$ whose support has measure $1/2^n$ for some $n \in \mathbb{N}$ is conjugate to an element of $\mathfrak{S}_{2^\infty}$, and hence $\rho(U)$ has support of full measure.

Now let $g \in \mathbb{G}$ be a nontrivial element, we can find $A \subseteq X$ non null such that $gA \cap A = \emptyset$. For large enough n, we then find an involution U_n supported on A such that its support has measure $1/2^n$. The commutator $[g, U_n]$ is an involution whose support has measure $2/2^n$, and so $\rho([g, U_n])$ has full measure support.

But the support of $\rho([g, U_n])$ is contained in $\mathrm{supp}\,\rho(g) \cup \rho(U_n)(\mathrm{supp}\,\rho(g))$. By automatic continuity (Cor. 7.1.8), $\rho(U_n)\,\mathrm{supp}\,g \to \mathrm{supp}\,g$ and hence we conclude that $\mathrm{supp}\,g = Y$ as wanted. □

We can now prove the announced result that the free parts of ρ and its restriction to $\mathfrak{S}_{2^\infty}$ agree.

Theorem 7.3.3. *Given a boolean action $\rho : \mathbb{G} \to \mathrm{Aut}(Y, \nu)$, we have that $A_\omega(\rho) = A_\omega(\rho_{\restriction \mathfrak{S}_{2^\infty}})$.*

Proof. By the previous proposition, the free part of ρ contains the free part of its restriction $\rho_{\restriction \mathfrak{S}_{2^\infty}}$, and since the reverse inclusion clearly holds they are actually equal. □

7.3.2 Continuity and support dependency on the non-free part

We will now show that $\mathrm{supp}\,\rho(g)$ only depends on $\mathrm{supp}\,g$. To this end, we first need to know that ρ is uniform-to-uniform continuous, which is an easy consequence of what we have done so far.

Proposition 7.3.4. *Given a boolean action $\rho : \mathbb{G} \to \mathrm{Aut}(Y, \nu)$, we have that the restriction of ρ to its non-free part is uniform to uniform continuous.*

Proof. By restricting to the non-free part, we may as well assume the free part of ρ is trivial. By the previous theorem, the free part of $\rho_{\restriction \mathfrak{S}_{2^\infty}}$ is trivial as well.

By Proposition 7.2.8, $\rho_{\restriction \mathfrak{S}_{2^\infty}}$ factors in a support preserving manner on a symmetric diagonal sum which by Proposition 7.2.6 is not discrete in the uniform topology. Since the factor map is support-preserving, this implies ρ has non-discrete image. We can now apply Corollary 7.1.10 to obtain that ρ is indeed uniform-to-uniform countionus. □

To see that $\mathrm{supp}\,\rho(g)$ only depends on $\mathrm{supp}\, g$, we also use again the character associated to our boolean action.

Proposition 7.3.5. *Assume the free part of ρ is trivial. Let $\chi : \mathbb{G} \to \mathbb{R}$ be defined by $\chi(g) = \nu(\mathrm{Fix}\,\rho(g))$. Then there is a unique convex sequence $(\alpha_i)_{i\in\omega}$ such that for every $g \in \mathbb{G}$ we have that $\chi(g) = \alpha_0 + \sum_{1\leqslant i<\omega} \alpha_i \mu(\mathrm{Fix}\, g)^i$.*

The proposition is also a corollary of the classification of character on full groups, see [CLM16, Prop. 5.3].

Proof. By Theorem 7.3.3, the free part of the restriction of ρ to $\mathfrak{S}_{2^\infty}$ is trivial. By Proposition 7.2.8 the desired result holds for elements of $\mathfrak{S}_{2^\infty}$.

Fix $g \in \mathbb{G}$. By Rokhlin's lemma, g is approximately conjugate to an element of the full group $[E_0]$ which is equal to the closure of $\mathfrak{S}_{2^\infty}$ by Proposition 7.2.7. So there are sequences $h_n \in \mathbb{G}$ and $\sigma_n \in \mathfrak{S}_{2^\infty}$ such that $h_n\sigma_n h_n^{-1}$ converges to g. By uniform-to-uniform continuity (Proposition 7.3.4), we have that $\mathrm{supp}\,\rho(h_n\sigma_n h_n^{-1})$ converges to $\mathrm{supp}\,\rho(g)$. Now observe that

$$\nu(\mathrm{supp}\,\rho(h_n\sigma_n h_n^{-1})) = \nu(\mathrm{supp}\,\rho(\sigma_n)) = \alpha_0 + \sum_i \alpha_i \mu(\mathrm{Fix}\,\sigma_n)^i = \alpha_0 + \sum_i \alpha_i \mu(\mathrm{Fix}\, h_n\sigma_n h_n^{-1})^i.$$

The sequence $(\mu(\mathrm{Fix}\, h_n\sigma_n h_n^{-1})^i)_{i\in\mathbb{N}}$ is bounded, so we can interchange taking convex combinations and limits to get the desired result. □

The exact formula in the previous proposition is not really important, all that matters is that we now know that $\nu(\mathrm{supp}\,\rho(g))$ depends continuously on $\mu(\mathrm{supp}\, g)$.

Corollary 7.3.6. *If T is aperiodic on its support, then so is $\rho(T)$.*

Proof. Observe that a measure-preserving transformation T is aperiodic on its support if and only if for every n, $\mu(\mathrm{Fix}(T^n)) = \mu(\mathrm{Fix}(T))$. Now if T is aperiodic on its support, then for every n we have $\nu(\mathrm{Fix}(\rho(T)^n)) = \chi(T^n) = \chi(T) = \nu(\mathrm{Fix}(\rho(T))$ so $\rho(T)$ is aperiodic on its support. □

Here is another useful observation. Note that the support of UT^n is always a subset of the union $\mathrm{supp}\, T \cup \mathrm{supp}\, U$.

Lemma 7.3.7. *Let T and U be measure-preserving transformations, with T aperiodic when restricted to its support. Then $\lim_n \mu\left(\mathrm{supp}(UT^n)\Delta(\mathrm{supp}\, T \cup \mathrm{supp}\, U)\right) = 0$.*

Proof. Observe that if $x \in \mathrm{supp}\, U \setminus \mathrm{supp}\, T$, then $UT^n(x) \neq x$. For each $n \in \mathbb{N}$, let $A_n := \{x \in \mathrm{supp}\, T : UT^n(x) = x\}$. Then $x \in A_n$ implies $T^n(x) = U^{-1}(x)$ and hence the aperiodicity of T implies that the measurable subsets $\{A_n\}_n$ are pairwise disjoint. Hence $\lim_n \mu(A_n) = 0$ and the lemma is proved. □

We can now prove that $\operatorname{supp}\rho(T)$ only depends on $\operatorname{supp} T$.

Proposition 7.3.8. *Let $T, U \in \mathbb{G}$ have the same support, then $\rho(T)$ and $\rho(U)$ also have the same support.*

Proof. As a first step, let us assume that T is aperiodic on its support. By the lemma above, we have $\lim_n \mu(\operatorname{supp}(UT^n)) = \mu(\operatorname{supp} T \cup \operatorname{supp} U) = \mu(\operatorname{supp} T)$. Proposition 7.3.5 then implies $\lim_n \nu(\operatorname{supp}(\rho(UT^n))) = \nu(\operatorname{supp}\rho(T))$. Corollary 7.3.6 tells us that $\rho(T)$ is aperiodic on its support. We can therefore use a second time Lemma 7.3.7 to get

$$\lim_n \nu(\operatorname{supp}\rho(UT^n)) = \nu(\operatorname{supp}\rho(T) \cup \operatorname{supp}\rho(U)).$$

Hence $\nu(\operatorname{supp}\rho(U) \setminus \operatorname{supp}\rho(T)) = 0$. On the other hand, using Proposition 7.3.5 again we have $\nu(\operatorname{supp}\rho(U)) = \nu(\operatorname{supp}\rho(T))$ and hence $\operatorname{supp}\rho(U) = \operatorname{supp}\rho(T)$.

Now if T is not aperiodic on its support, we can find an element $T' \in \mathbb{G}$ which is aperiodic on its support and has same support as T and U. But then, the above argument yields $\operatorname{supp}\rho(T) = \operatorname{supp}\rho(T') = \operatorname{supp}(\rho(U)$. □

7.3.3 End of the proof of the classification theorem

We now proceed to the proof of Theorem I. First observe that using the terminology of symmetric diagonal sums (Definition 7.2.4), it can be restated as follows.

Theorem 7.3.9. *Let $\mathbb{G} \leqslant \operatorname{Aut}(X,\mu)$ be an ergodic full group, let $\rho : \mathbb{G} \to \operatorname{Aut}(Y,\nu)$ be a boolean action. Then there is a unique symmetric diagonal sum $\tilde\iota$ such that the non-free part of ρ factors in a support-preserving manner onto $\tilde\iota$. Moreover, such a factor map itself is unique.*

Proof. Uniqueness of the symmetric diagonal sum $\tilde\iota$ comes from the fact that distinct symmetric diagonal sums have distinct characters, and support-preserving factor maps preserve characters. Uniqueness of the support-preserving factor map follows from the total non-freeness of symmetric diagonal sums and Lemma 7.2.3.

Let us now show existence. We may as well assume that the free part of our boolean action ρ is trivial, and by Theorem 7.3.3 this implies that the restriction of ρ to $\mathfrak{S}_{2^\infty}$ also has trivial free part. Proposition 7.2.8 provides a symmetric diagonal sum boolean action $\tilde\iota$ of $\mathfrak{S}_{2^\infty}$ on (Z,λ), and a support-preserving factor map $\Phi : \operatorname{MAlg}(Z,\lambda) \to \operatorname{MAlg}(Y,\nu)$ of $\rho_{\restriction \mathfrak{S}_{2^\infty}}$ onto $\tilde\iota$. Let us denote by $\tilde\iota_{\mathbb{G}}$ the natural extension of $\tilde\iota$ to $\mathbb{G}$.

We will show that Φ is still a measure-preserving support-preserving factor map of ρ onto $\tilde\iota_{\mathbb{G}}$. Let us first prove that it is support-preserving. If $g \in \mathfrak{S}_{2^\infty}$, we already have that $\Phi(\operatorname{supp}\tilde\iota(g)) = \operatorname{supp}\rho(g)$, and since ρ is uniform-to-uniform continuous (Proposition 7.3.4), the same is true of elements of the closure of $\mathfrak{S}_{2^\infty}$, which is the full group $[E_0]$ by Proposition 7.2.7. Now by Corollary 7.1.3 for every $g \in \mathbb{G}$, there exists some $g' \in [E_0]$ with the same support as g, and so using Proposition 7.3.8 we have

$$\Phi(\operatorname{supp}\tilde\iota(g)) = \Phi(\operatorname{supp}\tilde\iota(g')) = \operatorname{supp}\rho(g') = \operatorname{supp}\rho(g).$$

We conclude that Φ is indeed support-preserving. Let us now show it is a factor map.

Take $g \in \mathbb{G}$ and $T \in \mathbb{G}$, then we have

$$\begin{aligned}\varphi(\tilde\iota(g)\operatorname{supp}\iota(T)) &= \varphi(\operatorname{supp}\tilde\iota(gTg^{-1})) \\ &= \operatorname{supp}\rho(gTg^{-1}) \\ &= \rho(g)\operatorname{supp}\rho(T),\end{aligned}$$

so φ is equivariant on elements of the form $\operatorname{supp} \tilde{\iota}(g)$. Since such sets generate the measure algebra of (Z, λ) by total non freeness (Proposition 7.2.6) and φ is a morphism of measure algebras, we conclude that φ is indeed equivariant on all elements of $\mathrm{MAlg}(Z, \lambda)$, which finishes the proof. □

7.4 Connection to actions of equivalence relations

We will now explain how our Theorem 7.3.9 implies that for every measure-preserving boolean action of a full group $[\mathcal{R}]$, the non-free part of the action comes from measure-preserving actions of the equivalence relation and its symmetric tensor powers, yielding a statement more similar to the main result of Matte Bon [MB18]. For now, we can restate this result for separable full groups in the following version: the non-free part of every boolean action ρ of a separable ergodic full group $[\mathcal{R}]$ splits as a disjoint union of actions ρ_n, each such ρ_n factoring in a support-preserving manner on ι_n. We will now extend each ρ_n individually to an action of the full group generated by $\iota_n([\mathcal{R}])$, using the concept of a *full action*.

7.4.1 Extending ρ_n to a full action

Let $\mathcal{R}$ be a pmp equivalence relation on (X, μ). From now on, we identify the measured full group $[\mathcal{R}]$ defined in Definition 5.1.14 with the classical notion of full group of an equivalence relation, that is the subgroup of $\mathrm{Aut}(X, \mu)$ consisting of the pmp transformations of (X, μ) whose graph is contained in $\mathcal{R}$. We denote by ι the inclusion of $[\mathcal{R}]$ into $\mathrm{Aut}(X, \mu)$, and suppose we are given a measure-preserving inclusion $\varphi : \mathrm{MAlg}(X, \mu) \to \mathrm{MAlg}(Y, \nu)$.

We give a concise characterization of fullness in the case of equivalence relations:

Lemma 7.4.1. *Suppose $\rho : [\mathcal{R}] \to \mathrm{Aut}(Y, \nu)$ is a measure-preserving boolean action, then the following are equivalent:*

(i) ρ is full over φ;

(ii) φ is a support-preserving factor map from ρ to ι.

Proof. Suppose that φ is a support-preserving factor map from ρ to ι. If $T_{\restriction A} = S_{\restriction A}$ then TS^{-1} acts trivially on A, and since φ is support-preserving we get that $\rho(TS^{-1})$ acts trivially on $\varphi(A)$, which since ρ is an action means that $\rho(T)_{\restriction \varphi(A)} = \rho(S)_{\restriction \varphi(A)}$.

Conversely, suppose that ρ is full over φ. Let us show that φ is support-preserving. Let $T \in [\mathcal{R}]$, then since φ is a factor map, we have $\varphi(\operatorname{supp} T) \subseteq \operatorname{supp} \rho(T)$. To obtain the reversed inclusion, note that $T_{\restriction X \setminus \operatorname{supp} T} = \mathrm{id}_{\restriction X \setminus \operatorname{supp} T}$, so by fullness of ρ we have $\rho(T)_{\restriction Y \setminus \varphi(\operatorname{supp} T)} = \mathrm{id}_{Y \setminus \varphi(\operatorname{supp} T)}$ as desired. □

Given a pmp equivalence relation $\mathcal{R}$ on (X, μ), denote by $\mathcal{R}^{\odot n}$ the equivalence relation on $X^{\odot n}$ defined by $([x_1, \cdots, x_n], [y_1, \cdots, y_n]) \in \mathcal{R}^{\odot n}$ if and only if there is $\sigma \in \mathfrak{S}_n$ such that $(x_i, y_{\sigma(i)}) \in \mathcal{R}$ for all $i \in \{1, \ldots, n\}$.

Lemma 7.4.2. *Suppose $\mathcal{R}$ is an ergodic pmp equivalence relation on (X, μ). Then for all $n \in \mathbb{N}$, the full group generated by $\iota^{\odot n}([\mathcal{R}])$ is equal to $[\mathcal{R}^{\odot n}]$.*

Proof. By the Feldman-Moore theorem, we may and do fix a measure-preserving action of a countable group Γ inducing the equivalence relation $\mathcal{R}$.

Observe that every element of the measure algebra of $X^{\odot n}$ is covered by a countable family of elements of the form $[A_1, \cdots, A_n]$ where $(A_i)_{i=1}^n$ is a family of pairwise *disjoint* subsets of X. Let us call such elements *basic cylinders*.

Let $T \in [\mathcal{R}^{\odot n}]$. Applying the above observation, we can moreover find a countable cover $(C_k)_{k\in\mathbb{N}}$ of $X^{\odot n}$ consisting of basic cylinders whose T-image is contained in a basic cylinder.

It thus suffices to show that for every basic cylinder C such that $T(C)$ is contained in a basic cylinder, the restriction of T to C belongs to the full group generated by $\iota^{\odot n}([\mathcal{R}])$. Let C be such a basic cylinder, write it as $C = [A_1, ..., A_n]$, and suppose $T(C) \subseteq D$ where D is a basic cylinder of the form $[B_1, ..., B_n]$.

Note that given a basic cylinder of the form $[E_1, \ldots, E_n]$, the map $E_1 \times \cdots \times E_n \to [E_1, \ldots, E_n]$ which maps $(x_1, \ldots, x_n)$ to $[x_1, \ldots, x_n]$ is a measure-preserving bijection. For each $\gamma_1, ..., \gamma_n \in \Gamma$ and $\sigma \in \mathfrak{S}_n$, consider the set

$$C_{\gamma_1,...,\gamma_n,\sigma} = \{[x_1, ..., x_n] \in C \colon T([x_1, ..., x_n]) = [\gamma_1 x_1, ..., \gamma_n x_n] \text{ and } (x_i, \gamma_i x_i) \in A_i \times B_{\sigma(i)} \text{ for } i = 1, \ldots, n\}.$$

Note that $(C_{\gamma_1,...,\gamma_n,\sigma})_{(\gamma_1,...,\gamma_n,\sigma)\in\Gamma^n\times\mathfrak{S}_n}$ is a cover of C, so it actually suffices to show that the restriction of T to each $C_{\gamma_1,...,\gamma_n,\sigma}$ coincides with an element of $\iota^{\odot n}([\mathcal{R}])$.

For each $(\gamma_1, \ldots, \gamma_n, \sigma) \in \Gamma^n \times \mathfrak{S}_n$, let $\varphi_{\gamma_1,...,\gamma_n,\sigma}$ be the partial measure-preserving transformation of (X, μ) defined by

$$\varphi_{\gamma_1,...,\gamma_n,\sigma}(x) = \gamma_i(x) \text{ if } x \in A_i \cap \gamma_i^{-1}(B_{\sigma(i)}) \text{ for some } i \in \{1, \ldots, n\}.$$

Using Prop 7.1.2 we can extend $\varphi_{\gamma_1,...,\gamma_n,\sigma}$ to an element $T_{\gamma_1,...,\gamma_n\sigma} \in [\mathcal{R}]$. It is then straightforward to check that the restriction of T to $C_{\gamma_1,...,\gamma_n,\sigma}$ coincides with that of $\iota^{\odot n}(T_{\gamma_1,...,\gamma_n,\sigma})$, which finishes the proof. □

Remark 7.4.3. The result holds as well without any assumption on $\mathcal{R}$ other than being a pmp equivalence relation, because it is true in general that any element of the pseudo full group of $\mathcal{R}$ can be extended to an element of the full group of $\mathcal{R}$ (see for instance [LM16, Prop. 2.3]).

7.4.2 Some permanence properties

Recall that a pmp equivalence relation $\mathcal{R}$ has **property (T)** if whenever a unitary representation π of $\mathcal{R}$ has almost invariant unit sections, then it has an invariant unit section, i.e. we can find a section ξ such that for almost all $(x, y) \in \mathcal{R}$, we have $\pi(x, y)\xi(x) = \xi(y)$.

We will now show that if $\mathcal{R}$ is a pmp ergodic equivalence relation with property (T), then $\mathcal{R}^{\odot n}$ has property (T) as well. The following proposition already yields that $\mathcal{R}^n$ has property (T).

Lemma 7.4.4. *Let $\mathcal{R}$ and $\mathcal{S}$ be two pmp ergodic equivalence relations on (X, μ) and (Y, ν) respectively. Suppose that both $\mathcal{R}$ and $\mathcal{S}$ have property (T). Then the equivalence relation $\mathcal{R} \times \mathcal{S}$ on $X \times Y$ is ergodic and has property (T).*

Proof. The ergodicity of $\mathcal{R} \times \mathcal{S}$ is well-known, and follows from the fact that ergodic full groups act transitively on sets of the same measure.

Let $(F_\mathcal{S}, \epsilon_\mathcal{S})$ be a Kazhdan pair for $\mathcal{S}$. By Proposition 5.3.11, we find a Kazhdan pair $(F_\mathcal{R}, \epsilon_\mathcal{R})$ such that given any full unitary representation of $[\mathcal{R}]$, if there is an $(F_\mathcal{R}, \epsilon_\mathcal{R})$-invariant unit vector, then there is an invariant vector at distance at most $\min(\frac{1}{2}, \frac{\epsilon_\mathcal{S}}{3})$ from it. Consider the two commuting inclusions $\iota_{\mathcal{R}} : [\mathcal{R}] \to [\mathcal{R} \times \mathcal{S}]$ and $\iota_{\mathcal{S}} : [\mathcal{S}] \to [\mathcal{R} \times \mathcal{S}]$, which are full. Let $F := \iota_{\mathcal{R}}(F_{\mathcal{R}}) \cup \iota_{\mathcal{S}}(F_{\mathcal{S}})$ and $\epsilon < \min(\frac{\epsilon_{\mathcal{S}}}{3}, \epsilon_{\mathcal{R}})$. We will show that (F, ϵ) is a Kazhdan pair for $\mathcal{R} \times \mathcal{S}$.

Let $[\pi] : [\mathcal{R} \times \mathcal{S}] \to \mathcal{U}(\mathcal{H})$ be a full unitary representation with an (F, ϵ)-invariant unit vector ξ, then we can view $\mathcal{H}$ both as an $\mathrm{L}^\infty(X)$ and as an $\mathrm{L}^\infty(Y)$-module. Then $[\pi] \circ \iota_2$ is a full unitary representation of $[\mathcal{R}]$ and ξ is $(F_{\mathcal{R}}, \epsilon_{\mathcal{R}})$-invariant, so there is a $[\pi] \circ \iota_2$-invariant vector η at distance at most $\min(\frac{1}{2}, \frac{\epsilon_{\mathcal{S}}}{3})$ from ξ, in particular η is not zero. By the triangle inequality, η is $(F, \epsilon + \frac{2\epsilon_{\mathcal{S}}}{3})$ invariant. In particular, η is $(\iota_{\mathcal{S}}(F_{\mathcal{S}}), \epsilon_{\mathcal{S}})$-invariant.

Let $\mathcal{K}$ denote the space of $[\pi] \circ \iota_{\mathcal{R}}$-invariant vectors, note that $\mathcal{K}$ is an $\mathrm{L}^\infty(Y)$-module which is $[\pi] \circ \iota_{\mathcal{S}}$-invariant. This module is nonzero by the previous paragraph and contains an $(F_{\mathcal{S}}), \epsilon_{\mathcal{S}})$-invariant vector, so it contains a nonzero invariant vector for $[\pi] \circ \iota_{\mathcal{S}}$. We thus have found a nonzero vector which is both $[\pi] \circ \iota_{\mathcal{R}}$ and $[\pi] \circ \iota_{\mathcal{S}}$ invariant. By fullness and the fact that $[\mathcal{R} \times \mathcal{S}] = [\iota_{\mathcal{R}}([\mathcal{R}]) \cup \iota_{\mathcal{S}}([\mathcal{S}])]$, this vector is $[\pi]$-invariant as wanted. □

Proposition 7.4.5. *Let $\mathcal{R}$ be an ergodic pmp equivalence relation on (X, μ) and $n \in \mathbb{N}$. Then $\mathcal{R}^{\odot n}$ has property (T).*

Proof. Let $\pi : \mathcal{R}^{\odot n} \to \mathcal{U}(\mathcal{H})$ be a unitary representation with a sequence (ξ_k) of almost invariant unit sections. Each ξ_k is thus an element of $\mathrm{L}^2(X^{\odot n}, \mathcal{H})$. We denote by $\hat{\pi} : \mathcal{R}^n \to \mathcal{U}(\mathcal{H})$ the unitary representation defined by $\hat{\pi}((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \pi([x_1, \ldots, x_n], [y_1, \ldots, y_n])$. Its space of square integrable sections is equal to $\mathrm{L}^2(X^n, \mathcal{H})$, and the subspace of $\mathfrak{S}_n$-invariant sections naturally identifies to $\mathrm{L}^2(X^{\odot n}, \mathcal{H})$: every $\mathfrak{S}_n$-invariant section $X^n \to \mathcal{H}$ quotients down to a section $X^{\odot n} \to \mathcal{H}$ and conversely every section $\xi : X^{\odot n} \to \mathcal{H}$ yields a $\mathfrak{S}_n$-invariant section $\hat{\xi} : X^n \to \mathcal{H}$ given by $\hat{\xi}(x_1, \ldots, x_n) = \xi([x_1, \ldots, x_n])$.

With these identifications in mind, the orthogonal projection onto $\mathrm{L}^2(X^{\odot n}, \mathcal{H})$ is the map which takes $\xi \in \mathrm{L}^2(X^n, \mathcal{H})$ to the average $\frac{1}{n!}\sum_{\sigma \in \mathfrak{S}_n} \sigma\xi$, where $\sigma\xi(x_1, \ldots, x_n) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})$. Observe that $\mathcal{R}^n$ has property (T) as a consequence of the previous lemma. The sequence (ξ_k) is a sequence of almost invariant unit sections for $\hat{\pi}$ and they belong to the subspace $\mathrm{L}^2(X^{\odot n}, \mathcal{H})$, hence by Proposition 5.3.11 there is $\eta \in \mathrm{L}^2(X^n, \mathcal{H})$ which is $\hat{\pi}$-invariant, has norm 1 and is at distance < 1 from $\mathrm{L}^2(X^{\odot n}, \mathcal{H})$. Its orthogonal projection $\tilde{\eta} = \frac{1}{n!}\sum_{\sigma \in \mathfrak{S}_n} \sigma\eta$ is then non zero. Furthermore, observe that for every $\sigma \in \mathfrak{S}_n$,, the section $\sigma\eta$ is still $\hat{\pi}$-invariant. It follows that for every element $([x_1, \ldots, x_n], [y_1, \ldots, y_n]) \in \mathcal{R}^{\odot n}$, we have

$$\begin{aligned}
\pi([x_1, \ldots, x_n], [y_1, \ldots, y_n])\tilde{\eta}(x_1, \ldots, x_n) &= \frac{1}{n!}\sum_{\sigma \in \mathfrak{S}_n} \hat{\pi}((x_1, \ldots, x_n), (y_1, \ldots, y_n))\sigma\eta(x_1, \ldots, x_n) \\
&= \frac{1}{n!}\sum_{\sigma \in \mathfrak{S}_n} \sigma\eta(y_1, \ldots, y_n) \\
&= \tilde{\eta}(y_1, \ldots, y_n).
\end{aligned}$$

It follows that $\tilde{\eta}$ is a nonzero invariant section for π, which by ergodicity and rescaling yields an invariant unit section. We conclude that $\mathcal{R}^{\odot n}$ has property (T) as wanted. □

Remark 7.4.6. It is unclear to us how one could obtain an explicit a Kazhdan pair for $\mathcal{R}^{\odot n}$ from a Kazhdan pair for $\mathcal{R}^n$, which is why we have to go back to the original definition of property (T) in the proof.

7.4.3 Proof of Theorem III

In this section, we put together the previous results so as to obtain the following theorem, which is the direct implication from Theorem III.

Theorem 7.4.7. *Let $\mathcal{R}$ be an ergodic pmp equivalence relation, suppose $\mathcal{R}$ has property (T). Then every ergodic non-free boolean action of $[\mathcal{R}]$ is strongly ergodic.*

Proof. Let $\alpha : [\mathcal{R}] \to \mathrm{Aut}(Z, \lambda)$ be pmp boolean ergodic non-free action of $[\mathcal{R}]$. By our main result and ergodicity, there is some $n \geqslant 1$ such that α is full over ι_n, where ι_n is the inclusion of $[\mathcal{R}]$ in $[\mathcal{R}^{\odot n}]$. By Lemma 7.4.2 and Proposition 5.2.11, α extends uniquely to a full boolean action of $[\mathcal{R}^{\odot n}]$ which we denote by $\tilde{\alpha}$. The latter comes from a pmp action of $\mathcal{R}^{\odot n}$ as a consequence of Theorem 5.2.12. Since α is ergodic, the action of $\mathcal{R}^{\odot n}$ has to be ergodic as well, and since $\mathcal{R}^{\odot n}$ has (T) the equivalence relation generated by $\tilde{\alpha}$ (which is equal to the equivalence relation generated by α) must have (T) as a consequence of Proposition 5.3.12. It is thus strongly ergodic by Proposition 5.3.6 and we conclude that the action α itself is strongly ergodic. □

The other direction is a consequence of Theorem 5.3.28.

Theorem 7.4.8. *Let $\mathcal{R}$ be a pmp equivalence relation. Suppose that all non-free ergodic boolean actions of $[\mathcal{R}]$ are strongly ergodic. Then $\mathcal{R}$ has property (T).*

Proof. We use our characterization of property (T) obtained in Theorem 5.3.28. Let α be an ergodic measure-preserving $\mathcal{R}$-action. Then $[\alpha]$ is an ergodic boolean action of $[\mathcal{R}]$. Moreover, $[\alpha]$ is full and in consequence it is far from being free. Indeed, by Corollary 7.1.3 take $T \in [\mathcal{R}]$ such that $\operatorname{supp} T \notin \{\emptyset, X\}$. Then $T \neq \mathrm{Id}_X$, however, $[\alpha]$ is a full boolean action and so $\operatorname{supp}[\alpha](T) = \operatorname{supp} T \times Y \neq X \times Y$.

Therefore, $[\alpha]$ is strongly ergodic, or in other words, α is strongly ergodic. We conclude that $\mathcal{R}$ has property (T). □

Chapter 8

Conclusion and open questions

Many of the results in this thesis can be formulated in terms of comparison of some equivalence relations on the space of pmp Γ-actions with a given IRS. We recall the definitions of these equivalence relations before giving a summary of the current knowledge about the nature of the inclusions between them.

Definition 8.0.1. Let α, β be two pmp actions of a countable group Γ on a standard probability space (Y, ν). We say that α and β are:

- IRS equivalent, in symbols $\alpha \sim_{\mathrm{IRS}} \beta$, if $\theta_\alpha = \theta_\beta$
- weakly equivalent, in symbols $\alpha \sim^w \beta$, if $\overline{\mathrm{Aut}(Y,\nu)\cdot\alpha}^w = \overline{\mathrm{Aut}(Y,\nu)\cdot\beta}^w$
- elementarily equivalent, in symbols $\alpha \equiv \beta$, if α and β have the same first order theory
- approximately conjugate, in symbols $\alpha \sim^u \beta$, if $\overline{\mathrm{Aut}(Y,\nu)\cdot\alpha}^u = \overline{\mathrm{Aut}(Y,\nu)\cdot\beta}^u$
- stab-equivalent, in symbols $\alpha \underset{\mathrm{stab}}{\sim} \beta$, if $\mathrm{Aut}(Y,\nu)\cdot\overline{\mathrm{Aut}_{\mathrm{stab}_\alpha^{-1}}(Y,\nu)\cdot\alpha}^u = \mathrm{Aut}(Y,\nu)\cdot\overline{\mathrm{Aut}_{\mathrm{stab}_\beta^{-1}}(Y,\nu)\cdot\beta}^u$
- conjugate, in symbols $\alpha \simeq \beta$, if $\mathrm{Aut}(Y,\nu)\cdot\alpha = \mathrm{Aut}(Y,\nu)\cdot\beta$.

We have the following sequence of inclusions:

$$\simeq \;\subseteq\; \underset{\mathrm{stab}}{\sim} \;\subseteq\; \sim^u \;\subseteq\; \equiv \;\subseteq\; \sim^w \;\subseteq\; \sim_{\mathrm{IRS}}$$

The following tables regroup all the details on these inclusions that the author knows about.

Γ	Amenable	Not amenable	
		Property (T)	No property (T)
$\simeq \subseteq \sim^u$	$\subsetneq$	$=$	?
$\sim^u \subseteq \equiv$	$=$	?	?
$\equiv \subseteq \sim^w$	$=$	?	?
$\sim^w \subseteq$ Actions$(\Gamma)^2$	$=$	?	$\subsetneq$

Comparison of equivalence relations for free ergodic actions of an infinite group Γ

θ	Amenable	Not amenable	
		Property (T)	No property (T)
$\simeq \subseteq \underset{\text{stab}}{\sim}$	?	=	?
$\underset{\text{stab}}{\sim} \subseteq \sim^u$	$\subsetneq$	?	?
$\sim^u \subseteq \equiv$	=	?	?
$\equiv \subseteq \sim^w$	=	?	?
$\sim^w \subseteq \sim_{\text{IRS}}$	=	?	$\subsetneq$

Comparison of equivalence relations for ergodic actions with a given atomless IRS θ

Note that for a given ergodic atomless IRS θ and $\Lambda \leqslant \Gamma$, the sets $\text{stab}^{-1}(\Lambda)$ for two given stab-equivalent actions must have the same cardinality. Therefore there are infinitely many stab-equivalence classes of actions with IRS θ and in particular $\underset{\text{stab}}{\sim} \subsetneq \sim_{\text{IRS}}$, even though this could not fit with the format of the table.

Each question mark in those tables raises an open question. Furthermore, for each strict inclusion, one can ask for the Borel complexity of an equivalence relation in restriction to a given class of the weaker equivalence relations. Here are some notable questions:

Question 8.0.2. *Let θ be an ergodic IRS. Are there at least two non-conjugate ergodic pmp actions with IRS θ? Is there a continuum of pairwise non-conjugate ergodic pmp actions with IRS θ? Is the relation of conjugation on the space of actions with IRS θ not classifiable by countable structures?*

We believe the answer to those three questions is yes for θ "sufficiently nontrivial". The case of free actions of groups was treated with different methods by Glimm and later by Hjorth. The latter proved that for a non-amenable group Γ, the relation of conjugation on free pmp actions of Γ was not classifiable by countable structures.

The coset groupoid $\mathcal{G}_\Gamma$ defined in this thesis would allow to conclude for actions of a given IRS if Hjorth proof could be extended to non-amenable pmp groupoids.

Note that in [Bow14], L. Bowen proves that the measured entropy of Bernoulli shifts of a sofic IRS is an invariant of conjugation. This answers the two first questions for sofic IRSs. However, we do not know if every IRS is sofic or not.

Question 8.0.3. *Is there an ergodic IRS θ such that the relations of weak equivalence and elementary equivalence do not coincide for pmp ergodic actions of IRS θ?*

Is there an ergodic IRS θ such that the relations of elementary equivalence and approximate conjugation do not coincide for pmp ergodic actions of IRS θ?

Such an IRS should be non-amenable. Moreover, we know that at least one of these questions has a positive answer. Indeed, consider the case of free actions of a non-amenable group Γ with property (T). Then approximate conjugation implies conjugation. However, R. Tucker-Drob proved in [Tuc15, Remark 6.5] that conjugation of ergodic free actions in a given weak equivalence class is not classifiable by countable structures. Therefore, at least one between $\sim^u / \equiv$ or $\equiv / \sim^w$ is not classifiable by countable structures.

Question 8.0.4. *Let θ be an ergodic non-amenable IRS. Are there at least three ergodic pmp actions with IRS θ which are not weakly equivalent?*

Since the only example of two actions having same IRS and not being weakly equivalent comes from the dichotomy between strongly ergodic actions and not strongly ergodic actions, such a question requires new methods to approach it.

Chapter 9

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