30/05/2023

Belinskaya's theorem

Pb of conjugacy: If $T, U \in Aut(X;\mu)$ ergodic, $\exists ? S \in Aut(X;\mu)$, $ST = US$.

Thm of Dye: "orbit equivalence is too weak to distinguish between two such transformations"

Thm of Belinskaya: $T, U \in Aut(X;\mu)$ ergodic ; + cdt on the cocycle $\Rightarrow$ T ; U are flip-conjugate
(T ; U have the same orbits)

Definition: $T, U \in Aut(X;\mu)$ ergodic, S an o.e. between T and U, i.e. STS^{-1} has the same orbits as U.
Then we can define the cocycle associated with U: $c_U : X \longrightarrow \mathbb{Z}$

defined by $\forall x \in X$: $ST^{c_U(x)}(x) = US(x)$

Belinskaya's theorem ('68): Let $T, U \in Aut(X;\mu)$, ergodic, and let S be o.e. between them.

If we assume c_U to be integrable;

i.e. $\int_X |c_U(x)| \, d\mu(x) < +\infty$

Then T and U are flip-conjugate, i.e. either T or T^{-1} is conjugate to U.

Remark: We will work with T and U having the same orbits.

Notation: $T : X \to X$ bij° ; $I \subseteq \mathbb{Z}$

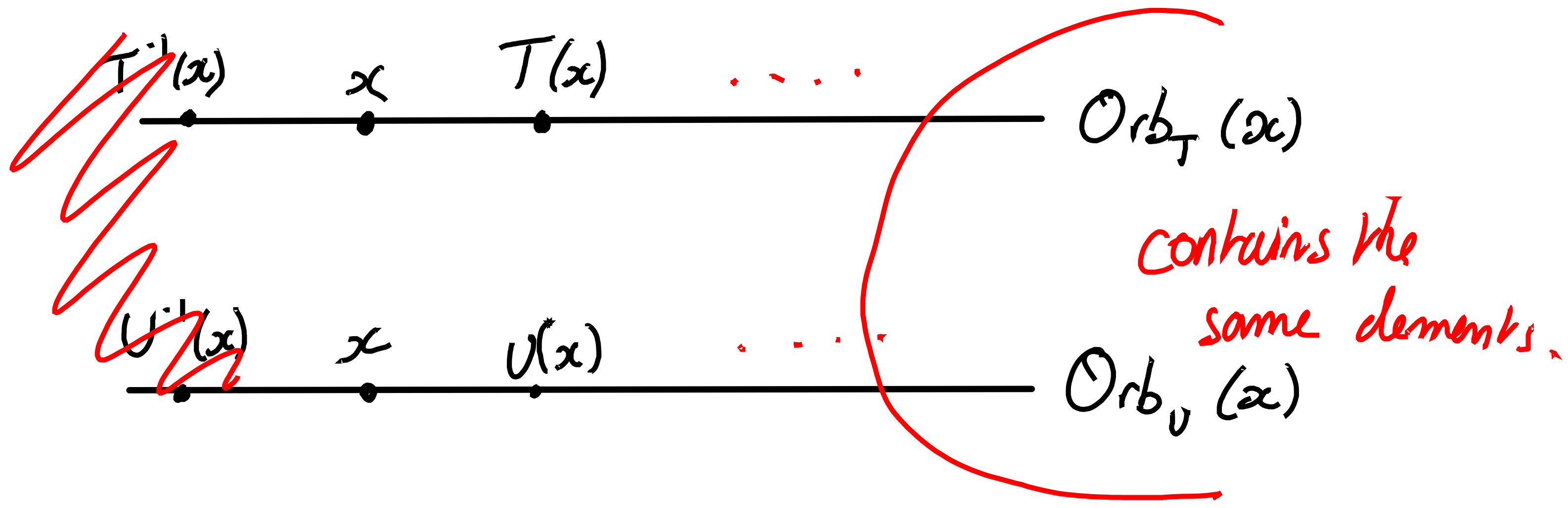
$$T^I(x) := \{T^n(x) \mid n \in I\}$$

Theorem 1: Let T be an aperiodic transformation in $\mathrm{Aut}(X;\mu)$; and let U be in $\mathrm{Aut}(X;\mu)$ with the same orbits as T.

If $|T^N(x) \triangle U^N(x)| < +\infty$, $\forall^* x \in X$:

then T and U are conjugate.

Proof: (Katznelson):



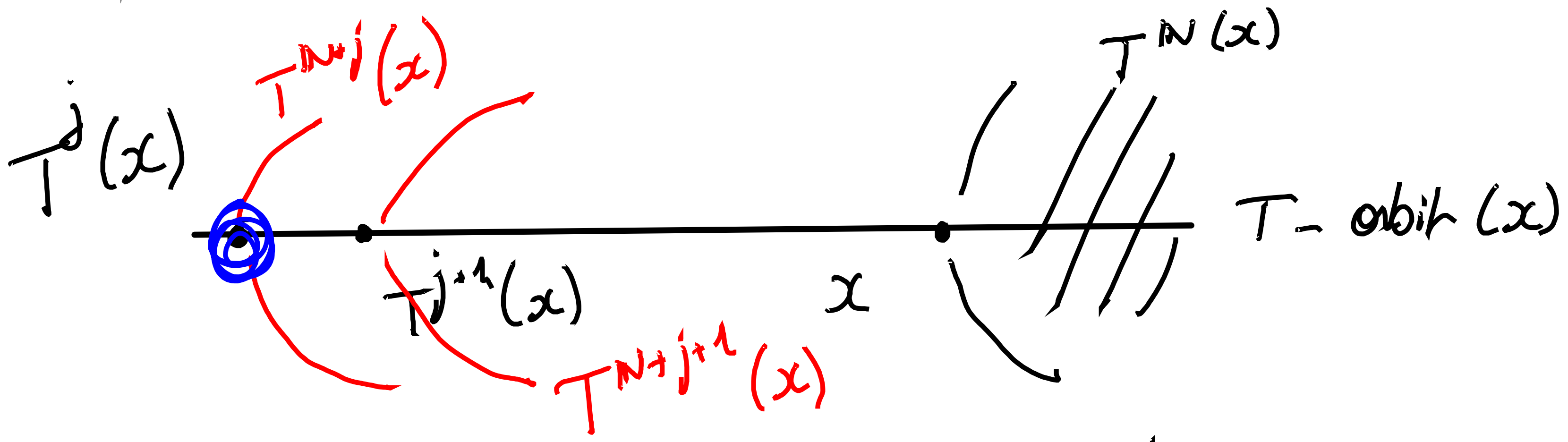
Claim: $\forall^* x \in X$: $\exists!\, j(x) \in \mathbb{Z}$,

$$|T^{N+j(x)}(x) \setminus U^N(x)| = |U^N(x) \setminus T^{N+j(x)}(x)|$$

Proof of the claim: For $x \in X$ let us define

$$\tau_x : \begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ j & \longmapsto & |T^{N+j}(x) \setminus U^N(x)| - |U^N(x) \setminus T^{N+j}(x)| \end{array}$$

We fix $x \in X$. Link: $\tau_x(j)$ and $\tau_x(j+1)$?



Case 1: $T^j(x) \in U^N(x)$

- $|T^{N+j+1}(x) \setminus U^N(x)| = |T^{N+j}(x) \setminus U^N(x)|$
- $|U^N(x) \setminus T^{N+j+1}(x)| = |U^N(x) \setminus T^{N+j}(x)| + 1$

Case 2: $T^j(x) \notin U^N(x)$

- $|T^{N+j+1}(x) \setminus U^N(x)| = |T^{N+j}(x) \setminus U^N(x)| - 1$
- $|U^N(x) \setminus T^{N+j+1}(x)| = |U^N(x) \setminus T^{N+j}(x)|$

in both cases; $\tau_x(j+1) = \tau_x(j) - 1$

So, $\tau_x(j) = \tau_x(0) - j$

for $j = \tau_x(0)$; $\tau_x(j) = 0$

$$\left(\forall^+ x \in \mathcal{X} : \exists!\, j(x) \in \mathbb{Z},\quad |T^{N+j(x)}(x) \setminus U^N(x)| = |U^N(x) \setminus T^{N+j(x)}(x)| \right. \qquad \boxed{C}$$

We define $S(x) = T^{j(x)}(x)$.

$\longrightarrow$ $\boxed{S(x) \text{ is the unique element of } Orb_T(x) \text{ s.t. } |T^N(S(x)) \setminus U^N(x)| = |U^N(x) \setminus T^N(S(x))|}$

4 different cases:

$x \in ?$ $\quad T^N(S(x)) \cap U^N(x)$

$S(x) \in ?$

$\hookrightarrow$ by similar arguments; "removing x and $S(x)$" does not change the equality:

$$|T^{N+1}(S(x)) \setminus U^{N+1}(x)| = |U^{N+1}(x) \setminus T^{N+1}(S(x))|$$

i.e. $|T^N(TS(x)) \setminus U^N(U(x))| = |U^N(U(x)) \setminus T^N(TS(x))|$

$x = U^{-1}(y)$

$\longrightarrow$ $|T^N(\underbrace{TSU^{-1}(y)}) \setminus U^N(y)| = |U^N(y) \setminus T^N(TSU^{-1}(y))|$

By uniqueness of $S(x)$ we have that

$$TSU^{-1} = S$$

i.e. $TS = SU$

We still have to prove that $S \in \mathrm{Aut}(X;\mu)$.

- $\underline{S \text{ bijection:}}$ $TS(x) = SU(x)$

  $\Leftrightarrow T^{-1}S(x) = SU^{-1}(x)$

  By induction: $SU^{n+1}(x) = SU^n(U(x))$
  $= SU^n(y)$
  $= T^nS(y)$
  $= T^nS(U(x))$
  $= T^nTS(x)$
  $= T^{n+1}S(x)$

  S induces a bijection on each orbit.

- $\underline{S \text{ pmp:}}$ $\left(A_n := \{x \in X \mid S(x) = T^n(x)\}\right)_{n \in \mathbb{Z}}$

  $(A_n)_{n\in\mathbb{Z}}$ is a partition of X ($\emptyset$ authorized)

  $B \subseteq X$ measurable;

$$\mu(S(B)) = \mu\Big(S\big(\bigsqcup_{\mathbb{Z}} A_n \cap B\big)\Big) = \sum_{\mathbb{Z}} \mu(T^n(A_n \cap B))$$
$$= \sum_{\mathbb{Z}} \mu(A_n \cap B)$$
$$= \mu(B).$$

$\blacksquare$

Lemma (Mass-transport principle):

Let $T \in \mathrm{Aut}(X;\mu)$, and $\mathcal{R}_T$ the equivalence relation which classes are the T-orbits.

$f : \mathcal{R}_T \longrightarrow \mathbb{N}$ measurable;

$$\int_X \sum_{\mathbb{Z}} f(x, T^n(x))\, d\mu(x) = \int_X \sum_{\mathbb{Z}} f(T^n(x), x)\, d\mu(x)$$

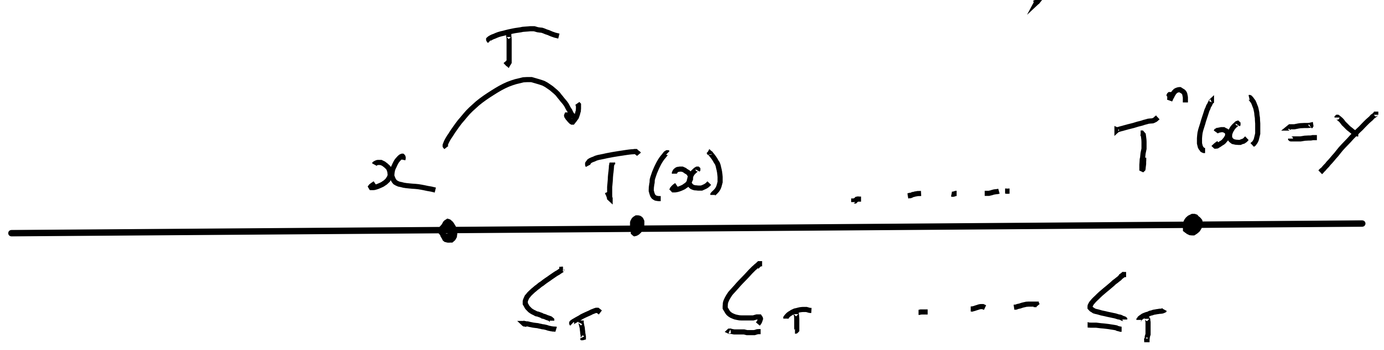
Proof: Fubini ▨

Theorem (Belinskaya) Let $T, U \in \mathrm{Aut}(X;\mu)$ ergodic with the same orbits. If c_U is integrable, then T and U are flip-conjugate.

Proof: Define on each T-orbit a total order $\leq_T$ by

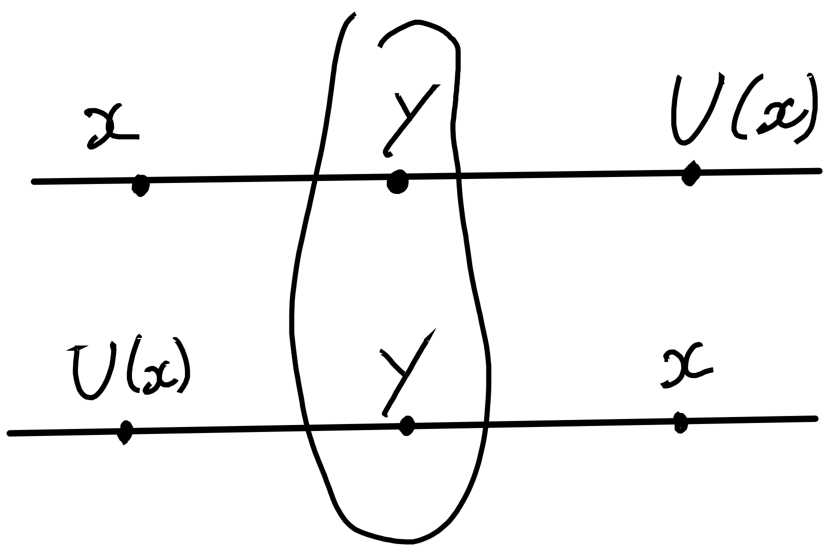
$$x \leq_T y \iff \exists n \in \mathbb{N},\ y = T^n(x)$$

$$<_T \iff \overline{\qquad\qquad} \quad x \neq y$$



We define also $f : \mathcal{R}_T \longrightarrow \mathbb{N}$

$$(x;y) \longmapsto \begin{cases} 1 & \text{if } \begin{array}{l} x \leq_T y <_T U(x) \\ U(x) <_T y \leq_T x \end{array} \\ 0 & \text{else} \end{cases}$$



Recall: $c_U : X \longrightarrow \mathbb{Z}$ def by $U(x) = T^{c_U(x)}(x)$

$$f(x, T^n(x)) = 1 \iff \begin{cases} 0 \leq n < c_U(x) \\ \text{or} \\ c_U(x) < n \leq 0 \end{cases}$$

$$\sum_{\mathbb{Z}} f(x, T^n(x)) = |C_U(x)|$$

Hyp.

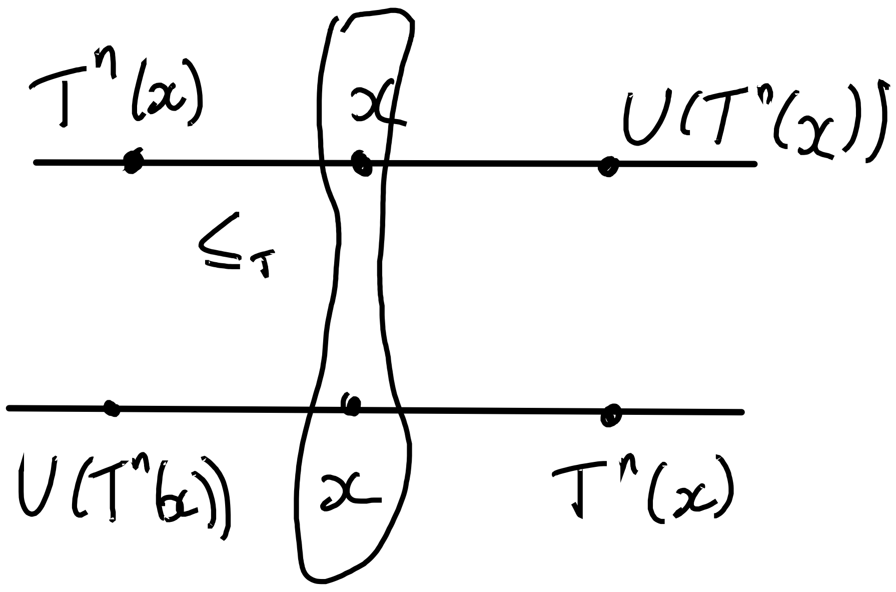
$$\int_X \sum_{\mathbb{Z}} f(x, T^n(x))\, d\mu(x) = \int_X |C_U(x)|\, d\mu(x) < +\infty$$

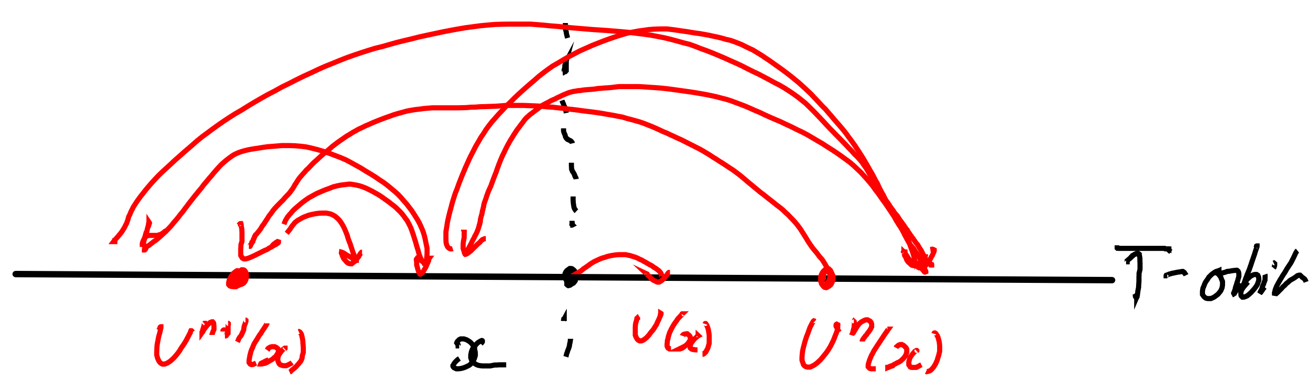
$\parallel$ Lemma. (money-transport)

$$\int_X \sum_{\mathbb{Z}} f(T^n(x), x)\, d\mu(x) < +\infty$$

$$\forall^* x \in X : \sum_{\mathbb{Z}} f(T^n(x), x) < +\infty$$

$$\boxed{\exists \text{ finite number of } n \in \mathbb{Z} \text{ s.t. } f(T^n(x), x) = 1}$$





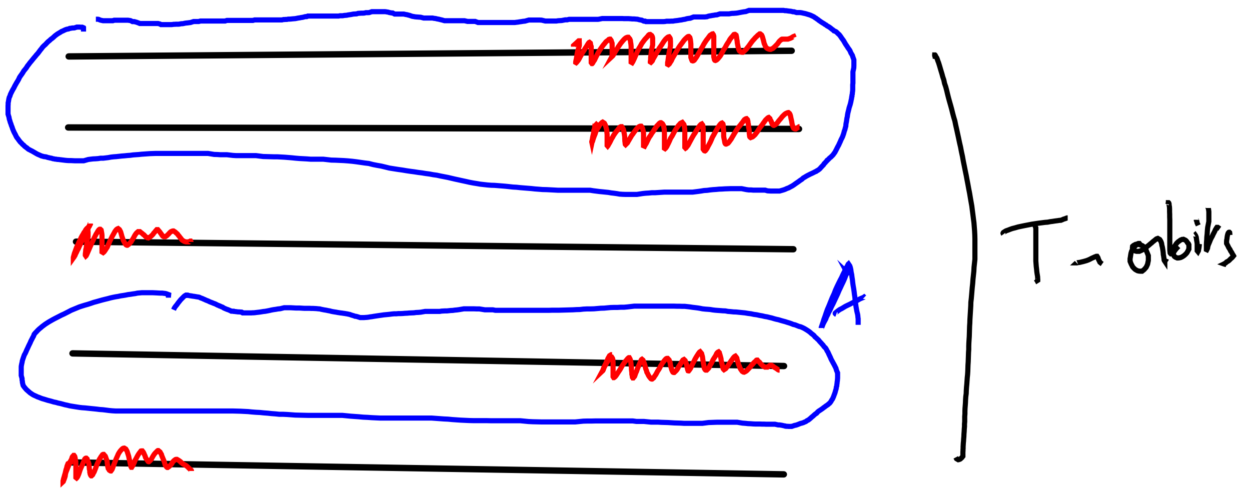
Those "jumps" over x can only occur finitely many times

This means that, except for a finite number of $n \in \mathbb{N}$

$$\left(\begin{array}{l} x \leq_T U^n(x) \\ \text{or} \\ U^n(x) \leq_T x \end{array} \right. \iff (U^n(x)) \xrightarrow[\leq_T]{} \pm\infty$$

This does not depend on x in $\mathrm{Orb}_U(x)$

This describes the behaviour of U on one orbit, so we now use ergodicity to understand the behaviour of U on X.



A is U-invariant; so by ergodicity; its measure is 0 or 1.

So up to replacing U by U^{-1}; we can assume that $\forall^* x \in X$: $x \leq_T U^n(x)$, except for a finite number of $n \in \mathbb{N}$. all but finitely many

By the symmetric argument; for $n \leq 0$; $\forall^* x \in X$:

$$\begin{cases} x \leq_T U^n(x) \\ \text{or} \\ U^n(x) \leq_T x \end{cases}$$

it is actually not possible that $x \leq_T U^n(x)$; because T and U have the same orbits; so the elements of $\mathrm{Orb}_T(x)$ that are $\leq_T x$ need to be reached by the negative powers of U.

In the end; $|T^{\mathbb{N}}(x) \,\triangle\, U^{\mathbb{N}}(x)| < +\infty$

so we can apply Theorem 1: T and U or U^{-1} are conjugated in $\mathrm{Aut}(X;\mu)$ ▨

(Carderi, Joseph, Le Maître, Tessera
Belinskaya's theorem is optimal $(ArXiv)$)