Exhaustive OE as a restricted OE, after Heicklen

$\Gamma = \bigcup_n \Gamma_n$, Γ_n finite, two pmp Γ-act$^\circ$ α and β on (X,μ) say that:

- α and β have <u>exhaustively the same</u> orbits if $\forall n$, $\forall x$,
$$\alpha(\Gamma_n)x = \beta(\Gamma_n)x$$

- α and β are <u>exhaustively OE</u> if α is conjugate to some α' which exhaustively has same orbits as β: $\exists S \in \mathrm{Aut}(X,\mu)$ st $\forall n$ $\forall x$ $S(\alpha(\Gamma_n)\cdot x) = \beta(\Gamma_n)\cdot S(x)$

<u>Restricted OE:</u>

$(M,d) \curvearrowleft G$ Polish by isometries. a <u>size</u> (for this act$^\circ$) is $\rho : M \times G \to \mathbb{R}^{\geq 0}$

Polish space

st if $\rho_\alpha : g \mapsto \rho(\alpha, g)$ $\to$ $d_\alpha(g,h) = \rho_\alpha(gh^{-1})$ right-invt

we have:

- ρ_α is a continuous pseudo-norm on G / refining $\rho^\circ_\alpha : g \mapsto d(\alpha,\alpha g)$
- ρ is equivariant: $\forall g, h \in G$,
$$\rho_{\alpha\cdot h}(g) = \rho_\alpha(hgh^{-1})$$
- ρ is <u>Cauchy compatible</u>: $\forall (g_i)$ d_α-Cauchy st. $\alpha g_i \to \beta$, $\forall h \in G$,
$$\rho_{\alpha g_i}(h) \to \rho_\beta(h)$$

Two elements $\alpha, \beta \in M$ are ρ-approx in the same G-orbit (<u>ρ-equivalent</u>) if $\exists (g_i)$ d_α Cauchy st. $\alpha g_i \to \beta$

<u>Setup:</u> $M = \{\Gamma\text{-arrangements}\} = \{\alpha : \Gamma \to [R] : \alpha \text{ is free } \& \ R = R_\alpha\}$

R hyperfinite ergodic pmp eq rel

$$\delta_u(\alpha,\beta) = \sum_i \frac{1}{2^i} d_u(\alpha(\gamma_i), \beta(\gamma_i))$$

$\Gamma = \{\gamma_i\}_{i\in\mathbb{N}}$

$d_u(T_1,T_2) = \mu(\{x \in X : T_1(x) \neq T_2(x)\})$

$$G = [R] := \{T \in \mathrm{Aut}(X,\mu) : (x, T(x)) \in R \ \forall x \in X\}$$

d_u is biinv complete on $[R]$

$M \curvearrowleft [R]$

$\alpha \cdot T(g) = T^{-1}\alpha(g)T$.

<u>Size for exhaustive:</u> $\rho_\alpha(T) = \rho^\circ_\alpha(T) + d_u(T, G_\alpha)$

where $G_\alpha = \{U \in [R] : \forall n \ \forall x \ U(\alpha(\Gamma_n)x) = \alpha(\Gamma_n)U(x)\}$

<u>Theorem</u> (Heicklen, Kammeyer-Rudolph) $\alpha, \beta \in M$ are ρ-equivalent iff $\exists T \in [R]$ st αT and β have exhaustively the same orbits.

Lemma: Suppose (T_i) is d_α-Cauchy and $\alpha T_i \to \beta$. Then up to taking a subsequence, there $\exists T \in [R]$, st. $\lim_{i\to\infty} d_\alpha(T_i, T) \to 0$

- αT and β have exhaustively same orbits
- $\rho_\beta = \lim_{i\to\infty} \rho_{\alpha T_i}$

Proof: G_α is a closed subgroup of $[R]$

⇝ define δ_α the quotient metric on $G_\alpha \backslash [R] = G_\alpha \backslash\backslash [R]$

let d_α be the metric assoc. to ρ

$$d_\alpha(T_1, T_2) = \delta_u(\alpha T_1, \alpha T_2) + \delta_\alpha(T_1, T_2)$$

$$(\delta_\alpha(T_1,T_2) = d_u(T_1 T_2^{-1}, G_\alpha) = d_u(T_2^{-1} T_1^{-1} G_\alpha)$$

$$\overset{^{-1}\text{ is an isometry}}{=} d_u(G_\alpha T_1, T_2))$$

since (T_i) is d_α Cauchy, it is δ_α Cauchy, so up to taking subsequence we find $U_i \in G_\alpha$ st $U_i T_i$ is d_u-Cauchy ⇝ $U_i T_i \xrightarrow{d_u} T$

$$\Big(\text{and } \lim_{i\to\infty} \underset{d_\alpha(U_iT_i,T)}{\underset{\parallel}{d_\alpha(T_i, T)}} = 0\Big)$$

observe that since $U_i \in G_\alpha$, $\alpha U_i T_i$ has exhaus. same orbits as αT_i

Since the set of pairs of arrangements which exhaustively have same orbits is closed

and $\alpha U_i T_i \to \alpha T$,

αT and $\underset{\lim \alpha T_i}{\underset{\parallel}{\beta}}$ have exhaustively same orbits

In particular $\rho_{\alpha T} = \rho_\beta$

Let us check $\rho_\beta = \lim \rho_{\alpha T_i}$, equivalently $d_\beta = \lim d_{\alpha T_i}$

Take $V_1, V_2 \in [R]$

$$\begin{aligned} d_\beta(V_1, V_2) &= d_{\alpha T}(V_1, V_2) \\ &= d_\alpha(TV_1, TV_2) \\ &= d_u(G_\alpha T V_1, G_\alpha T V_2) \\ &= \lim_{i\to\infty} d_u(G_\alpha U_i T_i V_1, G_\alpha U_i T_i V_2) \\ &= \lim_i (G_\alpha T_i V_1, G_\alpha T_i V_2) \\ &= \lim_i d_{\alpha T_i}(V_1, V_2) \end{aligned}$$

forget about the ρ_0 part which is OK!

because $d_u(U_i T_i, T) \to 0$

uses equivariance:

$$\rho_{\alpha T}(U) = \rho_\alpha(TUT^{-1})$$

$$d_u(U, G_{\alpha T}) = d_u(U, T^{-1} G_\alpha T) = d_u(TUT^{-1}, G_\alpha)$$

□

As a csq, ρ is a size and we prove the thm:

Theorem (Heicklen, Kammeyer–Rudolph) $\alpha, \beta \in M$ are ρ-equivalent iff $\exists T \in [R]$ st αT and β have exhaustively the same orbits.

Pf: $\Rightarrow$ By lemma, if $\alpha, \beta \in M$ are ρ-equiv, we have $T \in [R]$ st αT and β each have same orbits.

$\Leftarrow$ ρ equivalence is an eq rel and α is ρ-equiv to αT

So it suffices to show that if α, β have each same orbits, then they are ρ-eq

but they are via T_n constructed as follows:

let A_n be a fund domain of $\alpha(\Gamma_n)$ (and hence $\beta(\Gamma_n)$)

if $x \in \alpha(\gamma) A_n$, $\gamma \in \Gamma_n$

$$T_n(x) = \beta(\gamma)\left(\alpha(\gamma)^{-1} x\right)$$

T_n conjugates $\alpha_{\restriction \Gamma_n}$ to $\beta_{\restriction \Gamma_n}$

$T_n \in [\alpha(\Gamma_n)]$ then $T_n \in G_\alpha$ $\forall n$

if $m \leq n$, T_n conjugates $\alpha(\Gamma_m)$ to $\beta(\Gamma_m)$

$\to$ takes $\alpha(\Gamma_m)$ orbits to $\beta(\Gamma_m)$ orbits $= \alpha(\Gamma_m)$ orbits

for $m \geq n$, $T_n \in [\alpha(\Gamma_m)] \geq [\alpha(\Gamma_n)]$

$\to$ takes $\alpha(\Gamma_m)$ orbits to $\alpha(\Gamma_m)$-orbits $\square$

Sizes for restricted OE: a stronger assumption

We will replace Cauchy compatibility by something stronger, needed for later developments

Right(!) Def: A *size* is a continuous function $\rho : M \times G \to \mathbb{R}^{\geq 0}$ st $\forall \alpha$, ρ_α is a pseudonorm on G

and: $-\ \forall \alpha$, ρ_α refines ρ_0^α

$-\ \rho$ is equivariant: $\forall \alpha, \forall g, h$, $\rho_{\alpha h}(g) = \rho_\alpha(h g h^{-1})$

(To match the def in the K-R book, one could more generally ask ρ is upper semi continuous)

Observe that we have a natural G-action on $M \times G$

$(\alpha, g) \cdot h = (\alpha \cdot h, h^{-1} g h)$

if G has a biinv metric, this action is by isometries

moreover our equivariance condition on sizes is equivalent to

invariance under this action

so sizes can be seen as (semi) continuous functions $M \times G /\!/ G \longrightarrow \mathbb{R}^+$

Thm (Heicklen) Γ = lf $\Gamma = \bigcup \Gamma_n$, $M = \{\Gamma\text{-actions}\}$

$$M \times [R] /\!/ [R] = \text{The space of laws of random permutations of } G \text{ associated to } (\alpha, T)$$

$M \times [R] :=$ space of rearrangements

Every rearrangement defines a random permutation of Γ: $\forall x \in X$, $\mathrm{Orb}_\alpha(x) \xleftarrow{\sim} \Gamma$, $\alpha(\gamma) x \leftarrow\!\shortmid \gamma$

and T defines a permutation of $\mathrm{Orb}_\alpha(x)$ hence of Γ via this identification $\sigma_{T, \alpha, x} \in \mathrm{Sym}(\Gamma)$

$(\alpha, T) \mapsto \Phi_* \mu \qquad \left(\Phi(x) = \delta_{T,\alpha,x} \right)$

Φ_* ↑ Prob(Sym(Γ))

$\subseteq \Gamma^\Gamma \subseteq (\Gamma \cup \{\infty\})^\Gamma$

↑ one pt compactificatio

$\Phi_* \mu \in \text{Prob}((\Gamma \cup \{\infty\})^\Gamma)$

↑ compact space ⤳ weak * topology

What Heicklen proves is that ρ satisfies Axiom 3 from K-R :

ρ is semi continuous as a map from the space of laws of random perm associated to rearrangements to $\mathbb{R}^+$

In the book K-R state ρ is a 3^+ size i.e. it is <u>continuous</u>