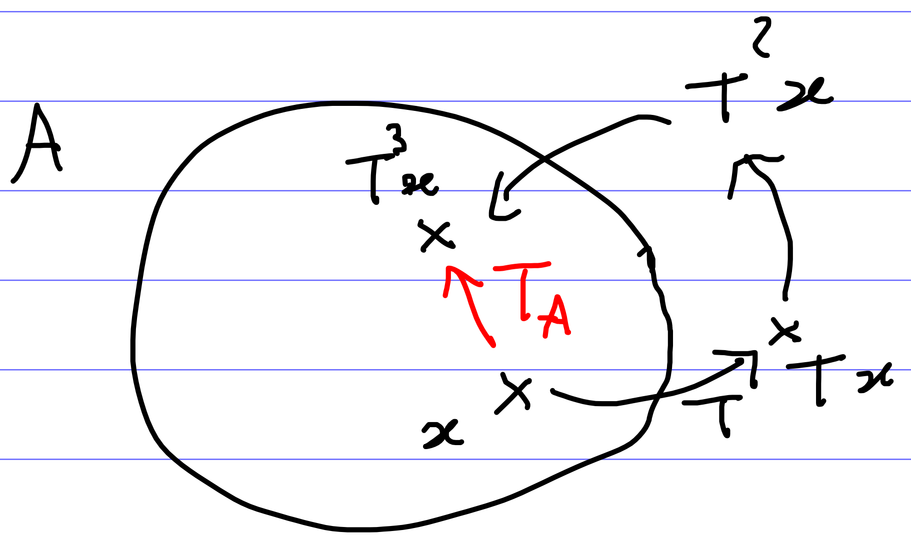
Introduction to Kakutani equivalence

I- Induced map, Kakutani tower
II- Kakutani equivalence for $\mathbb{Z}$-actions
III- Generalization to $\mathbb{Z}^d$-action

I-

$T \in Aut(X,\mu)$, $A \in \mathcal{A}$

$$r_{T,A} : A \to \mathbb{N}, \quad x \mapsto \inf\{k \geq 1 \mid T^k x \in A\}$$



$$T_A : A \to A, \quad x \mapsto T^{r_{T,A}(x)} x$$

induced map on A

Poincaré Recurrence theorem if $\mu(A) > 0$, then for a.e. $x \in A$ $\{k > 0,\ T^k x \in A\}$ is infinite

Remark: it is true for a.e. $x \in X$ when T is ergodic

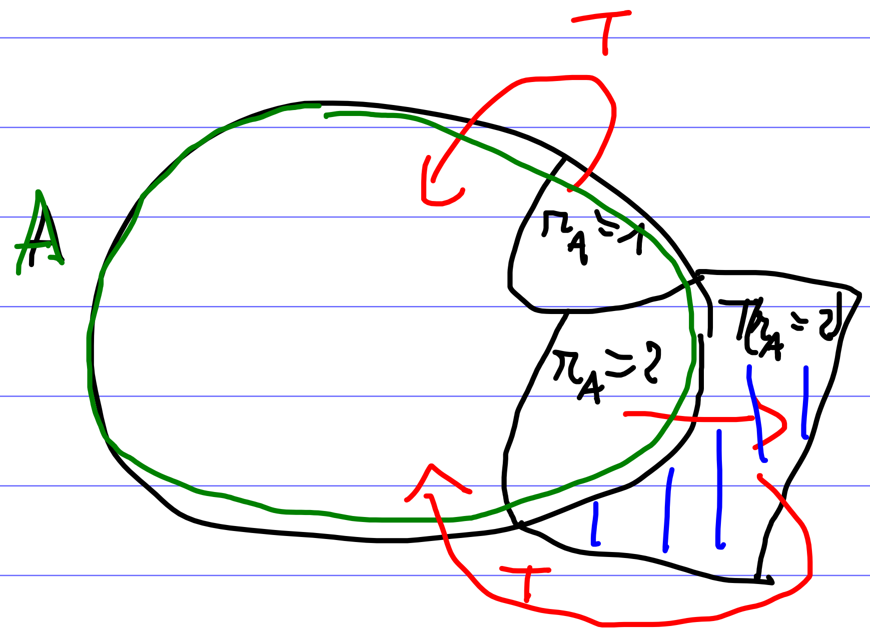
$\bigcup_{n\in\mathbb{Z}} T^n(A)$ is T-invariant and $\mu > 0$

$T_A \in Aut(A, \mathcal{A}_A, \mu_A)$ with $\mu_A(\cdot) = \dfrac{\mu(A \cap \cdot)}{\mu(A)}$

if T is ergodic (for μ), then T_A is ergodic (for μ_A)

Kac's theorem: $\int_A r_A \, d\mu_A \leq \dfrac{1}{\mu(A)}$ with equality when T is ergodic

Proof: $X \supseteq \bigsqcup_{k>0} \bigsqcup_{0\leq n < k} T^n(\{r_A = k\})$ with equality if T is ergodic and apply μ



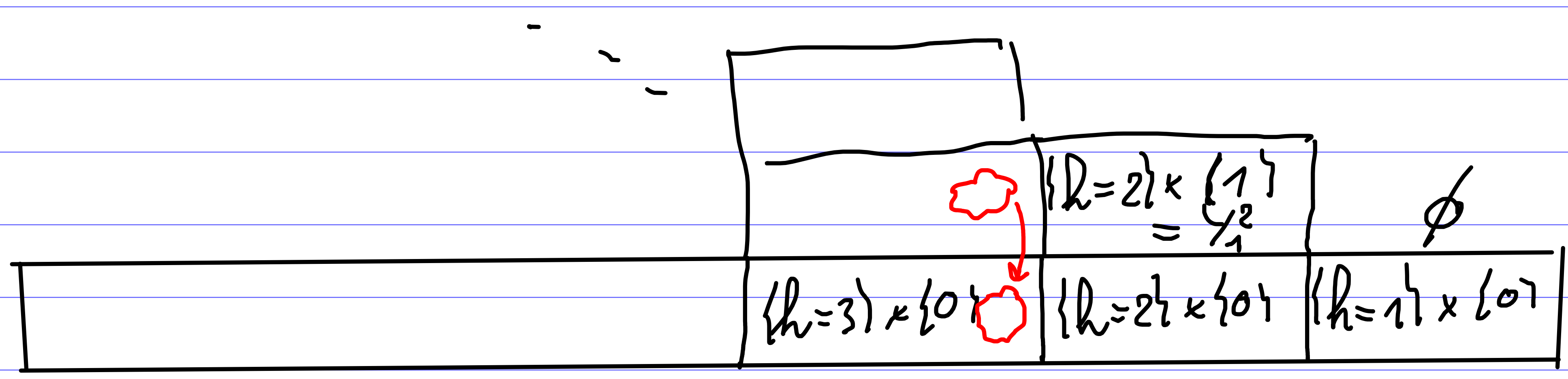
from T_A and r_A,
we can recover T by artificially build this blocks
$\to$ motivate Kakutani towers

$S \in Aut(Y, \mathcal{B}, \nu)$, $h : Y \to \mathbb{N}^*$ integrable
(ex : $S = T_A$, $h = r_A$ integrable Kac's lemma)

$$Y^h := \{(y,i) \mid y \in Y, 0 \leq i < h(y)\}$$

$$= \bigsqcup_{j>0} \bigsqcup_{0 \leq i < j} \underbrace{\{(y,i) \mid h(y) = j\}}$$

Y_i^j is a "copy" of $\{h = j\}$
via $f_i^j : \{h=j\} \to Y_i^j$
$y \mapsto (y,i)$

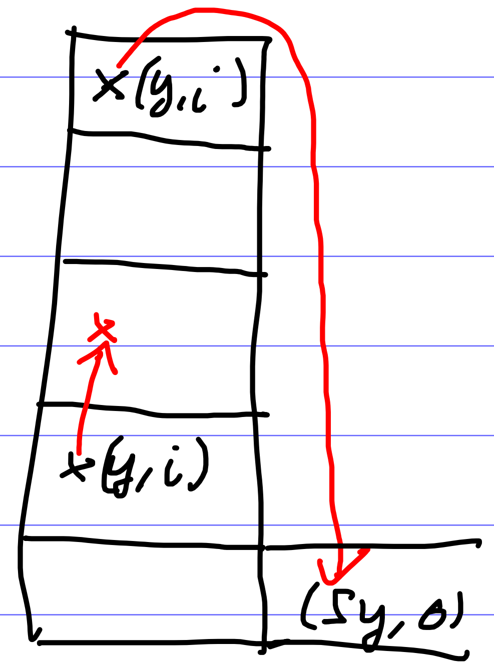


h : height function

μ^h is defined by $\forall B \in \underset{\substack{\shortparallel}}{\mathcal{B}^h}$, if $B \subset Y_i^j$, $\mu^h(B) = \mu\left((f_i^j)^{-1}(B)\right) \times \underbrace{\lambda}_{\text{normalization constant}}$

(σ-algebra of $Y \times \mathbb{N}$) $\cap$ Y^h

$$S^h : Y^h \to Y^h$$
$$(y,i) \mapsto \begin{cases} (y, i+1) & \text{if } i+1 < h(y) \\ (Sy, 0) & \text{if } i+1 = h(y) \end{cases}$$



$\{h = j\} \times \{0, 1, \dots, j-1\}$

$h(y) = j$

- if S is ergodic, then so is S^h.
- $h = r_A$, $Y = A$: $\quad (T_A)^{r_A} \underset{\text{isomorphic}}{\simeq} T$ when T is ergodic

- $(S^h)_{Y \times \{0\}} \simeq S \qquad r_{S^h, Y\times\{0\}} = h$

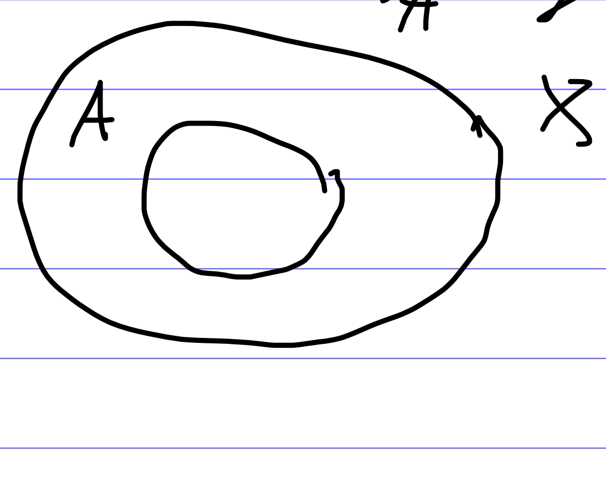
Value of λ :

$$1 = \nu^h(Y^h) = \sum_{\substack{j>0 \\ 0 \le i < j}} \underbrace{\nu^h(Y_i^j)}_{= \lambda \nu(\{h = j\})} = \lambda \sum_{j>0} j\, \nu(\{h = j\}) = \lambda \int h \, d\nu$$

$$\lambda = 1 / \int h \, d\nu$$

$$T_A \longrightarrow (T_A)^{r_A} \simeq T$$
$$\int r_A \, d\mu_A = 1/\mu(A)$$

$1/\mu(A) =$ how much you extend T_A to get T



$$S^h \longrightarrow (S^h)_{Y \times \{0\}} \simeq S$$
$$\nu^h(Y \times \{0\}) = \lambda = 1 / \int h \, d\mu$$

Some properties

a – $T \in Aut(X,\mu)$, $S \in Aut(Y,\nu)$

$$T \overset{\varphi : X \to Y}{\simeq} S \qquad (\text{meaning} : S = \varphi T \varphi^{-1})$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

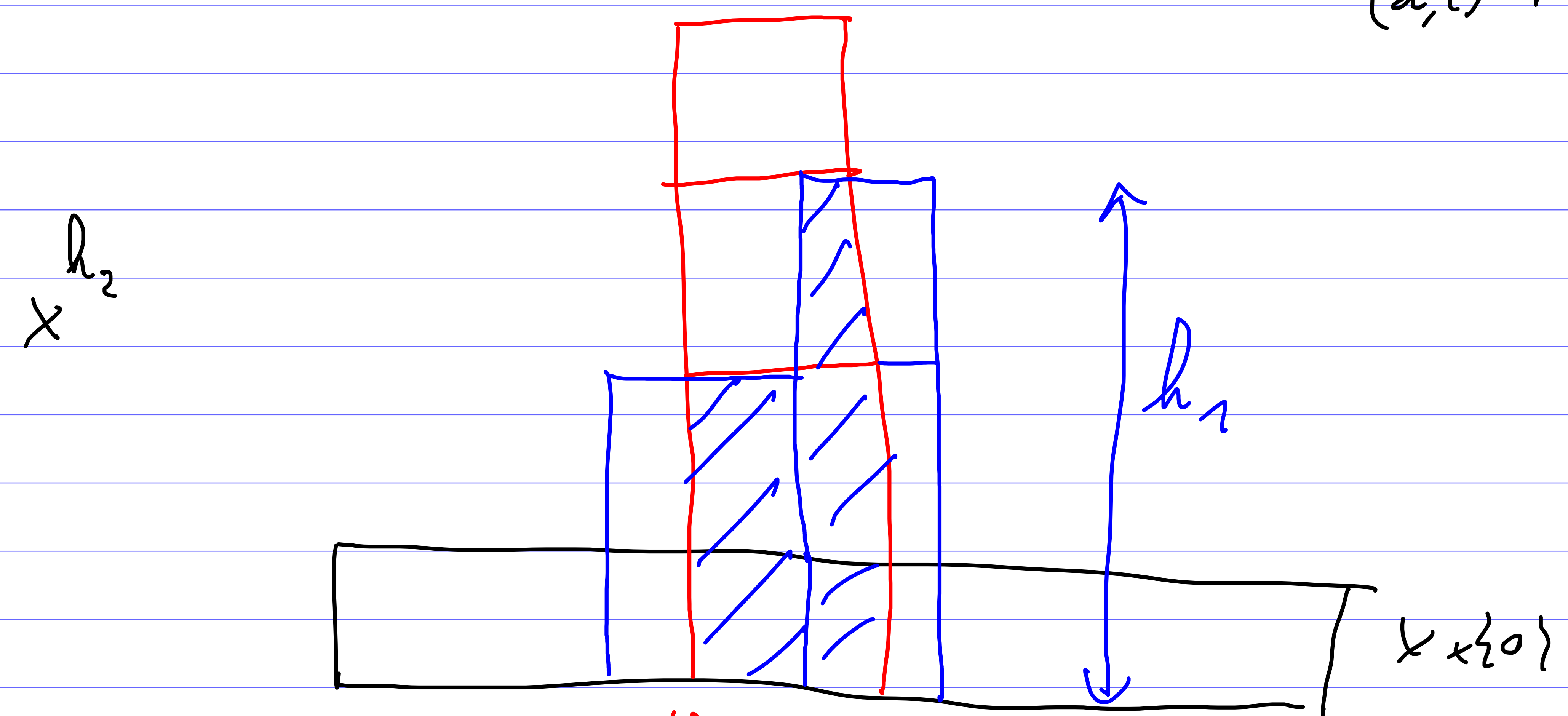
$$T_A \overset{\varphi : A \to \varphi(A)}{\simeq} S_{\varphi(A)} \qquad\qquad T^h \underset{\substack{\varphi : X^h \to Y^{h\circ\varphi^{-1}} \\ (x,i) \mapsto (\varphi(x), i)}}{\simeq} S^{h\circ\varphi^{-1}}$$

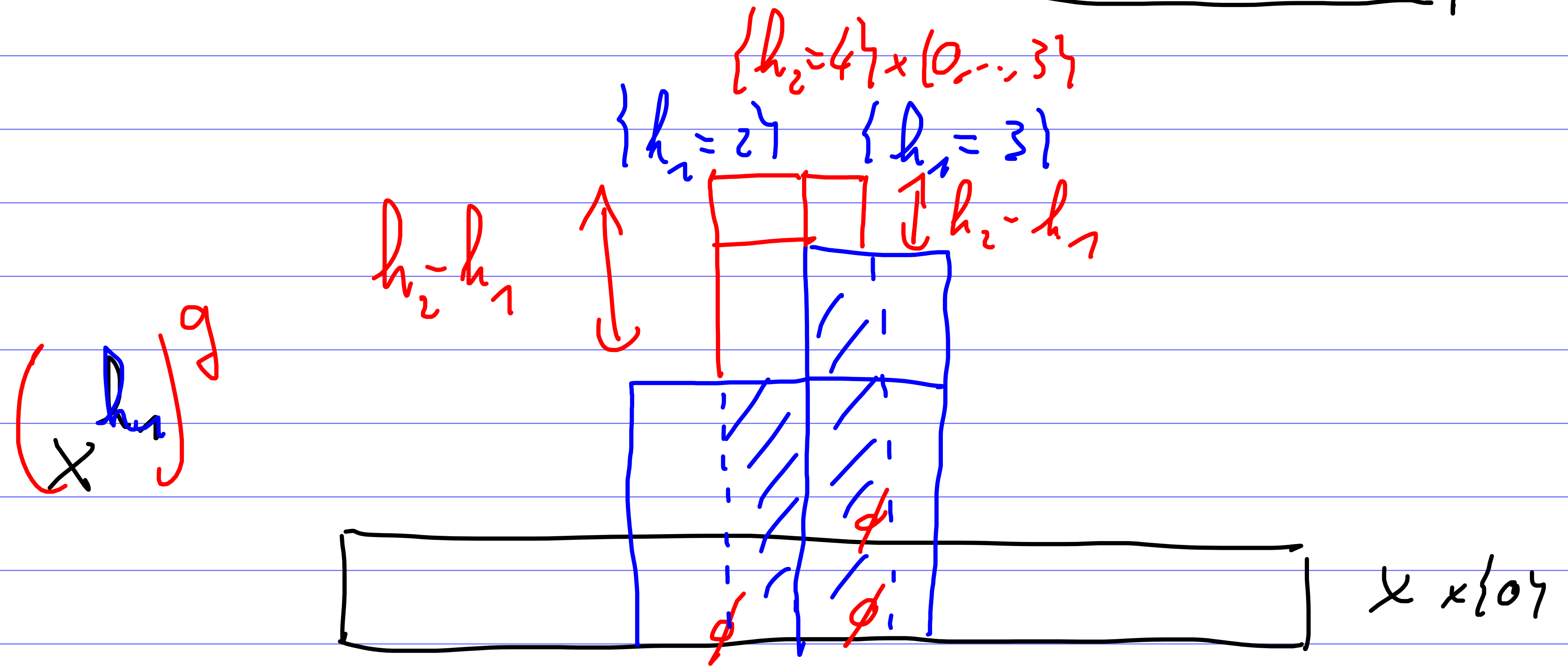
b) $T \in Aut(X,\mu)$, $A_1 \subset A_2 \subset X$

$$T_{A_1} = (T_{A_2})_{A_1}$$

$h_1, h_2 : X \to \mathbb{N}^*$ integrable, $h_1 \le h_2$

$$T^{h_2} = (T^{h_1})^g \quad \text{with } g : X^{h_1} \to \mathbb{N}^* \quad (x,i) \mapsto \begin{cases} 1 & \text{if } i < h_1(x) \\ h_2(x) - h_1(x) + 1 & \text{if } i = h_1(x) \end{cases}$$





c - T ergodic

if $\mu(A_1) < \mu(A_2)$: $T_{A_1} \simeq (T_{A_2})_{A_1'}$ with $\mu(A_1') = \mu(A_1)$, $A_1' \subseteq A_2$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad = T_{A_1'}$

if $\int h_1\, d\mu < \int h_2\, d\mu$: $T^{h_2} \simeq (T^{h_1})^g$

so both assertions are equivalent

for the second point:

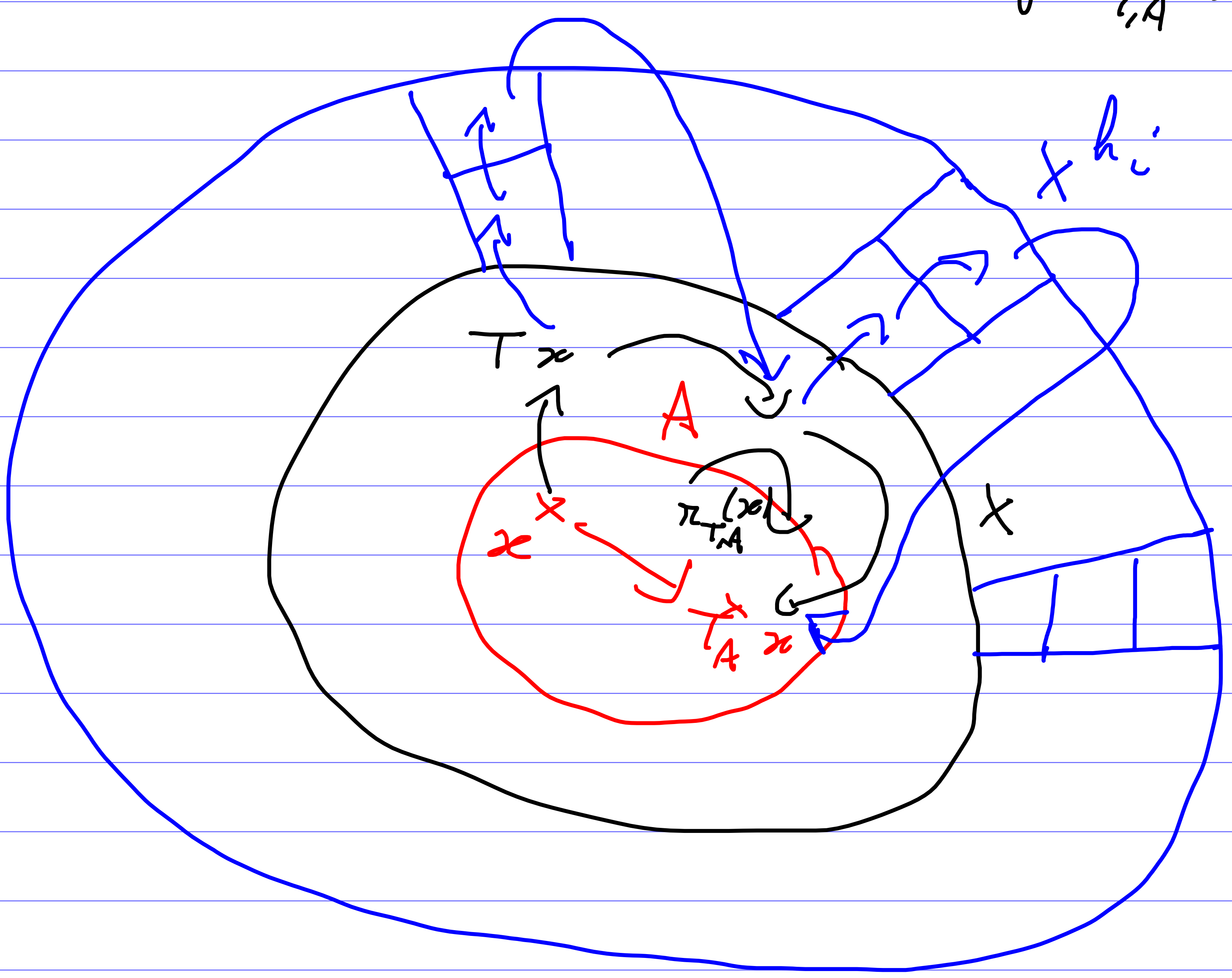
By ergodic theorem, $\exists N > 0$, $\forall n \geq N$,

$$\sum_{0 \leq j < n} h_1(T^j x) < \sum_{0 \leq j < n} h_2(T^j x)$$

for x in B with $\mu(B) > 0$

consider $A \subset B$ such that $r_{T,A} \geq N$

$T^{h_i} \simeq (T_A)^{h_i'}$ with $h_i' = \sum_{0 \leq j < r_{T,A}(x)} h_i(T^j x)$



$h_1' \leq h_2'$ on A. $(T_A)^{h_2'} \simeq \left((T_A)^{h_1'}\right)^g \simeq (T^{h_1})^g$

$(T_A)^{h_2'} = T^{h_2}$

II - Kakutani equivalence

Def: $T \in Aut(X,\mu)$, $S \in Aut(Y,\nu)$ ergodic

1) T and S are Kakutani equivalent, $T \sim_K S$, if $T_A \simeq S_B$ for some $A \subset X$, $B \subset Y$

2) ——— even K. eq., $T \sim_{eK} S$, if moreover $\mu(A) = \nu(B)$

Prop: they are equivalence relations

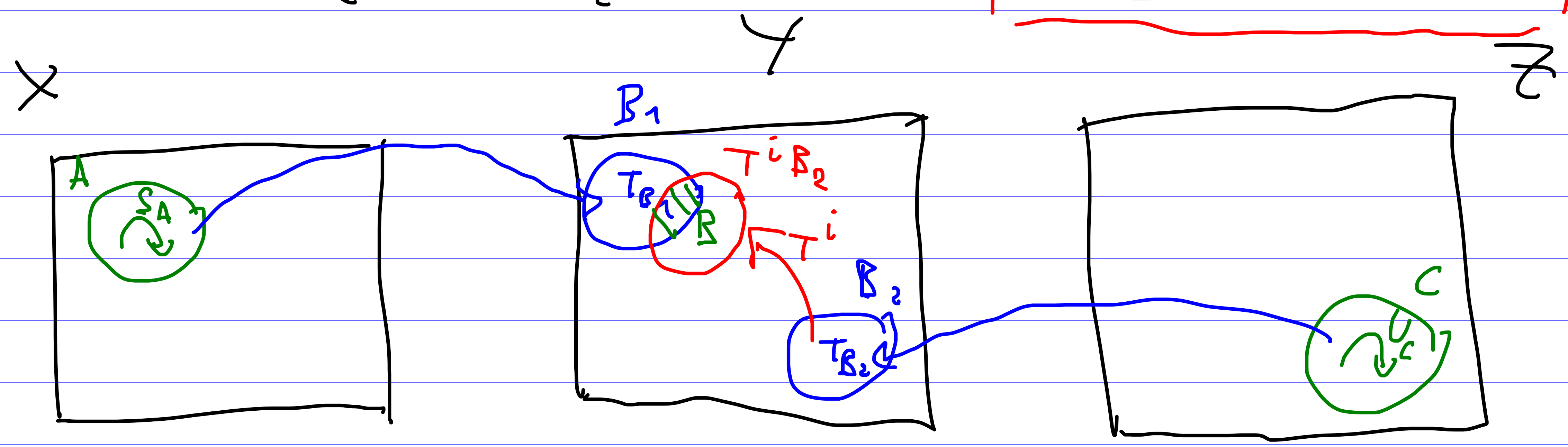
Proof: reflexivity, symmetry } OK

transitivity: $\sim_K$:

$S \in Aut(X,\mu)$, $T \in Aut(Y,\nu)$, $U \in Aut(Z,\rho)$

$S_A \simeq T_{B_1}$, $T_{B_2} \simeq U_C$

$T^i T_{B_2} = T_{T^i B_2} T^i$ then $\boxed{T_{B_2} \overset{T^i}{\simeq} T_{T^i B_2}}$



by ergodicity: $\exists\, i_0$, $\nu\Big(\underbrace{(T^{i_0} B_2) \cap B_1}_{B}\Big) > 0$

$$A \xrightarrow{\varphi_1} B_1 \supset B \subset T^{i_0} B_2 \xleftarrow{\varphi_2} C$$

$$(\cdot)_B \left(\underline{S_A \overset{\varphi_1}{\simeq} T_{B_1}} \qquad \underline{T_{T^i B_2} \overset{\tilde{\varphi}_2}{\simeq} U_C} \right.$$

$$S_{\varphi_1^{-1}(B)} \simeq T_B \simeq U_{\tilde{\varphi}_2(B)}$$

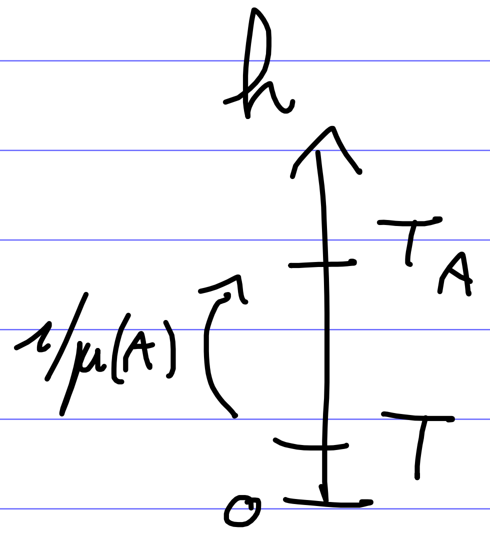
then $S \sim_K U$

$\sim_{eK}$: $\left.\begin{array}{l} \mu(A) = \nu(B_1), \ \nu(B_2) = \rho(C) \\ \varphi_1, \varphi_2 \text{ are measure isomorphism} \\ (A, \mu_A) \to (B_1, \nu_{B_1}) \\ (C, \mu_C) \to (B_2, \nu_{B_2}) \end{array}\right\} \Longrightarrow \mu(\varphi_1^{-1}(B)) = \rho(\tilde{\varphi}_2(B)),$

$S \sim_{eK} U$ ■

Theorem (Abramov's formula) T ergodic, $A \subset X$

$$\mu(A)\, h(T_A) = h(T)$$



Corollary: $\sim_{ek}$ preserves entropy

III - Generalization to $\mathbb{Z}^d$-action

Notation: T is a free $\mathbb{Z}^d$-action on (X,μ)

$\vec{T}(x,y)$ is $k \in \mathbb{Z}^d$ s.t. $T^k x = y$

"T-vector between x and y"

1) Stable orbit equivalence (SOE)

Def: G, H groups, $T : G \curvearrowright (X,\mu)$, $S : H \curvearrowright (Y,\nu)$ free ergodic pmp

$U \subset X$, $V \subset Y$ of positive measure

$\varphi : (U, \mu_U) \to (V, \nu_V)$ measure isomorphism

φ is a SOE between (X,μ,T) to (Y,ν,S) if for μ-a.e. $x \in X$, $\varphi(\mathrm{Orb}_T(x) \cap U) = \mathrm{Orb}_S(\varphi(x)) \cap V$

$\mathrm{dom}(\varphi) := U$, $\mathrm{rng}(\varphi) := V$, $\mathrm{comp}(\varphi) := \dfrac{\nu(\mathrm{rng}\,\varphi)}{\mu(\mathrm{dom}\,\varphi)}$

if U, V are of full measure, φ is a OE

Remark
- $A \subset \mathrm{dom}(\varphi)$, $\varphi_{|A} : A \to \varphi(A)$ then $\mathrm{comp}(\varphi_{|A}) = \mathrm{comp}(\varphi)$ (because $\varphi_* \nu_{\mathrm{rng}(\varphi)} = \mu_{\mathrm{dom}(\varphi)}$)
- $G = H = \mathbb{Z}$. If $x \in \mathrm{dom}(\varphi) = U$, $\mathrm{Orb}_T(x) \cap U = \mathrm{Orb}_{T_U}(x)$

$\mathrm{Orb}_S(\varphi(x)) \cap V = \mathrm{Orb}_{S_V}(\varphi(x))$

S is an OE between T_U and S_V

2) M-Kakutani equivalence

$\|.\|$ norm on $\mathbb{R}^d$, M $d\times d$ real matrice

Def free ergodic $\mathbb{Z}^d$-actions T on (X,μ)
S on (Y,ν)

$T \overset{M}{\rightsquigarrow} S$ if there exist an SOE φ between T and S
$\varphi : X \to Y$

- $\operatorname{dom}\varphi = X$
- $\forall \varepsilon > 0$, $\exists N_\varepsilon \geq 0$, $X_\varepsilon \subset X$, $\mu(X) > 1-\varepsilon$,

(*) $\forall x,y \in X_\varepsilon$, $y \in \mathrm{Orb}_T(x)$,
$\|\vec{T}(x,y)\| \geq N_\varepsilon \Rightarrow \|M\vec{T}(x,y) - \vec{S}(\varphi(x),\varphi(y))\|$
$\leq \varepsilon \|\vec{T}(x,y)\|$

$y = T^k x$

(*) $\forall x \in X_\varepsilon$, $\forall k \in \mathbb{Z}^d$ s.t. $T^k x \in X_\varepsilon$,
$\|k\| \geq N_\varepsilon \Rightarrow \|Mk - \underbrace{\vec{S}(\varphi(x), \varphi(T^k x))}_{\text{"partial" cocycle}}\| \leq \varepsilon \|k\|$

$\underline{k'} \in \mathbb{Z}^d$
$S^{k'}\varphi(x) = \varphi(T^{\underline{k}} x)$

if $T \overset{M_1}{\rightsquigarrow} S$, $S \overset{M_2}{\rightsquigarrow} U$, then $T \overset{M_2 M_1}{\rightsquigarrow} U$

e.g. : $d=1$ $(\mathbb{Z}^d = \mathbb{Z})$, $M=(2)$

$$k' \simeq 2k$$

$$\varphi(\mathrm{Orb}_T(x)) \simeq \mathrm{Orb}_{S^2}(\varphi(x)) \quad \text{"} \tfrac{1}{2}\, \mathrm{Orb}_S(\varphi(x)) \text{"}$$

Proposition : $\mathrm{comp}(\varphi) \; \big(= \nu(\mathrm{rng}\,\varphi)\big) = \dfrac{1}{|\det M|}$

1) $|\det M| \geq 1$ (invertible)

2) φ is an OE $\iff |\det M| = 1$

if $|\det M| = 1$, $\quad T \overset{M}{\rightsquigarrow} S \iff S \overset{M^{-1}}{\rightsquigarrow} T$

[I_d - Kak eq is an equivalence relation

Theorem : $T \in \mathrm{Aut}(X,\mu)$, $S \in \mathrm{Aut}(Y,\nu)$, $\underbrace{m \in \mathbb{R}}_{M=(m)}$, $\underbrace{|m| \geq 1}_{|\det M| \geq 1}$

$\mathrm{sgn}(m)$ its sign

$$T \overset{m}{\rightsquigarrow} S \iff \left(T^{\mathrm{sgn}(m)}\right)_A \simeq S_B \text{ for } A \subset X, B \subset Y,$$
$(T \overset{(m)}{\rightsquigarrow} S)$
$$\frac{\nu(B)}{\mu(A)} = \frac{1}{|m|}$$

Consequences • the eq rel generated by $\overset{m}{\rightsquigarrow}$ for $m \geq 1$ is Kak equivalence

• ———————————— $|m| \geq 1$, flip-Kak equivalence

• 1-Kak eq = even Kak equivalence

Proof: WLOG $m \geq 1$ $\quad \left(\overleftarrow{T}^{\geq 1}(x,y) = -\vec{T}(x,y) \right)$

$\boxed{\Longrightarrow}$ $T \overset{m}{\rightsquigarrow} S$

consider φ SOE, $\varepsilon \in]0,1[$, N_ε, X_ε

take $A \subset X_\varepsilon$ such that $A, T(A), \dots, T^{N_\varepsilon - 1}(A)$ are pairwise disjoint.

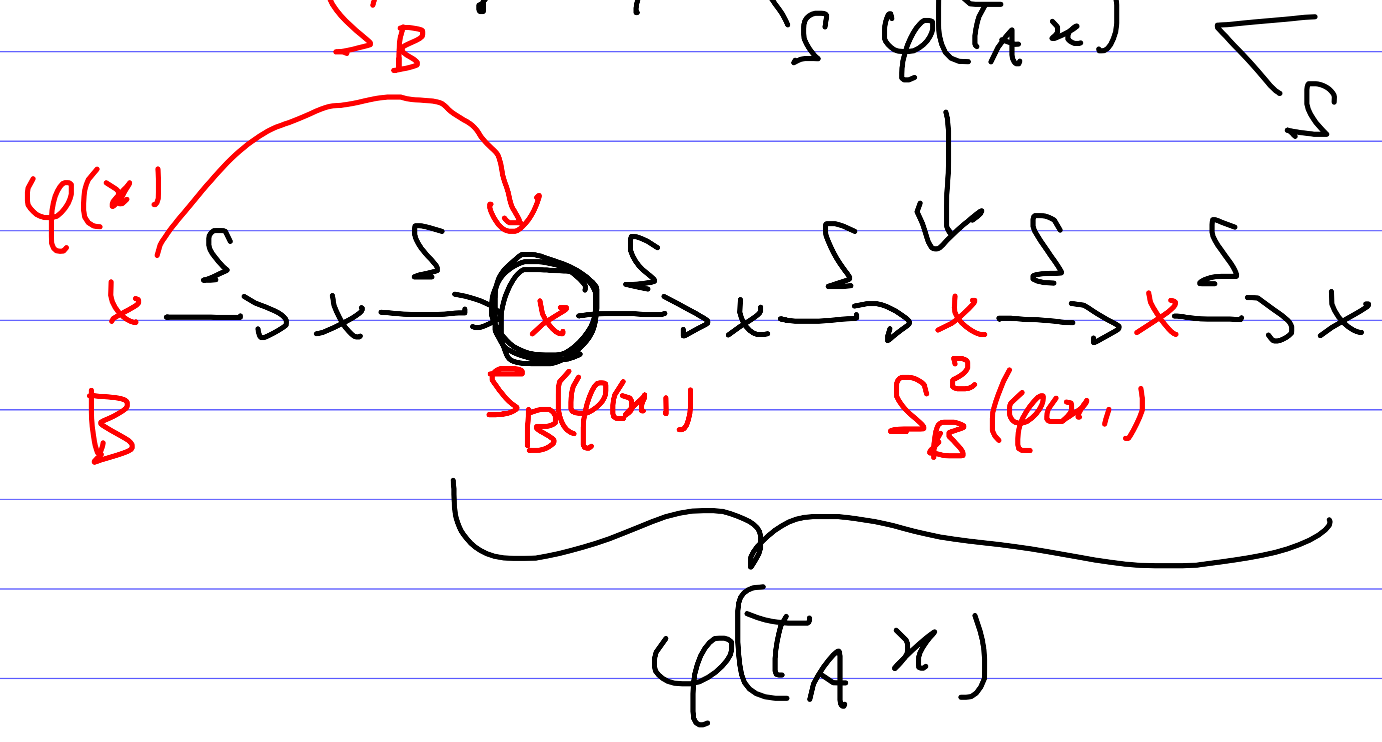
$x \in A$, $\vec{T}(x, T_A(x)) \geq N_\varepsilon$

then $\left| \underbrace{m}_{\geq 1} \underbrace{\vec{T}(x, T_A(x))}_{>0} - \vec{S}(\varphi(x), \varphi(T_A(x))) \right| \leq \underset{0<\cdot<1}{\varepsilon} \left| \vec{T}(x, T_A(x)) \right|$

then $\underbrace{\vec{S}(\varphi(x), \varphi(T_A x)) > 0}$

$$\boxed{\varphi\left(\mathrm{Orb}_{T_A}(x)\right) = \mathrm{Orb}_{S_B}(\varphi(x))}$$

$\underset{j<0}{\varphi(T_A^j x)} <_S \varphi(x) <_S \varphi(T_A x) <_S \varphi(T_A^2 x) <_S \varphi(T_A^i x)$ $\quad B = \varphi(A)$ $\quad i \geq 2$



$$\varphi(T_A x) = S_B(\varphi(x))$$

∎

Theorem (Nadler) if $T \overset{M}{\rightsquigarrow} S$, then $h(S) = \dfrac{h(T)}{|\det M|}$

$d = 1$: consequence of Abramov's formula and "$\mathrm{comp}(\varphi) = \dfrac{1}{|\det M|}$"