# Introduction to Kakutani equivalence (2/2)

Induced map : $T_A(x) = T^{r_A(x)} x$

Kakutani equivalence : $T_A$ isomophic $S_B$ for some $A \subset X$, $B \subset Y$

even k.e : $\quad + \quad \mu(A) = \nu(B)$

equivalence relations

<u>Goal</u> : generalizing to $\mathbb{Z}^d$-actions for $d \geq 2$

<u>Definition</u> : (M-Ke) $M$ $d \times d$ real matrix, $T : \mathbb{Z}^d \curvearrowright (X, \mu)$
$S : \mathbb{Z}^d \curvearrowright (Y, \nu)$

$T \overset{M}{\rightsquigarrow} S$ iff $\exists$ a $\underbrace{\text{SOE}}_{\text{stable orbit equivalence}}$ $\varphi$ between $T$ and $S$ s.t.

1) $\operatorname{dom}(\varphi) = X$

2) $\forall \varepsilon > 0$, $\exists N_\varepsilon \geq 0$, $X_\varepsilon$ $\mu(X_\varepsilon) > 1 - \varepsilon$,

(*) $\forall x, y \in X_\varepsilon$ in the same $T$-orbit,
$\|\vec{T}(x,y)\| \geq N_\varepsilon \implies \|M\vec{T}(x,y) - \vec{S}(\varphi(x), \varphi(y))\| \leq \varepsilon \|\vec{T}(x,y)\|$

(maybe see Cantrell)

$\vec{T}(x,y) = k \in \mathbb{Z}^d$ s.t. $T^k x = y$ (unique because free actions)
+ ergodic, pmp

$\|.\|$ norm on $\mathbb{R}^d$ : $\|v\| = \max(|v_i|)$

SOE $\varphi$ : $U := \operatorname{dom}(\varphi)$, $V = \operatorname{rng}(\varphi)$
measure isomorphism $(U, \mu_U) \to (V, \nu_V)$

for $\mu$-a.e. $x \in X$, $\varphi(\operatorname{Orb}_T(x) \cap U) = \operatorname{Orb}_S(\varphi(x)) \cap V$

($d = 1$ : $\varphi$ is OE between $T_U$ and $S_V$ )

" $y = T^n x$ "

$$(*) = \begin{cases} \forall x \in X_\varepsilon,\ \forall n \in \mathbb{Z}^d \text{ s.t. } T^n x \in X_\varepsilon \\ \|n\| \geq N_\varepsilon \Rightarrow \|\underbrace{Mn - \vec{S}(\varphi(x), \varphi(T^n x))}\| \leq \varepsilon \|n\| \end{cases}$$

for $d = 1$, $M = (m)$

ex: $m = 2$

$$2n \simeq \vec{S}(\varphi(x), \varphi(T^n x))$$

$$\varphi(\mathrm{Orb}_T(x)) \simeq \mathrm{Orb}_{S^2}(\varphi(x)) \quad \text{" } \tfrac{1}{2}\, \mathrm{Orb}_S(x) \text{ "}$$

**Proposition**: if $T \overset{M}{\rightsquigarrow} S$ and $\varphi$ is an associated SOE, then $M$ is invertible and $\mathrm{comp}(\varphi) = 1/|\det(M)|$

$$\mathrm{comp}(\varphi) = \frac{\nu(\mathrm{rng}(\varphi))}{\mu(\mathrm{dom}\,\varphi)} = \nu(\mathrm{rng}\,\varphi)$$

**Proof**: $\varepsilon > 0$, $N_\varepsilon, X_\varepsilon$ as in the def

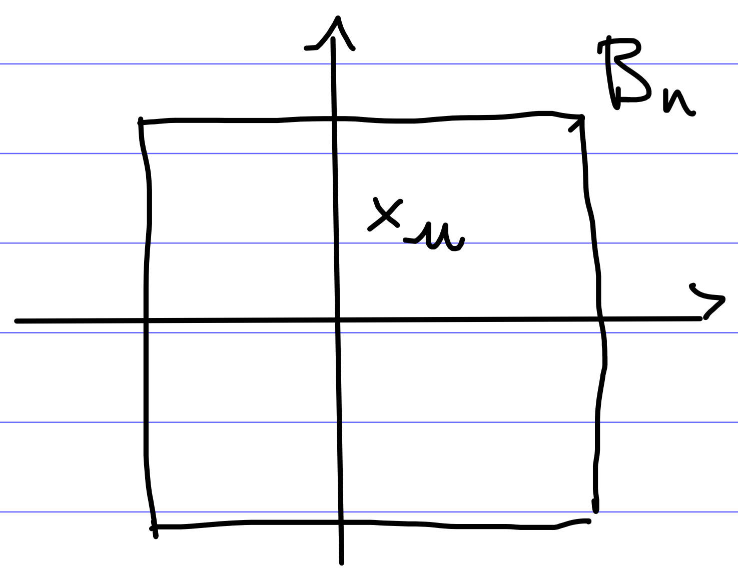
$$B_n = \{ v \in \mathbb{R}^d : \|v\| \leq n \}$$

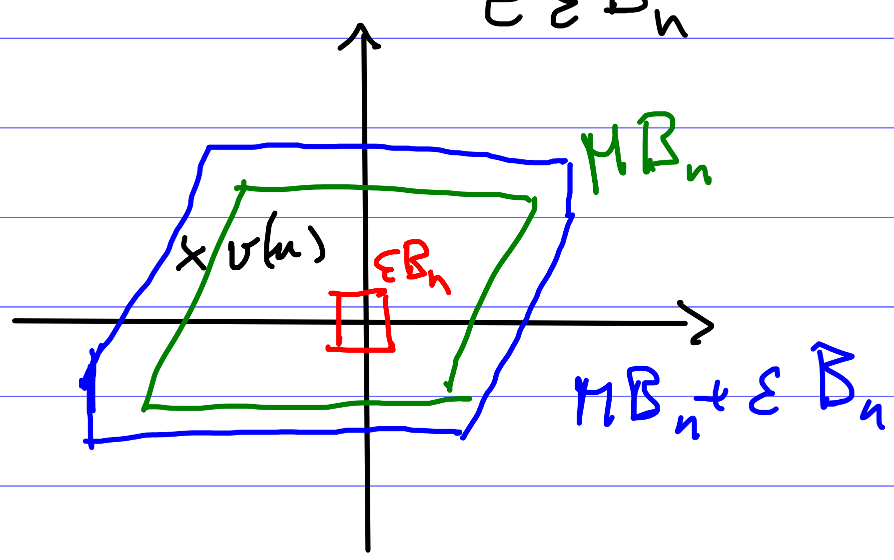
$$F_n = B_n \cap \mathbb{Z}^d$$

Fix $x \in X_\varepsilon$. If $u \in B_n$ and $\begin{cases} T^u x \in X_\varepsilon \\ \|u\| \geq N_\varepsilon \end{cases}$

then $v(u) := \vec{S}(\varphi(x), \varphi(T^u x))$ satisfies $\|Mu - v(u)\| \leq \varepsilon \|u\|$

in particular $v(u) = \underbrace{(v(u) - Mu)}_{\in \varepsilon B_n} + \underbrace{Mu}_{\in M B_n} \in MB_n + \varepsilon B_n$



$$MB_n + \varepsilon B_n \subset \alpha(\varepsilon) MB_n \qquad \alpha(\varepsilon) \underset{\varepsilon \to 0}{\searrow} 1$$

$v(\cdot)$ is injective (actions are free)

$$\left| \left\{ u \in \underbrace{B_n \setminus B_{N_\varepsilon}}_{1} : T^u x \in X_\varepsilon \right\} \right| \leq \left| \left\{ v \in \overbrace{\alpha(\varepsilon) MB_n \cap \mathbb{Z}^d}^{G_n} : S^v(\varphi(x)) \in \varphi(X_\varepsilon) \right\} \right| \quad (2)$$

$(F_n)$ and $(G_n)$ are Følner sequences

By erg theorem,
$$(1) \underset{n \to +\infty}{\sim} |F_n| \mu(X_\varepsilon)$$
$$(2) \underset{n \to +\infty}{\sim} |G_n| \nu(\varphi(X_\varepsilon))$$

$$|G_n| \sim |\det(\alpha(\varepsilon) M)| \, |F_n|$$

$$\longrightarrow \quad \mu(X_\varepsilon) \leq |\det(\alpha(\varepsilon) M)| \, \nu(\varphi(X_\varepsilon))$$

$\varepsilon \to 0$ :
$$\underline{\mu(X) \leq |\det(M)| \, \nu(\varphi(X))}$$

same idea for $\geq$. $\blacksquare$

Consequences :

1) $|\det(M)| \geq 1$
2) $\varphi$ is an OE $\iff |\det(M)| = 1$
3) if $|\det(M)| = 1$, then $T \overset{M}{\rightsquigarrow} S \iff S \overset{M^{-1}}{\rightsquigarrow} T$
4) $I_d$ - Kakutani equivalence is an equivalence relation

(reflexive : OK
symmetric : by 3)
transitive :
($T \overset{M_1}{\rightsquigarrow} S$ and $S \overset{M_2}{\rightsquigarrow} U$
then $T \overset{M_2 M_1}{\rightsquigarrow} U$))

**Theorem:** $d = 1$

$T \in Aut(X,\mu)$, $S \in Aut(Y,\nu)$

$m \in \mathbb{R}$, $|m| \geq 1$, $sgn(m) :=$ sign of $m$ $\qquad (M = (m))$

$T \overset{m}{\rightsquigarrow} S \iff \left(T^{sgn(m)}\right)_A$ is isomorphic to $S_B$ with $A \subset X$, $B \subset Y$ s.t. $\dfrac{\nu(B)}{\mu(A)} = \dfrac{1}{|m|}$

**Consequences**

1) the eq rel generated by $m$-Kak eq for $m \geq 1$ is Kakutani equivalence

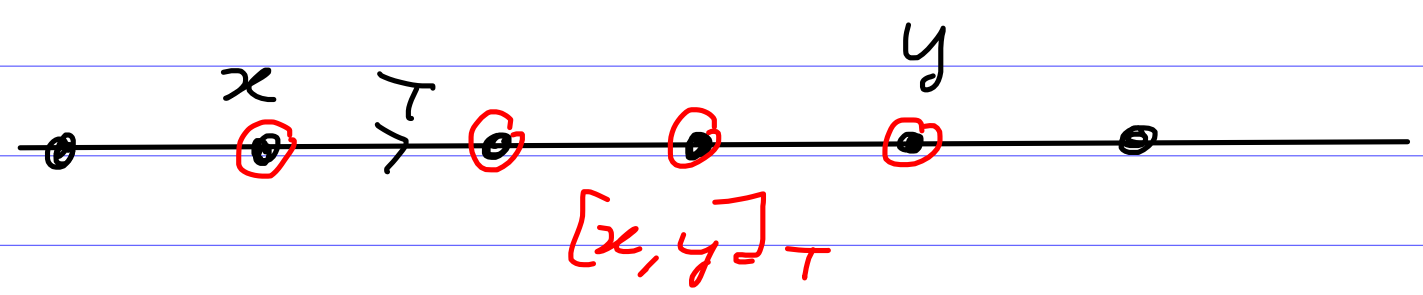
2) ______ for $m \geq 1$ and $m \leq -1$ is flip-Kak equ

3) $1$-Ke $=$ even Ke
($I_d$-Ke for $d=1$)

notation: $x <_T y \iff \exists n > 0,\ T^n x = y$

$[x,y]_T = \{ z \in Orb_T(x) : x \leq_T z \leq_T y \}$ $\quad x \leq_T y$

$y \leq_T z \leq_T x$ $\quad y \leq_T x$




**Proof:** Assume $m \geq 1$ $\qquad \left( -\vec{T}(x,y) = \vec{T}^{-1}(x,y) \right)$

$\boxed{\Longrightarrow}$ Assume $T \overset{m}{\rightsquigarrow} S$. Let $\varphi$ SOE given by the definition.
Let $0 < \varepsilon < 1$ and $N_\varepsilon, X_\varepsilon$ given by the def

Find $A \subset X_\varepsilon$ s.t $\begin{cases} r_{T,A} \geq N_\varepsilon \text{ on } A. \\ \mu(A) > 0 \end{cases}$ $\qquad B = \varphi(A)$

$\mathrm{comp}(\varphi_{|A}) = \mathrm{comp}(\varphi) = 1/m$, where $\mathrm{comp}(\varphi_{|A}) = \frac{\nu(B)}{\mu(A)}$

<u>goal</u> : $T_A = \varphi^{-1} S_B \varphi$

$x \in A$.

$\to x \in X_\varepsilon$ and $T_A(x) \in X_\varepsilon$

$\vec{T}(x, T_A(x)) \geq N_\varepsilon$, where $\vec{T}(x, T_A(x)) = r_{T,A}(x)$

then $\left| m \vec{T}(x, T_A(x)) - \vec{S}(\varphi(x), \varphi(T_A(x))) \right| \leq \varepsilon \vec{T}(x, T_A(x))$

$\vec{S}(\varphi(x), \varphi(T_A(x))) \geq \underbrace{(m - \varepsilon)\, \vec{T}(\ \ ,\ \ )}_{>0}$ $\qquad m \geq 1$, $\varepsilon < 1$, $\vec{T} > 0$

$\left(\varphi(T_A^i(x))\right)_{i \in \mathbb{Z}}$ is $<_S$-increasing

so in $\{\varphi(T_A^i(x)) : i \in \mathbb{Z}\}$, the $<_S$-least element $>_S \varphi(x)$ is $\varphi(T_A(x))$

$\{\varphi(T_A^i(x)) : i \in \mathbb{Z}\} = \varphi(\mathrm{Orb}_{T_A}(x)) = \mathrm{Orb}_{S_B}(\varphi(x))$ the $<_S$-least element $>_S \varphi(x)$ is $S_B(\varphi(x))$

$\to \varphi(T_A(x)) = S_B(\varphi(x))$ $\qquad$ this holds for every $x \in A$

$\to \varphi T_A = S_B \varphi$

$\boxed{\Leftarrow}$ Assume $T_A \simeq S_B$ with $\frac{\nu(B)}{\mu(A)} = \frac{1}{m}$.

to show that $T \overset{m}{\rightsquigarrow} S$ :

1) find a SOE $\varphi$ of full domain

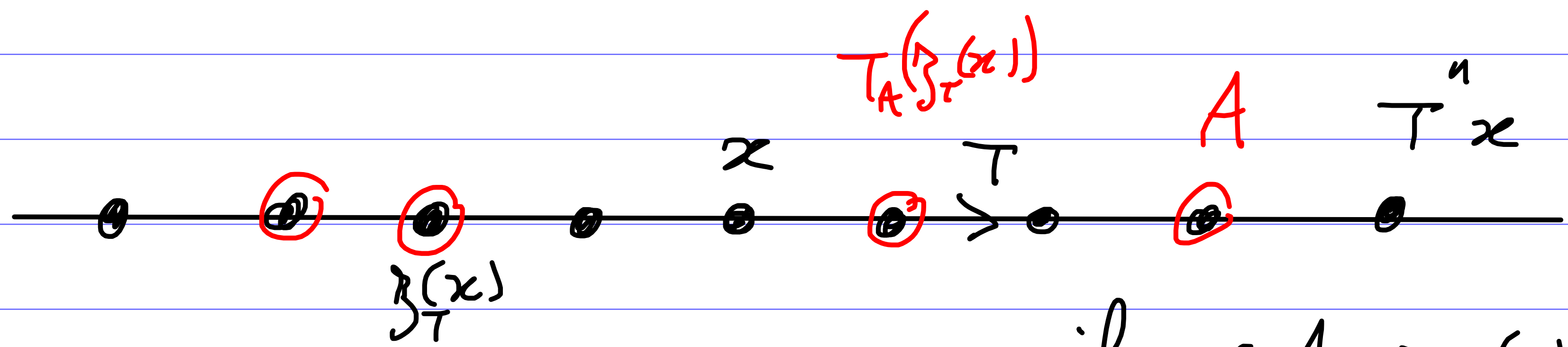
2) try to write $\vec{S}(\varphi(x), \varphi(T^n x)) = mn +$ terms

1) $\exists\, \varphi : A \to B$ measure isomorphism s.t. $T_A = \varphi^{-1} S_B \varphi$

<u>Furman</u> : $\varphi$ can be extended to a $\underbrace{\text{SOE}}_{\text{between } T \text{ and } S}$ of $\underbrace{\text{full domain}}_{m \geq 1}$

then we have a SOE $\varphi$ of full domain

2) if $x \in X$, $\zeta_T(x) = \max_{\leq_T} \{ \zeta \in A : \zeta \leq_T x \}$




$k_n(x) = 3$

if $x \in A$, $\zeta_T(x) = x$

$$k_n(x) = \Big| \underbrace{A \cap [\zeta_T(x), T^n x]_T}_{\zeta_T(x),\ T_A(\zeta_T(x)),\ \dots,\ \underbrace{T_A^{k_n(x)-1}(\zeta_T(x))}_{\zeta_T(T^n x)}} \Big|$$

$$\vec{S}(\varphi(x), \varphi(T^n x)) = \underbrace{\vec{S}(\varphi(x), \varphi(\zeta_T(x))) + \vec{S}(\varphi(\zeta_T(T^n x)), \varphi(T^n x))}_{\text{bound on large set}} + \underbrace{\sum_{i=0}^{k_n(x)-2} \vec{S}\left(\varphi(T_A^i(\zeta_T(x))), \varphi(T_A^{i+1}(\zeta_T(x)))\right)}_{\text{erg th}}$$

$\Rightarrow \underset{n \to +\infty}{\sim} mn$

$\varphi T_A = S_B \varphi$ then $\vec{S}\left(\varphi(T_A^i(\zeta_T(x))), \varphi(T_A^{i+1}(\zeta_T(x)))\right)$

$= \vec{S}\left(S_B^i(\varphi(\zeta_T(x))), S_B^{i+1}(\varphi(\zeta_T(x)))\right)$

$$= r_{S,B}\left( S_B^i \left( \varphi(\mathfrak{Z}_T(x)) \right) \right)$$

$S_B$ preserves $\nu_B$

$r_{S,B}$ is $\nu_B$- intégrable

then erg th:
$$\sum_{i=0}^{k_n(x)-2} (\dots) \underset{n\to+\infty}{\sim} k_n(x) \int r_{S,B}\, d\nu_B \qquad \text{if } k_n(x) \to +\infty$$
$$= \frac{k_n(x)}{\nu(B)}$$

$$k_n(x) = 1 + \sum_{i=1}^{n} \mathbb{1}_{T^i x \in A} \underset{n\to+\infty}{\sim} n\, \mu(A)$$

then
$$\sum_{i=0}^{k_n(x)-2} (\dots) \underset{n\to+\infty}{\sim} n \frac{\mu(A)}{\nu(B)} = m n$$

Let $\alpha > 0$. $\exists N(x)$, $\forall n \geq N(x)$,
$$(1-\alpha) m n \leq \sum_{i=0}^{k_n(x)-2} (\dots) \leq (1+\alpha) m n$$

$$X_\varepsilon := \{ N(x) \leq N_\varepsilon \} \cap \left\{ \left| \vec{S}\left(\varphi(x), \varphi(\mathfrak{Z}_T(x))\right) \right| \leq \alpha m N_\varepsilon \right\}$$

and I choose $N_\varepsilon$ s.t. $\mu(X_\varepsilon) \geq 1-\varepsilon$

If $x \in X_\varepsilon$, $n \geq N_\varepsilon$ $T^n x \in X_\varepsilon$,
$$\left| \vec{S}\left(\varphi(x), \varphi(T^n x)\right) - m n \right| \leq 2\alpha m N_\varepsilon + \alpha m n \leq 3 \alpha m n$$

$\to \alpha = \varepsilon/3$

$n \leq -N_\varepsilon$ $\quad \vec{S}\left(\varphi(x), \varphi(T^n(x))\right) = \vec{S}\left(\varphi(y), \varphi(T^{|n|} y)\right)$

with $y = T^n x$, $|n| \geq N_\varepsilon$

$\boxed{}$

$x \in A \cap \gamma^{-1}A$

on $A$, $\mu(A) > 0$, $c(\gamma, x) = M(\gamma)$ $M$ automorphism of $G$

$\| c(\gamma, x) M(\gamma)^{-1} \| \leq \varepsilon \|\gamma\|$ ?