Exhaustive OE and sizes, after Heicklen

$\Gamma = \bigcup_n \Gamma_n$

$\Gamma_n \leq \Gamma_{n+1}$, Γ_n finite.

Two pmp free Γ-act° α, β on (X, μ) are exhaustively OE if $\exists\, \psi \in \mathrm{Aut}(X, \mu)$

$$\forall x \in X,\ \forall n \in \mathbb{N},\quad \psi(\alpha(\Gamma_n)x) = \beta(\Gamma_n)\psi(x)$$

Last time: defined $G_\alpha = \{T \in [R_\alpha],\ T$ is exhaustive OE from α to itself: $\forall n\ \forall x\ T(\alpha(\Gamma_n)x) = \alpha(\Gamma_n)T(x)\}$

associated a size $\rho^{Ex}_\alpha(T) = \underbrace{\rho^0_\alpha(T)}_{d_u(\alpha, \alpha\cdot T)} + d_u(T, G_\alpha)$

Proved that α, β are ρ-approx in the same orbit iff $\exists T \in [R]$ st αT & β have exhaustively same orbits.

arrangements: $R = R_\alpha = R_\beta$ $\qquad \to \exists T_i$ d_α-Cauchy, $\alpha T_i \to \beta$

Today: • Another point of view on ρ

(• if $A :=$ space of arrangements, describe $A \times [R] /\!/ [R]$ by conjugacy)

On A we have another metric

(Heicklen) $$d(\alpha, \beta) = d_u(\alpha, \beta) + \left(1 - \sup\{\mu(B) : R_{\alpha(\Gamma_n)}\restriction B\times B = R_{\beta(\Gamma_n)}\restriction B \times B\}\right)$$

$$\rho_\alpha(T) := d(\alpha, \alpha\cdot T) \leq \rho^{Ex}_\alpha(T)$$

if $T(x) = \underbrace{U(x)}_{\in G_\alpha}$ $\forall x \in B$

then $R_{\alpha(\Gamma_n)\restriction B\times B} = R_{\alpha T(\Gamma_n)\restriction B \times B}$

We will show that $\underline{\rho_\alpha = \rho_\alpha^{Ex}}$ (Kammeyer–Rudolph)

Lemma (flattening lemma, Heicklen)

Take $\alpha, \beta \in A$. Let $[[\alpha \to \beta]] = \{\varphi \in [[R]]$ st $\varphi \times \varphi(R_{\alpha(\Gamma_n)}\restriction \mathrm{dom}\,\varphi^2) = R_{\beta(\Gamma_n)}\restriction \mathrm{rng}\,\varphi^2\}$

Then every element of $[[\alpha \to \beta]]$ extends to a bijection $\in [[\alpha \to \beta]]$

Why does this imply $\rho_\alpha^{Ex} \leq \rho_\alpha$

Take α, T with $\sup\{\mu(B) : \overbrace{R_{\alpha(\Gamma_n)}\restriction B^2 = R_{\alpha T(\Gamma_n)}\restriction B^2\ \forall n}^{T\restriction B \text{ is in } [[\alpha \to \alpha]]}\} > \delta$

$R_{\alpha T(\Gamma_n)}\restriction B^2 = ((T \times T)R_\alpha)\restriction B^2$

Then fix B st $\mu(B) > \delta$ & ______

Apply lemma to $\alpha' = \alpha$, $\beta' = \alpha$ $\quad \varphi = T_{\restriction B} \rightsquigarrow$ extension $U \in G_\alpha$

$T = U$ on B : $d_u(T, G_\alpha)$ $\underline{< 1-\delta}$

$R_{\alpha T} = T \times T(R_\alpha)$

$\forall x, y \in B,\ y \in \alpha(\Gamma_n)\, x$ iff $T(y) \in \alpha(\Gamma_n)\, T(x)$

$\Downarrow$

$T_{\restriction B} \in [[\alpha \to \alpha]]$

Lemma (flattening lemma, Hiedler)

Take $\alpha, \beta \in A$. Let $[[\alpha \to \beta]] = \{ \varphi \in [[R]]$ st $\varphi \times \varphi(R_{\alpha(\Gamma_n) \restriction \operatorname{dom} \varphi^2}) = R_{\beta(\Gamma_n) \restriction \operatorname{im} \varphi^2}$

Then every element of $[[\alpha \to \beta]]$ extends to a bijection $\in [[\alpha \to \beta]]$.

Proof: $\Gamma_0 = \{1\}$

$\varphi_0 = \varphi$

inductively build φ_n st $\operatorname{dom} \varphi_n$ $\alpha(\Gamma_n)$-inv

$\operatorname{im} \varphi_n$ $\beta(\Gamma_n)$-inv

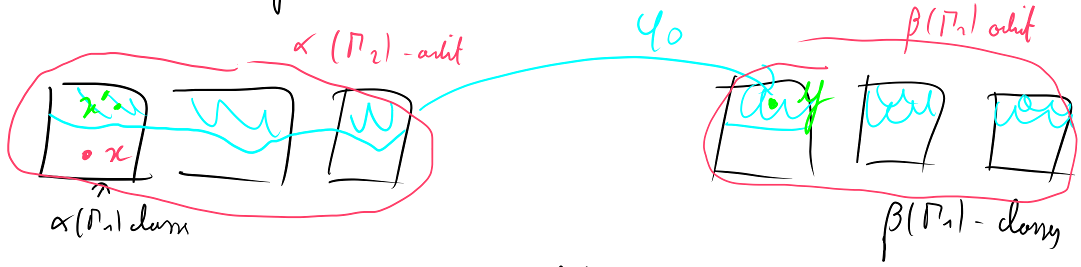
φ_{n+1} extends φ_n & $\varphi_n \times \varphi_n(R_{\alpha(\Gamma_k) \restriction (\operatorname{dom} \varphi_n)^2}) = R_{\beta(\Gamma_k) \restriction \operatorname{im} \varphi_n^2}$ $\boxed{\forall k}$

$\varphi = \bigcup_n \varphi_n \to \operatorname{dom} \varphi$ is R-inv $\to \operatorname{dom} \varphi = X$ by ergodicity & φ is as wanted.

$\varphi_n \in [[\alpha \to \beta]]$

inductive step: at each step, $\operatorname{dom} \varphi_{n+1} = \alpha(\Gamma_{n+1})$-saturation of $\operatorname{dom} \varphi_n$

$0 \to 1$: For every $x \in \operatorname{dom} \varphi_1 \setminus \operatorname{dom} \varphi_0$



$\varphi_0 \restriction \alpha(\Gamma_1)\, x$ is a bijection from

$\alpha(\Gamma_1)\, x \cap \operatorname{dom} \varphi_0$ to $\beta(\Gamma_1)\, y \cap \operatorname{im} \varphi_0$

where y is the φ_0-image of any element in $\alpha(\Gamma_1)\, x \cap \operatorname{dom} \varphi_0$

(since $\varphi_0 \in [[\alpha \to \beta]]$)

so φ_0 can be extended to a bijection $\varphi_1 : \alpha(\Gamma_1)\, x \to \beta(\Gamma_1)\, y$

Why is φ_1 still in $[[\alpha \to \beta]]$ $\quad n = 0, n = 1$: OK

$n \geq 2$

$x' \in \alpha(\Gamma_n)\, x$

$\operatorname{dom} \varphi_0 \qquad \operatorname{dom} \varphi_0$

$\to x'_1 \in \alpha(\Gamma_1)\, x' \cap \operatorname{dom} \varphi_0 \qquad \to x_1 \in \alpha(\Gamma_1)\, x \cap \operatorname{dom} \varphi_0$

since $\varphi_0 \in [[\alpha \to \beta]]$ and $x'_1 \in \alpha(\Gamma_n) x_1$

$$\begin{array}{ccc} \varphi_0(x'_1) & \in \beta(\Gamma_n) & \varphi_0(x_1) \\ \wr \beta(\Gamma_1) & & \wr \beta(\Gamma_1) \\ \varphi_1(x') & & \varphi_1(x) \end{array}$$

For the general case $\underset{n \to n+1}{}$: same idea, replacing points by $\alpha(\Gamma_n)$ orbits $\overline{\beta(\Gamma_n)}$

When entering $\alpha(\Gamma_n) x \to \beta(\Gamma_n) y$

do it equivariantly $\alpha(\Gamma_n) \to \beta(\Gamma_n)$

$\square$

Define a rearrangement as a couple (α, T), $\alpha \in A$, $T \in [R]$

To a rearrangement we can associate a random permutation of Γ : $\sigma_{\alpha,T} = (\sigma^x_{\alpha,T})_{x \in X}$

$T \in [R_\alpha]$ so T acts by permutation on every $[x]_{R_\alpha}$

& we have a bij $\varphi^\alpha_x : \Gamma \to [x]_{R_\alpha}$
$\gamma \mapsto \alpha(\gamma) x$

$$\sigma^x_{\alpha,T} = (\varphi^\alpha_x)^{-1} T_{\restriction [x]_{R_\alpha}} \varphi^\alpha_x$$

in other words,

$$T \alpha(\gamma) x = \alpha(\sigma^x_{\alpha,T}(\gamma)) \cdot x$$

Let $\nu_{\alpha,T}$: law of $(\sigma^x_{\alpha,T})_{x \in X}$: pushforward of μ by the map $x \mapsto \sigma^x_{\alpha,T} \in \mathrm{Sym}(\Gamma)$

$\underbrace{A \times [R]}_{\text{endowed with}} \curvearrowleft [R]$ by diagonal conj:

$$(\alpha, U) \cdot T = (\alpha T, T^{-1} U T)$$

$$d((\alpha_1, U_1), (\alpha_2, U_2)) = d_u(\alpha_1, \alpha_2) + d_u(U_1, U_2).$$

Thm (Hindley) : Γ loc finite, then $A \times [R] /\!/ [R] \underset{\text{homeo}}{\cong} \{\nu_{\alpha,T} : (\alpha, T) \in A \times [R]\}$

on Prob $(\mathrm{Sym}(\Gamma))$ topology :

$\nu_n \to \nu$ iff $\forall \varphi : \Gamma \to \Gamma$ partial injective with finite domain

$$\nu_n(\{\sigma \in \mathrm{Sym}(\Gamma) : \sigma_{\restriction \mathrm{dom}\,\varphi} = \varphi\}) \to \nu(\{\text{———}\})$$

Proof : $(\alpha, T) \mapsto \nu_{\alpha,T}$ is continuous, $[R]$-invariant

so it quotients down to a continuous map $A \times [R] /\!/ [R] \to \mathrm{Prob}(\mathrm{Sym}(\Gamma))$

We have have to show that if $\nu_{\alpha,T}$ and $\nu_{\beta,U}$ are close

then we can conjugate (α, T) close to (β, U).

Given $\varphi : \Gamma \to \Gamma$ partial with finite domain

$$X_{\alpha,\varphi,T} := \{x \in X : T(\alpha(g) x) = \alpha(\varphi(g)) x \quad \forall g \in \mathrm{dom}\,\varphi\}$$

Fix (α, T) once and for all. Let $\mathcal{I}(\Gamma_n,\Gamma) := \{\text{injections } \Gamma_n \hookrightarrow \Gamma\}$

a basic nbhd of $\nu_{\alpha,T}$ is given by

$$\Big\{\nu_{\beta,U} : \underbrace{\sum_{\iota\in \mathcal{I}(\Gamma_n,\Gamma)} \big|\mu(X_{\alpha,\iota,T}) - \mu(X_{\beta,\iota,U})\big| < \varepsilon}_{(*)}\Big\}$$

Our goal is to show that if (β, U) satisfies $(*)$ it can be conjugated ε-close to α.

Consider a maximal element $\varphi \in [[R_T]]$ with $\alpha(\Gamma_n)$-inv domain, which is $\alpha(\Gamma_n) \to \beta(\Gamma_n)$ equivariant

and $\forall x \in \operatorname{dom}\varphi$, $\forall g \in \Gamma_n$

$$\boxed{\varphi(T\alpha(g)x) = U(\beta(g)\varphi(x))}$$

$\hookrightarrow \varphi(\operatorname{dom}\varphi \cap X_{\alpha,\iota,T}) = \operatorname{rng}\varphi \cap X_{\beta,\iota,U} \quad \forall \iota \in \mathcal{I}(\Gamma_n,\Gamma)$

Assume by contradiction $\mu\operatorname{dom}\varphi < 1-\varepsilon$

$\to$ there is $\iota \in \mathcal{I}(\Gamma_n,\Gamma)$ such that

$$\mu(X_{\alpha,\iota,T}\setminus \operatorname{dom}\varphi) > 0 \quad \& \quad \mu(X_{\beta,\iota,U}\setminus \operatorname{rng}\varphi) > 0$$

We can then extend φ as follows: find $X_0 \subseteq X_{\alpha,\iota,T}\setminus\operatorname{dom}\varphi$

intersecting every $\alpha(\Gamma_n)$ orbit in one pt exactly

$Y_0 \subseteq X_{\beta,\iota,U}\setminus \operatorname{im}\varphi$

——— $\beta(\Gamma_n)$ orbit in exactly 1 pt

by ergo, there is $\psi_{X_0} \in [[R_T]]$ $\quad \operatorname{dom}\psi \subseteq X_0$, $\operatorname{rng}\psi \subseteq Y_0$

extend ψ in the unique $\alpha(\Gamma_n) \to \beta(\Gamma_n)$ equivariant way $\to \tilde\psi$

$\varphi \sqcup \tilde\psi$ contradicts maximality.

equivariance condition
$$\boxed{\tilde\psi(T\alpha(g)x) = U(\beta(g)\tilde\psi(x))}$$

if $x \in X_0$, OK because $\tilde\psi(\underset{X_{\alpha,\iota,T}}{\underset{\in}{x_0}}) \subseteq Y_0 \subseteq X_{\beta,\iota,U}$

if not use equivariance: write $x = \alpha(\gamma)x'$ where $x' \in X_0$

$$\begin{aligned}
\tilde\psi(T\alpha(g)x) &= \psi(T(\alpha(g\gamma)x')) \\
&= U(\beta(g\gamma)\psi(x')) \\
&= U(\beta(g)\beta(\gamma)\psi(x')) \\
&= U(\beta(g)\psi(\alpha(\gamma)x')) \\
&= U(\beta(g)\psi(x))
\end{aligned}$$

$\square$