Séance 3

Growth of cocycles of Isom. Banach representations

Prop: (de la Salle, Marrakchi 2019)

Let $p > 2$. G compactly generated loc. compact group

For any $\pi : G \to U(H)$ unitary rep and any cocycle $b \in Z^1(G,\pi)$, there exists E an L^p-space, $\overline{\pi} : G \to O(E)$ $\overline{b} \in Z^1(G,\overline{\pi})$ s.t.

$$\| \overline{b}(g) \|_E = \| b(g) \|_H \quad \forall g \in G$$

$$L^2 \subset L^p$$

Thm (Lafforgue) $G = \langle S \rangle$, S compact, $S = S^{-1}$, $|g|_S$ = word metric

If G doesn't have prop. (T), then $\exists\, \pi,\ b \in Z^1(G,\pi)$ such that

$$\sup_{g \in S^n} \| b(g) \| \geq \left(\frac{\sqrt{n}}{2} - 2 \right) \sup_{g \in S^1} \| b(g) \|$$

If G doesn't have prop (T), then $\exists\, \pi : G \to O(E)$ E L^p-space, and $b : G \to E$ s.t.

$$\sup_{|g| \leq n} \| b(g) \| \geq \frac{\sqrt{n}}{2}$$

Thm (P, Lopez-Neumann)

$p > 2$

G w/o FL^p

$\exists\, b, \pi$, $b \in Z^1(G,\pi)$

$$\sup_{|g| \leq n} \| b(g) \| \geq n^{1/p}$$

So the theorem is only relevant when G has prop. (T)

Examples:

- $G = Sp(n,1)$ $n \geq 2$
  no FL^p, for $p > 4n+1$
- F_4^{-20}
- hyp. groups that have prop. (T)
- random groups.

Given a cocycle $b: G \to E$, $\pi: G \to \mathcal{O}(E)$
Let m_G be a Haar measure (left)
$\mu \in Prob(G)$

$$\frac{d\mu}{dm_G} = \frac{1}{m_G(S)} 1_S$$

$$\|b\|_{L^p(\mu^{*n})} = \left(\int_G \|b(g)\|^p \, d\mu^{*n}(g) \right)^{1/p}$$

$$\|b\|_{L^p(\mu^{*n})} \leq \sup_{|g| \leq n} \|b(g)\|$$

E real Banach space

E^* continuous dual

(Chidume)
Def: E is smooth if there exists
$*: E \to E^*$ uniquely determined by
$x \mapsto x^*$
the conditions $\|x^*\|_{E^*} = \|x\|_E$ and
$\langle x, x^* \rangle = \|x\|^2$ $x \in E$

$\langle x, f \rangle = f(x)$, $f \in E^*$, $x \in E$

$*$ is linear iff E is a Hilbert space

$$(\lambda x)^* = \lambda x^* \qquad \lambda \in \mathbb{R}$$

$E = L^p(\Omega,\mu)$ $\qquad \frac{1}{q}+\frac{1}{p} = 1$

$$* : L^p(\Omega,\mu) \longrightarrow L^q(\Omega,\mu)$$
$$f \longmapsto Mf$$

$$Mf(x) = \operatorname{sgn}(f(x))\, |f(x)|^{p-1} \, \|f\|_p^{2-p}$$

Let's denote by

$$x^{*r} = \begin{cases} \|x\|^{r-2} x^* \in E^* & x \neq 0 \\ 0 & x = 0 \end{cases}$$

PROP:

E is (p, C_E)- unif convex Banach space
smooth
For all $x, y \in E$:

$$\|x+y\|^p \geq \|x\|^p + p\langle y, x^{*p}\rangle + C_E\|y\|^p$$

DEF: $b \in Z^1(G,\pi)$ is μ-harmonic if
$$\int_G b(g)\, d\mu(g) = 0 = b(1)$$

If $b \in Z^1(G,\pi)$

$$b(gh) = b(g) + \pi(g)\, b(h)$$

$$b(1) = 2b(1) \Rightarrow b(1) = 0$$

$$\int_G b(gh)\,d\mu(h) = b(g)$$

$$b(1) = b(g) + \pi(g)\, b(g^{-1})$$

$$\pi(g)\, b(g^{-1}) = -b(g)$$

PROP: E smooth, (p, C_E)-unif. convex Banach space
$\pi : G \to \mathcal{O}(E)$ isom. rep. and $b \in Z^1(G,\pi)$
If b is μ-harmonic, then:
G unimodular

$$\|b\|_{L^p(\mu^{*n})} \geq \left(C_E(n-1)+1\right)^{1/p} \|b\|_{L^p(\mu)}$$

Pf: For any two "nice" measures $\mu, \nu \in \mathrm{Prob}(G)$

$$\|b\|^p_{L^p(\mu * \nu)} = \int_G \int_G \|b(gh)\|^p \, d\mu(g)\, d\nu(h)$$

$$= \int_G \int_G \|\pi(g^{-1}) b(g) + b(h)\|^p \, d\mu(g)\, d\nu(h)$$

$$= \int_G \int_G \|-b(g^{-1}) + b(h)\|^p \, d\mu(g)\, d\nu(h)$$

μ symmetric $\quad i_* \mu = \mu \quad$ $i : G \to G$, $g \mapsto g^{-1}$

$$= \int_G \int_G \|b(h) - b(g)\|^p \, d\mu(g)\, d\nu(h)$$

$$\geq \int_G \int_G \|b(h)\|^p + C_E \|b(g)\|^p - p\langle b(g), b(h)^{*p}\rangle \, d\mu(g)\, d\nu(h)$$

$$\geq C_E \|b\|^p_{L^p(\mu)} + \|b\|^p_{L^p(\nu)} - p\Big\langle \underbrace{\int_G b(g)\,d\mu(g)}_{0}, \int_G b(h)^{*p}\, d\nu(h) \Big\rangle$$

Since $\int_G b(g)\,d\mu(g) = 0$

So $\|b\|^p_{L^p(\mu*\nu)} \geqslant C_E \|b\|^p_{L^p(\mu)} + \|b\|^p_{L^p(\nu)}$

$\nu = \mu^{*(n-1)}$

$$\|b\|^p_{L^p(\mu^{*n})} \geqslant C_E \|b\|^p_{L^p(\mu)} + \|b\|^p_{L^p(\mu^{*(n-1)})}$$
$$\geqslant (1 + C_E(n-1)) \|b\|^p_{L^p(\mu)}$$
$$\|b\|_{L^p(\mu^{*n})} \geqslant (1 + C_E(n-1))^{1/p} \|b\|_{L^p(\mu)}$$

Def:
E Banach space
$\pi: G \to \mathcal{O}(E)$ continuous isom. rep.
We say π has spectral gap if $\exists\ \varepsilon > 0$
s.t. $\forall v \in E$ we have that

$$\sup_{g \in S} \| v - \pi(g).v \| \geqslant \varepsilon \|v\|$$

PROP:
E unif. convex Banach space
$\pi: G \to \mathcal{O}(E)$ cont. isom. rep.
If π has spectral gap, every $b \in Z^1(G,\pi)$
can be written as

$$b = b_{harm} + b_0$$

b_{harm} is μ-harmonic and $b_0(g) = v_0 - \pi(g)v_0$

$b_{harm} \in Z^1(G,\pi)$ $\qquad b_0 \in Z^1(G,\pi)$

$\|b_0(g)\| \leqslant 2\|v_0\|$

Thm [Bader, Furman, Gelander, Monod 07]
$p > 2$

G has prop.(T), then all w/o inv. vectors $\pi : G \to \mathcal{U}(E)$
E L^p-space, have spectral gap

"$(T) \Rightarrow (T)_{L^p}$"

Pf (Thm C) $p > 2$

Suppose G has prop.(T) w/o prop FL^p
$\pi : G \to \mathcal{U}(E)$ w/o inv. vectors
$b \in Z^1(G, \pi)$
$\exists$ μ-harmonic $b_h \in Z^1(G, \pi)$

$$b_0 = b - b_h \in B^1(G, \pi)$$
$$\|b_0(g)\| \leq 2\|v_0\|$$

$$\|b\|_{L^p(\mu^{*n})} \geq \|b_h\|_{L^p(\mu^{*n})} - 2\|v_0\|$$

$$\|b\|_{L^p(\mu^{*n})} \geq (c_p(n-1)+1)^{1/p} \|b_h\|_{L^p(\mu^{*1})} - 2\|v_0\|$$

$$\underbrace{\sup_{g \in S^n}}\ \|b(g)\| \geq \|b\|_{L^p(\mu^{*n})}$$

- Polish gps

$b : G \to E$, $\|b\| : G \to \mathbb{R}$

coarsely Haagerup (Rosendal)

$(G, \mathcal{E})$