Around the commensurating full group II: involutions everywhere

Reminders
- $Sym(\mathbb{Z}, \mathbb{N}) = \{\sigma \in Sym(\mathbb{Z}) \text{ s.t. } |\sigma\mathbb{N} \Delta \mathbb{N}| < \infty\}$
- $[T]_{com} = [T] \cap \mathcal{L}^\infty(X, \mu, Sym(\mathbb{Z}, \mathbb{N}))$

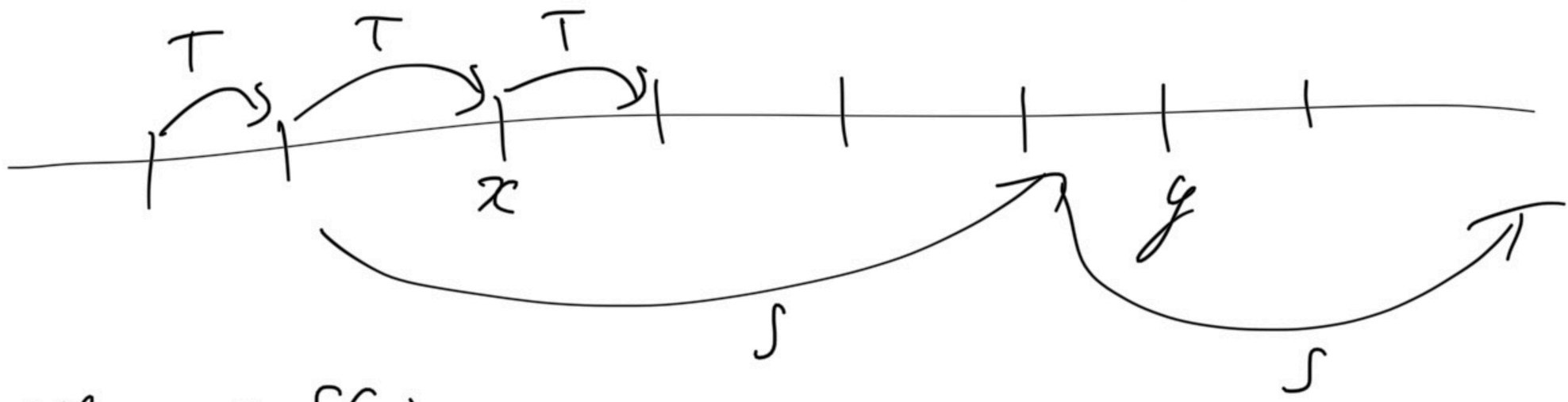
properties: finitely full
induction friendly

Any $S \in [T]_{com}$ is written $S = S_p S_+ S_-$ (periodic, almost positive, almost negative)

$c_{S^k}(x) \xrightarrow[k\to\infty]{} +\infty$

Def $S \in [T]_{com}$ is positive if $c_S \geq 0$

Def $\leq_T$ is the partial order on X s.t. $x \leq_T y$ if $\exists n \geq 0, y = T^n(x)$



(S is positive iff $x \leq_T S(x)$

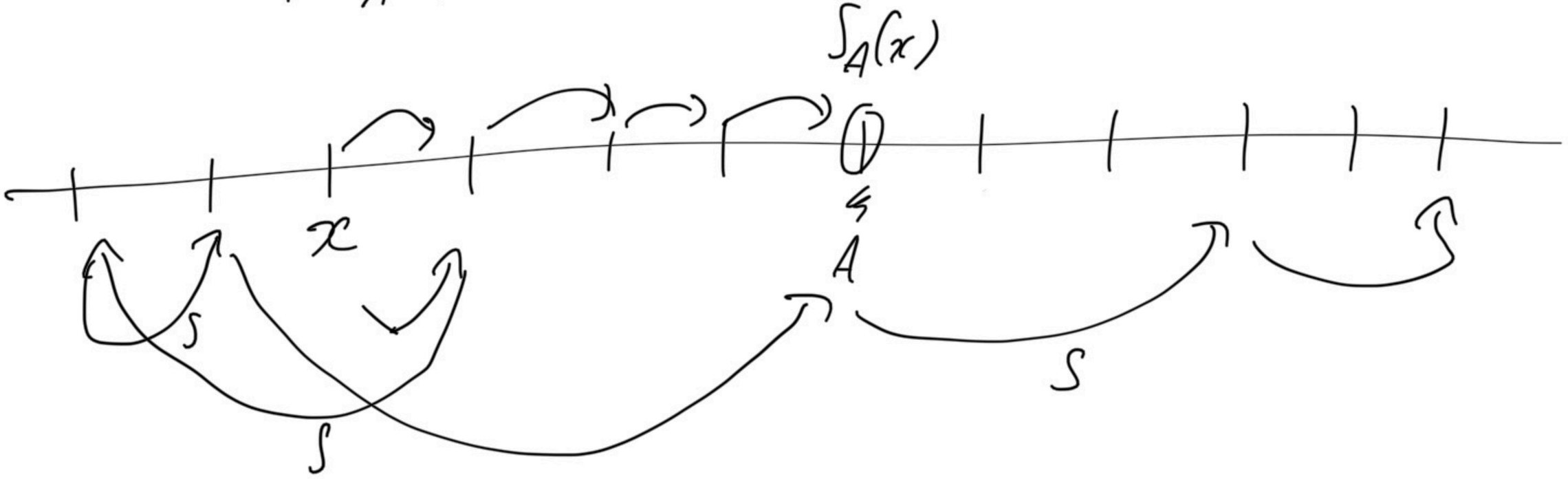
Prop $S \in [T]_{com}$ almost positive, there exists $S' \in [T]_{com}$ positive st $\operatorname{supp} S' \subset \operatorname{supp} S$ and $S(S')^{-1}$ and $(S')^{-1}S$ are periodic

Proof Let $A = \{x \in X,\ C_{S^k}(x) > 0 \ \forall k > 0\}$

$x <_T S^k(x)$

A intersects every (positive) S-orbit (because $C_{S^k}(x) \xrightarrow[k]{} \infty$)

$T_A(x) \leq_T S_A(x)$



$S_A(x) \geq_T T_A(x) \geq_T x$, so S_A is positive, and

$S(S_A)^{-1}$, $(S_A)^{-1}S$ are periodic ∎

Lemma: Let $S \in [T]_{com}$ be positive, let $A = \operatorname{supp}(S)$. Then $S(T_A)^{-1}$ is positive.

Proof $A = \{x,\ S(x) \geq_T x\}$, so $S(x) \geq_T T_A(x)$, and $S(T_A)^{-1}(x) \geq_T x$ (for $x \in \operatorname{supp} S$)

for $x \notin \operatorname{supp} S$, $x \notin \operatorname{supp} T_A$, so $x \notin \operatorname{supp} S(T_A)^{-1}$

for $x \notin \operatorname{supp} S$, $x \notin \operatorname{supp} T_A$, so $x \notin \operatorname{supp} S(T_A)^{-1}$.

(for $x \in \operatorname{supp} S$)

Prop Let $S \in [T]_{con}$ (positive), there are $A_1, \dots, A_k \subset X$ st

$$S = T_{A_1} T_{A_2} \cdots T_{A_k}$$

Proof: Induction on $I(S)$

($I(S)=0$ and S positive $\Rightarrow S = id$)

Induction step: $I(S(T_A)^{-1}) = I(S) - \underbrace{I(T_A)}_{\uparrow}$

Corollary $[T]_{con}$ is generated by periodic maps and T.

$S = S_\uparrow S_+ S_-$ (almost positive)$^{-1}$

$\downarrow$

periodic $\times$ positive

$T(T_A)^{-1}$ is periodic so ok.

Lemma $\overline{\underbrace{G_{inv}}} \supset G_{per}\}$ —— periodic maps

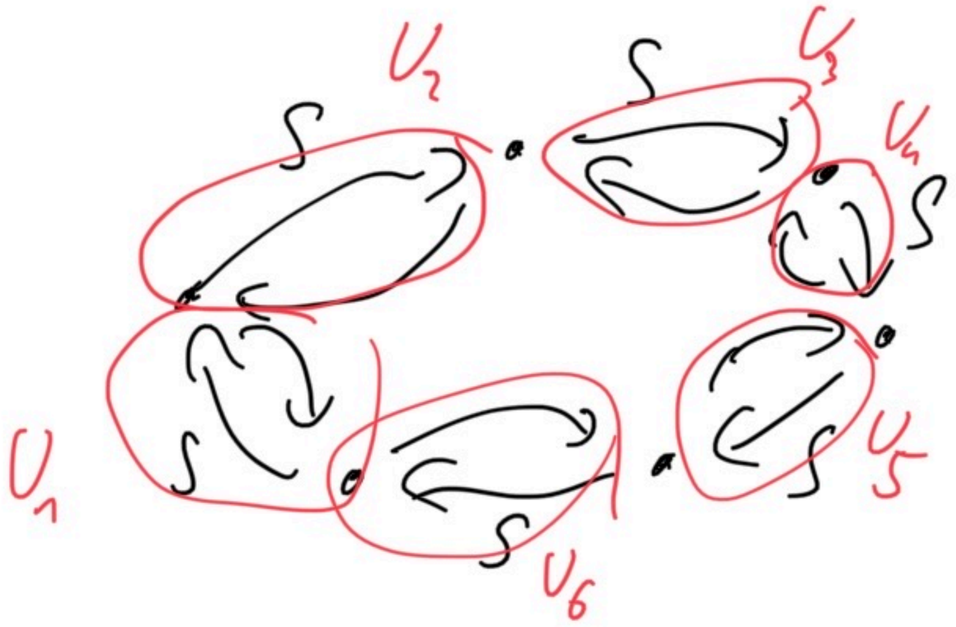
group generated by involutions

Proof S periodic, let $A_n = \{x \in X,\ |S^{\mathbb{Z}}(x)| = n\}$

$$\bigsqcup_{n \geq 1} A_n = X$$

$$\bigsqcup_{n=1}^{N} S_{A_n} = S_{\bigsqcup_n^N A_n} \longrightarrow S_X = S$$

Now for S an n-cycle, $(1\ 2\ 3\ 4 \dots\ n) = (1\ 2)(2\ 3) \cdots (n-1\ n)$



All the V_i are involutions, $S = V_1 V_2 V_3 \cdots V_{n-1}$

Lemma: $V \in [T]_{com}$ an involution, $V \in \overline{\underbrace{D([T]_{com})}_{\langle [S,U] \rangle}}$

We need some other lemmas

Lemma: If $\nu \sim \mu$ is st $T_* \nu = \nu$, if $A, B \subset X$ are st $\nu(A) = \nu(B)$

then there exists a partial bijection φ which preserves T-orbits

st $\varphi(A) = B$

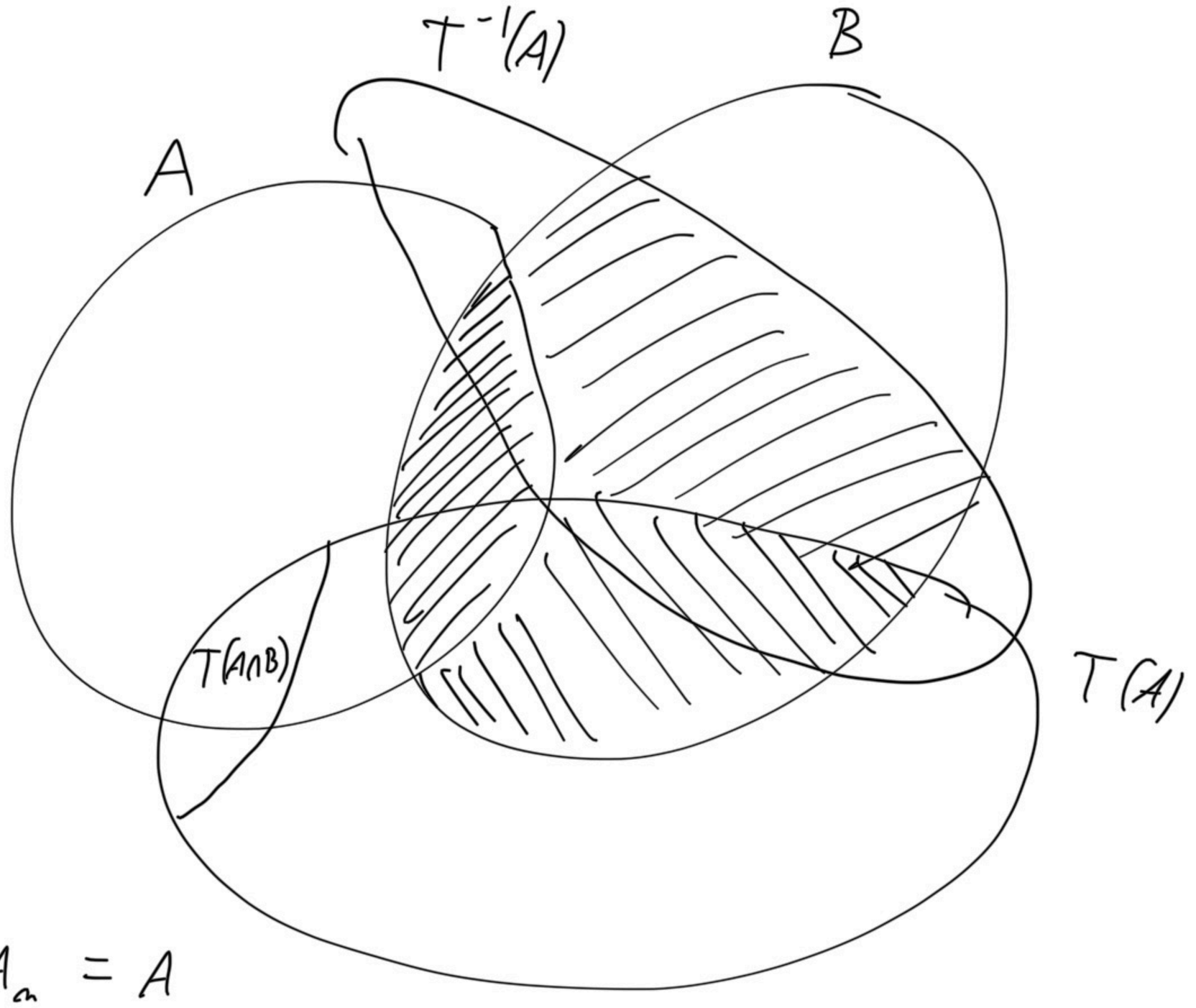
If no such ν exist, then the result holds for every A, B of positive measure

Proof

First, try to send as many people from A into B using T^0.

Next, try the same with T

Next, try T^{-1}



$\varphi_n : A_n \to B_n$ then $\bigcup A_n = A$

$\nu(A_n) = \nu(B_n)$ since $\nu(A \setminus A_n) \to 0$,

$\nu(B \setminus B_n) = \nu(B) - \nu(B_n)$

$= \nu(A) - \nu(\tilde{A}) \to 0$

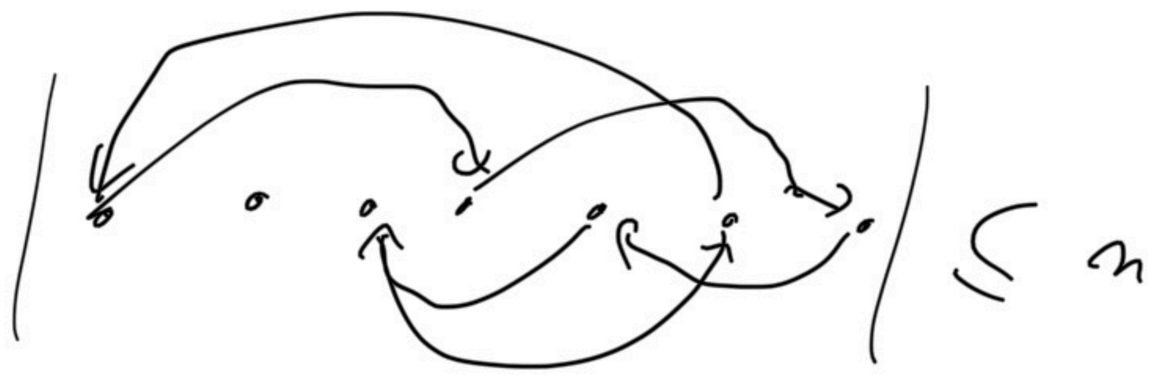
Lemma If $A, B \subset X$, same hypotheses as before and A, B disjoint there is some $U \in [T]$ such that $U(A) = B$ and $\operatorname{supp}(U) = A \cup B$

$$U(x) = \begin{cases} \varphi(x) & \text{on } A \\ \varphi^{-1}(x) & \text{on } B \\ x & \text{otherwise} \end{cases}$$

Lemma If $U \in [T]$ is periodic, there is a sequence $A_n \subset X$ of U-invariant sets

st $\bigcup_n A_n = \operatorname{supp} U$ and $U_{A_n} \in [T]_{fin}$

Proof Let $A_n = \{x \in \operatorname{supp} U \text{ st } \operatorname{diam}_{d_T}(U^{\mathbb{Z}}(x)) \leq n\}$



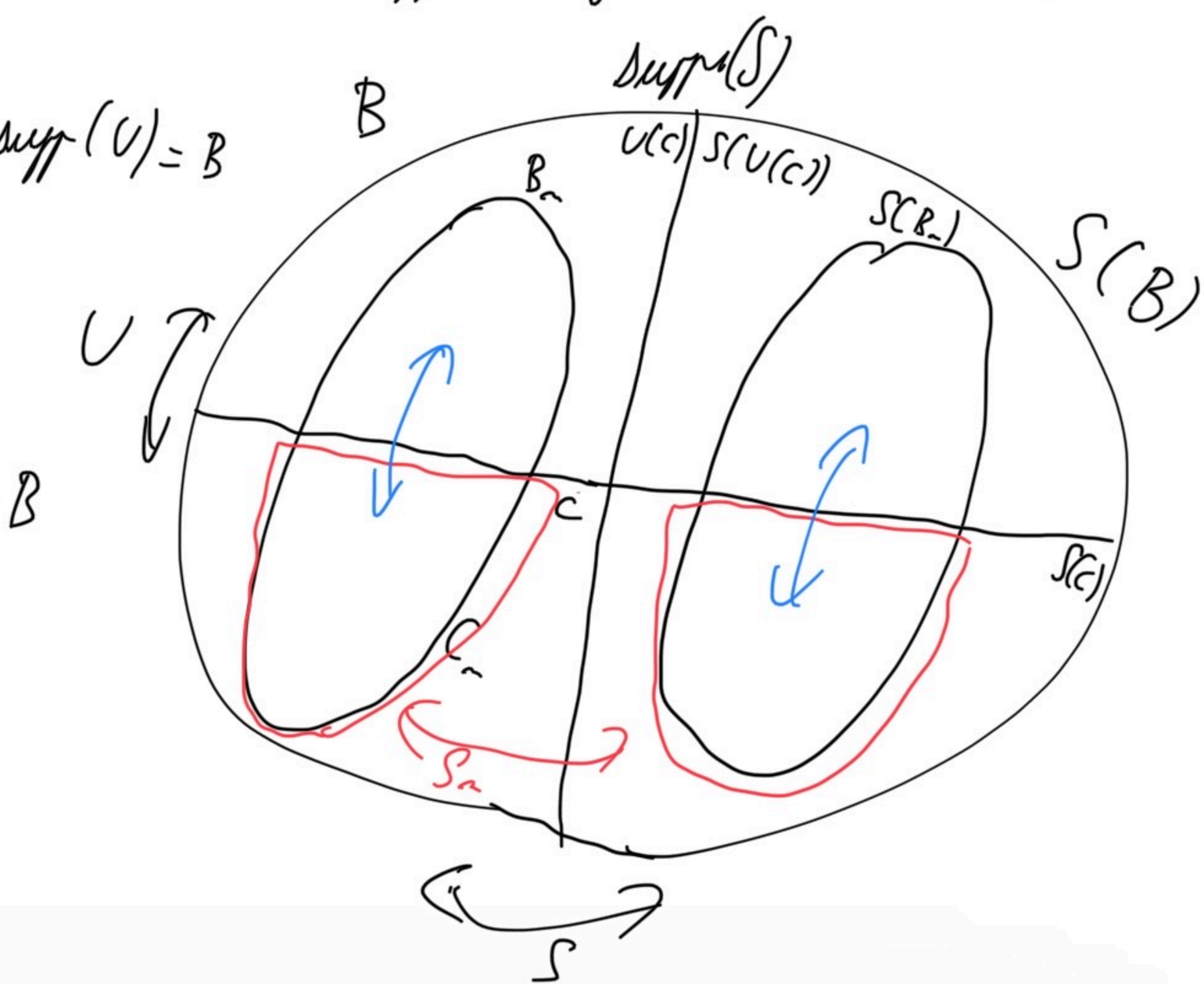
Corollary $[T]_{con}$ has many involutions: $\forall A \subset X$, $\exists V \in [T]_{con}$ involution st $\operatorname{supp} V \subset A$ (and $V \neq id$)

Proof of lemma (V involution is in $\overline{D([T]_{con})}$)

Let $S \in [T]_{con}$ be an involution, $B \subset \operatorname{supp} S$ a fondamental domain.

Take $V \in [T]$ involution st $\operatorname{supp}(V) = B$

(take C a f.d. for V)



Take $B_n \subset B$ st $\bigcup_n V^n B_n = B$

and $V_{B_n} \in [T]_{con}$

$C_n = C \cap B_n$

I want some commutator to equal $S_{B_n \sqcup S(B_n)}$

Let $S_n = S_{C_n \sqcup S(C_n)}$

Let $V_n = V_{C_n \sqcup V(C_n)} \circ S V S_{S(C_n \sqcup V(C_n))}$

Check $\underbrace{[S_n, V_n]}_{S_n V_n S_n V_n} = S_{B_n \sqcup S(B_n)}$

Thm $G_{per} = \overline{G_{inv}} = \overline{D([T]_{com})} = \ker I$

$\leqslant \qquad \leqslant \qquad \leqslant$

We are left to prove that $\ker I \leqslant G_{per}$

G_{per} is normal

$$[T]_{com}/G_{per} = \langle G_{per}, T \rangle / G_{per} = \langle T \rangle$$

ie if $S \in [T]_{com}$, $S = T^n V$, where $V \in G_{per}$

$I(S) = n$

So now if $S \in \ker I$, $S = T^n V$ ——————, and $n = 0$

so $S = V$

So $\ker I \leqslant G_{per}$. ∎

$$[T]_{con}/_{bnI} = \mathbb{Z}$$

III quasi-isometry types of Polish groups

d compatible left-inv distance d on G (Polish) is <u>maximal</u> if for any metric d' compatible left-inv, we have $d' \leq Cd + C$

If a maximal distance exists, they are all quasi-isometric to each other
↳ this defines a well defined quasi isometry type on G.

When do they exist?

A CG is <u>bounded</u> if for any continuous action by isometries $G \curvearrowright (M, d)$, any $A \cdot x$ is of bounded diameter.

Thm (Rosendal): A maximal distance exist iff G is boundedly generated.

Def A Polish group G is <u>bounded</u> if for any nbhd V of 1, $\exists N$ and F finite st $(VF)^N = G$. If G is bounded, any left-inv compatible distance is bounded and the BI type of G is $\{*\}$

Prop $\{$ periodic maps $\}$ is dense in $\overline{D([T]_{con})}$

Proof R equivalence relation, $[R] = \{S \in Aut(X,[\mu]) \text{ st } S(x)\, R\, x\}$

$$\left(\begin{array}{l} [T] = [R_T] \\ \text{i.e. } x R_T y \text{ iff } \exists n \in \mathbb{Z},\ x = T^n(y) \end{array} \right)$$

$R_T = \bigcup_{n \geq 0} R_n$ with R_n increasing, and with finite classes

$S \in [R_n]$ is periodic

Claim: $\bigcup_n [R_n] \cap \overline{D([T]_{con})}$ is dense in $\overline{D([T]_{con})}$.

We only have to check that we can approximate involutions

$U \in [T_{con}]$ be an involution.

$A_n = \{x \in X,\ x\, R_n\, U(x)\}$ A_n is increasing, U not

$U_{A_n} \in [R_n]$ and $U_{A_n} \in \overline{D([T]_{con})}$

$\bigcup_n A_n = X$