Around the commensurating full group III

Complete invariant of flip conjugacy.

Prop If $U \in [T]_{com}$ has the same orbits as T, then T and U are flip-conjugate (T ergodic)

($U = STS^{-1}$ or $U^{-1} = STS^{-1}$)

Proof $U = U_R U_+ U_-$. Since $\mathcal{R}_T = \mathcal{R}_U$, $U_R = id$.

Moreover supp $U_\pm$ are U-invariant and disjoint.

U is ergodic, so one of them has to be null.

Wlog, assume $U = U_+$ is almost positive

$|T^N(x) \Delta U^N(x)| < \infty$ (for a.e. x)

By Thm A1 of CJLMT23), T and U are conjugate

Katznelson, (see

Ideas of Katznelson's result

We explicitly construct S st $U = STS^{-1}$.

$$j(x) := |T^{\mathbb{N}}(x) \setminus V^{\mathbb{N}}(x)| - |V^{\mathbb{N}}(x) \setminus T^{\mathbb{N}}(x)|$$

$S(x) := T^{j(x)}(x)$, check it works.

How to get $|T^{\mathbb{N}}(x) \Delta V^{\mathbb{N}}(x)|$ for $V \in [T]_1$?

Use "mass transport principle": for $f : R_T \to \mathbb{R}_+$

$$\int_X \sum_{n \in \mathbb{Z}} f(T^n(x), x)\, d\mu(x) = \int_X \sum_{n \in \mathbb{Z}} f(x, T^n(x))\, d\mu(x)$$

with $f : (x,y) \longmapsto \begin{cases} 1 & \text{if } x \leq_T y < V(x) \text{ or } V(x) <_T y \leq x \\ 0 & \text{otherwise} \end{cases}$

Recall: $G \leq Aut(X, [\mu])$ is said to have many involutions

if $\forall A \subset X$ of >0 measure, $\exists V \in G$ st V is a non-trivial involution and $\mathrm{supp}(V) \subset A$

Recall: $[T]$, $[T]_{con}$ have many involutions

Thm (Frenlin) If $G, H \leq Aut(X, [\mu])$ have many involutions, abstract any isomorphism comes from a conjugacy:

if $\rho : G \xrightarrow{\sim} H$, there exists $S \in Aut(X, [\mu])$ st $\rho(T) = STS^{-1}$

Thm Let T_1, T_2 be ergo $\in Aut(X, [\mu])$, TFAE:

(i) $[T_1]_{con}$ and $[T_2]_{con}$ are abstractly isomorphic

(ii) ——— topologically ———

(iii) T_1 and T_2 are flip-conjugate.

Proof (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i)

(i) $\Rightarrow$ (iii) Let S be such that $[T_1]_{con} = S[T_2]_{con} S^{-1}$

T_1 and $S T_2 S^{-1}$ have the same orbits: $S T_2 S^{-1} \in [T_1]_{con}$

If the inclusion of the orbits is strict, then $S^{-1} T_1 S \notin [T_2]_{con}$

We can now apply the previous proposition to T_1 and $S T_2 S^{-1}$. ▨

Thm: The same holds if we replace $[T]_{con}$ by $\overline{D([T]_{con})}$

Idea of proof (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) is ok, we just need to prove that if $S \overline{D([T_1]_{con})} S^{-1} = \overline{D([T_2]_{con})}$, then $S[T_1]_a S^{-1} = [T_2]_{con}$.

↳ We have a bounded complete invariant of flip-conjugacy!

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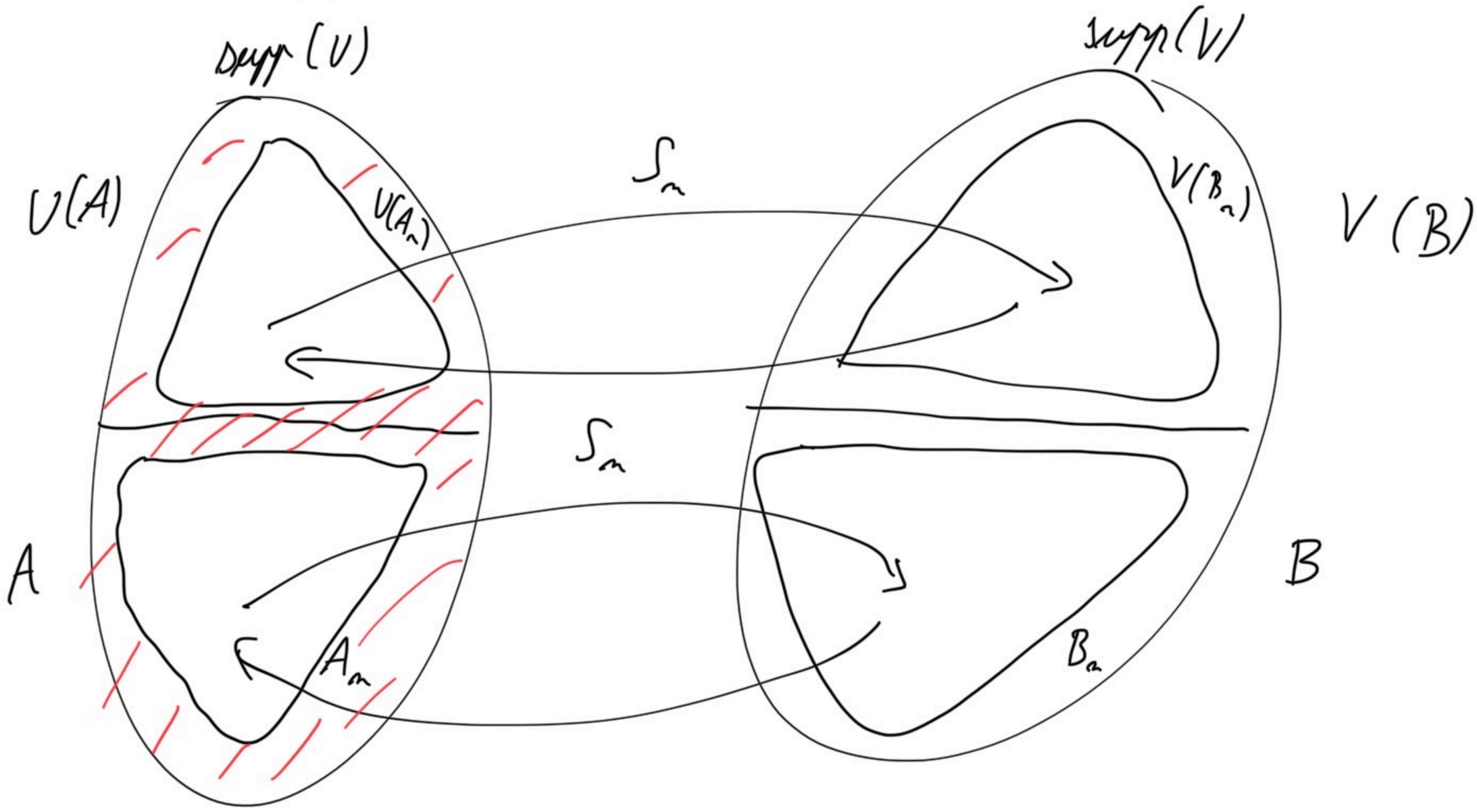
topological simplicity (any closed normal subgp is trivial)

Lemma If $\nu \sim \mu$ is T-invariant, and $U, V \in [T]_{as}$ are involutions with disjoint support st $\nu(\mathrm{supp}(U)) = \nu(\mathrm{supp}(V))$, then U and V are approximately conjugated:

$$S_n U S_n^{-1} \xrightarrow[n\to\infty]{} V$$

If T is of type III any U, V of disjoint nonempty support are A.C.

Proof Let A, B be fund. doms for U, V, let $S \in [T]$ be st $S(A) = B$ and $\mathrm{supp}(S) = A \sqcup B$.



$A_n \subset A$ be $A_n = \{x \in A : |C_S(x)| < n\}$, B_n the same, note that $S(A_n) = B_n$

$$S_n(x) = \begin{cases} S(x) \text{ if } x \in A_n \cup B_n \\ USV(x) \text{ if } x \in V(B_n) \\ VSV(x) \text{ if } x \in U(A_n) \\ x \text{ otherwise} \end{cases}$$

$$S_n \cup S_n(x) = \begin{cases} U(x) & \text{if } x \in (A \cup U(A)) \setminus (A_n \cup U(A_n)) \} C_n \\ V(x) & \text{if } x \in B_n \cup V(B_n) \} D_n \\ x & \text{otherwise} \end{cases}$$

By induction friendliness,

$$S_n \cup S_n = V_{C_n} V_{D_n}, \quad \text{and } \bigcup C_n = \emptyset$$

$$\underbrace{\quad}_{\to 1} \quad \underbrace{\quad}_{\to V} \qquad \bigcup D_n = \operatorname{supp} V, \text{ so}$$

$$\xrightarrow[n\to\infty]{} V$$

Lemma If T is of type III and $U \in [T]_{con}$ with $0 < \mu(\operatorname{supp} U) < 1$, involution then $\overline{\langle\langle U \rangle\rangle} = \overline{D([T]_{con})}$ in $\overline{D([T]_{con})}$

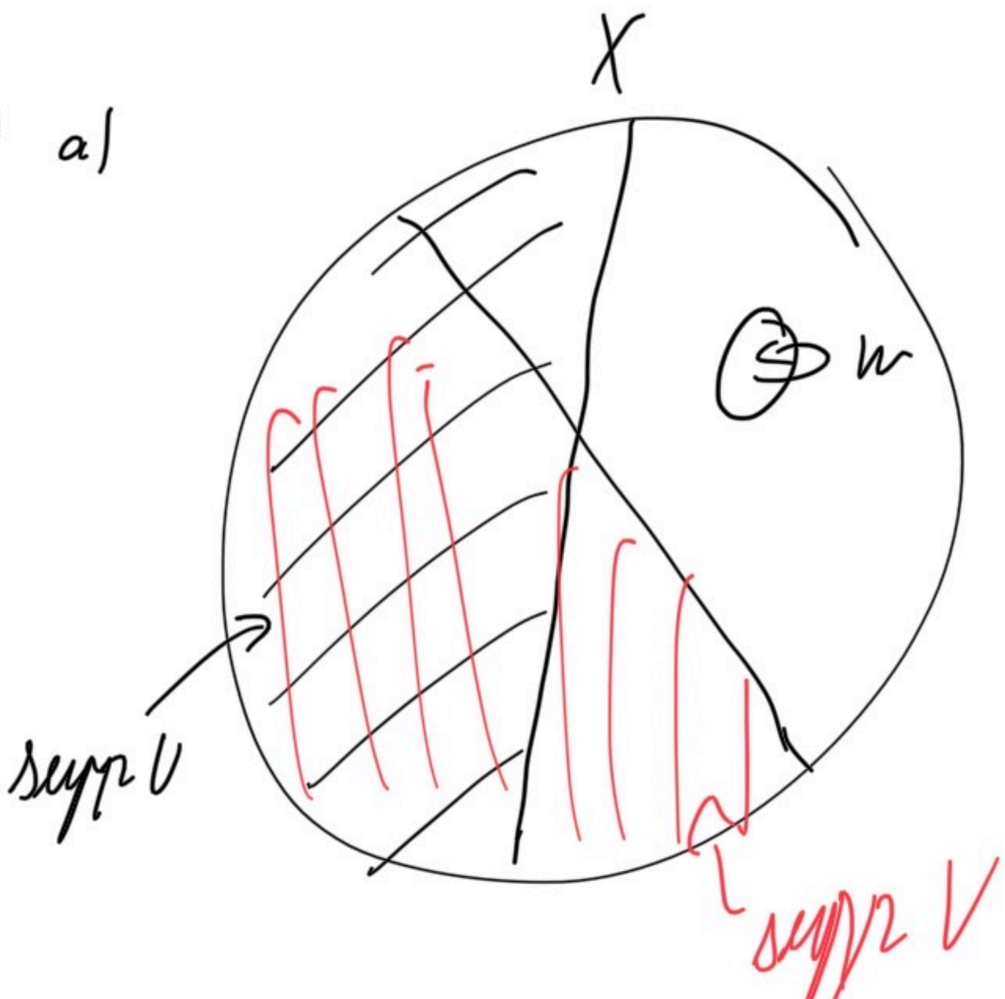
Proof Write $V_1 \sim V_2$ if V_1, V_2 are approx. conj., let $G = \overline{\langle\langle U \rangle\rangle}$.

If $V \sim S_1 \sim \cdots \sim S_k \sim V$, then $V \in G$.

Since $\overline{D([T]_{con})} = \overline{G_{inv}}$, we only need to prove that any involution $V \in [T]_{con}$ is in G.

- Two cases: 1) $\operatorname{supp}_2(V) \neq X$ or 1) $\operatorname{supp}_2(V) = X$.

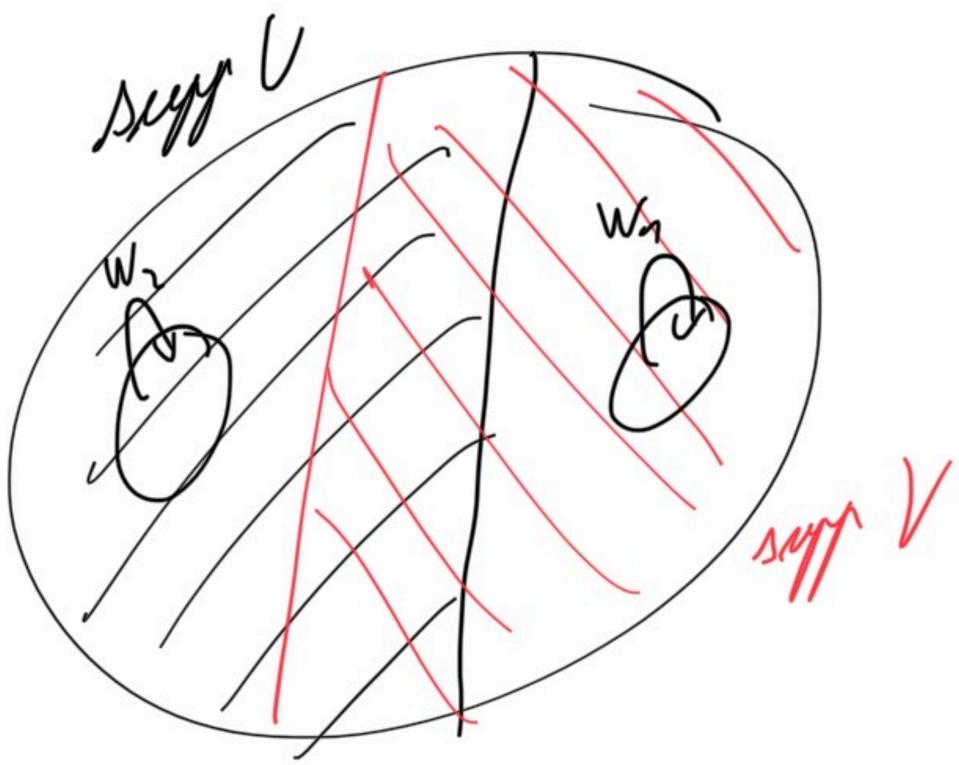
1) a)



(Find W s.t. $\operatorname{supp} W \cap (\operatorname{supp} U \cup \operatorname{supp} V) = \emptyset$,

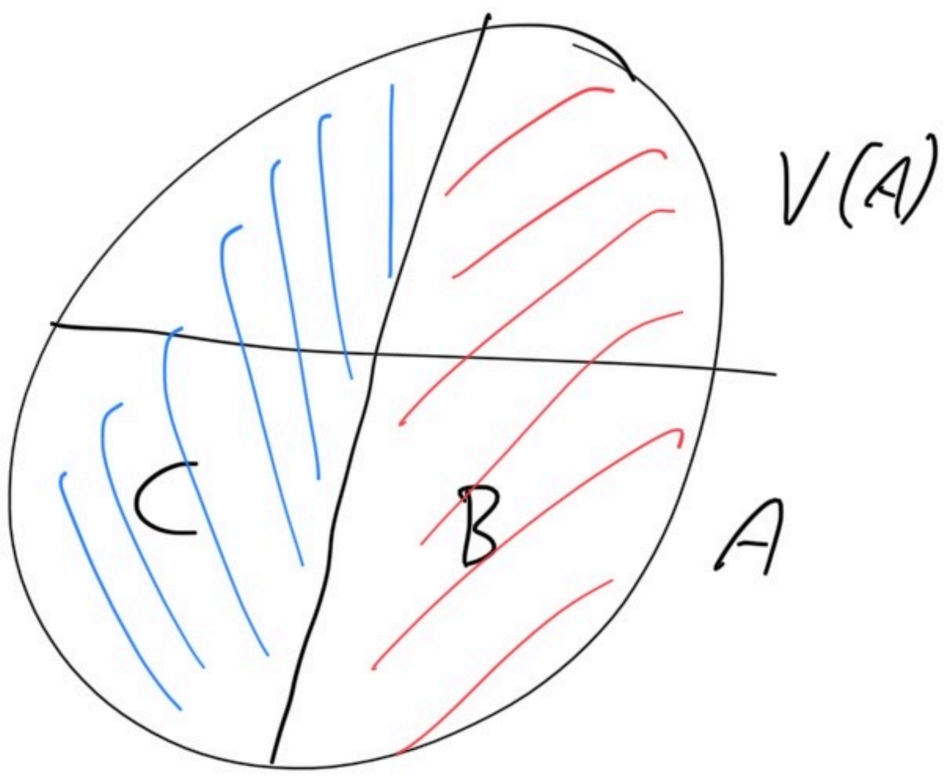
then $U \sim W \sim V$ so $V \in G$

b) if $\operatorname{supp} U \cup \operatorname{supp} V = X$



then $U \sim W_1 \sim W_2 \sim V$ so $V \in G$.

2) if supp $V = X$ A f.d. for V. split $A = B \sqcup C$,



$V \simeq V_{B \sqcup V(B)} V_{C \sqcup V(C)}$. Apply 1) to ▪ and ▪.

So $G = \overline{D([T]_{com})}$. ▪

If T is of type II, i.e. see [LM 18, Lemma 3.22].

Thm $\overline{D([T]_{com})}$ is top. simple

Let $S \in \overline{D([T]_{com})}$ and we want to find some involution $U \in \langle\langle S \rangle\rangle$ for which the hypotheses apply.

Let $A \subset X$ be "small enough" ($\nu(A) \leq \frac{1}{4}$ if $\nu(X) \in \{1, \infty\}$ and $\mu(A) < 1$ if type II)

such that $A \cap S(A) = \emptyset$, let U be supported on A an involution

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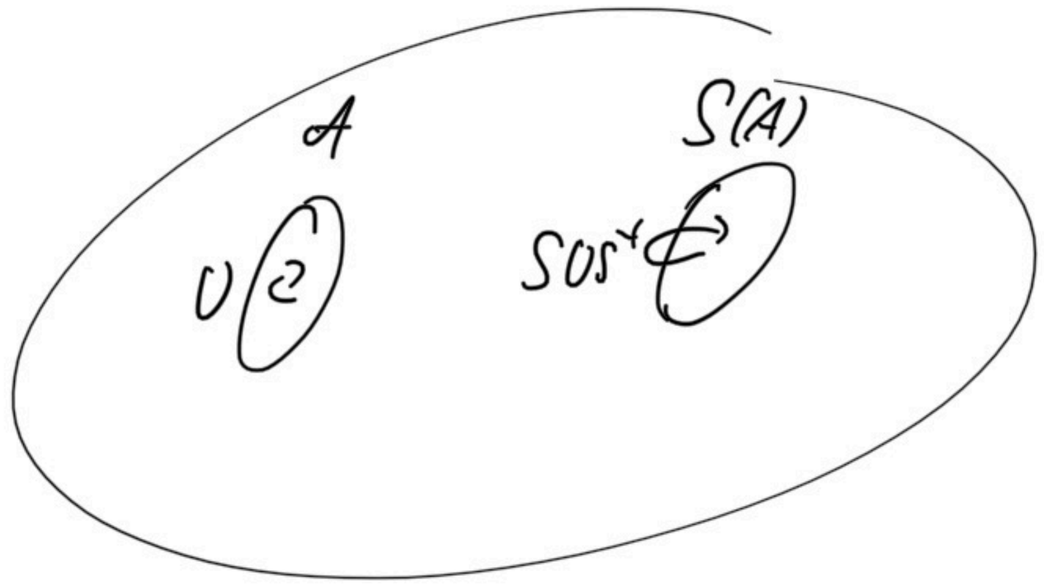


U and SUS^{-1} are involutions of disjoint support

$USUS^{-1}$ is an involution, supported on $\underbrace{A \cup S(A)}_{\text{"small enough"}}$.

$USU \in \langle\langle S \rangle\rangle$ and $S^{-1} \in \langle\langle S \rangle\rangle$, so $USUS^{-1} \in \langle\langle S \rangle\rangle$ ☒

Question: is $\overline{D([T]_{conj})}$ simple?

Lemma: Let R be a finite class Borel eq. rel. on X. Let $A, B \subset X$ be

st $\forall^* x \in X$, $\quad |[x]_R \cap A| \leq |[x]_R \cap B|$

then $\exists\, \varphi \in [R]$ st $\varphi(A) \subset B$.

define $\varphi : A \to B$ however you want, say order every R-class and send the first element of A to the first element of B.

Now do the same for $\varphi : X \setminus A \to X \setminus \varphi(A)$.

Prop Let $T \in Aut(X, \mu)$ ergo,

Let A, B be st $\mu(A) < \mu(B)$ then $\exists S \in \overline{D([T]_{con})}$ st $S(A) \subset B$

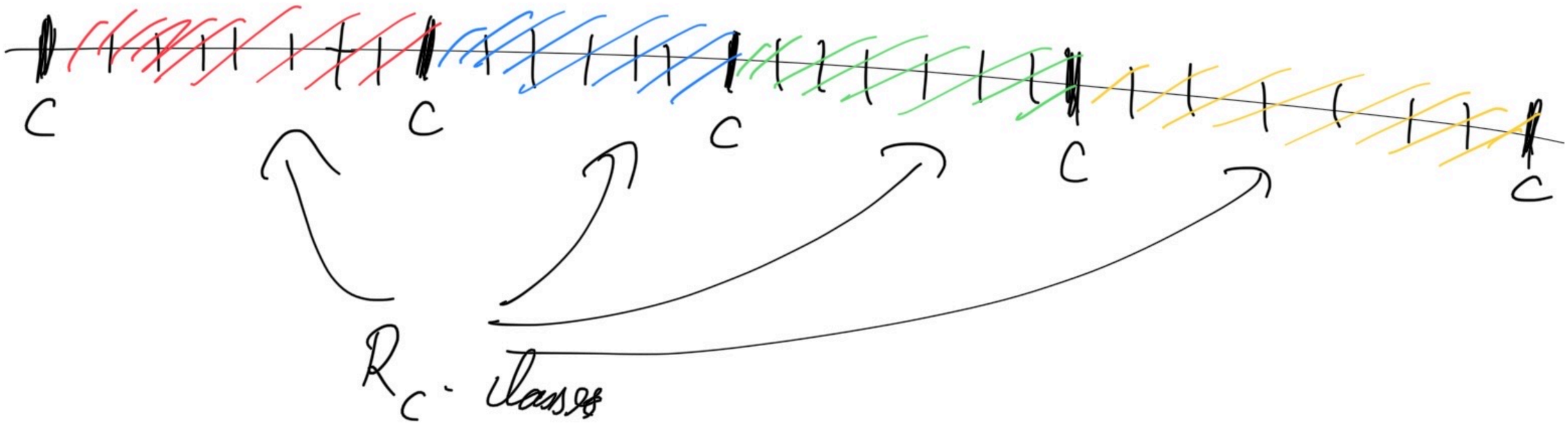
Proof By Birkhoff, $\forall^{ae}$, $\exists N = N(x)$ st $\forall n \geq N$,

$$\sum_{k=0}^{n-1} 1_A(T^k(x)) < \sum_{k=0}^{n-1} 1_B(T^k(x)) \quad (*)$$

Let N be st the set X_N for which $(*)$ holds for $N(x) = N$ is of positive measure.

Let $C \subset X_N$ be st $C, T(C), \dots, T^{N-1}(C)$ are all disjoint.

By ergodicity of T, C separates the T-orbits into finite segments:



The hypotheses of the previous lemma hold for A, B, R_C.

We get $S \in [R_C] \subset [T]$ st $S(A) \subset B$, and S acts on T-orbits by stabilizing intervals, so $S \in [T]_{con}$.

If T is of type II_∞, we can still do the same proof for $\nu(A) < \nu(B) < \infty$ by the ratio ergodic theorem.

If T is of type III, ?????

$$\frac{\sum f(T^k)}{\sum g(T^k)} \longrightarrow \frac{\int f}{\int g}$$

ratio ergodic theorem