The existence of ergodic hyperfinite subgraphs, part 1

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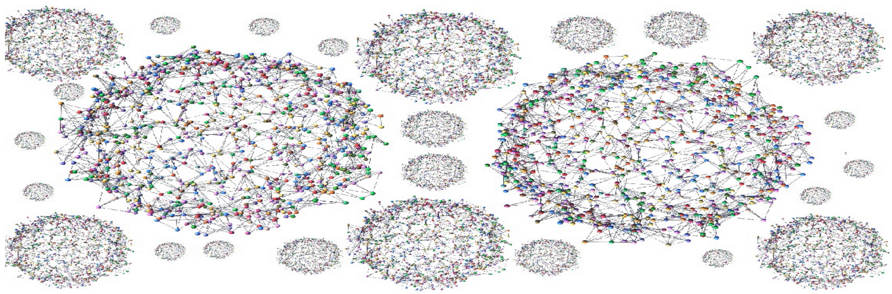
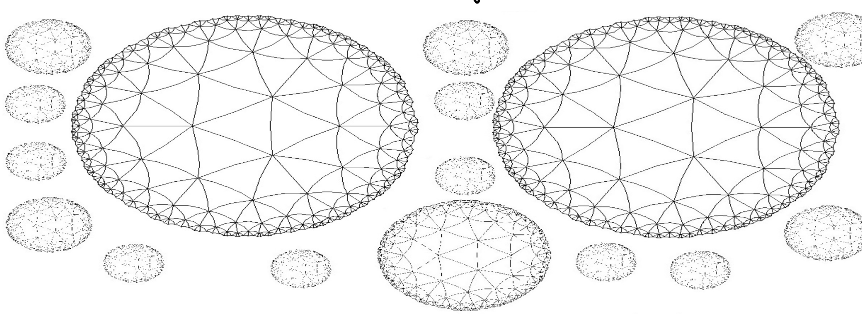
Measured equivalence relations, graphs, and group actions.

- Let (X, μ) be a standard probability space, e.g. $([0,1), \lambda)$.
- A (locally) ctbl Borel equivalence relation on X is an equiv. rel. on X such that $R \subseteq X^2$ is Borel and each R-class is ctbl. Abbreviate: cBer.
- A locally ctbl Borel graph on X is a symmetric Borel subset $G \subseteq X^2$ (we identify the graph with its set of edges) with G_x ctbl for all $x \in X$. Denote by R_G its connectedness eq. rel.



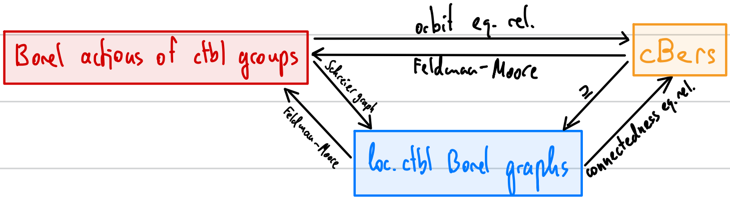
- Every Borel group action $\Gamma \curvearrowright X$ of a ctbl group Γ gives:
  - a cBer R_Γ, its orbit equivalence relation;
  - a loc. ctbl Borel graph G_S, its Schreier graph: for any symmetric set S of generators of Γ,

$$(x,y) \in G_S \;:\Leftrightarrow\; \gamma x = y \text{ for some } \gamma \in S.$$



Feldman–Moore (really just **Luzin–Novikov**). Every loc. ctbl Borel graph $G = \bigcup_{n \in \mathbb{N}} \operatorname{graph}(\gamma_n)$, where each $\gamma_n : X \to X$ is a partial Borel involution (with Borel domain and image). In particular:

- every cBer is the orbit eq. rel. of a Borel action of a ctbl group.
- every loc. ctbl Borel graph is a Schreier graph of a Borel action of a ctbl group.

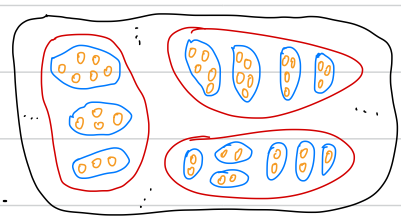


— A cBer R on (X, μ) is *probability-measure-preserving* (resp. *measure-class-preserving*) if it is induced by a Borel action $\Gamma \curvearrowright X$ of a ctbl group Γ which preserves the measure μ (resp. the measure-class of μ, i.e. maps μ-null to μ-null).

— A cBer R is *ergodic* if every R-invariant Borel set is null or conull.

$\iff$ *measure-density*: for every positive measure set $B \subseteq X$, a.e. R-class meets B.

Hyperfinite/amenable cBers and graphs.



— A Borel eq. rel. R is called

- *finite* if each R-class is finite.
- *hyperfinite* if $R = \bigcup\limits_{n \in \mathbb{N}}\!\!\!\uparrow R_n$, where each R_n is a finite Borel eq. rel.

Weiss/Slaman–Steel. A cBer R is hyperfinite $\iff$ $R = R_{\mathbb{Z}}$ for some Borel $\mathbb{Z} \curvearrowright X$.

Connes–Feldman–Weiss. A cBer R on (X,μ) is hyperfinite μ-a.e. $\Leftrightarrow$ R is μ-amenable.
$\Leftrightarrow$ R is induced μ-a.e. by a Borel action of an amenable group.

— We say that a loc. ctbl Borel graph G is pmp/mcp/ergodic/hyperfinite/amenable if R_G is.
Note: G is hyperfinite $\Leftrightarrow$ $G = \bigcup\uparrow_{n \in \mathbb{N}} G_n$, where each G_n is a Borel graph with finite components.

Ergodic hyperfinite subgraphs.

Hyperfiniteness is a smallness notion (closed downward), while ergodicity is a largeness notion (closed upward), and when these two meet, good things happen.

Theorem (Miller ?). Every ergodic cBer R contains a hyperfinite ergodic subeq. rel. $S \subseteq R$

Theorem (Tucker-Drob, 2016 unpublished). Every ergodic pmp loc. ctbl Borel graph G contains an ergodic hyperfinite subgraph $H \subseteq G$.

Tucker-Drob's ingeneous proof used a major result of Hutchcroft and Nachmias from percolation theory, while after knowing that such a subgraph exists, I really wanted to build it by hand using only Feldman-Moore and Borel-Cantelli, and prove it for mcp graphs...

Theorem (Ö. 2017-22). Every ergodic loc. ctbl Borel graph G contains an ergodic hyperfinite subgraph $H \subseteq G$.

Applications.

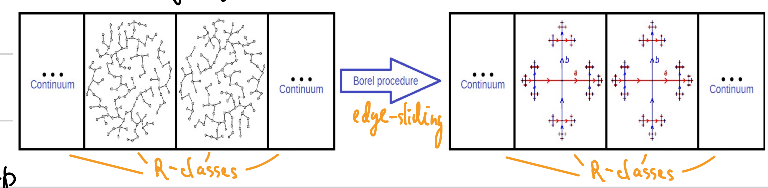
① An answer to the question of L. Bowen on ergodic free factors:

Corollary. Every ergodic treeable eq. rel. R admits an ergodic hyperfinite free factor.

Proof. Take a treeing T of R (i.e. T is an acyclic Borel graph with $R_T = R$) and let $H \subseteq T$ be an ergodic hyperfinite subgraph. Then R_H is an ergodic free factor of R, precisely: $R = R_H * R_{T\setminus H}$, because any alternating $R_H - R_{T\setminus H}$ cycle would yield a nonbacktracking cycle in T. $\square$

② An ergodic strengthening of Hjorth's lemma for cost attained:

Theorem (Miller – Ť 2017+, unpublished). Every ergodic treeable pmp equivalence relation R of cost $n \in \omega+1 := \mathbb{N} \cup \{\infty\}$ is induced by an essentially free Borel action of the free group $\mathbb{F}_n$ (Hjorth 2006) so that each of the n standard generators of $\mathbb{F}_n$ acts ergodically.



③ A strengthening and generalization of the Gaboriau – Ghys theorem — a Day – von Neumann style statement:

Theorem (Gaboriau 2000 + Ghys 1995). Let G be a locally finite ergodic pmp graph G. If a.e. G-component has >2 ends then G is nowhere amenable, in fact, R_G contains a nowhere amenable ergodic forest $T \subseteq R_G$.

Theorem (Chen–Terlov–Ť. 2023+). Let G be loc. finite ergodic mcp graph. If a.e. G-component contains > 2 nonvanishing ends then G is nowhere amenable, in fact, G contains a nowhere amenable ergodic subforest $T \subseteq G$.

④ An answer to a question of Gaborian, Tucker-Drob, and Ť:

Theorem (Poulin 2024+). Every ergodic nonamenable treeable type III mcp cBer is induced by an essentially free action of the free group $\mathbb{F}_n$ of for every $n \in \omega + 1 := \mathbb{N} \cup \{\infty\}$. Moreover, the Schreier graph of this action is obtained by edge-sliding any given treeing.

⑤ Measure equivalence classification of Baumslag–Solitar groups: for nonzero $r, s \in \mathbb{Z}$,

$$B(r,s) := \langle a, t : t a^r t^{-1} = a^s \rangle.$$

The following completes the classification of these groups up to measure equivalence:

Theorem (Poulin–Gaboriau–Ť–Tucker-Drob–Wróbel 2025+). All $BS(r,s)$ with $|s| > |r| > 1$ are measure equivalent to each other.

Rephrasing as an ergodic theorem for graphs.

Recall the main result:

Theorem (Ť. 2017-22). Every ergodic loc. ctbl Borel graph G contains an ergodic hyperfinite subgraph $H \subseteq G$.

— One can build hyperfinite subequiv. relations or subgraphs iteratively, by iteratively building larger and larger finite subequiv. relations or component-finite subgraphs.

- But how to ensure that the limit-object is ergodic? How do we "make progress" toward ergodicity at every step?
- This is exactly what pointwise ergodic theorems do: translate the global property of ergodicity into a limit of local finitary properties.

The relevant ergodic theorem here is:

Theorem (folklore, Miller–T., Bowen–Nevo for a stronger version in pmp). A hyperfinite mcp cBer S on (X,μ) is ergodic $\iff$ for every hyperfinite exhaustion $S = \bigcup_{n\in\mathbb{N}} S_n$, we have

(pmp) $$\lim_{n\to\infty} \left(\text{average of } f \text{ over } [x]_{S_n}\right) = \int f \, d\mu \quad \text{a.e. } x \in X.$$

(mcp) $$\lim_{n\to\infty} \big(\underbrace{\text{Radon-Nikodym cocycle-weighted}}_{\text{RN-weighted}} \text{ average of } f \text{ over } [x]_{S_n} = \int f \, d\mu \quad \text{a.e. } x \in X.$$

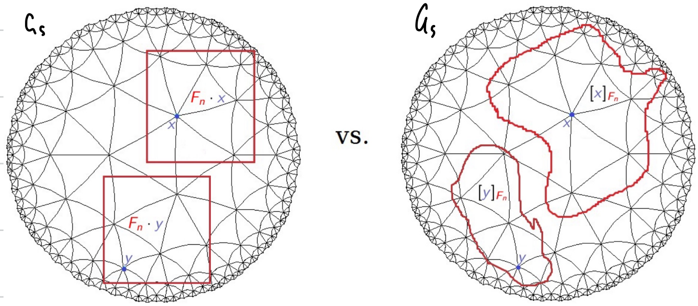
Using this ergodic theorem, the main result translates into a general pointwise ergodic theorem for graphs (e.g. Schreier graphs of actions):

Ptwise ergodic version. Let G be a loc. ctbl ergodic mcp Borel graph on (X,μ). Then there is an increasing sequence (H_n) of component-finite ergodic subgraphs of G (typically $G \supsetneq \bigcup_n H_n$) such that for each $f \in L^1(X,\mu)$,

$$\lim_{n\to\infty} \left(\text{RN-weighted average of } f \text{ over } [x]_{H_n}\right) = \int f \, d\mu \quad \text{a.e. } x \in X.$$

- For an ergodic mcp action $\Gamma \curvearrowright X$ of a ctbl group, pointwise ergodic theorems are of the following form: for some sequence (F_n) of finite subsets of the group Γ, for each $f \in L^1(X,\mu)$
$$\lim_{n\to\infty} (\text{RN-weighted average of } f \text{ over } F_n\cdot x) = \int f \, d\mu \quad \text{a.e. } x\in X.$$

- Thinking of these $F_n \Subset \Gamma$ as deterministic test windows for each point $x \in X$, the ptwise ergodic theorem for a Schreier graph G_S of the action $\Gamma\curvearrowright X$ can be thought of as having random test windows:



Deterministic test windows

Random test windows

- Deterministic ergodic theorems don't hold for the ergodic actions of $\mathbb{F}_n, n\geq 2$ in the pmp setting (Tao), and not even for the abelian groups $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}$ in the mcp setting (Hochman). Thus, the generality of the above theorem is necessary.

Reduction to the main lemma.

A standard approximation argument reduces proving the existence of erg. hyperfinite subgraph to the following finitary version:

Main Lemma. Let G be a loc. ctbl ergodic mcp Borel graph and let $H_0 \subseteq G$ be a component-finite Borel subgraph. Then for each $f \in \mathcal{L}^\infty(X, \mu)$ and $\varepsilon > 0$, there is a component-finite Borel subgraph $H_1 \supseteq H_0$ of G such that

$$(\text{RN-weighted average of } f \text{ over } [x]_{H_1}) \approx_\varepsilon \int f \, d\mu$$

for all x in a set of measure $\geq 1-\varepsilon$.

Proof of theorem from lemma. Just a diagonalization + Borel-Cantelli. Let $\mathcal{D} \subseteq L^\infty(X,\mu)$ be a ctbl set dense in $L^1(X,\mu)$ and enumerate it $\mathcal{D} = \{f_n : n \in \mathbb{N}\}$ so that each $f \in \mathcal{D}$ appears infinitely many times. Let $(\varepsilon_n) :=$ summable sequence. Let $H_0 := \emptyset$ and iteratively apply the main lemma to get an increasing sequence (H_n) of component-finite Borel subgraphs of G such that for each $n \geq 1$:

$$(\text{RN-weighted average of } f_n \text{ over } [x]_{H_n}) \approx_{\varepsilon_n} \int f_n \, d\mu.$$

for all x in a Borel set X_n of measure $\geq 1-\varepsilon_n$. By Borel-Cantelli, a.e. $x \in X$ is eventually in X_n. Thus, for any $f \in \mathcal{D}$, letting (n_k) be the subsequence with $f_{n_k} = f$, the Dominated Convergence thm gives

$$\lim_{k\to\infty} (\text{RN-weighted average of } f \text{ over } [x]_{H_{n_k}}) = \int f \, d\mu \quad \text{a.e. } x \in X.$$

Convexity of averages lets us extend this to the whole sequence (H_n). The density of $\mathcal{D}$ in $L^\infty(X,\mu) \cap L^1(X,\mu)$ and another application of Domin. Conv. thm yields: for all $f \in L^\infty(X,\mu)$

$$\lim_{n\to\infty} \left(\text{RN-weighted average of } f \text{ over } [x]_{H_n}\right) = \int f \, d\mu \quad \text{a.e. } x \in X.$$

Applying this to the indicator function $\mathbb{1}_B$ of any R_H-invariant Borel set shows that $\mu(B) = 0$ or 1, hence H is ergodic. $\square$

We will sketch the first half of the proof of the Main Lemma next time, in the pmp setting.