The existence of ergodic hyperfinite subgraphs, part 2

Anush Tserunyan

(McGill University, visiting Université Paris-Cité)

Recap.

We reduced proving the existence of hyperfinite ergodic subgraphs to the following:

Main Lemma. Let G be a loc. ctbl ergodic mcp Borel graph and let $H_0 \subseteq G$ be a component-finite Borel subgraph. Then for each $f \in \mathcal{L}^\infty(X, \mu)$ and $\varepsilon > 0$, there is a component-finite Borel subgraph $H_1 \supseteq H_0$ of G such that

$$(\text{RN-weighted average of } f \text{ over } [x]_{H_1}) \approx_\varepsilon \int f \, d\mu$$

for all x in a set of measure $\gtrsim 1 - \varepsilon$.

We will prove this in the pmp setting and then discuss in the end what extra modifications and ingredients are needed for the general mcp setting. Even when G is pmp, to prove the Main Lemma, it is convenient to work with the quotient $X/_{H_0} := X/_{R_{H_0}}$ in which every point $[x]_{H_0} \in X/_{X_0}$ has weight $w([x]_{H_0}) := |[x]_{H_0}|$.

In other words, if $\pi : X \to X/_{H_0}$ is the quotient map, then $\tilde{R} := R/_{H_0}$ is no longer pmp on $(X/_{H_0}, \pi_* \mu)$, it is a very nice mcp.

So let's introduce the mcp setting properly, and then specify it to our weighted case.

Mcp via Radon–Nikodym cocycle.

Recall that we called a cBer R on (X,μ) **mcp** if every Borel action $\Gamma \curvearrowright X$ of a ctbl group inducing R preserves the measure class of μ, i.e. its null sets.

More intrinsically: R is mcp $\iff \exists$ μ-essentially unique Borel $\rho : R \to \mathbb{R}_{>0}$, called the **Radon–Nikodym cocycle** of R with resp. to μ, such that: $(x,y) \mapsto \rho^y(x) = \frac{\text{weight}(x)}{\text{weight}(y)}$

(i) ρ is a **cocycle**: $\rho^y(x) \cdot \rho^z(y) = \rho^z(x)$ for all R-equivalent $x,y,z \in X$,

(ii) ρ satisfies **mass transport**: for every Borel (transport) $t : R \to [0,\infty]$, we have

$$\underbrace{\int \sum_{y \in [x]_R} t(x,y)\, d\mu(x)}_{\text{mass } x \text{ sends out}} = \underbrace{\int \sum_{y \in [x]_R} t(y,x)\, \rho^x(y)\, d\mu(x)}_{\text{mass } x \text{ receives}}$$

Prop. R is pmp $\iff \rho \equiv 1$ a.e., i.e. $\int \sum_{y \in [x]_R} t(x,y)\, d\mu(x) = \int \sum_{y \in [x]_R} t(y,x)\, d\mu(x)$.

Proof. $\Leftarrow$. Enough to show that for each Borel bijection $\gamma : A \to B$ with $\text{graph}(\gamma) \subseteq R$ we have $\mu(A) = \mu(B)$. Let $t := \mathbb{1}_{\text{graph}(\gamma)}$. Then mass transport gives $\mu(A) = \mu(B)$.

$\Rightarrow$. By Feldman–Moore, we write $R = \bigsqcup_{n \in \mathbb{N}} \text{graph}(\gamma_n)$ where each $\gamma_n : X \to X$ is a Borel partial involution, which allows us to replace $\sum_{y \in [x]_R} \cdots y$ with $\sum_{n \in \mathbb{N}} \cdots \gamma_n x$. Details are left as exercise. $\square$

When $\rho(x,y) = \frac{w(x)}{w(y)}$ for some Borel (weight) function $w : X \to \mathbb{R}_{>0}$, then we say that ρ is the **differential** of w.

Prop. Let R be a pmp cBer on (X,μ) and $F \subseteq R$ be a finite Borel subeq. rel. Let $\pi : X \to X/_F =: \tilde{X}$, $\tilde{\mu} := \pi_* \mu$, $\tilde{R} := R/_F$. Then $\tilde{R}$ is mcp on $(\tilde{X}, \tilde{\mu})$ and its Radon-Nikodym cocycle is the differential of the weight function $w : \tilde{X} \to \mathbb{R}_{>0}$ defined by
$$w([x]_F) := |[x]_F|.$$

Proof. Exercise.

Local-global bridge.

When proving pointwise ergodic theorems, one has to establish a connection between the global average $\int f \, d\mu$ and the local finite averages, and this is done via change of variable / mass transport. In our context, it is the following:

Local-global bridge lemma. Let F be an mcp finite Borel eq. rel. on (X,μ) with RN-cocycle $\rho : F \to \mathbb{R}_{>0}$. Then for each $f \in L^1(X,\mu)$,
$$\int f \, d\mu = \int A_F^\rho f \, d\mu,$$
where $A_F^\rho f(x) :=$ the ρ-weighted average of f over $[x]_F := \frac{1}{\rho^x([x]_F)} \sum_{y \in [x]_F} f(y) \cdot \rho^x(y)$.

Note. If ρ is a differential of $w : X \to \mathbb{R}_{>0}$, then
$$A_F^\rho f(x) = A_F^w f(x) := \frac{1}{w([x]_F)} \sum_{y \in [x]_F} f(y) \, w(y).$$

Proof. If F was pmp, then we can define a transport $t(x,y) := f(x) / |[x]_F|$. Then
$$\int f \, d\mu = \int \sum_{y \in [x]_F} f(x) \cdot \frac{1}{|[x]_F|} \, d\mu(x) = \int \sum_{y \in [x]_F} f(y) \cdot \frac{1}{|[y]_F|} \, d\mu(x) = \int \frac{1}{|[x]_F|} \sum_{y \in [x]_F} f(y) \, d\mu(x) = \int A_F f \, d\mu.$$
For general mcp F, we replace $1/|[x]_F|$ with $\rho^x(y) / \rho^x([x]_F)$, so $t(x,y) := f(x) \cdot \rho^x(y) / \rho^x([x]_F) =$

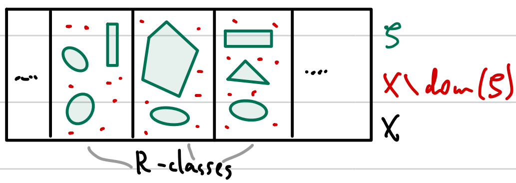
$= f(x)\, \rho^y(y) \big/ \rho^y([x]_F) = f(x)\, 1 \big/ \rho^y([y]_F)$. By mass transport:

$$\int f\, d\mu = \int \sum_{y\in[x]_F} f(x)\cdot \rho^x(y) \big/ \rho^x([x]_F)\, d\mu(x) = \int \sum_{y\in[x]_F} \underbrace{f(x)\cdot 1 \big/ \rho^y([y]_F)}_{t(x,y)}\, d\mu(x) = \int \sum_{y\in[x]_F} \underbrace{f(y)\cdot \frac{1}{\rho^x([x]_F)}\cdot \rho^x(y)}_{t(y,x)}\, d\mu(x)$$

$$= \int A^\rho_F f\, d\mu. \qquad \square$$

Tiling.

For a cBer R on a standard Borel X, the space $[X]^{<\infty}_R$ of all finite nonempty subsets of R-related points is also standard Borel. For a collection $\mathcal{C} \subseteq [X]^{<\infty}_R$, we call any pairwise disjoint subcollect $\mathcal{S} \subseteq \mathcal{C}$ a *tiling* with $\mathcal{C}$. Denote:



- $\mathrm{dom}(\mathcal{S}) := \bigcup \mathcal{S} = \bigcup_{S\in\mathcal{S}} S$.
- $R(\mathcal{S}) :=$ the equiv. rel. induced by $\mathcal{S}$,
  i.e. the classes inside $\mathrm{dom}(\mathcal{S})$ are exactly the elements of $\mathcal{S}$,
  and the classes outside of $\mathrm{dom}(\mathcal{S})$ are just singletons.

Note. If $\mathcal{S}$ is Borel, then $\mathrm{dom}(\mathcal{S})$ and $R(\mathcal{S})$ are also Borel.

The following is the most basic tiling theorem:

Maximal tilings (Kechris–Miller). Every Borel collection $\mathcal{C} \subseteq [X]^{<\infty}_R$ admits a Borel $\subseteq$-maximal tiling $\mathcal{S} \subseteq \mathcal{C}$.

Proof. Feldman–Moore says that $R \setminus \mathrm{Id}_X$, as a graph, admits a ctbl Borel edge-colouring. Using this, one gets that the intersection graph on $[X]_R^{<\infty}$ admits a ctbl Borel vertex-colouring, i.e. $\exists$ Borel $c : [X]_F^{<\infty} \to \mathbb{N}$ such that $U \cap V \neq \emptyset \Rightarrow c(U) \neq c(V)$. Then observe that a tiling is just an independent set in the intersection graph on $\mathcal{C}$, but every Borel graph that admits a ctbl Borel vertex-colouring, also admits a Borel maximal independent set. (Exercise) $\square$

How averacages work.

Let $w : X \to \mathbb{R}_{>0}$ be a weight function and fix a bdd $f : X \to \mathbb{R}$. For a finite nonempty set, put

$$A_U^w f := \frac{1}{w(U)} \sum_{y \in U} f(y) \cdot w(y).$$

Prop. For any disjoint nonempty finite sets U, V, and $x \in X \setminus U$, we have:

(a) $A^w_{U \sqcup V} f =$ convex combination of $A_U^w f$ and $A_V^w f = \dfrac{w(U)}{w(U)+w(V)} A_U^w f + \dfrac{w(V)}{w(U)+w(V)} A_V^w f$.

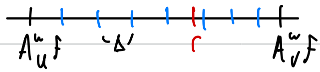
(b) $A^w_{U \cup \{x\}} f \approx_\Delta A_U^w f$, where $\Delta := 2\|f\|_\infty \cdot \max\limits_{y \in X} w(y) \cdot \dfrac{1}{w(U)}$. (Think that $w(U)$ is huge.)

Proof. (a) is by definition, and (b) follows from (a):

$$|A^w_{U \cup \{x\}} f - A^w_U f| \leq \frac{w(x)}{w(U)+w(x)} |A^w_{\{x\}} f - A^w_U f| \leq \Delta. \qquad \square$$

Intermediate value property. Let $U \subseteq V \subseteq X$ be finite nonempty sets. Then for every real r between $A^w_U f$ and $A^w_V f$, there is $U \subseteq I \subseteq V$ with $A^w_I f \approx_\Delta r$, where $\Delta := 2\|f\|_\infty \cdot \max_{y \in X} w(y) \cdot \frac{1}{w(U)}$.

Proof. Adding the points of $V \setminus U$ to U one-by-one, we get
$$U =: I_0 \subseteq I_1 \subseteq \dots \subseteq I_n = V, \quad \text{where } |I_{i+1} \setminus I_i| = 1$$
so $A^w_U f = A^w_{I_0} f \approx_\Delta A^w_{I_1} f \approx_\Delta \dots \approx A^w_{I_n} f = A^w_V f$.



$\square$

Main Lemma for cBers.

We will first prove the following much easier lemma:

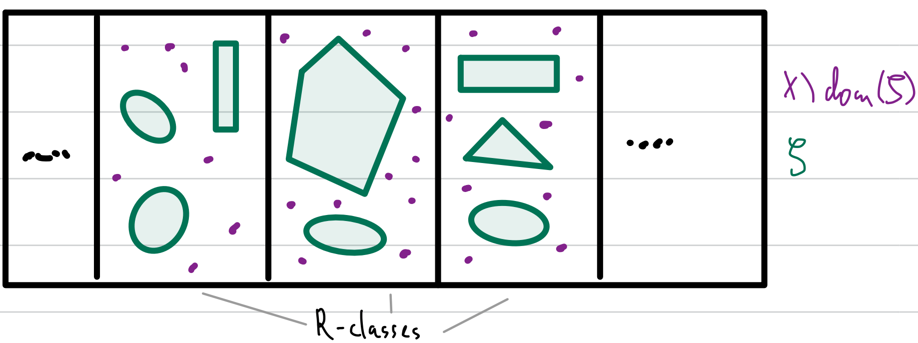
Main Lemma for cBers. Let R be an ergodic pmp cBer on (X,μ) and $F_0 \subseteq R$ be a finite Borel eq. rel. Then for each $f \in \mathcal{L}^\infty(X,\mu)$ and $\varepsilon > 0$, there is a finite Borel subeq. rel. $F_1 \supseteq F_0$ of R such that
$$A_{F_1} f(x) := (\text{average of } f \text{ over } [x]_{F_1}) \approx_\varepsilon \int f \, d\mu$$
for all x in a set of measure $\geq 1-\varepsilon$.

Note that it is enough to prove this assuming F_0-classes have bounded size. Indeed, we can then return an F_1 with bounded classes by taking a large enough $N \in \mathbb{N}$ so that classes of F_1 of size $\leq N$ form a $\geq 1-\varepsilon$ measure set, and we can keep F_0 on the remaining part.

By taking the quotient $\pi : X \to X/_{F_0} =: \tilde{X}$, we can replace X, μ, R, f with $\tilde{X}$, $\tilde{\mu} := \pi_* \mu$, $\tilde{R} := R/_{F_0}$, and $\tilde{f} := A_{F_0} f$. We also replace $A_{F_1} f$ with the weighted average $A^w_{F_1} f$, where $w : \tilde{X} \to \mathbb{R}_{>0}$ defined by $w([x]_{F_0}) := |[x]_{F_0}|$.
Thus, it's enough to prove:

<u>Quotient Main Lemma for cBers.</u> Let R be an ergodic mcp cBer on (X, μ) whose RN-cocycle is the differential of a bdd Borel function $w : X \to \mathbb{N}_{>0}$.
Then for every $f \in L^\infty(X, \mu)$ and $\varepsilon > 0$, there is a finite Borel subeq. rel. $F \subseteq R$ such that $A^w_F f \approx_\varepsilon \int f \, d\mu$ for all x in a set of measure $\geq 1 - \varepsilon$.

Proof. By subtracting $\int f \, d\mu$ from f, we may assume that $\int f \, d\mu = 0$. For any $\delta > 0$, call a finite nonempty set $U \subseteq X$ δ-negative / δ-zero / δ-positive if $A^w_U f$ is $< -\delta$ / $\in [-\delta, \delta]$ / $> \delta$. Assuming $\varepsilon < 1$, put $\delta := \varepsilon^2$ and let $\mathcal{S} \subseteq [X]^{<\infty}_R$ be a Borel maximal tiling with δ-zero sets. We will show that $F := R(\mathcal{S})$ works.



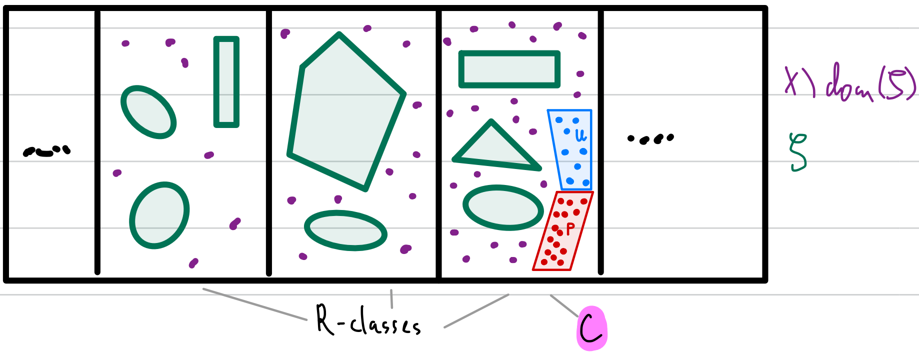
What kinds of points (δ-negative/zero/positive) are left out of $\mathrm{dom}(\mathcal{S})$?

Claim. After discarding an R-invariant null set, all points in $X \setminus \mathrm{dom}(\xi)$ are δ-negative or all points in $X \setminus \mathrm{dom}(\xi)$ are δ-positive.

Proof. Suppose not. Then the ergodicity of R yields that, after discarding a null set, each R-class contains both δ-negative and δ-positive points.

The R-classes which have only finitely many of one or the other form a smooth, hence null (by ergodicity) set, which we ignore. So all R-classes have infinitely many δ-negative and δ-positive points. Fix an R-class C and let $U \subseteq C \setminus \mathrm{dom}(\xi)$ be a finite finite set of δ-negative points such that

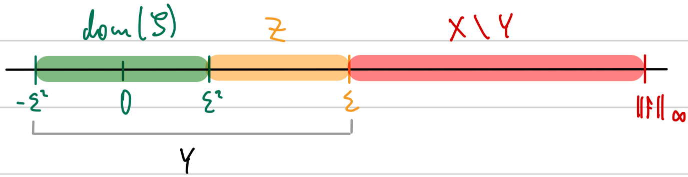
$$\Delta := 2\|f\|_\infty \cdot \|w\|_\infty / w(U) < \delta.$$



Let $V = U \sqcup P$ where P is a large enough subset of $C \setminus \mathrm{dom}(\xi)$ of δ-positive points so that $A_V^w f > 0$ (we can do this because $w \geq 1$).

Then by the intermediate value property for $U \subseteq V$, we get a set $U \subseteq I \subseteq V$ with $A_I^w f \approx_\Delta 0$ hence I is δ-zero, contradicting the maximality of ξ. $\square$

Let $Y :=$ the union of all F-classes that are ε-zero, so it is enough to show that $\mu(Y) \geq 1-\varepsilon$. By the claim, after discarding an R-invariant null set, we may assume that each point $x \in X \setminus \mathrm{dom}(\mathcal{S})$ is δ-positive (the δ-negative case is handled analogously). Then $Y = \mathrm{dom}(\mathcal{S}) \sqcup Z$, where $Z := \{x \in X \setminus \mathrm{dom}(\mathcal{S}) : f(x) \in (\varepsilon^2, \varepsilon]\}$.



We simply compute using the local-global bridge lemma:

$$0 = \int_X f \, d\mu = \int_X A^w_F f \, d\mu = \int_{\mathrm{dom}(\mathcal{S})} A^w_F f \, d\mu + \int_Z A^w_F f \, d\mu + \int_{X\setminus Y} A^w_F f \, d\mu$$

$$(*) \qquad \geq -\varepsilon^2 \cdot \mu(\mathrm{dom}(\mathcal{S})) + \varepsilon^2 \cdot \mu(Z) + \varepsilon \cdot \mu(X \setminus Y)$$

$$\geq -\varepsilon^2 \cdot 1 + \varepsilon^2 \cdot 0 + \varepsilon \, \mu(X \setminus Y),$$

so $\mu(X \setminus Y) \leq \varepsilon$, as desired. $\square$

Remark. The inequality $(*)$ above also shows that $\mathrm{dom}(\mathcal{S})$ must have positive measure. Achieving the analogue of this for graphs would be our first challenge next time, when we take as $\mathcal{S}$ a Borel maximal tiling of δ-zero connected sets. It's not even clear that this $\mathcal{S}$ is nonempty.