The existence of ergodic hyperfinite subgraphs, part 2

Anush Tserunyan

(McGill University, visiting Université Paris-Cité)

Recap.

We reduced proving the existence of hyperfinite ergodic subgraphs to the following:

Main Lemma. Let G be a loc. ctbl ergodic mcp Borel graph and let $H_0 \subseteq G$ be a component-finite Borel subgraph. Then for each $f \in L^\infty(X,\mu)$ and $\varepsilon > 0$, there is a component-finite Borel subgraph $H_1 \supseteq H_0$ of G such that

$$(\text{RN-weighted average of } f \text{ over } [x]_{H_1}) \approx_\varepsilon \int f d\mu$$

for all x in a set of measure $\geq 1-\varepsilon$.

We will prove this in the pmp setting and then discuss in the end what extra modifications and ingredients are needed for the general mcp setting. Even when G is pmp, to prove the Main Lemma, it is convenient to work with the quotient $X/H_0 := X/R_{H_0}$ in which every point $[x]_{H_0} \in X/H_0$ has weight $w([x]_{H_0}) := |[x]_{H_0}|$.
In other words, if $\pi : X \to X/H_0$ is the quotient map, then $\tilde{R} := R/H_0$ is no longer pmp on $(X/H_0, \pi_*\mu)$, it is a very nice mcp.

So let's introduce the mcp setting properly, and then specify it to our weighted case.

Mcp via Radon-Nikodym cocycle.

Recall that we called a cBer R on (X,μ) mcp if every Borel action $\Gamma \curvearrowright X$ of a ctbl group inducing R preserves the measure class of μ, i.e. its null sets.

More intrinsically: R is mcp $\iff$ $\exists$ μ-essentially unique Borel $\rho : R \to \mathbb{R}_{>0}$, called the Radon-Nikodym cocycle of R with resp. to μ, such that: $(x,y) \mapsto \rho^y(x) = \frac{\text{weight}(x)}{\text{weight}(y)}$

(i) ρ is a cocycle: $\rho^y(x) \cdot \rho^z(y) = \rho^z(x)$ for all R-equivalent $x,y,z \in X$.

(ii) ρ satisfies mass transport: for every Borel (transport) $t : R \to [0,\infty]$, we have

$$\int \sum_{y \in [x]_R} t(x,y)\, d\mu(x) = \int \sum_{y \in [x]_R} t(y,x)\, \rho^x(y)\, d\mu(x)$$

(mass x sends out) (mass x receives)

Prop. R is pmp $\iff$ $\rho \equiv 1$ a.e., i.e. $\int \sum_{y \in [x]_R} t(x,y)\, d\mu(x) = \int \sum_{y \in [x]_R} t(y,x)\, d\mu(x)$.

Proof. $\Leftarrow$. Enough to show that for each Borel bijection $\gamma : A \to B$ with $\text{graph}(\gamma) \subseteq R$ we have $\mu(A) = \mu(B)$. Let $t := \mathbb{1}_{\text{graph}(\gamma)}$. Then mass transport gives $\mu(A) = \mu(B)$.

$\Rightarrow$. By Feldman-Moore, we write $R = \bigsqcup_{n \in \mathbb{N}} \text{graph}(\gamma_n)$ where each $\gamma_n : X \to X$ is a Borel partial involution, which allows us to replace $\sum_{y \in [x]_R} \cdots$ with $\sum_{n \in \mathbb{N}} \cdots \gamma_n x$. Details are left as exercise. □

When $\rho(x,y) = \frac{w(x)}{w(y)}$ for some Borel (weight) function $w : X \to \mathbb{R}_{>0}$, then we say that ρ is the differential of w.

Prop. Let R be a pmp cBer on (X,μ) and $F \subseteq R$ be a finite Borel subeq. rel. Let $\pi: X \to X/F =: \tilde{X}$, $\tilde{\mu} := \pi_* \mu$, $\tilde{R} := R/F$. Then $\tilde{R}$ is mep on $(\tilde{X}, \tilde{\mu})$ and its Radon-Nikodym cocycle is the differential of the weight function $w: \tilde{X} \to \mathbb{R}_{>0}$ defined by

$$w([x]_F) := |[x]_F|.$$

Proof. Exercise.

Local–global bridge.

When proving pointwise ergodic theorems, one has to establish a connection between the global average $\int f \, d\mu$ and the local finite averages, and this is done via change of variable / mass transport. In our context, it is the following:

Local-global bridge lemma. Let F be an mep finite Borel eq. rel. on (X,μ) with RN-cocycle $\rho: F \to \mathbb{R}_{>0}$. Then for each $f \in L^1(X,\mu)$,

$$\int f \, d\mu = \int A_F^\rho f \, d\mu,$$

where $A_F^\rho f(x) :=$ the ρ-weighted average of f over $[x]_F := \dfrac{1}{\rho^x([x]_F)} \displaystyle\sum_{y \in [x]_F} f(y) \cdot \rho^x(y)$.

Note. If ρ is a differential of $w: X \to \mathbb{R}_{>0}$, then

$$A_F^\rho f(x) = A_F^w f(x) := \frac{1}{w([x]_F)} \sum_{y \in [x]_F} f(y) \, w(y).$$

Proof. If F was pmp, then we can define a transport $t(x,y) := f(x) / |[x]_F|$. Then

$$\int f \, d\mu = \int \sum_{y \in [x]_F} f(x) \cdot \frac{1}{|[x]_F|} \, d\mu(x) = \int \sum_{y \in [x]_F} f(y) \cdot \frac{1}{|[y]_F|} \, d\mu(x) = \int \frac{1}{|[x]_F|} \sum_{y \in [x]_F} f(y) \, d\mu(x) = \int A_F f \, d\mu.$$

For general mep F, we replace $1/|[x]_F|$ with $\rho^x(y)/\rho^x([x]_F)$, so $t(x,y) := f(x) \cdot \rho^x(y)/\rho^x([x]_F) =$

$= f(x)\ \rho^y(y) / \rho^y([x]_F) = f(x)\ 1/\rho^y([y]_F)$. By mass transport:

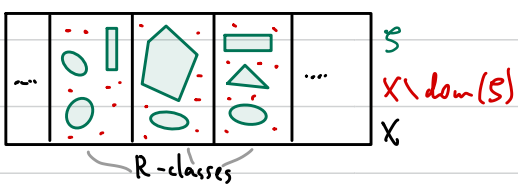
$$\int f\,d\mu = \int \sum_{y\in[x]_F} f(x)\cdot \frac{\rho^x(y)}{\rho^x([x]_F)}\,d\mu(x) = \int \sum_{y\in[x]_F} \underbrace{f(x)\cdot \frac{1}{\rho^y([y]_F)}}_{t(x,y)}\,d\mu(x) = \int \sum_{y\in[x]_F} \underbrace{f(y)\cdot \frac{1}{\rho^x([x]_F)}}_{t(y,x)}\cdot \rho^x(y)\,d\mu(x)$$

$$= \int A_F^\rho f\,d\mu. \qquad \square$$

Tiling.

For a cBer R on a standard Borel X, the space $[X]_R^{<\infty}$ of all finite nonempty subsets of R-related points is also standard Borel. For a collection $\mathcal{C} \subseteq [X]_R^{<\infty}$, we call any pairwise disjoint subcollect $\mathcal{S} \subseteq \mathcal{C}$ a tiling with $\mathcal{C}$. Denote:

- $\mathrm{dom}(\mathcal{S}) := \bigcup \mathcal{S} = \bigcup_{S\in\mathcal{S}} S$.
- $R(\mathcal{S})$:= the equiv. rel. induced by $\mathcal{S}$,
  i.e. the classes inside $\mathrm{dom}(\mathcal{S})$ are exactly the elements of $\mathcal{S}$,
  and the classes outside of $\mathrm{dom}(\mathcal{S})$ are just singletons.



Note. If $\mathcal{S}$ is Borel, then $\mathrm{dom}(\mathcal{S})$ and $R(\mathcal{S})$ are also Borel.

The following is the most basic tiling theorem:

Maximal tilings (Kechris-Miller). Every Borel collection $\mathcal{C} \subseteq [X]_R^{<\infty}$ admits a Borel $\subseteq$-maximal tiling $\mathcal{S} \subseteq \mathcal{C}$.

Proof. Feldman–Moore says that $R \setminus Id_X$, as a graph, admits a ctbl Borel edge-colouring. Using this, one gets that the intersection graph on $[X]^{<\infty}_R$ admits a ctbl Borel vertex-colouring, i.e. $\exists$ Borel $c : [X]^{<\infty}_F \to \mathbb{N}$ such that $U \cap V \neq \emptyset \Rightarrow c(U) \neq c(V)$. Then observe that a tiling is just an independent set in the intersection graph on $\mathcal{C}$, but every Borel graph that admits a ctbl Borel vertex-colouring, also admits a Borel maximal independent set. (Exercise) $\square$

Properties of averages.

Let $w : X \to \mathbb{R}_{>0}$ be a weight function and fix a bdd $f : X \to \mathbb{R}$. For a finite nonempty set, put

$$A^w_U f := \frac{1}{w(U)} \sum_{y \in U} f(y) \cdot w(y).$$

Prop. For any disjoint nonempty finite sets U, V, and $x \in X \setminus V$, we have:

(a) $A^w_{U \cup V} f$ = convex combination of $A^w_U f$ and $A^w_V f$ = $\frac{w(U)}{w(U)+w(V)} A^w_U f + \frac{w(V)}{w(U)+w(V)} A^w_V f$.

(b) $|A^w_{U \cup V} f - A^w_V f| \leq 2\|f\|_\infty \, w(U)/w(V)$. In particular:
$A^w_{V \cup \{x\}} f \approx_\Delta A^w_V f$, where $\Delta := 2\|f\|_\infty \|w\|_\infty / w(V)$.

Proof. (a) is by definition, and (b) follows from (a):

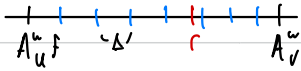
$$|A^w_{U \cup V} f - A^w_V f| \leq \frac{w(U)}{w(U)+w(V)} \, |A^w_U f - A^w_V f| \leq 2\|f\|_\infty \, w(U)/w(V). \quad \square$$

Intermediate value property. Let $U \subseteq V \subseteq X$ be finite nonempty subs. Then for every real r between $A^w_U f$ and $A^w_V f$, there is $U \subseteq I \subseteq V$ with $A^w_I f \approx_\Delta r$, where $\Delta := 2\|f\|_w \cdot \|w\|_\infty \cdot \frac{1}{w(U)}$.

Proof. Adding the points of $V \setminus U$ to U one-by-one, we get

$$U =: I_0 \subseteq I_1 \subseteq \dots \subseteq I_n := V, \quad \text{where } |I_{i+1} \setminus I_i| = 1$$

so $A^w_U f = A^w_{I_0} f \approx_\Delta A^w_{I_1} f \approx_\Delta \dots \approx_\Delta A^w_{I_n} f = A^w_V f$.



□

Main Lemma for cBers.

We will first prove the following much easier lemma:

Main Lemma for cBers. Let R be an ergodic pmp cBer on (X,μ) and $F_0 \subseteq R$ be a finite Borel eq. rel. Then for each $f \in L^\infty(X,\mu)$ and $\varepsilon > 0$, there is a finite Borel subeq. rel. $F_1 \supseteq F_0$ of R such that

$$A_{F_1} f(x) := (\text{average of } f \text{ over } [x]_{F_1}) \approx_\varepsilon \int f \, d\mu$$

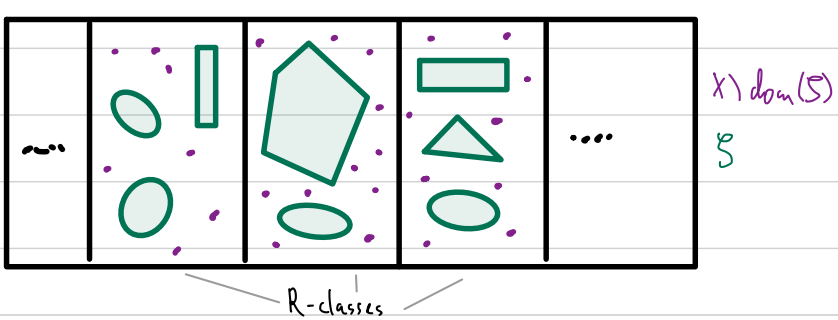
for all x in a set of measure $\geq 1-\varepsilon$.

Note that it is enough to prove this assuming F_0-classes have bounded size. Indeed, we can then return an F_1 with bounded classes by taking a large enough $N \in \mathbb{N}$ so that classes of F_1 of size $\leq N$ form a $\geq 1-\varepsilon$ measure set, and we can keep F_0 on the remaining part.

By taking the quotient $\pi: X \to X/_{F_0} =: \tilde{X}$, we can replace X, μ, R, f with $\tilde{X}$, $\tilde{\mu} := \pi_* \mu$, $\tilde{R} := R/_{F_0}$, and $\tilde{f} := A_{F_0} f$. We also replace $A_{F_1} f$ with the weighted average $A^w_{F_1} f$, where $w: \tilde{X} \to \mathbb{R}_{>0}$ defined by $w([x]_{F_0}) := |[x]_{F_0}|$. Thus, it's enough to prove:

Quotient Main Lemma for cBers: Let R be an ergodic mcp cBer on (X, μ) whose RN-cocycle is the differential of a fdd Borel function $w: X \to \mathbb{N}_{>0}$. Then for every $f \in L^\infty(X, \mu)$ and $\varepsilon > 0$, there is a finite Borel subeq. rel. $F \subseteq R$ such that $A^w_F f \approx_\varepsilon \int f \, d\mu$ for all x in a set of measure $\geq 1 - \varepsilon$.

Proof. By subtracting $\int f \, d\mu$ from f, we may assume that $\int f \, d\mu = 0$. For any $\delta > 0$, call a finite nonempty set $U \subseteq X$ δ-negative / δ-zero / δ-positive if $A^w_U f$ is $< -\delta$ / $\in [-\delta, \delta]$ / $> \delta$. Assuming $\varepsilon < 1$, put $\delta := \varepsilon^2$ and let $S \subseteq [X]^{<\infty}_R$ be a Borel maximal tiling with δ-zero sets. We will show that $F := R(S)$ works.



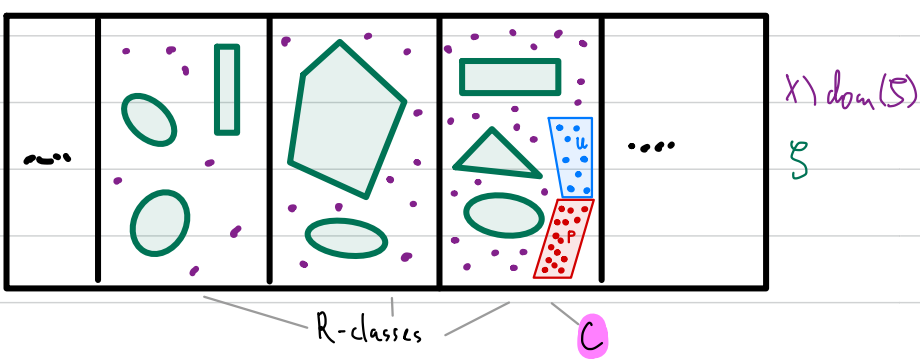
What kinds of points (δ-negative/zero/positive) are left out of $\mathrm{dom}(S)$?

Claim. After discarding an R-invariant null set, all points in $X \setminus \mathrm{dom}(\xi)$ are δ-negative or all points in $X \setminus \mathrm{dom}(\xi)$ are δ-positive.

Proof. Suppose not. Then the ergodicity of R yields that, after discarding a null set, each R-class contains both δ-negative and δ-positive points.

The R-classes which have only finitely many of one or the other form a smooth, hence null (by ergodicity) set, which we ignore. So all R-classes have infinitely many δ-negative and δ-positive points. Fix an R-class C and let $U \subseteq C \setminus \mathrm{dom}(\xi)$ be a finite finite set of δ-negative points such that

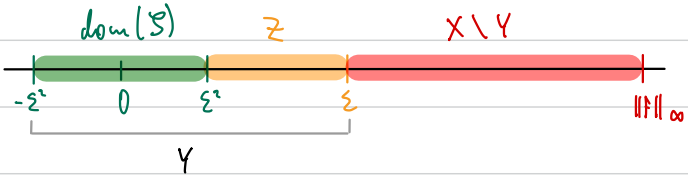
$$\Delta := 2\|f\|_\infty \cdot \|w\|_\infty / w(U) < \delta.$$



Let $V = U \sqcup P$ where P is a large enough subset of $C \setminus \mathrm{dom}(\xi)$ of δ-positive points so that $A_V^w f > 0$ (we can do this because $w \geq 1$).

Then by the intermediate value property for $U \subseteq V$, we get a set $U \subseteq I \subseteq V$ with $A_I^w f \approx_\Delta 0$ hence I is δ-zero, contradicting the maximality of ξ. □

Let $Y :=$ the union of all F-classes that are ξ-zero, so it is enough to show that $\mu(Y) \geq 1-\xi$. By the claim, after discarding an R-invariant null set, we may assume that each point $x \in X \setminus \mathrm{dom}(S)$ is δ-positive (the δ-negative case is handled analogously). Then $Y = \mathrm{dom}(S) \sqcup Z$, where $Z := \{x \in X \setminus \mathrm{dom}(S) : f(x) \in (\xi^2, \xi]\}$.



We simply compute using the local-global bridge lemma:

$$0 = \int_X f\,d\mu = \int_X A_F^\omega f\,d\mu = \int_{\mathrm{dom}(S)} A_F^\omega f\,d\mu + \int_Z A_F^\omega f\,d\mu + \int_{X\setminus Y} A_F^\omega f\,d\mu$$

$$(*) \quad \geq -\xi^2 \cdot \mu(\mathrm{dom}(S)) + \xi^2 \cdot \mu(Z) + \xi \cdot \mu(X \setminus Y)$$

$$\geq -\xi^2 \cdot 1 + \xi^2 \cdot 0 + \xi\,\mu(X \setminus Y),$$

so $\mu(X \setminus Y) \leq \xi$, as desired. □

Remark. The inequality $(*)$ above also shows that $\mathrm{dom}(S)$ must have positive measure. Achieving the analogue of this for graphs would be our first challenge next time, when we take as S a Borel maximal tiling of δ-zero connected sets. It's not even clear that this S is nonempty.