The existence of ergodic hyperfinite subgraphs, part 3

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Recap.

We reduced proving the existence of hyperfinite ergodic subgraphs to the following:

Main Lemma. Let G be a loc. ctbl ergodic pmp Borel graph and let $H_0 \subseteq G$ be a component-finite Borel subgraph. Then for each $f \in \mathcal{L}^\infty(X,\mu)$ and $\varepsilon > 0$, there is a component-finite Borel subgraph $H_1 \supseteq H_0$ of G such that

$$A_{H_1} f(x) := (\text{average of } f \text{ over } [x]_{H_1}) \approx_\varepsilon \int f \, d\mu$$

for all x in a set of measure $\geq 1-\varepsilon$.

Also recall that by modifying H_0 on an R_{H_0}-invariant set of measure $\leq \varepsilon/2$, we may assume that the H_0-components have bounded size. Thus, taking the quotient by the G-connected bounded eq. rel. R_{H_0}, it is enough to prove the following:

Quotient Main Lemma for graphs. Let G be an ergodic mcp loc. ctbl graph on (X,μ), whose RN-cocycle is the differential of a bounded Borel function $w : X \to \mathbb{N}_{>0}$. Then for every $f \in L^\infty(X,\mu)$ and $\varepsilon > 0$, there is a component-finite Borel subgraph $H \subseteq G$ such that $A_H^w f(x) := \frac{1}{w([x]_H)} \sum_{y \in [x]_H} f(y) \cdot w(y) \approx_\varepsilon \int f \, d\mu$ for all x in a set of measure $\geq 1-\varepsilon$.

Setup of proof of Main Lemma.

Call a subset $V \subseteq X$ G-connected if the induced subgraph $G|_V := G \cap (V \times V)$ is a connected graph on V. Call a cBer F on X G-connected if each F-class is G-connected, equivalently, the connected components of the graph $G \cap F$ are exactly the F-classes.

To prove the main lemma, we need to find a G-connected finite Borel subeq. rel. $F \subseteq R_G$ with $A_F^w f(x) \approx_\varepsilon \int f \, d\mu$ for all x in a set of measure $\geq 1-\varepsilon$.

We again may assume that $\int f \, d\mu = 0$ and for any $\delta > 0$, call a finite set $U \subseteq X$ δ-negative / δ-zero / δ-positive if $A_U^w f := \frac{1}{w(U)} \cdot \sum_{y \in U} f(y) w(y)$ is $< -\delta$ / $\in [-\delta, \delta]$ / $> \delta$.

We will construct a desired G-connected eq. rel. F by taking some kind of Borel maximal tiling with G-connected δ-zero sets. But how do we know such sets even exist???

Asymptotic averages along G (0.)

Let G, w, f be as in the quotient main lemma. For a point $x \in X$, call a real $r \in \mathbb{R}$ an asymptotic average of f along G at x if r can be approximated by averages over arbitrarily w-large finite G-connected $U \ni x$, i.e. $\forall \varepsilon > 0 \ \forall N > 0$ there is a finite G-connected $U \ni x$ such that $w(U) \geq N$ and $A_U^w f \approx_\varepsilon r$.

In other words, these are all possible averages which can be approximated by arbitrarily large connected neighbourhoods of x.

Denote by $\mathcal{A}_G^w f(x)$ the set of all asymptotic averages of f along G at x.

Observation. For each $x \in X$, $r \in \mathcal{A}_G^w(x)$, finite G-connected set $U \ni x$, and $\varepsilon > 0$, there are arbitrarily w-large $V \supseteq U \ni x$ with $A_V^w f \approx_\varepsilon r$.

Proof. Take $\tilde{V} \ni x$ large enough so the impact of U in $V := U \cup \tilde{V}$ is negligable. □

Properties of $\mathcal{A}_G^w$.

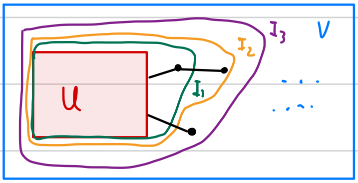
(a) $A_G^w(x)$ is a closed interval $\subseteq [-\|f\|_\infty, \|f\|_\infty]$.

(b) $\mathcal{A}_G^w(x) \neq \emptyset$ for a.e. $x \in X$.

(c) $x \mapsto A_G^w(x) : X \to F(\mathbb{R})$ is R_G-invariant Borel, hence consant a.e. by ergodicity.

Proof. (a) The closedness of $\mathcal{A}_G^w(x)$ follows from the asymptotic nature of the def. of asymptotic averages. That $\mathcal{A}_G^w(x)$ is convex follows from:

Connected intermediate value property. Let $U \subseteq V \Subset X$ be finite nonempty G-connected sets. Then for each real r between $A_U^w f$ and $A_V^w f$, there is a G-connected set $U \subseteq I \subseteq V$ with $A_I^w \approx_\Delta r$, where $\Delta := 2\|f\|_\infty \cdot \|w\|_\infty / w(U)$.



Proof. Same as before, but we add vertices to U in such order so that the intermediate sets $U \subseteq I_1 \subseteq I_2 \subseteq \dots \subseteq I_n \subseteq V$ are G-connected. ⊠

Now if $a, b \in \mathcal{A}_G^w f(x)$, $a \le b$, and $a \le r \le b$, then let $U \ni x$ be an arbitrarily w-large

G-connected set with $A^w_U f \approx_{\varepsilon/3} a$, and by the Observation above, get a G-connected finite $V \supseteq U \ni x$ such that $A^w_V \approx_{\varepsilon/3} b$. Then the intermediate value property gives a G-connected $U \subseteq I \subseteq V$ with $A^w_f I \approx_{2\varepsilon + \Delta} r$, where $\Delta := 2\|f\|_\infty \|w\|_\infty / w(U) \leq \varepsilon/3$.

(b) By ergodicity, a.e. R_G-class is w-infinite (w-finite classes form a smooth set) so the compactness of $[-\|f\|_\infty, \|f\|_\infty]$ finishes the proof.

(c) Let $x \in X$ and $r \in \mathcal{A}^w_G f(x)$. We show that $r \in \mathcal{A}^w_G(y)$ for any $y \in [x]_G$. Let $P := \{x = x_0, x_1, \dots, x_n = y\}$ be a G-path, then Observation gives an arbitrarily large $V \supseteq P \supseteq \{x, y\}$ with $A^w_V f \approx_\varepsilon r$, witnessing $r \in \mathcal{A}^w_G(y)$. □

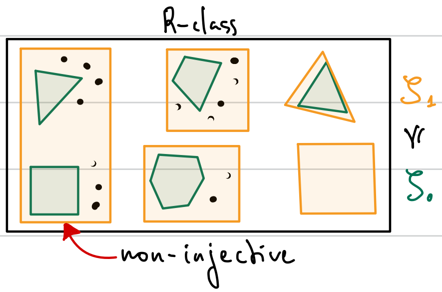
Discarding an R_G-invariant null set, we get that $\mathcal{A}^w_G f$ is constant.
Now if the main lemma is at all true, then we'd at least have $\int f \, d\mu \in \mathcal{A}^w_G f$.
Thus, we better be able to prove $\int f \, d\mu \in \mathcal{A}^w_G f$. This will indeed follow from the local-global bridge lemma and the following:

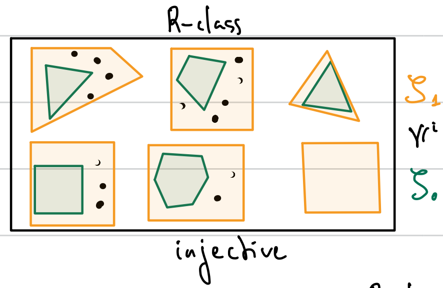
Asymptotic averages (AA) tiling lemma. For each $\varepsilon > 0$, there is a Borel tiling $\mathcal{S} \subseteq [X]^{<\infty}_G$:= finite nonempty G-connected sets such that $\operatorname{dom}(\mathcal{S})$ is conull and $A^w_{\mathcal{S}} f \in_\varepsilon \mathcal{A}^w_G$, i.e. $\operatorname{distance}(A^w_{\mathcal{S}} f, \mathcal{A}^w_G f) \leq \varepsilon$.

To prove this we need slightly better Borel maximal tilings, which we now discuss.

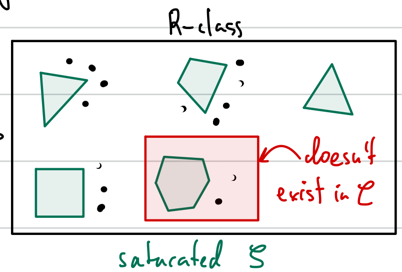
Saturated tilings (Miller – Ö.)

Let R be a cBer on a standard Borel space X. For tilings $\mathcal{S}_0, \mathcal{S}_1 \subseteq [X]^{<\infty}_R$, we say that $\mathcal{S}_1$ **extends** $\mathcal{S}_0$, and write $\mathcal{S}_0 \preccurlyeq \mathcal{S}_1$, if each tile $S \in \mathcal{S}_0$ is contained in a tile $\tilde{S} \in \mathcal{S}_1$. If $S \mapsto \tilde{S}$ is injective, we say that $\mathcal{S}_1$ **injectively extends** $\mathcal{S}_0$, and write $\mathcal{S}_0 \preccurlyeq^i \mathcal{S}_1$.





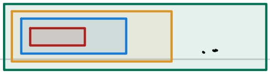
Def. For $\mathcal{C} \subseteq [X]^{<\infty}_R$, call a tiling $\mathcal{S} \subseteq \mathcal{C}$ **saturated within $\mathcal{C}$** if there is no proper injective extension with $\mathcal{C}$, i.e. $\nexists\, \tilde{\mathcal{S}} \succcurlyeq^i \mathcal{S}$ and $\tilde{\mathcal{S}} \subseteq \mathcal{C}$ and $\tilde{\mathcal{S}} \neq \mathcal{S}$.



Note. saturated $\Rightarrow$ maximal.

Theorem. Let R be an mcp cBer on (X, μ) whose RN-cocycle is the differential of a Borel function $w : X \to \mathbb{N}_{>0}$. Every Borel collection $\mathcal{C} \subseteq [X]^{<\infty}_R$ admits a Borel saturated tiling $\mathcal{S} \subseteq \mathcal{C}$, after discarding an R-invariant null set.

Proof. Let $c : \mathcal{C} \to \mathbb{N}$ be a Borel colouring of the intersection graph on $\mathcal{C}$. Iteratively construct a sequence $\mathcal{S}_0 \preccurlyeq^i \mathcal{S}_1 \preccurlyeq^i \mathcal{S}_2 \preccurlyeq^i \dots$ of tilings with $\mathcal{C}$ so that $\mathcal{S}_n$ contains all sets $U \in \mathcal{C}$

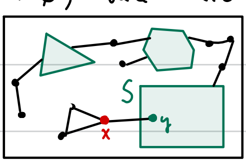
of colour n which contain ≤ 1 tile from $\mathcal{S}_{n-1}$. Then $F_\infty := \bigcup^{\uparrow}_{n\in\mathbb{N}} R(\mathcal{S}_n)$ is smooth because each F_∞-class looks like this: . This implies that each F_∞-class is w-finite a.e. (Exercise: smooth eq. rel. have w-finite classes a.e. by mass transport.) Hence each F_∞-class is finite a.e. because $w \geq 1$. Discarding an R-invariant null set, F_∞ is finite and one checks that $\mathcal{S} := \lim_{n\to\infty} \mathcal{S}_n := \{S \in X/F_\infty : S \subseteq \bigcup_n \mathrm{dom}(\mathcal{S}_n)\}$ is saturated. □

Asymptotic averages along G (continued).

AA tiling lemma. For each $\varepsilon > 0$, there is a Borel tiling $\mathcal{S} \subseteq [X]^{<\infty}_G$ with conull domain such that $A^w_{\mathcal{S}} f \in_\varepsilon \mathcal{A}^w_G$.

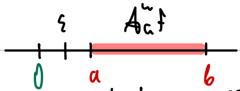
Proof. Note that for each $x \in X$ there is $N_x > 0$ such that if $U \ni x$ is a G-connected finite set with $w(U) \geq N_x$ then $A^w_U f \in_\varepsilon \mathcal{A}^w_G f$. (Indeed, otherwise a sequence (U_n) of counter-examples and compactness of $[-\|f\|_\infty, \|f\|_\infty]$ will witness the existence of an $r \in \mathcal{A}^w_G f(x)$ with $r \notin_\varepsilon \mathcal{A}^w_G f(x)$.) Let $\mathcal{C}$:= all $U \subseteq [X]^{<\infty}_G$ such that $w(U) \geq N_x$ for some $x \in U$, and discarding an R_G-invariant null set, get a Borel saturated tiling $\mathcal{S} \subseteq \mathcal{C}$.

Because $\mathcal{C}$ has tiles in every R_G-class and $\mathcal{S}$ is maximal, $\mathcal{S}$ too must have tiles in every R_G-class. This and saturation imply that $\mathrm{dom}(\mathcal{S}) = X$. Indeed, if $X \setminus \mathrm{dom}(\mathcal{S}) \neq \emptyset$, then there is $x \in X \setminus \mathrm{dom}(\mathcal{S})$ that is G-adjacent to some $y \in \mathrm{dom}(\mathcal{S})$. Letting S be the tile of $\mathcal{S}$ containing y, we have $S \cup \{x\} \in \mathcal{C}$, contradicting saturation. □



R_G-class

Corollary. $0 = \int f \, d\mu \in \mathcal{A}^\omega_G f$.

Proof. Suppose not. Because $A^\omega_G f = [a, b]$, we must have $0 < a$ or $b < 0$. Suppose for concreteness that $0 < a$, i.e. 

. Let $\varepsilon := a/2$. Then the AA tiling lemma gives, after discarding a null set, a Borel tiling $\mathcal{S} \subseteq [X]^{<\infty}_G$ with G-connected finite sets S with $A^\omega_S f \in_\varepsilon [a, b]$, so $A^\omega_S f \geq \varepsilon$, and $\mathrm{dom}(\mathcal{S}) = X$. But this contradicts the bridge lemma:

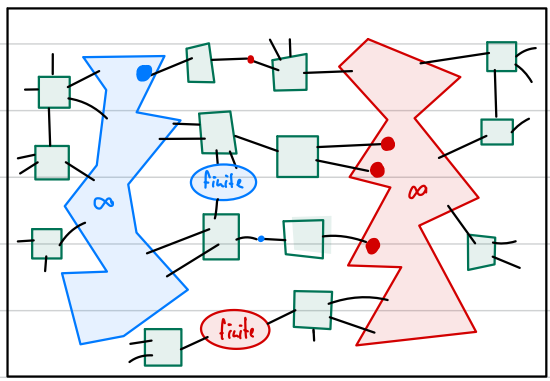
$$0 = \int f \, d\mu = \int A^\omega_{R(\mathcal{S})} f \, d\mu \geq \varepsilon > 0.$$

□

Proof of Main Lemma

Okay, now set $\delta := \varepsilon^2$ and take a Borel maximal (even saturated, why not) tiling $\mathcal{S}$ with G-connected δ-zero sets. Now we know that $\mathcal{S}$ has tiles in every R_G-class. Can we deduce, as before, that $X \setminus \mathrm{dom}(\mathcal{S})$ must contain only δ-negative or only δ-positive points a.e.? No:

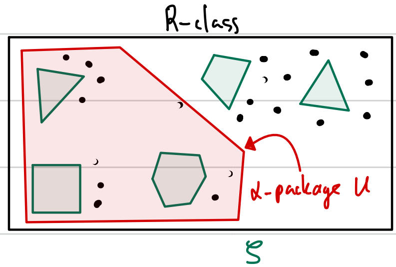
Sure, we can combine several $\mathcal{S}$-tiles together with points outside of $\mathrm{dom}(\mathcal{S})$ to form G-connected δ-zero sets, but this doesn't contradict saturation.



$\mathcal{S}$, δ-negative, δ-positive R_G-class

Idea: we need even more maximal/saturated tilings to prevent such "packages"...

Packed tilings (Ũ.)



Let R be a cBer on a standard Borel space X.

For a tiling $\mathfrak{S} \subseteq [X]^{<\infty}_R$ and $\alpha > 0$, a set $U \in [X]^{<\infty}_R$ is called an **α-package over $\mathfrak{S}$** if it is $R(\mathfrak{S})$-invariant and

$$w(U \setminus \mathrm{dom}(\mathfrak{S})) \geq \alpha \cdot w(U \cap \mathrm{dom}(\mathfrak{S})).$$

For tilings $\mathfrak{S}_0, \mathfrak{S}_1 \subseteq [X]^{<\infty}_R$, we say that $\mathfrak{S}_1$ is an **α-packing extension** of $\mathfrak{S}_0$, written $\mathfrak{S}_0 \preceq_\alpha \mathfrak{S}_1$, if each tile $\tilde{S} \in \mathfrak{S}_1$ is an α-package over $\mathfrak{S}_0$.

For a collection $\mathcal{C} \subseteq [X]^{<\infty}_R$, a tiling $\mathfrak{S} \subseteq \mathcal{C}$ is called **α-packed within $\mathcal{C}$** if there is no proper α-packing extension $\tilde{\mathfrak{S}} \succneq_\alpha \mathfrak{S}$ with tiles from $\mathcal{C}$.

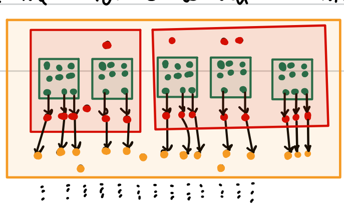
Note. α-packed $\Rightarrow$ maximal. Exercise: Any extension $\tilde{\mathfrak{S}} \subseteq \mathcal{C}$ of an α-packed tiling $\mathfrak{S} \subseteq \mathcal{C}$ is α-packed.

Theorem. Let R be an mcp cBer on (X, μ) whose RN-cocycle is the differential of a Borel function $w : X \to \mathbb{N}_{>0}$. Then for each $\alpha > 0$, each Borel collection $\mathcal{C} \subseteq [X]^{<\infty}_R$ admits an α-packed Borel tiling $\mathfrak{S} \subseteq \mathcal{C}$, after discarding an R-invariant null set.

Proof. As before, use a Borel ctbl colouring of the intersection graph on $\mathcal{C}$ to get a sequence

$$\mathfrak{S}_0 \preceq_\alpha \mathfrak{S}_1 \preceq_\alpha \mathfrak{S}_2 \preceq_\alpha \dots$$

of tilings with $\mathcal{C}$ and show that $\mathfrak{S} := \lim_{n\to\infty} \mathfrak{S}_n$ is a desired tiling. The main point is to show that the tiles of $\mathfrak{S}$ are w-finite a.e. (hence finite a.e.). Letting $F := R(\mathfrak{S})$, this is done via violating mass transport on the union Z of all w-infinite F-classes, hence forcing Z to be null. Left as exercise. □



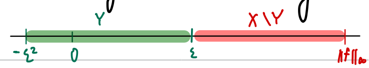
Exercise: In the last theorem, we can ensure that $\mathcal{S}$ is both α-packed and saturated.

Proof of Main Lemma (continued).

Blanket assumption. For every G-connected finite Borel eq. rel. $F \subseteq R_G$, the set $\mathcal{A}^{w/F}_{G/F}(A^w_F f)$ of asymptotic averages of the quotient function $f/F := A^w_F f$ meets both $(-\infty, -\varepsilon^2]$ and $[\varepsilon^2, \infty)$.

Justification. Firstly, note that because the F-classes may not have bounded size, the quotient weight function w/F may not be bounded, hence the Intermediate Value Property may fail, so $\mathcal{A}^{w/F}_{G/F}(f/F)$ may not be convex. However, it is still a compact set and $x \mapsto \mathcal{A}^{w/F}_{G/F}(f/F)(x)$ is (R_G/F)-invariant, hence constant a.e.

Now suppose that, say, $A^{w/F}_{G/F}(A^w_F f) \subseteq (-\varepsilon^2, \infty)$. Then the AA tiling lemma gives a Borel tiling $\mathcal{S} \subseteq [X]^{<\infty}_G$ with conull domain and tiles U satisfying $A^w_U f \in (-\varepsilon^2, \infty)$.

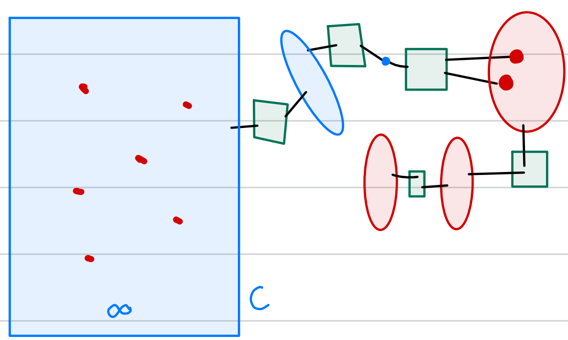


Let Y be the union of all tiles $U \in \mathcal{S}$ with $A^w_U f \in [-\varepsilon^2, \varepsilon]$. The the bridge lemma yields $0 \geq -\varepsilon \cdot \mu(Y) + \varepsilon \cdot \mu(X \setminus Y)$, so $\mu(X \setminus Y) \leq \varepsilon$. Thus, $R(\mathcal{S})$ satisfies the conclusion of the main lemma. □

Now fix any positive $\delta < \varepsilon^2$ and $\alpha \leq \varepsilon^2 / 2\|f\|_\infty$, and let $\mathcal{S}$ be an α-packed tiling with finite G-connected δ-zero sets. Let's analyze $X \setminus \mathrm{dom}(\mathcal{S})$.

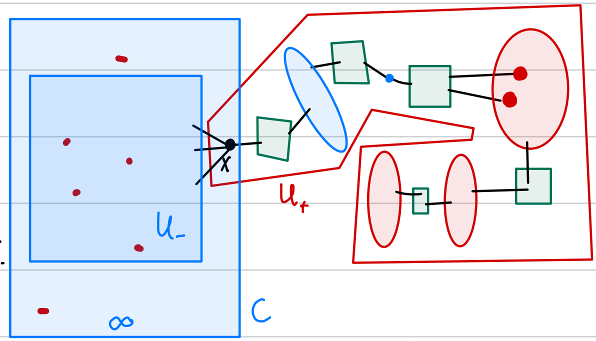
Claim. $G|_{X \setminus \mathrm{dom}(\mathcal{S})}$ is component-finite.

Proof. Suppose towards a contradiction that $G|_{X \setminus \mathrm{dom}(\mathcal{S})}$ has an infinite component C, so $w(C) = \infty$ because $w \geq 1$. Then there don't exist arbitrarily large G-connected

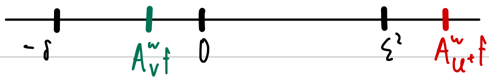


that are δ-negative and that are δ-positive because the Intermediate Value Property would give a G-connected δ-zero set in C, contradicting the maximality of $\mathfrak{S}$. Thus, suppose C only has δ-negative arbitrarily large G-connected sets.

Fix an $x \in C$. By the blanket assumption, there exists a G-connected $R(\mathfrak{S})$-invariant finite set $U_+ \ni x$ such that $A^w_f U_+ \geq \varsigma^2$ and U_+ is large enough that $\Delta := 2\|f\|_\infty \|w\|_\infty / w(U_+) \leq \delta$. By the intermediate value property, there is a



finite $U_- \subseteq C$ disjoint from U_+ such that $V := U_- \sqcup U_+$ is G-connected and $-\delta \leq A^w_V f \leq 0$.

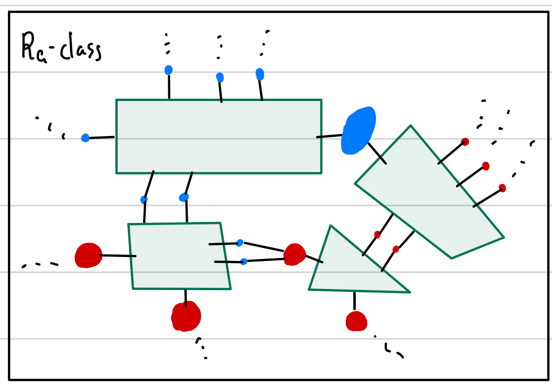


Thus, V is a G-connected δ-zero set. Moreover, because $A^w_V f \leq 0$ while $A^w_{U_-} f \geq \varsigma^2$, the set U_- must be sufficiently large relative to U_+ to change the average by $\geq \varsigma^2$, namely:

$$\varsigma^2 \leq |A^w_{U_+} f - A^w_V f| \leq 2\|f\|_\infty w(U_+)/w(U_-),$$

so $w(U_+)/w(U_-) \geq \varsigma^2 / 2\|f\|_\infty \geq \alpha$. But $\mathrm{dom}(\mathfrak{S}) \cap V \subseteq U^+$, so V is an α-package over $\mathfrak{S}$, contradicting $\mathfrak{S}$ being α-packed. □

Okay, but does this somehow say that $\mathrm{dom}(\mathfrak{S})$ has large measure? Yes, if we assume, as we may, that G is not μ-hyperfinite.



$\mathfrak{S}$, δ-negative, δ-positive

Finitizing cuts and finitizing price.

Let G be a loc. ctbl mcp graph on (X, μ).

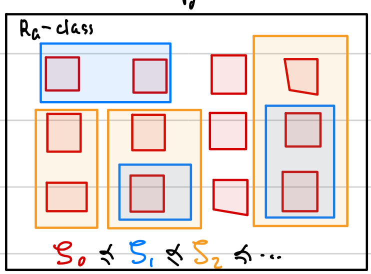
Def (♡). A set $C \subseteq X$ is called a *finitizing cut* for G if $G|_{X \setminus C}$ is component-finite. The *finitizing price* of G is the number $fp_\mu(G) := \inf\{\mu(C) : C \subseteq X \text{ is a Borel finitizing cut for } G\}$.

Prop. If G is not μ-hyperfinite, then $fp_\mu(G) > 0$. (The converse holds for locally finite G.)

Proof. Suppose $fp_\mu(G) = 0$ and show that G is μ-hyperfinite. Let C_n be a finitizing cut of measure $< 2^{-n}$, so replacing C_n with $\bigcup_{k \geq n} C_k$ (the proof of Borel-Cantelli), we may assume that the C_n are decreasing and $\mu(C_n) \to 0$, so $\bigcap_{n \in \mathbb{N}}^{\Downarrow} C_n$ is null. Then $G := \bigcup_{n \in \mathbb{N}}^{\Uparrow} G_n$, where $G_n := G|_{X \setminus C_n}$, witnesses the μ-hyperfiniteness of G. □

Proof of Main Lemma (continued).

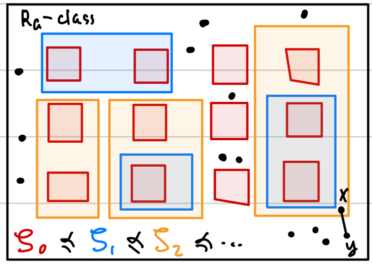
Thus whenever we take an α-packed tiling $\mathcal{S}$ with δ-zero sets for any small enough α, δ, we get that $\mathrm{dom}(\mathcal{S})$ is a finitizing cut for G, so $\mu(\mathrm{dom}(\mathcal{S})) \geq fp_\mu(G) > 0$. But $fp_\mu(G)$ might be tiny, much smaller than $1-\varepsilon$.



Maybe we iterate this, letting $\alpha_n, \delta_n \searrow 0$, and get a sequence of tilings $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \dots$, where each $\mathcal{S}_n$ is both α_n-packed and saturated with G-connected δ_n-zero tiles that are R_{n-1}-invariant, where $R_n := \bigcup_{i \leq n}^{\Uparrow} R(\mathcal{S}_i)$, and have size $\geq n$. Put $R_\infty := \bigcup_{n \in \mathbb{N}}^{\Uparrow} R(\mathcal{S}_n)$.

Last Claim. $\bigcup_{n \in \mathbb{N}} \mathrm{dom}(\mathcal{S}_n)$ is conull.

This would indeed finish the proof because the classes of R_k on $\bigcup_{n \le k} \mathrm{dom}(\mathcal{S}_n)$ are δ_0-zero and $\delta_0 \le \varepsilon^2 \le \varepsilon$ so taking k large enough we would get that the R_k-classes are ε-zero G-connected on a $\geq 1-\varepsilon$ measure set.



Proof-idea of Last Claim. What I didn't see for half a year was the following triviality (dual Borel–Cantelli):

Measure pigeonhole. For a prob. space (X,μ) and $\lambda > 0$, if sets $D_n \subseteq X$ have measure $\geq \lambda$, then $\limsup_n D_n := \{x \in X : \forall_n^\infty \, x \in D_n\} = \bigcap_m \bigcup_{n \geq m} D_n$ has positive measure (in fact, $\geq \lambda$).

Thus, $D_\infty := \limsup_n \mathrm{dom}(\mathcal{S}_n)$ has positive measure, hence meets a.e. G-component.

Note. For each $x \in D_\infty$, $\lim_{n \to \infty} |[x]_{R_n}| = \infty$ and $[x]_{R_n}$ is δ_n-packed and saturated.

So any point $y \in X \setminus D_\infty$ that is G-adjacent to $x \in D_\infty$ has less and less excuse for not joining the $\mathcal{S}_n$-tile of x, as $n \to \infty$. And indeed, another packing-style mass transport on R_∞-classes in D_∞ G-adjacent to $X \setminus D_\infty$ shows that these shameless points $y \in D_\infty$ form a null set. This implies that D_∞ is conull, hence so is $\bigcup_{n \in \mathbb{N}} \mathrm{dom}(\mathcal{S}_n)$. ☐