Perfect kernel of generalized Baumslag Solitar groups

Setting: Γ a countable group

$$\mathrm{Sub}\,\Gamma = \{ \Lambda \leq \Gamma \text{ sbg} \}$$

I / Space of subgroups

- $\underbrace{\mathrm{Sub}\,\Gamma}_{\text{closed subset of } \{0,1\}^\Gamma} \hookrightarrow \underbrace{\{0,1\}^\Gamma}_{\substack{\text{Cantor space} \\ \text{compact, metrizable}}}$

- Basis of neighbourhoods: $\mathcal{V}(\underset{\text{finite subsets of }\Gamma}{O, I}) = \{ \Lambda \leq \Gamma \mid I \subseteq \Lambda, \; O \cap \Lambda = \emptyset \}$.

  (basis of clopen sets)

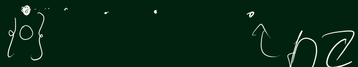
- Topology of simple convergence: $\Lambda_n \xrightarrow[n\to\infty]{} \Lambda$

  iff $\begin{cases} \forall \delta \in \Lambda, \; \exists n_0(\delta) \; \delta \in \Lambda_n \quad \forall n \geq n_0 \\ \quad\;\, \notin \qquad\qquad\qquad\quad \notin \end{cases}$

Ex : - $\overline{\{\text{finitely generated sbgps}\}} = \text{Sub}\,\Gamma$

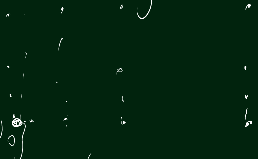
$$\left(\Lambda = \langle \lambda_i, \ i \in \mathbb{N} \rangle = \lim_{N \to +\infty} \langle \lambda_i, \ i \in [\![1, N]\!] \rangle \right)$$

- In $\text{Sub}\,\mathbb{Z}$: - the only non isolated point is $\{0\} = \lim_{n \to +\infty} n\mathbb{Z}$
  - all the other ones have finite index.



$= \{0\} \sqcup \{ \frac{1}{n}, n \in \mathbb{N}^* \}$

- In $\text{Sub}\,\mathbb{Z}^2$



$\simeq$ ~~$\{0,1\}$~~ $\times (\text{Sub}\,\mathbb{Z})^2$

Proposition: If Γ finitely generated, then f.i. subgps are isolated.

[Proof: f.i. subgp of fg groups are f.g ⊠]

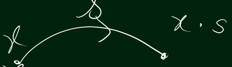
Obs: $\left\{ \begin{array}{l} \text{isomorphism class of} \\ \text{(pointed) transitive right actions } (X, x) \curvearrowleft \Gamma \end{array} \right\} \longleftrightarrow$ conjugacy class of $\mathrm{Sub}\,\Gamma$

$$(X, x) \curvearrowleft \Gamma \longmapsto \mathrm{Stab}_\Gamma(x)$$

$$(\underline{\Lambda} \backslash \Gamma, \underline{\Lambda}) \curvearrowleft \Gamma \longleftarrow\!\shortmid \underline{\Lambda}$$

To any transitive right action $(X, x_0) \curvearrowleft \Gamma = \langle S \rangle$ is associated a Schreier graph:

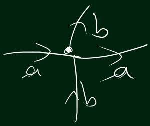
- vertices: X
- $\forall x \in X$



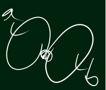
$\forall s \in S$

- based at x_0

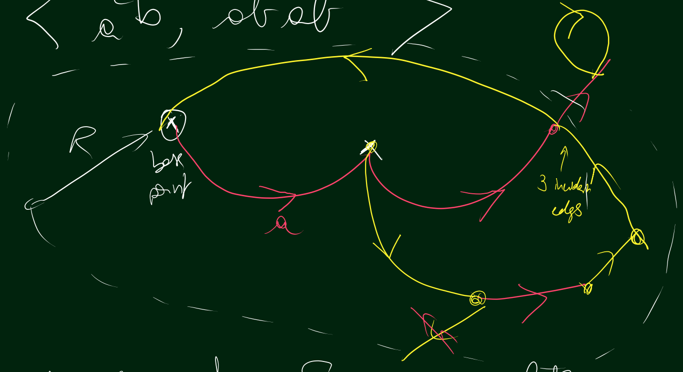
Ex: $\mathbb{F}_2 = \langle a, b \mid \emptyset \rangle$



Schreier graphs = coverings of



$\Lambda = \langle a^2 b, abab^3 \rangle$



Two subgroups of Γ are close

$\Leftrightarrow \exists R > 0$ big, the R-balls around the origin of the associated Schreier graphs are the same

Λ non isolated $\boxtimes$

Th (Cantor-Bendixson): $\exists!$ decomposition $\mathrm{Sub}\,\Gamma = \underbrace{K(\Gamma)}_{\text{closed subset without isolated}} \sqcup \underbrace{C}_{\text{countable set}}$

Point
↝ "perfect kernel" of Γ

Ex: - Γ f.g. abelian group
↝ $K(\Gamma) = \emptyset$

- Γ abelian [Cornulier–Guyot–Pittet]

- Γ lamplighter groups [Bowen–Grigorchuk–Kravchenko]

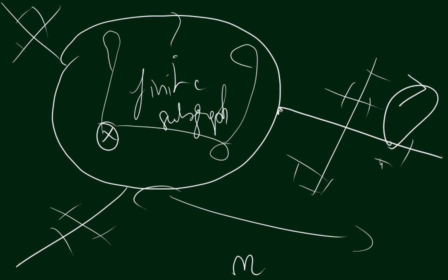
Th (folklore): $K(\mathbb{F}_2) = \mathrm{Sub}_{[\infty]}(\mathbb{F}_2)$.

Proof: $\boxed{\subseteq}$ f.i. sbgps are isolated ☺

$\boxed{\supseteq}$ - $\mathrm{Sub}_{[\infty]}(\mathbb{F}_2)$ is closed
- f.g. sbg are dense

↝ prove that $\left\{\begin{array}{l}\infty.\text{i}\\ \text{f.g}\end{array}\right.$ sbgps of $\mathbb{F}_2$

are limits ∞.i sbgps of $\mathbb{F}_2$.



$\rightsquigarrow \Lambda = \lim \Lambda_n$
$\in \mathrm{Sub}_{fg}(F_2)$

$\blacksquare$

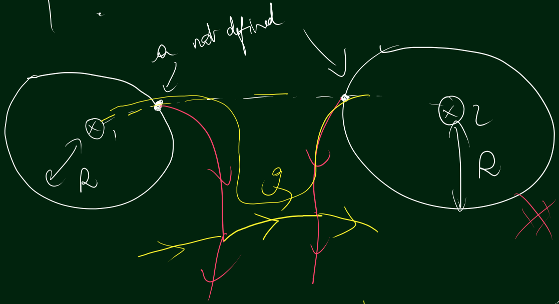
Obs: $\Gamma \curvearrowright \Gamma$ by conjugation
$\rightsquigarrow \Gamma \curvearrowright \mathrm{Sub}\,\Gamma$ by homeomorphisms which preserve $K(\Gamma)$.

Def: $\Gamma \curvearrowright X$ by homeomorphisms,
top space
is topologically transitive (T.T.)
if $\forall\, \mathcal{U}, V \subseteq X$, $\exists\, g \in \Gamma$ such
open, non empty
that $g\mathcal{U} \cap V \neq \emptyset$.

Rk: if X is Polish, this is equivalent to the existence of a dense orbit.

Thm (folklore) : $\mathbb{F}_2 \curvearrowright K(\mathbb{F}_2) = \mathrm{Sub}_{[\infty]}(\mathbb{F}_2)$ is TT.

Proof :



$\Lambda \in \mathcal{U} \qquad g\Lambda g^{-1} \in V$

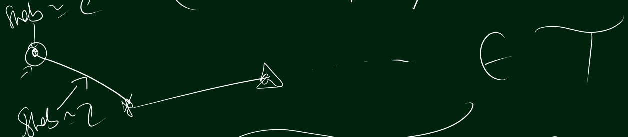
Th (Azuelos, Gaboriau '23) : $\Gamma \curvearrowright T$ tree (minimally and irreducibly) such that $\exists\, e_1, e_2 \in \mathcal{E}(T)$, $\mathrm{Stab}(e_1) \cap \mathrm{Stab}(e_2)$ is finite. Then :

- $\mathrm{Sub}_{|\cdot\backslash T|_\infty}(\Gamma) \subseteq K(\Gamma)$ ⟵ $\{\Lambda \leq \Gamma \mid \Lambda\backslash T \text{ is } \infty\text{d}\}$
- Γ is T.T.

II / Generalized Baumslag-Solitar groups (GBS group)

Def A GBS group is a group Γ that acts cocompactly on an oriented tree T with vertex and edge stabilizers $\simeq \mathbb{Z}$.

($\Gamma \backslash T$ is finite)



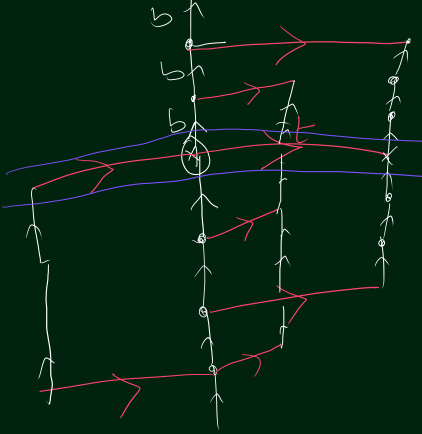
$\mathrm{Stab}(e) \hookrightarrow$ f.i. in Stab ⊙

Ex: Baumslag-Solitar groups!

$$BS(m,n) = \langle t, b \mid t b^m t^{-1} = b^n \rangle$$

$m,n \in \mathbb{Z}^*$

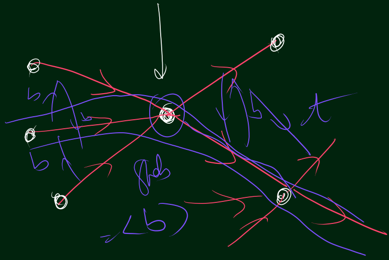
Cay BS(3,2) :
$\{b,t\}$



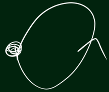
$\gamma \cdot x$
s
$\gamma x \cdot s$

$(s = b^{\pm 1}, t^{\pm 1})$

$\xrightarrow{\text{shrink the}}$
$\langle b \rangle$-orbits



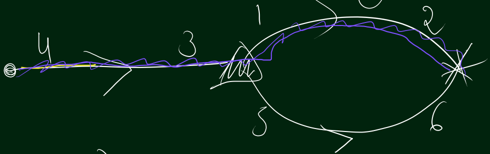
Quotient graph :



Bass – Serre theory : a GBS group Γ is encoded by a "graph of groups"

⇝ finite oriented graph all of whose half-edges are labelled by ≠0 integers

Ex : $\mathcal{H} =$



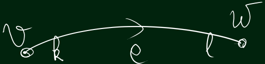
$\Gamma = \pi_1(\mathcal{H})$ Choose a spanning tree A

- generators : $\cdot\ a_v$, $\forall\ v \in \mathcal{V}(\mathcal{H})$

  $\cdot\ t_e$, $\forall\ e \in \mathcal{E}(\mathcal{H})$

- Relations : $\cdot\ t_{\bar{e}} = t_e^{-1}$

  $\cdot\ \forall$ edge

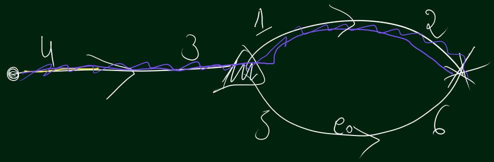


$$t_e\, a_v^k\, t_e^{-1} = a_w^l$$

  $\cdot\ \forall$ edge $e \in \mathcal{E}(A)$,

$t_e = 1$.

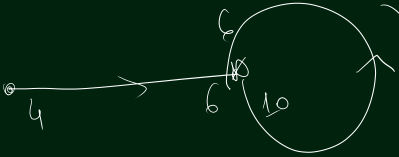
Ex: $\mathcal{H} =$



$$\Gamma = \pi_1(\mathcal{H})$$

$$= \left\langle a_\circ, a_\triangle, a_\ast, t_{e_0} \;\middle|\; \begin{array}{l} a_\circ^4 = a_\triangle^3 \\ \boxed{a_\triangle^1 = a_\ast^2} \\ t_{e_0} a_\triangle^5 t_{e_0}^{-1} = a_\ast^6 \end{array} \right\rangle$$

$= \pi_1\Big($



$\Big)$

From now we assume that Γ is a GBS group defined by a reduced graph of group.

(i.e $\forall$ edge e that has a label $= \pm 1$, e is a loop)

Bass – Serre tree T defined

by $\mathcal{H}$.

(generalizes results of Carderi, Gaboriau, Le Maître, Stalder

Th. 1 (B., 24) : Γ non amenable ($\not\simeq BS(1,n)$)

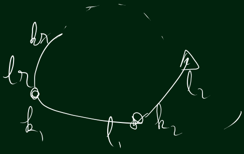
Then $K(\Gamma) = \underline{\mathrm{Sub}_{|\cdot\backslash\Gamma|_\infty}(\Gamma)}$

$$= \begin{cases} \mathrm{Sub}_{[\infty]}(\Gamma) & \text{if } \nexists N \underset{\mathbb{Z}}{\overset{}{\triangleleft}} \Gamma \\ \text{otherwise, i.e. if } \exists N \triangleleft \Gamma,\ N \simeq \mathbb{Z}, & \pi^{-1}(\mathrm{Sub}_{[\infty]}(\Gamma/N)) \text{ where } \pi : \Gamma \to \Gamma/N \end{cases}$$

Remark :

$\exists N \triangleleft \Gamma$, $N \simeq \mathbb{Z}$ $\iff$ $\Gamma \simeq F_n \times \mathbb{Z}$

$\iff$ $\forall$ cycle



,

$$\left|\prod_{i=1}^{n} k_i\right| = \left|\prod_{i=1}^{n} l_i\right| .$$

Th. 2 (B.24'): Γ GBS group.

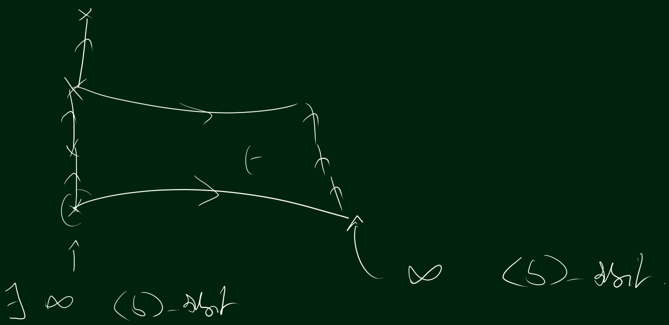
Then $\exists$ $K(\Gamma) = \coprod_{n \in \mathbb{N}^* \amalg \{\infty\}} K_n$ s.t.:

invariant partition

- K_n is open $\forall n \in \mathbb{N}^*$ (and also closed iff Γ unimodular);
- K_∞ is closed;
- $\Gamma \curvearrowright K_n$ is TT for every $n \in \mathbb{N}^* \amalg \{\infty\}$.

Idea of the proofs:

$BS(3,2) \cong \langle b, t \mid t b^3 t^{-1} = b^2 \rangle$

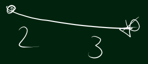


obs: in a Schreier graph
if $\exists$ an ∞ $\langle b \rangle$- orbit (in a Schreier graph)

then all $\langle b \rangle$ -orbits are ∞.

⇝ We will introduce the notion of "$\mathcal{H}$- graphs".

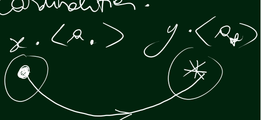
$\Lambda \leq \Gamma$. The $\mathcal{H}$ - graph of Λ is the Schreier graph of Λ "all of those $\langle a_v \rangle$- orbits have been shrunk $\forall\ v \in V(\mathcal{H})$".

$\mathcal{H} =$ 

$\langle a, b \mid a^2 = b^3 \rangle$

More formally: - vertices : $\langle a_v \rangle$- orbits $\forall\ v \in V(\mathcal{H})$, labelled by their cardinalities.

- edges : $x \cdot \langle a_\bullet \rangle$ $y \cdot \langle a_* \rangle$



$$\text{iff } \exists \; \circ \xrightarrow{e} \circ \in \mathcal{H} \quad e'$$

$$\begin{cases} e \in A \text{ and } x\cdot\langle a_0\rangle \cap y\cdot\langle a_\infty\rangle \neq \emptyset \\ e \notin A \text{ and } x t_e\cdot\langle a_0\rangle \cap y\cdot\langle a_\infty\rangle \neq \emptyset. \end{cases}$$

Ex: - trivial action of Γ on a single point: $\mathcal{H}$.

- $\Gamma \curvearrowright \Gamma$ by translation: $\boxed{\mathcal{T}}$

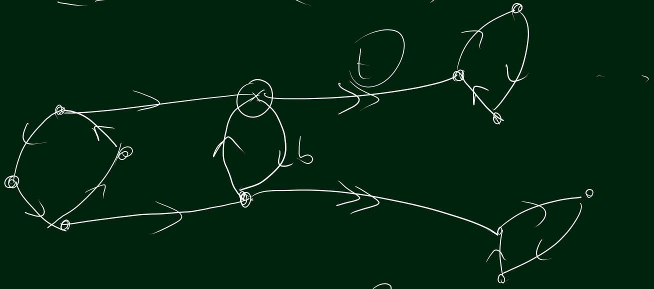
$$\begin{tikzcd}
 & \mathrm{Cay}\,\Gamma \arrow[dl] \arrow[dr, "\text{shrink } \langle b\rangle\text{-orbits}"] & \\
\underset{\substack{\parallel \\ \mathrm{Sch}(\Lambda)}}{\overset{BS(m,n)}{\Lambda \backslash \mathrm{Cay}\,\Gamma}} \arrow[dr, "/\langle b\rangle"] & & \mathrm{Cay}\,\Gamma/\langle b\rangle \cong \mathcal{T} \arrow[dl] \\
 & \mathcal{H}\text{-graph of } \Lambda = \Lambda\backslash\mathcal{T}. &
\end{tikzcd}$$

$\Lambda \leq \Gamma$

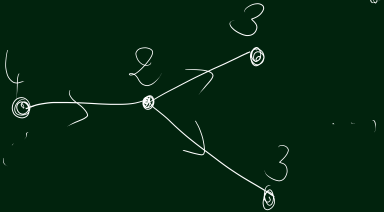
- $\Gamma = BS(3,2)$

$$\Lambda' = \langle b^2 \rangle$$

Sch(Λ)



H-graph:

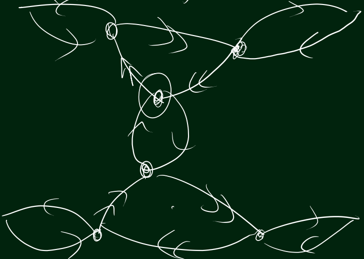


$$- \quad \Gamma = \pi_1\left(\underset{2}{\circ} \longrightarrow \underset{3}{\circ} \right)$$

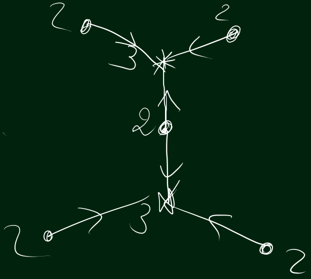
$$= \langle a, b \mid a^2 = b^3 \rangle$$

$$\Lambda = \langle a^2 \rangle$$

Sch(Λ)



$\mathcal{H}$-graph :



Proposition : Let $\overset{N}{\underset{V}{\circledcirc}} \xrightarrow[\in \mathcal{E}(\mathcal{H})]{e} \overset{M}{\ast} \in \mathcal{V}(\mathcal{G})$ $\mathcal{H}$-graph of $\underline{\Lambda}$

if $\underset{k}{\circ} \xrightarrow{e} \underset{l}{\ast} \in \mathcal{E}(\mathcal{H})$

Then • $\boxed{\dfrac{N}{N\wedge k} = \dfrac{M}{M\wedge l}}$.

• there are exactly $\dfrac{N}{N\wedge k}$ edges labeled $\leq$ e with source V in $\mathcal{G}$.

Def : An abstract $\mathcal{H}$-graph is a labelled graph that satisfies the conditions of the last proposition.

Assume $\underline{\Lambda} \subseteq \Omega$ s.t. $\underline{\Lambda} \not\top \infty$,

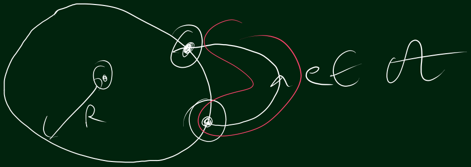
$\iff$ the $\mathcal{H}$ graph of $\underline{\Lambda}$ is ∞.

Lemma : $\mathcal{G}$ $\mathcal{H}$-graph of $\underline{\Lambda}$

FALSE

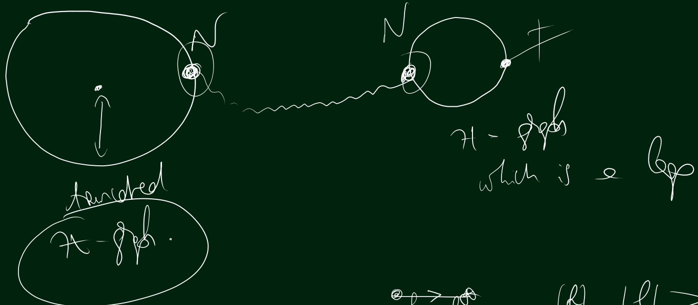
- assume $\exists'$ $B_{\mathcal{G}}(0,R) \subseteq \mathcal{G}$ abstract $\mathcal{H}$-graph
  very big
- then $\exists$ a subgroup $\underline{\Lambda}'$ whose $\mathcal{H}$-graph is $\mathcal{G}'$ s.t. $\underline{\Lambda}'$ and Λ are close.

s.t. $\mathcal{G}/B_{\mathcal{G}}(0,R)$ is a tree

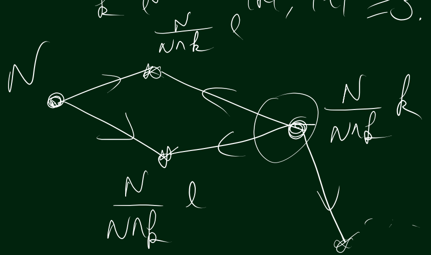


$B_{\mathcal{G}}(0,R)$

Idea of the proof of thm 1 :



∎

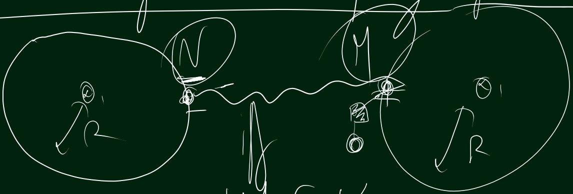


This proves

& $\mathrm{Sub}_{|\setminus \mathrm{Max}}(\Gamma) \subseteq K(\Gamma)$

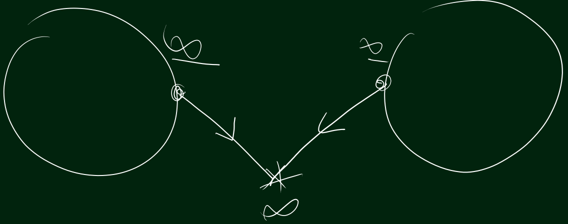
$\supseteq$ general

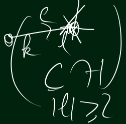
Idea of the proof of thm 2:

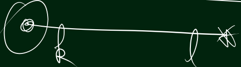


$N, M \in K_n$

if $N = M = \infty$





Ex: $\mathcal{H} =$ 

$\Gamma = \langle a, b \mid a^k = b^\ell \rangle$

$\langle a^h \rangle$

$\underline{\mathrm{Ph}_{\circledcirc,\mathcal{H}}}$: $\mathbb{N}^* \sqcup \{\infty\} \to \mathbb{N}^* \sqcup \{\infty\}$

$N \mapsto \prod_{\substack{p \in \mathcal{P} \\ |N|_p > |k|_p}} p^{|N|_p}$

$(\infty \mapsto \infty)$

$\Lambda_0, \Lambda_1 \in K(\Gamma)$

$\Leftrightarrow \mathrm{Ph}_{\circledcirc,\mathcal{H}}(N_0) = \mathrm{Ph}_{\circledcirc,\mathcal{H}}(N_1)$.

$$\begin{cases} \Lambda_0 \cap \langle a_\circ \rangle = \langle a_\circ^{N_0} \rangle \\ \Lambda_1 \cap \langle a_\circ \rangle = \langle a_\circ^{N_1} \rangle \end{cases}$$