Ergodic theory of permutation groups.

I Generalized Bernoulli shifts

Theorem (with JAHEL and ROY):

Γ ctble acting on a countable set Ω, transitive

(SIO) $\forall a \in \Omega$, $\Gamma_a \curvearrowright \Omega \setminus \{a\}$ has only infinite orbits.

Take (Z_1, ζ_1) and (Z_2, ζ_2) std prob spaces. Then the associated generalized Bernoulli shifts $\Gamma \curvearrowright (Z_1, \zeta_1)^\Omega$, $\Gamma \curvearrowright (Z_2, \zeta_2)^\Omega$ are isomorphic iff $(Z_1, \zeta_1) \simeq (Z_2, \zeta_2)$.

$$\Gamma \curvearrowright (Z, \zeta)^\Omega : \quad \gamma \cdot (z_a)_{a \in \Omega} = (z_{\gamma^{-1}(a)})_{a \in \Omega}$$

isomorphic: $\Gamma \curvearrowright (X, \mu)$ and $\Gamma \curvearrowright (Y, \nu)$ are isomorphic iff

$\exists \varphi : (X, \mu) \xrightarrow{\sim} (Y, \nu)$ st $\forall \gamma \in \Gamma$, $\forall^* x \in X$, $\varphi(g \cdot x) = g \cdot \varphi(x)$.

Two examples of (SIO):

- $\mathrm{Sym}_f(\mathbb{N}) \curvearrowright \mathbb{N}$.
- $PL_+(\mathbb{Q})$ = group of $\underbrace{\text{piecewise linear}}_{\text{finitely many pieces.}}$ order preserving bijections of $\mathbb{Q}$.

  $PL_+(\mathbb{Q}) \curvearrowright \mathbb{Q}$ is (SIO).

- Thompson's group $F < \mathrm{Homeo}([0,1])$

  F is the group of PL orientation preserving homeo of $[0,1]$

with $\left|\begin{array}{l}\text{break pts in } \mathbb{Z}[1/2] \\ \text{slopes}\end{array}\right.$

$F \curvearrowright \mathbb{Z}[1/2] \cap]0,1[$ (SIO).

Corollary: $G \leq \mathrm{Sym}(\Omega)$ closed subgroup. Ω ctble set, $G \curvearrowright \Omega$ transitively (SIO): $G_a \curvearrowright \Omega \setminus \{a\}$ has only infinite orbits. Fix two std prob spaces (Z_1, Ξ_1), (Z_2, Ξ_2). Then TFAE:

i) The spatial prop action $G \curvearrowright (Z_i, \Xi_i)^\Omega$ are spatially isom.

ok $\Downarrow$

ii) The Boolean prop actions $G \curvearrowright (Z_i, \Xi_i)^\Omega$ are Booleanly isom

$\Downarrow$

iii) The prob. spaces (Z_i, Ξ_i) are isom.

ok

Boolean actions	Spatial actions
G top group.	G top group.
continuous morphism $G \to \underbrace{\mathrm{Aut}(X,\mu)}$ initial topology associated with $T \mapsto T(A)$ $\forall A \in \mathrm{MAlg}(X,\mu)$	Borel action $G \times X \to X$ with a $\mu \in \mathrm{Prob}_G(X)$
Boolean isomorphism: α_1, α_2 are Booleanly isomorphic if there exists a isomorphism Φ between the measure algebras $\mathrm{MAlg}(X_i, \mu_i)$ st. $\forall A \in \mathrm{MAlg}(X_1, \mu), \forall g \in G, \Phi(\alpha_1(g)A) = \alpha_2(g)\Phi(A)$	Spatial isomorphism: α_1, α_2 spatially isom if $\exists \Phi : X_1 \to X_2$ Borel bijection, $\Phi_* \mu_1 = \mu_2$ and $\exists X_0 \subseteq X_1$, $\mu(X_0) = X_1$, st $\forall g \in G$ $\forall x \in X_0$ satisfying $\alpha_1(g) x \in X_0$, $\Phi(\alpha_1(g)x) = \alpha_2(g)\Phi(x)$.

From the definition, one sees that if α_1, α_2 spatial pmp actions, Φ spatial isomorphism from α_1 to α_2 induces $\Phi^*: MAlg(X_2, \mu_2) \to MAlg(X_1, \mu_1)$ Boolean isomorphism from (the Boolean action induced by) α_2 to (——) α_1. Hence $\Phi^*([A]) := [\Phi^{-1}(A)]$

Remark: every spatial $\underset{\text{pmp}}{\text{action}}$ induces a Boolean pmp action.

The converse is not always true in general.

But for countable groups / loc. compact / closed subgroups of $Sym(\Omega)$, [GW] for every Boolean pmp action α, $\exists$ a spatial pmp action whose induced Boolean pmp action is α.

Remark: For $Sym(\Omega)$, $\exists$ two spatial pmp actions α_1, α_2 which are NOT spatially isomorphic but whose induced Boolean pmp actions are Booleanly isomorphic.

Proof of the corollary:

ii) $\Rightarrow$ iii) $\Phi: MAlg((Z_2, \zeta_2)^\Omega) \to MAlg((Z_1, \zeta_1)^\Omega)$

Boolean isom between α_2 and α_1.

Claim: $\exists\, \Gamma \leq G$ ctble group and dense in G.

Proof: G is Polish: Take S ctble $\underset{\text{dense}}{\subseteq} G$, let $\Gamma = \langle S \rangle$ $\boxed{S}$

Γ and G have the same orbits on Ω^n, $\forall n \in \mathbb{N}^*$

Since G has (SIO) then Γ has (SIO)

Since Γ is ctble, Φ descends a spatial isomorphism between $\alpha_1\restriction_\Gamma$ and $\alpha_2\restriction_\Gamma$. (because Γ is dense and ctble).

Use the theorem to get that (Z_i, Σ_i) are isomorphic. $\blacksquare$

Let's prove the theorem.

Γ acts on Ω.

For $\alpha : \Gamma \curvearrowright (X, \mu)$ pmp, and $a \in \Omega$, define

$$\mathcal{F}_a(\alpha) = \{Y \text{ measurable subset of } X \mid \forall g \in \Gamma_a, \mu(\alpha(g)Y \Delta Y) = 0\}$$

sub σ-alg of the σ-alg on X.

I can also define

$$\widetilde{\mathcal{F}}_a(\alpha) := \{Y \in \mathrm{MAlg}(X, \mu) \mid \forall g \in \Gamma_a, \alpha(g)Y = Y\}$$

Fact: $\alpha_i : \Gamma \curvearrowright (X_i, \mu_i)$ pmp. Then every $\Phi : (X_1, \mu_1) \to (X_2, \mu_2)$ pmp isomorphism from α_1 to α_2 induces a measure algebra isomorphism between $\mathcal{F}_a(\alpha_1)$ and $\mathcal{F}_a(\alpha_2)$, $\forall a \in \Omega$.

Proof of the theorem:

If (Z_i, Σ_i) are isom. then $\Gamma \curvearrowright (Z_i, \Sigma_i)^\Omega$ are isomorphic.

Assume that $\Gamma \curvearrowright (Z_i, \Sigma_i)^\Omega$ are isomorphic.

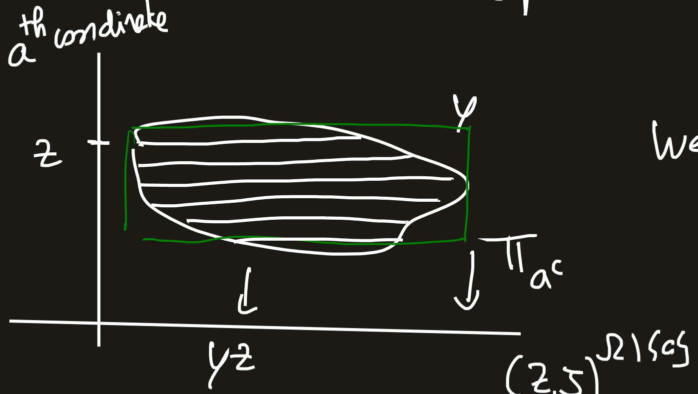
By the above fact, $\forall a \in \Omega$, $\widetilde{\mathcal{F}}_a(\alpha_1) \simeq \mathcal{F}_a(\alpha_2)$.

Lemma: $\mathcal{F}_a(\alpha_i) \simeq \mathrm{MAlg}(Z_i, \Sigma_i)$.

Proof: fix $a \in \Omega$. $\pi_a : (Z,\zeta)^\Omega \to (Z,\zeta)$ projection onto the a^{th} coordinate. This induces $\pi_a^* : \mathrm{MAlg}(Z,\zeta) \to \mathrm{MAlg}((Z,\zeta)^\Omega)$ morphism (always injective). Let's prove that $\pi_a^*(\mathrm{MAlg}(Z,\zeta))$
$= \mathcal{F}_a(\alpha)$

- $\subseteq$ straightforward: take $Y \in \mathrm{MAlg}(Z,\zeta)$. Then $\pi_a^{-1}(Y)$ is Γ_a-invariant up to null set.

- $\supseteq$: take $Y \in \mathcal{F}_a(\alpha)$. Up to null set, Y can be chosen to be truely Γ_a-inv.



We want Y as a product of someone in $\mathrm{MAlg}(Z,\zeta)$ in the a^{th} coordinate and $(Z,\zeta)^{\Omega\setminus\{a\}}$

$$\forall z \in Z, \quad Y^z = \pi_{a^c}\left(\{(z_n) \in Y \mid z_a = z\}\right).$$

Y^z is Γ_a-inv for the action $\Gamma_a \curvearrowright (Z^{\Omega\setminus a}, \zeta^{\Omega\setminus a})$.

(SIO) implies that this action is ergodic.

So $\zeta^{\Omega\setminus\{a\}}(Y^z) \in \{0,1\}$.

Use Fubini to say that up to null set, $Y \in \pi_a^*(\mathrm{MAlg}(Z,\zeta))$. $\blacksquare$

Examples: for $\mathrm{Sym}(\Omega) = S_\infty$

- $S_\infty \curvearrowright (\{0,1\}^\Omega, \mathrm{Ber}(p)^\Omega)$ are never isomorphic (either spatially or Boolean-ly)

- $S_\infty \curvearrowright (\{0,1\}^{\mathcal{P}_2(\Omega)}, \mathrm{Ber}(p)^{\mathcal{P}_2(\Omega)})$ are never isomorphic (use the Gottlay with $S_\infty \curvearrowright \mathcal{P}_2(\Omega)$ which is (SIO)).

- ...

Question: Is $\alpha : S_\infty \curvearrowright (\{0,1\}^{\Omega}, \mathrm{Ber}(p)^{\Omega})$ isom to $\beta : S_\infty \curvearrowright (\{0,1\}^{\mathcal{P}_2(\Omega)}, \mathrm{Ber}(p)^{\mathcal{P}_2(\Omega)})$?

No : neither spatially, nor Booleanly.

Argument : $\mathcal{F}_{G_a}(\alpha)$ is non trivial whereas $\mathcal{F}_{G_a}(\beta)$ is trivial for $a \in \Omega$.

II Kolmogorov and Hewitt-Savage 0-1 Laws.

with Rémi BARRITAULT and Colin JAHEL.

<u>Theorem</u> : Let $\alpha : S_\infty \to \mathrm{Aut}(X,\mu)$ be a Boolean pmp action. The the tail measure algebra of α is equal to the invariant measure algebra of α.

Take $(X_n)_{n\in\mathbb{N}}$ iid, real valued r.v. Then Kolmogorov 0-1 laws say that

$$\mathcal{T} = \bigcap_{n\geq 0} \sigma(X_n, X_{n+1}, \ldots) = \{\emptyset, E\} \text{ a.s.}$$

$\nu \in \mathrm{Prob}(\mathbb{R}^{\mathbb{N}})$ law of $(X_n)_{n\geq 0}$. Look at $\mathcal{T} = \bigcap_{n\geq 0} \sigma(\pi_k \mid k \geq n)$ (projection)

$\mu = \nu^{\mathbb{N}}$ because (X_n) are iid.

$$= \bigcap_{n\geq 0} \sigma(\mathcal{F}_n, \mathcal{F}_{n+1}, \ldots)$$

Then $\mathcal{T} = \{\emptyset, \mathbb{R}^{\mathbb{N}}\}$ μ.a.s

$(X_n)_{n\in\mathbb{N}}$ iid, real valued. Hewitt Savage say that
$\mathrm{Law}(X_n)=\nu$

$$\mathcal{E}=\{A\subseteq\mathbb{R}^{\mathbb{N}} \text{ measurable, exchangeable}: \forall\sigma\in\mathrm{Sym}_{fin}(\mathbb{N}),\ (z_n)\in A \iff (z_{\sigma(n)})\in A \text{ a.s.}\} = \{\emptyset,\mathbb{R}^{\mathbb{N}}\} \text{ a.s.}$$

Remark: $\mathcal{E} \underset{a.s.}{=} \sigma\text{-alg of } \mathrm{Sym}_{fin}(\mathbb{N})\text{-invariant subsets of } \mathbb{R}^{\mathbb{N}}$

$$\underset{a.s.}{=} \sigma\text{-alg of } S_\infty\text{-invariant subsets of } \mathbb{R}^{\mathbb{N}}.$$

because $S_\infty \to \mathrm{Aut}(\mathbb{R}^{\mathbb{N}},\nu^{\mathbb{N}})$ is continuous.

Take $\alpha: S_\infty\to\mathrm{Aut}(X,\mu)$ Boolean pmp action

Define $\mathcal{E}=\{Y \text{ measurable in } X \mid \forall g\in S_\infty,\ \mu(gY\Delta Y)=0\}$.

Define the tail σ-algebra of α:

$$\mathcal{T}=\bigcap_{\substack{B \text{ cofinite}\\ \text{subset of } \mathbb{N}}}\sigma(\mathcal{F}_A \mid A\subseteq B \text{ finite})$$

where $\mathcal{F}_A=\{Y \text{ measurable in } X \mid \forall g\in(S_\infty)_A,\ \mu(\alpha(g)Y\Delta Y)=0\}$

$$(S_\infty)_A = \{g\in S_\infty,\ g\restriction_A=\mathrm{id}\}$$

Theorem: $\alpha: S_\infty\to\mathrm{Aut}(X,\mu)$ Boolean pmp. Then $\mathcal{T}\underset{a.s.}{=}\mathcal{E}$.

One inclusion is easy: if $Y\in\mathcal{E}$ then $Y\in\mathcal{F}_A$ for every A finite.

so $\mathcal{E} \underset{a.s.}{\subseteq} \mathcal{T}$.

Remark:

we can prove very easily (without the theorem) that $\forall A, B \subseteq \mathbb{N}$ finite, disjoint, then $\mathcal{F}_A$ and $\mathcal{F}_B$ are independent conditionally on $\mathcal{T}$.

This last theorem is useful: $\alpha : S_\infty \to \mathrm{Aut}(X, \mu)$ is ergodic, then $\mathcal{E} = \{\emptyset, X\}$ a.s. we get from the remark independence: $\mathcal{F}_A \perp\!\!\!\perp \mathcal{F}_B$.

(JAHEL, TSANKOV)

$S_\infty \curvearrowright (\{0,1\}^{\mathbb{N}}, \mathrm{Ber}(\frac{1}{2})^{\mathbb{N}})$

$$\phi : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$$
$$(x_n) \mapsto (y_n) \qquad y_n = \begin{cases} 1 & \text{if } x_n = 0 \\ 0 & \text{else} \end{cases}$$

$\underbrace{\text{compact factor associated to } \phi}$, not isomorphic to $S_\infty \curvearrowright (7,5)^{\mathbb{N}}$.

$$S_\infty \curvearrowright (\{0,1\}^{\mathbb{N}}, \mathrm{Ber}(\tfrac{1}{2})^{\mathbb{N}}) / (x_n) \sim \phi((x_n)).$$