Odomutants and flexibility results for quantitative orbit equivalence

I - Basics on quantitative OE, overview

↳ 3 flexibility results provided by "odomutants"
Th A, Th B, Th C

II - Odometers

III - Odomutant

IV - Sketchs of proofs of theorems

I. Basics

(X,μ) prob space standard atomless $\cong ([0,1], \mathrm{Leb})$

$$\mathrm{Aut}(X,\mu) = \{T : X \to X \text{ bimeasurable bij}, \ \mu(T^{-1}(\cdot)) = \mu\}/_\mu$$

Def: $S, T \in \mathrm{Aut}(X,\mu)$ are OE if
$\exists\, \Theta \in \mathrm{Aut}(X,\mu), \forall^* x \in X, \ \mathrm{Orb}_T(x) = \mathrm{Orb}_{\Theta^{-1}S\Theta}(x)$

(conjugacy : $T = \Theta^{-1} S \Theta$)

Th (Dye 59): S, T ergodic $\Rightarrow$ OE

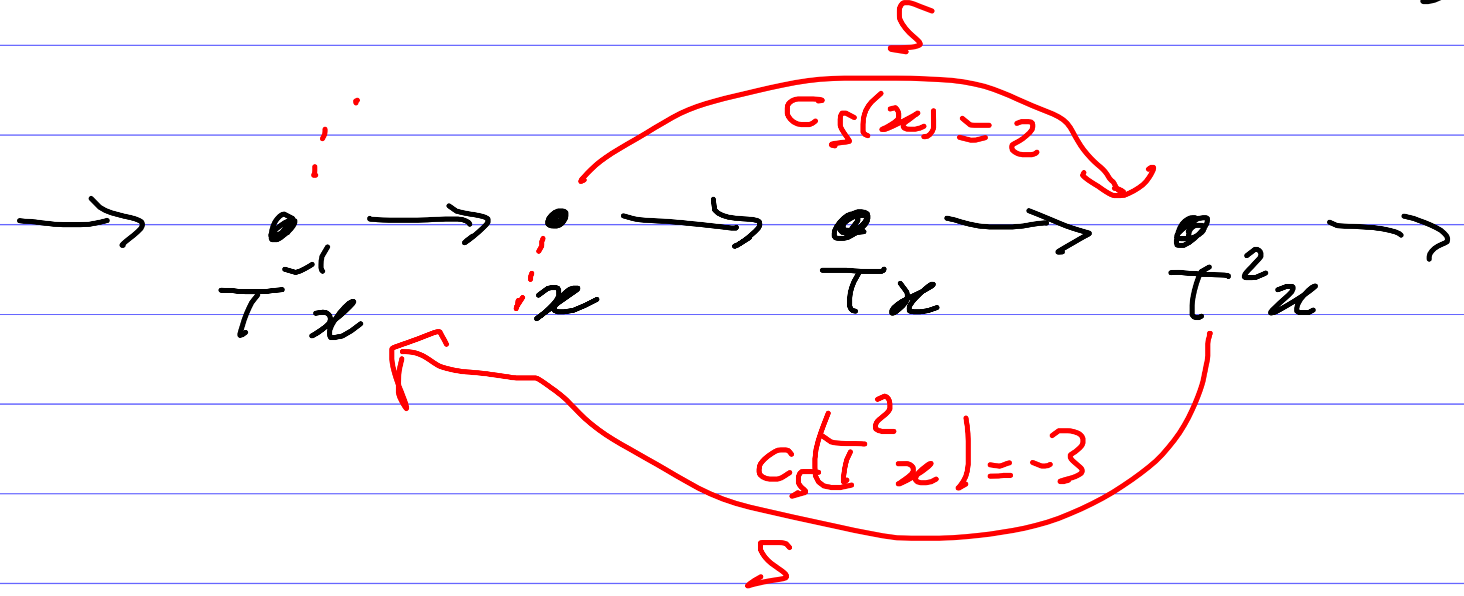
Def: Assume S, T aperiodic $(\forall^* x \in X \ \forall n \in \mathbb{Z} \setminus \{0\}, \ T^n x \neq x)$
(erg $\Rightarrow$ aperiodicity)

$$T x = \Theta^{-1} S^{c_T(x)} \Theta x \qquad c_T, c_S : X \to \mathbb{Z}$$

$$S x = \Theta T^{c_S(x)} \Theta^{-1} x$$

"cocycle associated to the OE Θ"

Example: $\theta = id_X$ $\qquad \forall^* x,\ \mathrm{Orb}_T(x) = \mathrm{Orb}_S(x)$



Th: (Belinskaya 69) if c_S or c_T is integrable

$$\left(\int |c_{T/S}(x)|\, d\mu(x) < \infty \right)$$

then S is conjugate to T or T^{-1} (flip-conjugacy).

Def: • $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ $\qquad$ Delabie, Koivisto, Le Maître, Tessera

T, S are φ-integrably orbit equivalent (φ-OE) if c_T, c_S are φ-int

$$\int \varphi(|c_T(x)|)\, d\mu(x) < +\infty, \quad \int \varphi(|c_S(x)|)\, d\mu(x) < +\infty$$

• T, S are Shannon OE if c_S, c_T are Shannon $\qquad$ Kerr, Li

$$H_\mu(\{c_T^{-1}(n) \mid n \in \mathbb{Z}\}) < +\infty, \quad H_\mu(\{c_S^{-1}(n) \mid n \in \mathbb{Z}\}) < +\infty$$

partition $\mathcal{P}$, $\quad H_\mu(\mathcal{P}) = -\sum_{P \in \mathcal{P}} \mu(P) \log \mu(P)$

$\varphi \neq 0$, linear: $\quad \varphi$-int $=$ int

what about sublinear map φ? $\qquad \dfrac{\varphi(t)}{t} \xrightarrow[t \to +\infty]{} 0$

1) Flip-conjugacy

a) rigidity: Belinskaya (integrability)

b) flexibility:

Th: (Carderi, Joseph, Le Maître, Tessera) CJLMT

$\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ sublinear, $S \in \mathrm{Aut}(X,\mu)$ ergodic s.t.

(*) $\exists\, n \geq 2$, S^n ergodic

then $\exists\, T \in \mathrm{Aut}(X,\mu)$,
- T, S are φ-OE
- T, S not flip-conjugate

(*) Bernoulli shifts, irrational rotations, some odometers (but not all of them)

ThA (C.): φ sublinear, S odometer

then same conclusion

S odometer $\rightsquigarrow$ T "odomutant"

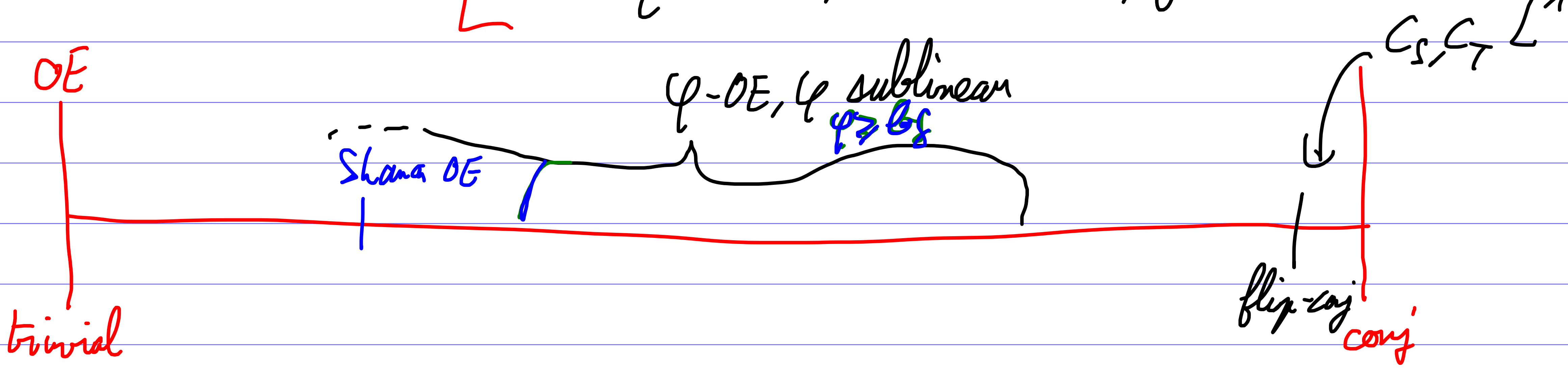
These th can be stated in terms of Shannon OE:

Th: CJLMT: if $\varphi \geq \log$:
- φ-int $\Rightarrow$ Shannon
- φ-OE $\Rightarrow$ Shannon OE

2) Entropy

a) rigidity   **Th (Kerr, Li):** Shannon OE preserves entropy

Cor: $\varphi \geq \log$
φ-OE preserves entropy.



Question: $\exists ? \varphi$, any $S, T \in \mathrm{Aut}(X,\mu)$ ergodic are φ-OE ?

(would be $< \log$)

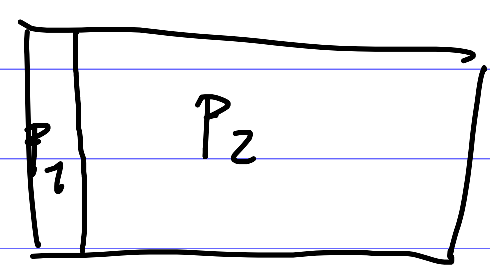
b) flexibility: ThB (C-): $\alpha \geq 0$ or $+\infty$. $\exists\, S, T \in \mathrm{Aut}(X,\mu)$,

- $h_\mu(S) = 0$, $h_\mu(T) = \alpha$
- S, T are $\underbrace{\dfrac{\log}{\log^{\circ m}}}$-OE $\forall m \geq 0$

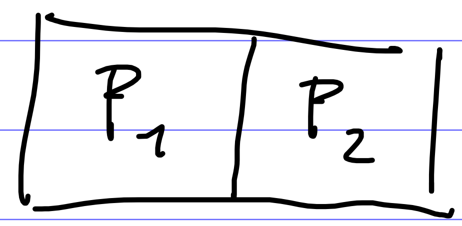
in particular $\log^\beta$-OE $\forall \beta < 1$

stronger: $\exists$ OE which $\dfrac{\log}{\log^{\circ m}}$-OE $\forall m \geq 0$

$\mathcal{P} \rightsquigarrow H_\mu(\mathcal{P})$ $\qquad |\mathcal{P}| = 2$



X $\qquad H_\mu(\mathcal{P})$ small



$H_\mu(\mathcal{P})$ large

$T \in \mathrm{Aut}(X,\mu)$ $\qquad T$ acts on $\mathcal{P}$

$$\mathcal{P}^n = \bigvee_{i=0}^{n-1} T^{-i}(\mathcal{P}) = \left\{ \bigcap_{i=0}^{n-1} T^{-i}(P_i) \;\middle|\; P_0, \dots, P_{n-1} \in \mathcal{P} \right\}$$

$$h_\mu(T, \mathcal{P}) = \text{asymptotic of } H_\mu(\mathcal{P}^n) = \lim_{n \to +\infty} \frac{H_\mu(\mathcal{P}^n)}{n}$$

$$h_\mu(T) = \sup_{\mathcal{P}} h_\mu(T, \mathcal{P})$$

Examples: zero entropy: odometers, irrational rotations

positive entropy: Bernoulli shifts $\qquad |\Sigma| = k$

$$T : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}, \quad (x_n)_{n \in \mathbb{Z}} \mapsto (x_{n+1})_{n \in \mathbb{Z}}$$

$h_\mu(T) = H_\mu(\nu)$, $\mu = \nu^{\mathbb{Z}}$

Outlines of the proof (Th B')

- S is an odometer (in a large class that we can describe — see later class - up to conjugacy and supernatural number)

  T "odomutant"

- topological entropy is easier (combinatorial)

  if T is uniquely erg (unique T-inv μ), then $h_{top}(T) = h_\mu(T)$ (variational principle)

- $h_{top}(T)$ well-defined? Topological context!

  we can extend some odomutants to homeomorphisms on a Cantor space (minimal)

  $\rightsquigarrow$ OE is a strong OE

- S (odometer) uniquely erg, S, T are strongly OE $\Rightarrow$ T uniquely erg

Def: X Cantor set

S, T are minimal Cantor homeomorphisms

S, T are strongly OE if $\exists \theta : X \to X$ homeo $\forall x \in X$, $Orb_T(x) = Orb_{\theta^{-1}S\theta}(x)$ $(\ast\ast)$

and c_T, c_S have each at most 1 point of discontinuity

Lemma: if $(\ast\ast)$ is satisfied and if T preserves μ, then S preserve μ

Proof: $\mu(S(A)) = \sum_{n\in\mathbb{Z}} \mu(\underbrace{S(A \cap \{c_S = n\})}_{T^n(A\cap\{c_S=n\})}) \overset{T\text{-inv}}{=} \sum_n \mu(A \cap \{c_S = n\}) = \mu(A)$ $\blacksquare$

The more general statement is

Th B'' : $\alpha > 0$ or $+\infty$,

S odometer whose associated supernatural number $\prod_{p \in \mathcal{P}} p^{k_p}$ satisfies : $\exists p,\ k_p = +\infty$

(dyadic odometer, p-adic odometers, universal odometer)

then $\exists T$ minimal Cantor homeo s.t.

- $h_{top}(T) = \alpha$
- $\exists$ strong OE between S, T which is $\frac{\log}{\log^{om}}$ - int $\forall m \geq 0$

I will prove a weaker statement :

Th B : S universal odometer

$\exists T$ minimal Cantor homeo :

- $h_{top}(T) > 0$
- $\exists$ strong OE which is $\frac{\log}{\log^{om}}$-int $\forall m \geq 0$

3) Kakutani equivalence

"even Ke" preserves entropy

Question : connexion with Shannon OE or φ-OE $\varphi \geq \log$?

Th C (C_0) : S dyadic odometer

$\exists T \in \mathrm{Aut}(X,\mu)$, S,T are L^p OE $\forall p < 1/2$

are not evenly Ke

$\underset{LB}{S} \rightsquigarrow T$ an odomutant which is not loosely Bernoulli

$\downarrow$ (construction of Feldman)

loosely Bernoulli $\downarrow$ class of systems closed by Ke

- "even Ke" well-understood (entropy : complete invariant)

4) Other dynamical properties :

weak / strong mixing properties, point spectrum (eigenvalues)

$\longrightarrow$ we only have flexibility results

II- Odometers (and motivation behind the definition of odomutats)

up to conjugacy, this the adding machine

$$X = \prod_{n\geq 0} \{0, \dots, q_n - 1\} \qquad q_n \geq 2$$

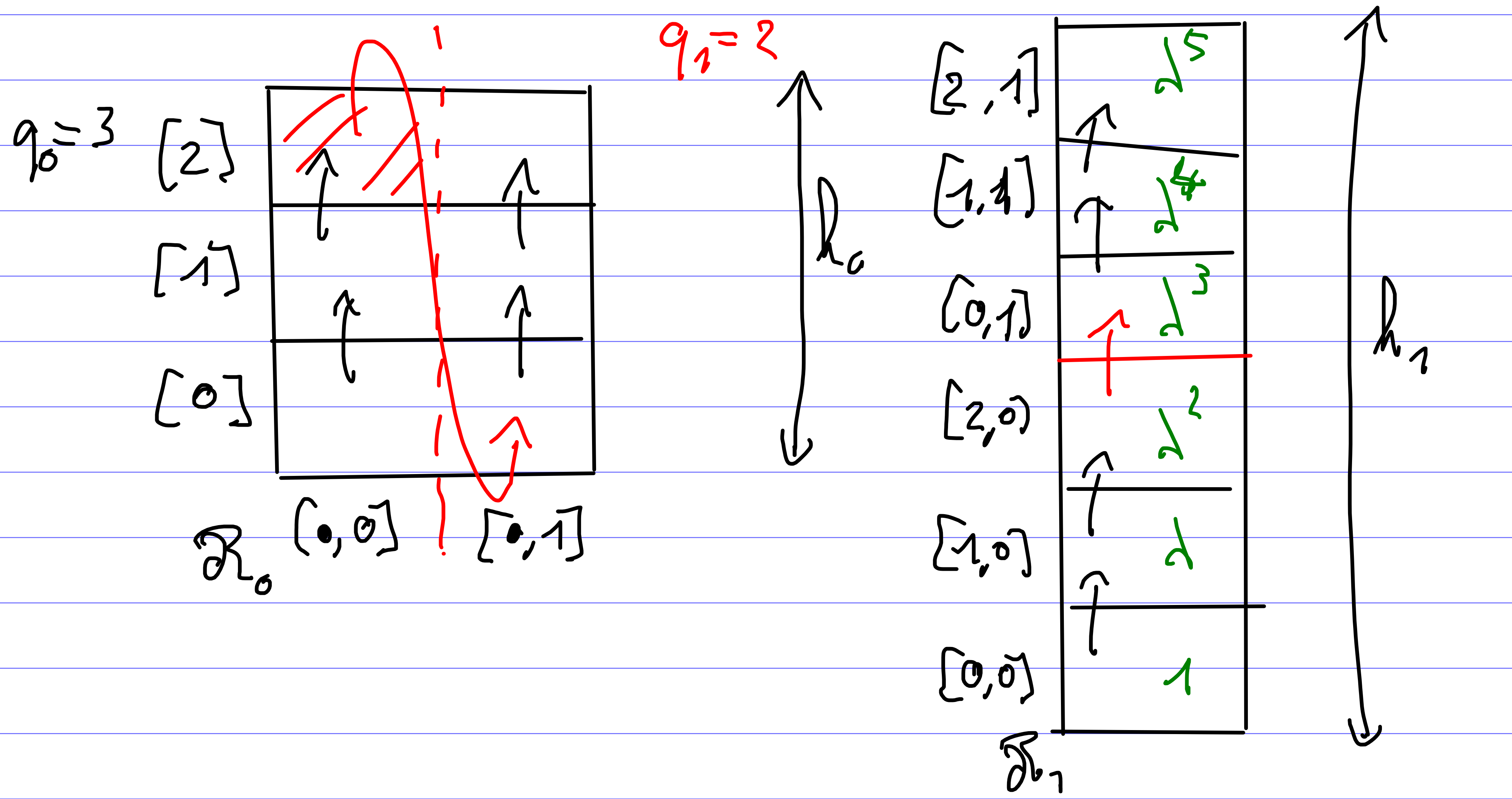
$$S((x_n)_{n\in\mathbb{N}}) = (x_n)_n + (1, 0, 0, \dots)$$

$$S(q_0 - 1, q_1 - 1, \dots) := (0, 0, 0, \dots)$$

ex : $q_n = 2$

$$\begin{array}{rcccccl} & 1 & 0 & 1 & 0 & \dots & (=x) \\ + & 1 & 0 & 0 & 0 & \dots & \\ \hline & 0 & 1 & 1 & 0 & & (=S\,x) \end{array}$$

n-cylinder : $[y_0, \dots, y_{n-1}] = \left\{ (x_n) \in X \;\middle|\; \begin{array}{l} x_0 = y_0 \\ \vdots \\ x_{n-1} = y_{n-1} \end{array} \right\}$



$h_n = q_0 q_1 \cdots q_{n-1}$

f

Def : $\lambda \in \mathbb{C}$ is an eigenvalue of S if $\exists \underbrace{f}_{\text{eigenfunction}} \in L^2(X, \mu) \setminus \{0\}$, $f \circ S = \lambda f$ a.e

unitary operator on L^2 : $f \mapsto f \circ S$ (Koopman operator)

$\longrightarrow |\lambda| = 1$

[point spectrum = {eigenvalues} $\quad Sp(S)$

Th: let S be the odometer on $\prod\{0,\dots,q_n-1\}$

- $Sp(S) = \left\{ \underbrace{\exp\left(\frac{2i\pi k}{h_n}\right)}_{\lambda} \;\middle|\; n\geq 0,\ k\in\{0,\dots,h_n-1\} \right\}$

  (see drawing for an eigenfunction)

- $\underbrace{\langle \text{eigenfunction} \rangle \text{ is dense in } L^2(X,\mu)}_{\text{having discrete spectrum}}$

Th (Halmos–Von Neumann): point spectrum is a complete invariant of $\underbrace{\text{conjugacy}}_{\substack{\text{even stronger}\\ \text{flip-conjugacy}}}$ among ergodic systems with discrete spectrum

$Sp(S)$ is completely determined by the amount of prime numbers in $\{h_n \mid \cancel{n\in\mathbb{N}}\}$ $\{q_0, q_1, q_2, \dots\}$

$p\in\Pi \rightsquigarrow k_p = \sum_{n\in\mathbb{N}} \nu_p(q_n)$

$\prod p^{k_p}$: supernatural number associated to S

↳ complete invariant of conjugacy

dyadic odometer: $k_2=\infty$, $k_p=0\ \forall p\neq 2$

$q_0=q_1=\dots=2$

$q_0'=2^{10}, q_1'=2, q_2'=4$

p-adic " : $k_p=\infty$, $k_q=0\ \forall q\neq p$

universal " : $\forall p$, $k_p=\infty$

S do not satisfy $(\ast)$ $(\exists n\geq 2, S^n \text{ erg})$ when $\forall p$, $k_p\neq 0$

S, S' odometers $\quad$ S' is a factor of S if $\forall p \in \Pi$, $k'_p \leq k_p$ $\Big)$ $\exists \psi : X \to X$ onto, $\mu(\psi^{-1}(\cdot)) = \mu$, $\psi \circ S = S' \circ \psi$

$$\begin{array}{ccc} \cdot & \xrightarrow{S} & \cdot \\ \psi \downarrow & & \downarrow \psi \\ \cdot & \xrightarrow[S']{} & \cdot \end{array}$$

Odometers have a poor dynamic

$\mathcal{P}^* = \{1\text{-cylinders}\}$

if $x \in [i]$, then $Sx \in [i+1]$, $S^2 x = [i+2]$

$[q_0 - 1]$
$[0]$
$[1]$
$\vdots$

Def : partition $\mathcal{P}$: coding map $[\mathcal{P}]_n$

$$[\mathcal{P}]_n(x) = \underbrace{(P_0, \dots, P_{n-1})}_{\text{word}} \text{ if } S^i x \in P_i \quad (P_i \in \mathcal{P})$$
$(0 \leq i \leq n-1)$

$x \in P_0$
$Sx \in P_1$
$\vdots$

$\mathcal{P} = \mathcal{P}^*$: if you know "$x \in [i]$" : only one word $[\mathcal{P}]_n(x)$

topological entropy and LB property are related to the diversity of words produced by a system through the time

$\longrightarrow h_{top}(S) = 0$
$\longrightarrow S$ is LB $\Big)$ see my notes

Topological entropy : X compact, $\mathcal{U}$ open cover of X

(if X is Cantor, $\mathcal{U}$ can be a partition)

$$\mathcal{U}^n = \bigvee_{i=0}^{n-1} T^{-i}\mathcal{U} = \{\cap T^{-i}(U_i) \mid U_i \in \mathcal{U}\}$$

$h_{top}(T, \mathcal{U})$ = asymptotic of $N(\mathcal{U}^n)$ $\qquad$ $N(\mathcal{U}) = \min\{|\mathcal{U}'|, \mathcal{U}' \text{ open subcover}\}$

$$= \lim_{n \to +\infty} \frac{\log N(\mathcal{U}^n)}{n}$$

$$h_{top}(T) = \sup_{\mathcal{U}} h_{top}(T, \mathcal{U})$$

if $\mathcal{U}$ is partition: $N(\mathcal{U}^n)$ = number of words of the form $[\mathcal{U}]_n(x)$, $x \in X$

Variational principle : $h_{top}(S) = \sup_{\mu} h_\mu(S)$

$\left(= h_\mu(S) \text{ if } \mu \text{ is unique} \right)$

III - odomutant

<u>Goal</u>: distort the orbits of the odometer to enrich the dynamic
(produce more words)

- increase entropy
- loose the LB property