Odomutants and flexibility results for quantitative orbit equivalence

(2/2)

III – Odomutants

Goal : distort the orbits of an odometer
to enrich the dynamics (produce more words)
- increase entropy
- loose the LB property

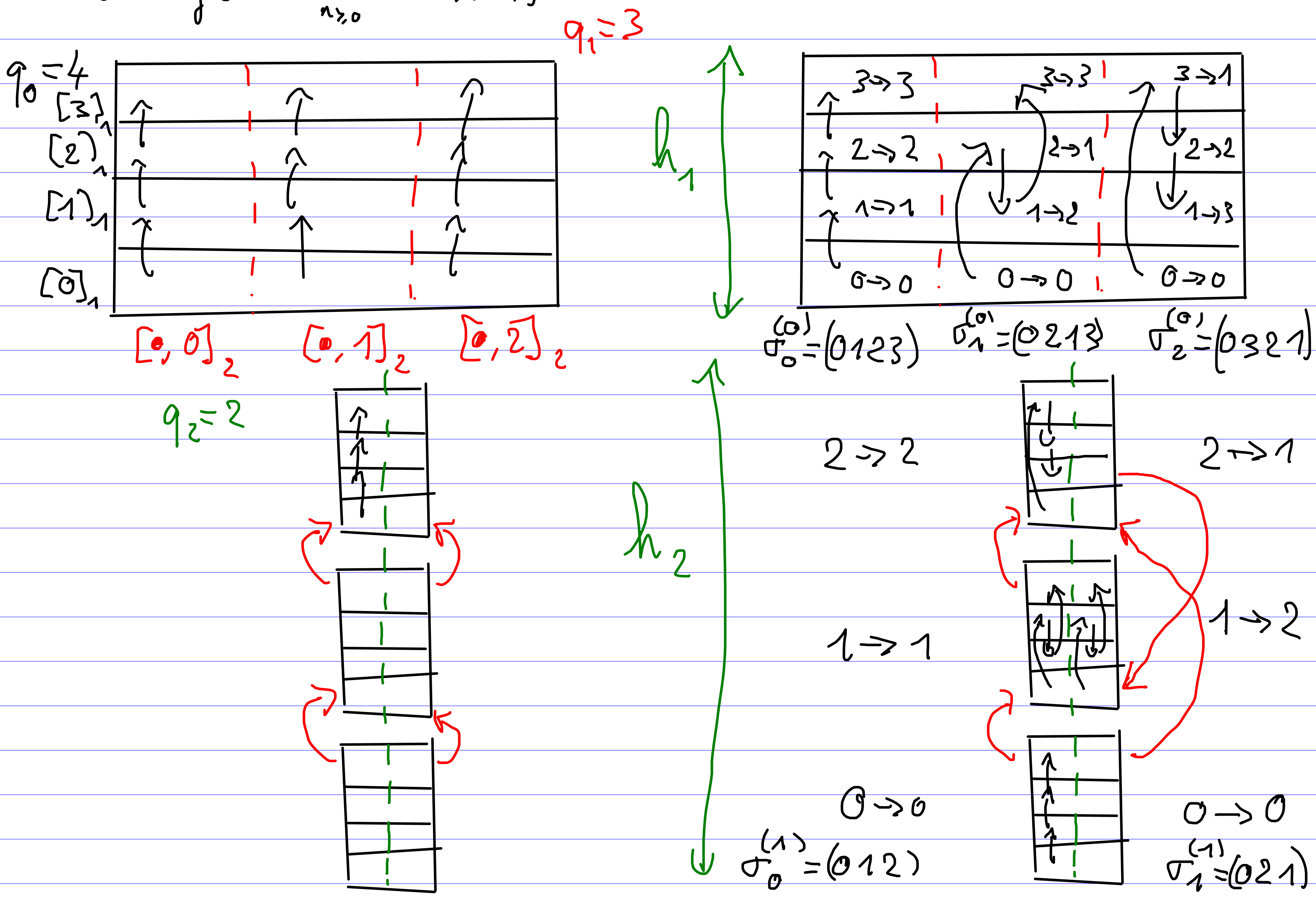
given a partition $\mathcal{P}$

$[\mathcal{P}]_n(x) = (P_0, \dots, P_{n-1}) \in \mathcal{P} \times \dots \times \mathcal{P}$
if $x \in P_0$,
$Sx \in P_1, \dots$
$S^{n-1}x \in P_{n-1}$

$[\cdot]_k$: k-cylinder

S odometer
(adding on $X = \prod_{n \geq 0} \{0, \dots, q_n - 1\}$)

T odomutant



$\sigma = (x_0 \cdots x_{n-1}) : \; i \mapsto x_i$

$h_0 = 0$

$h_i = q_0 \cdots q_{i-1}$ height of the i-th tower

Odomutat $=$ odometer S on $\prod \{0, \dots, q_n - 1\}$

$+ \; \forall n \geq 0, \; \underbrace{\left(\sigma^{(n)}_{x_{n+1}}\right)_{0 \leq x_{n+1} \leq q_{n+1}-1}}_{\mathrm{Sym}(\{0, \dots, q_n - 1\})}$

the $(n+1)$-th coordinate x_{n+1} determines the permutation on the n-th coordinate

$$\psi_n(x) = \left(\sigma^{(0)}_{x_1}(x_0), \sigma^{(1)}_{x_2}(x_1), \dots, \sigma^{(n)}_{x_{n+1}}(x_n), x_{n+1}, x_{n+2}, \dots\right)$$

$$\psi(x) = \lim_{n \to +\infty} \psi_n(x) = \left(\sigma^{(i)}_{x_{i+1}}(x_i)\right)_{i \geq 0}$$

Properties : a) ψ_n and $\psi : X \to X$ are pmp

b) $\psi_i : X \to X$ is a homeomorphism

c) ψ is continuous and onto

Idea of proof : look at cylinders

hard point c) "onto" $\quad y = (y_0, y_1, \dots) \in X$

$\forall n \in \mathbb{N}$, I can find $x^{(n)} \in X$ such that the n first coordinates of $\psi(x^{(n)})$ are $(y_0, \dots, y_{n-1})$

$\psi(x^{(n)}) \to y$

x : an accumulation point of $(x^{(n)})$ (X is compact)

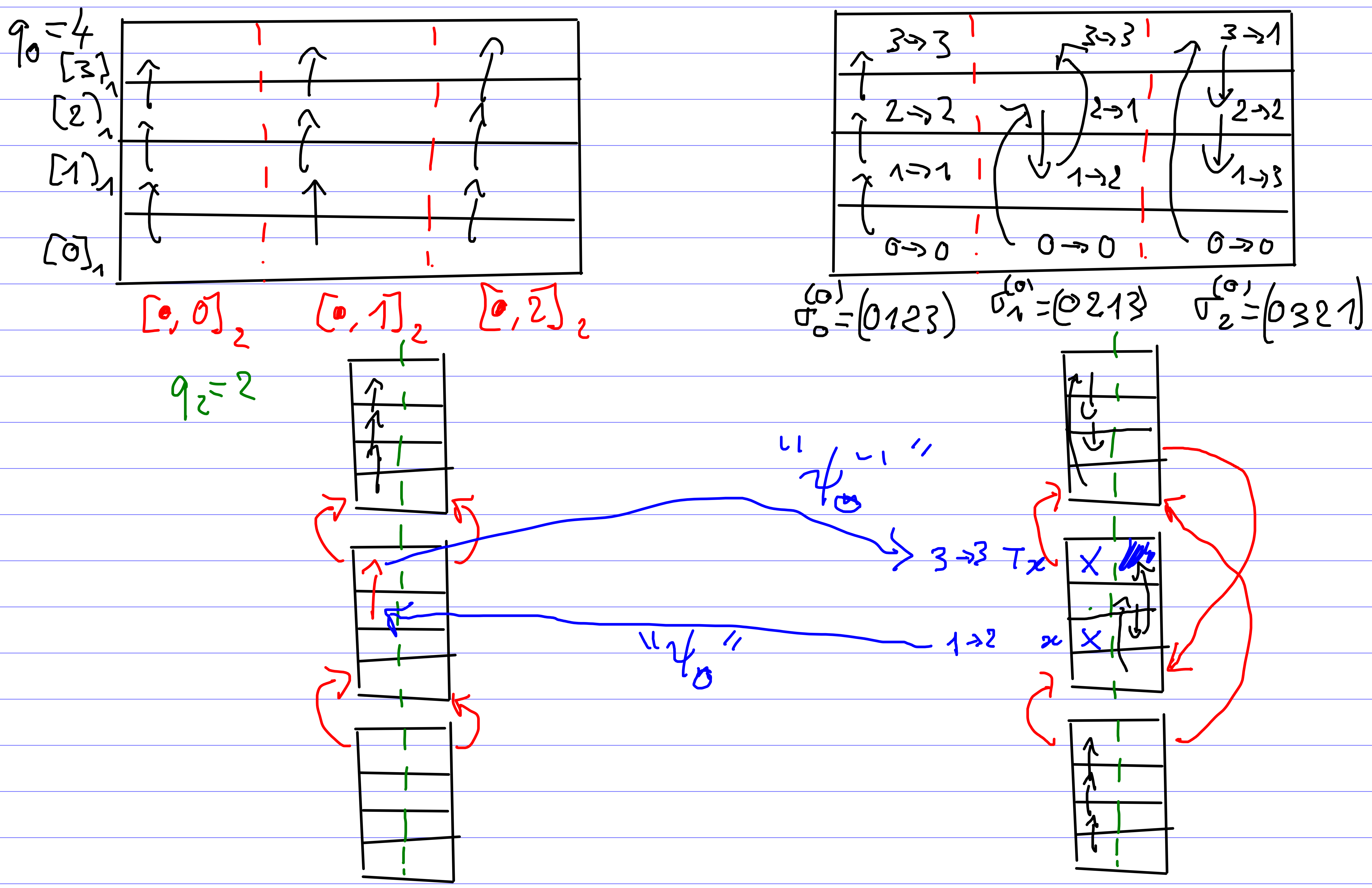
Def: $x^+ = (q_0 - 1, q_1 - 1, \dots)$

$x^- = (0, 0, 0, \dots)$

$N^+(x) = \min\{ i \geq 0 \mid (\psi(x))_i \neq q_i - 1 \} \in \mathbb{N} \cup \{+\infty\}$

$\forall n \geq N^+(x)$, $Tx := \psi_n^{-1} S \psi_n(x)$ $\quad (N^+(x) < \infty)$

T "odomutant" (does not depend on $n \geq N^+(x)$)



$\psi_n \circ T(x) = S \circ \psi_n(x)$ when x is "nice" $N^+(x) < \infty$

$\downarrow n \to +\infty$

$\psi \circ T(x) = S \circ \psi$

Prop: T is a bijection $\underbrace{\psi^{-1}(X\setminus\{x^+\})}_{\{x \mid N^+(x)<\infty\}}$ to $\psi^{-1}(X\setminus\{x^-\})$

$T\in \mathrm{Aut}(X,\mu)$

T is a factor its associated odometer S via ψ

$Sp(S) \subset Sp(T)$

Th: $Sp(S) = Sp(T)$

by Halmos - Von Neumann theorem:

T is an odometer

$\Updownarrow$

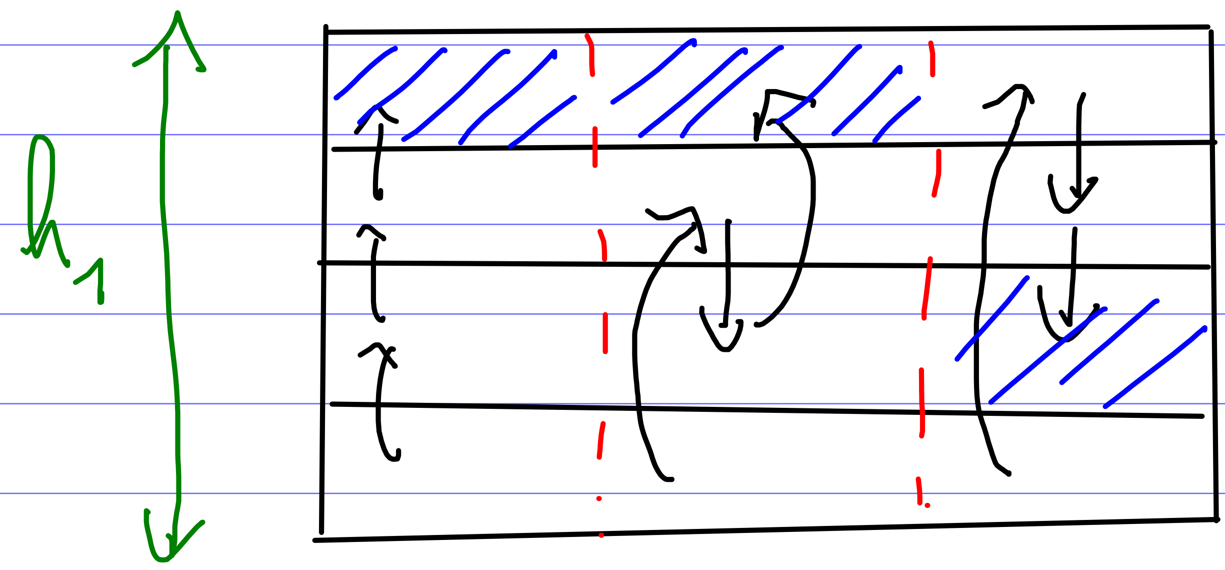
T is conjugate to its associated odometer S

Th:
- id_X is an OE between S and T
  $(\forall^{\mu} x,\ \mathrm{Orb}_T(x) = \mathrm{Orb}_S(x))$
- $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$
  if (C1) $\sum \frac{\varphi(h_{n+1})}{h_n} < \infty$, then it is a φ-integrable OE

Proof: real proof: computing exactly the cocycles (I don't want)

intuitive proof: with the drawing

1^{st} step

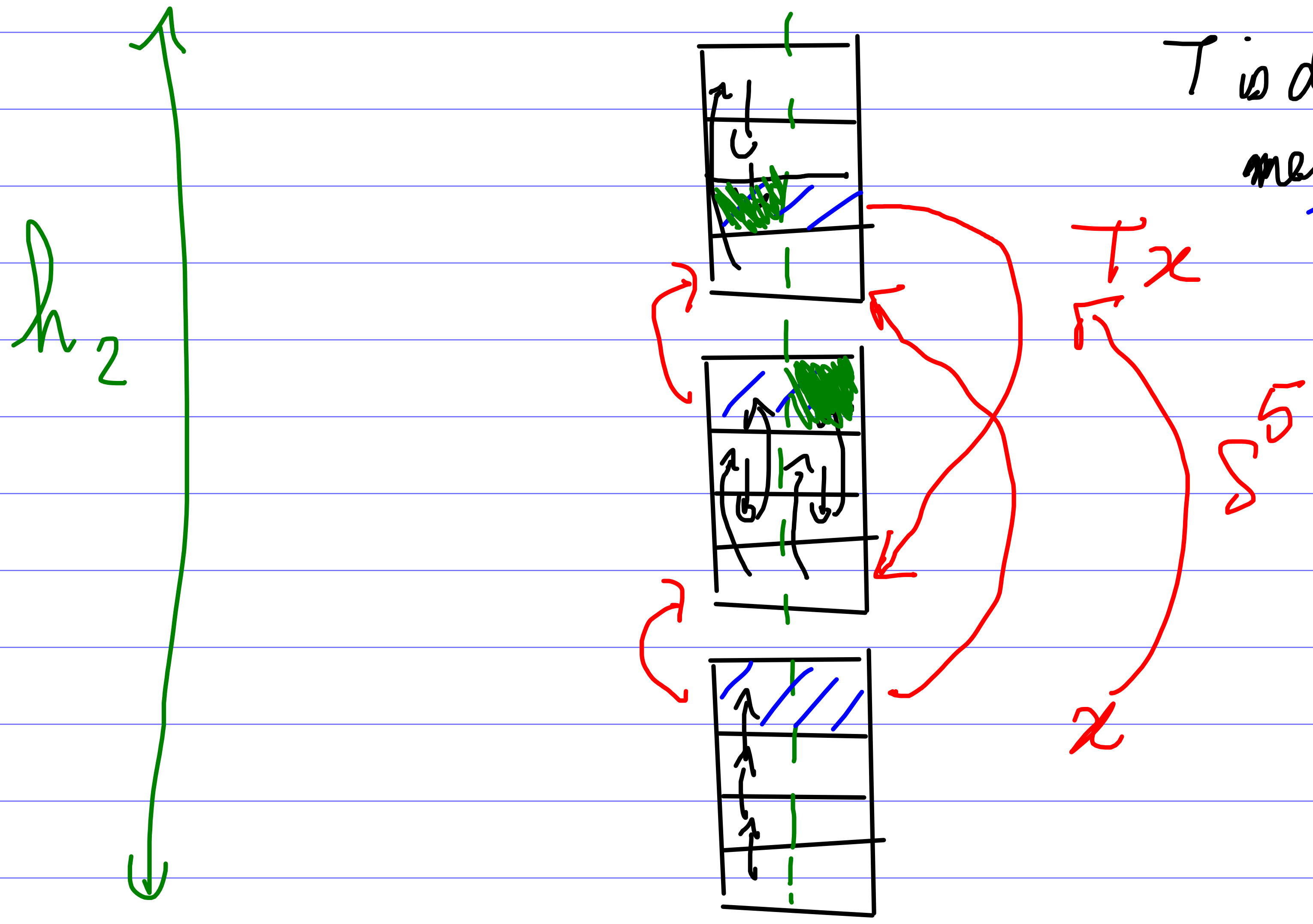


T defined on $X \setminus \square$

$|C_T| \leq h_1$

$\mu(X \setminus \square) \leq \mu(X) = 1$

2^{nd} step:



T is defined on new points in $\square \setminus \square$

$|C_T| \leq h_2$

$\mu(\square \setminus \square) \leq \mu(\square) \leq 1/h_1$

$Tx = S^{C_T(x)} x$

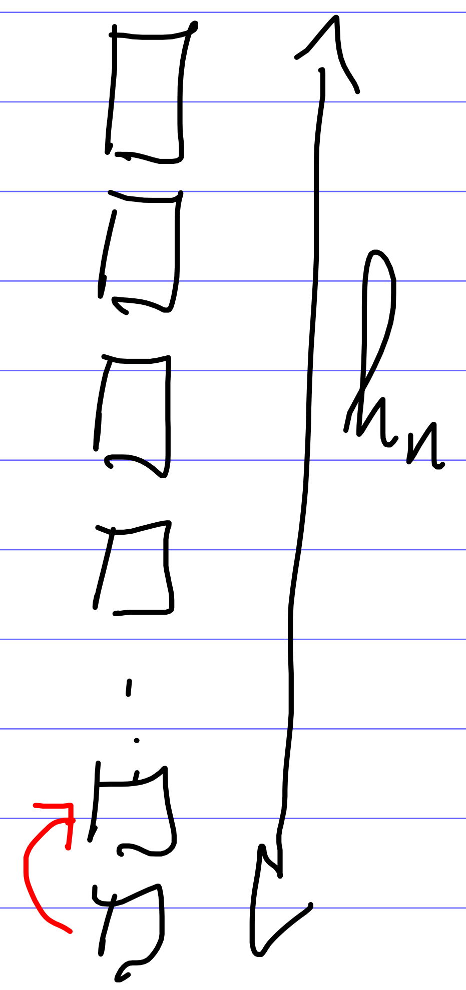
length of the arrows is exactly $|C_T|$

$$\int_X \varphi(|C_T(x)|) = \sum \int_{\substack{\text{step } n \\ \mu \leq 1/h_n}} \underbrace{\varphi(|C_T(x)|)}_{\leq \varphi(h_{n+1})} \leq \sum \frac{\varphi(h_{n+1})}{h_n}$$

Remark: $|C_T| \leq h_n$ is maybe coarse

there exists a finer condition (C2)

and we will apply it for ThA



Topological context :

Th : Assume :
$\forall n \geq 0, \forall 0 \leq x_{n+1} \leq q_{n+1}-1, \sigma^{(n)}_{x_{n+1}}$ fix 0 and $q_n - 1$

then : • T has an extension to a homeomorphism
• the OE is a strong OE

Proof : $\psi^{-1}(\{x^+\}) = \{x^+\}$

if $\psi(x) = x^+$

$\sigma^{(i)}_{x_{i+1}}(x_i) = q_i - 1 = \sigma^{(i)}_{x_{i+1}}(q_i - 1)$

$x_i = q_i - 1$

$\psi^{-1}(\{x^-\}) = \{x^-\}$

$T : X \setminus \{x^+\} \to X \setminus \{x^-\}$

$Tx^+ := x^-$ $\rightsquigarrow$ homeomorphism minimal

c_T, c_S are continuous on $X \setminus \{x^+\}$

$c_T(x^+) = c_S(x^-) = 1$ ∎

III - Sketchs of proof

1) **Th C** : S dyadic odometer, $\exists T \in \mathrm{Aut}(X, \mu)$
• T and S are $L^{<1/2}$ - OE
• T and S are not Kakutani equivalent

- write the non LB system of Feldman as an odomutat associated to the dyadic odo S
- check (C1) for $\varphi(x) = x^p$ $\forall p < 1/2$

S is LB so S and T are not Ke

2) **Th A**: $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ sublinear, S an odometer.
then $\exists\, T \in Aut(X,\mu)$,
- S, T are φ-int OE
- S, T not flip-conjugate

given a factor T of a system S, with factor map ψ

$$\psi \circ T = S \circ \psi$$

- if ψ is an isomorphism, then T, S are isomorphic
- if T, S are isomorphic, ψ is an isomorphism?

NO

Def: S is coalescent if the answer is YES!

Th: An ergodic system with discrete spectrum (ex: odometer) is coalescent

Proof: S discrete spectrum, T conjugate to S, ψ factor map

$E_S(\lambda) = \{$ eigenfunctions of S associated to $\lambda \}$

$E_T(\lambda) = \{$ T $\}$

ergodicity $\Rightarrow$ $E_S(\lambda)$ and $E_T(\lambda)$ are lines or $\{0\}$

ψ factor map $\Rightarrow$ if $f \in E_S(\lambda)$, then $f \circ \psi \in E_T(\lambda)$
$(\psi T = S \psi)$

conjugate $\Rightarrow$ $E_S(\lambda) \neq \{0\} \iff E_T(\lambda) \neq \{0\}$

$E_T(\lambda) = \{f \circ \psi \mid f \in E_S(\lambda)\}$

$L^2(X,\mu) = \{f \circ \psi \mid f \in L^2(X,\mu)\}$

$\Rightarrow$ ψ is injective a.e

Corollary: S odometer, T an associated odomutant

TFAE

- T is an odometer
- T is conjugate to S
- $\psi : x \mapsto \left(\sigma^{(i)}_{x_{i+1}}(x_i)\right)_{i\geq 0}$ is injective a.e

($\iff$ isomorphism)

Goal to thm A: find permutations s.t.:

- ψ is not injective a.e.
- weak distortions of the orbit to get φ-int$^{\text{d}}$

(use finer condition (C2))

3) **Th B** : S universal odometer. $\exists T$ minimal Cantor homeo s.t.

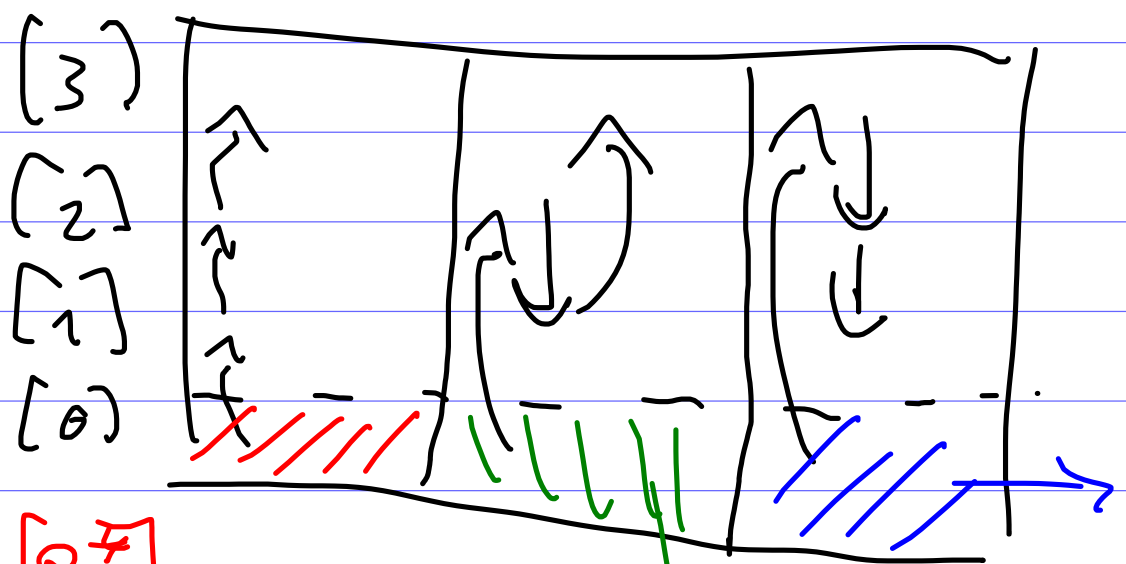
- $h_{top}(T) > 0$ $\quad (h_{top}(S) = 0)$
- S, T are strongly OE
  $\frac{\log}{\log^{\circ m}}$ - int $\forall m \geq 0$

$\sigma^{[n]}_{x_{n+1}}$ fixes 0 and $q_n - 1$ $\to$ T homeo
$h_{top}(T)$ well-defined
$\to$ strong OE between T, S
$\to$ T is uniquely erg

[T is an extension of S via φ]

$\mathcal{P}^* = \{1\text{-cylinders}\}$ $\qquad h_{top}(T) \geq h_{top}(T, \mathcal{P}^*)$

<u>Goal</u>: $h_{top}(T, \mathcal{P}^*) > 0$



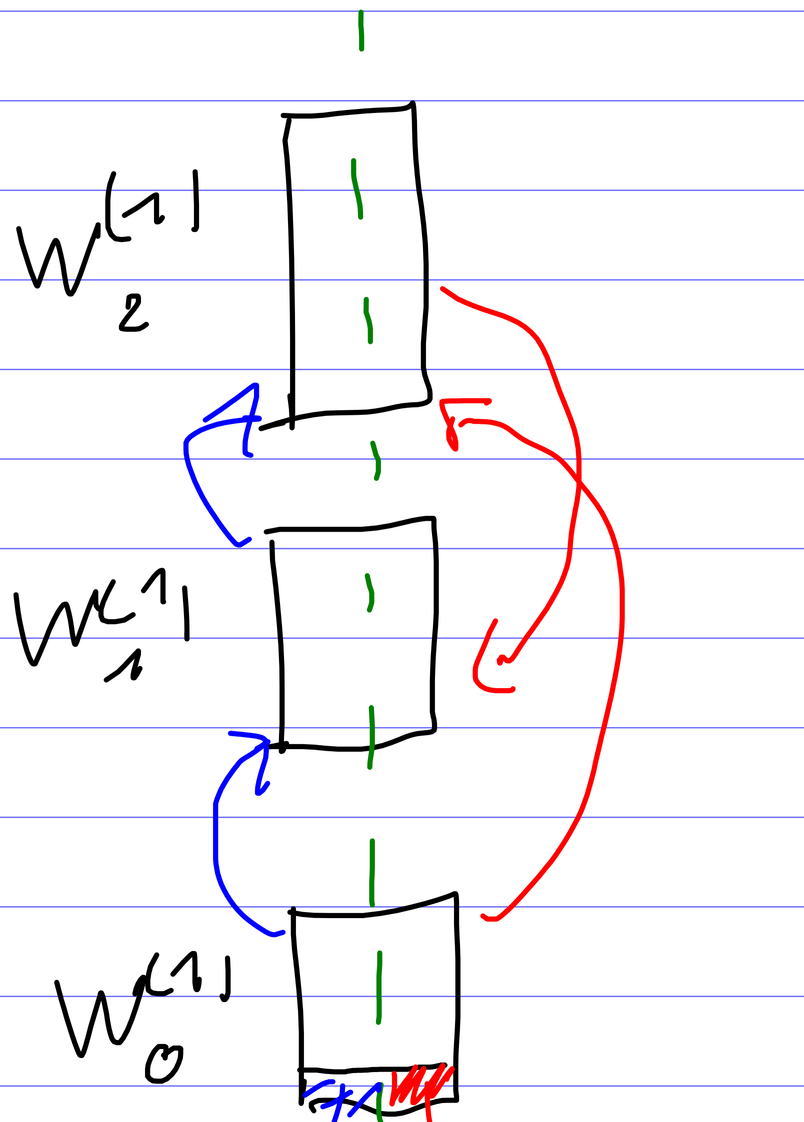
$\to [0,3,2,1]$ $= W_2^{(1)}$

$[\mathcal{P}^*](x) = (0,1,2,3) = W_0^{(1)}$, $\varphi = \pi_1$

$[0],[2],[1],[3] = W_1^{(1)}$

$[0,0]_2 \qquad [0,1]_2 \qquad [0,2]_2$

x_n with $n=1$



$W_0^{(2)} = [\mathcal{P}^*]_{h_2} = W_0^{(1)} \cdot W_1^{(1)} \cdot W_2^{(1)}$

$W_1^{(2)} = W_0^{(1)} \cdot W_2^{(1)} \cdot W_1^{(1)}$

① $\left\{ [\partial^{*}]_{h_n}(x) \mid x \in [0,\dots,0,x_n]_{n-n+1} \right\} = \left\{ W^{(n)}_{x_n} \right\}$

② $W^{(n+1)}_{x_{n+1}} = W^{(n)}_{\left(\sigma^{(n)}_{x_{n+1}}\right)^{-1}(0)} \cdot \dots\dots \cdot W^{(n)}_{\left(\sigma^{(n)}_{x_{n+1}}\right)^{-1}(q_n-1)}$

if $W^{(n)}_0, \dots, W^{(n)}_{q_n-1}$ are pairwise different
and $\sigma^{(n)}_0, \dots, \sigma^{(n)}_{q_{n+1}-1}$ too,

$q_{n+1} = (q_n-2)!$.

then $W^{(n+1)}_{(0)}, \dots, W^{(n+1)}_{q_{n+1}-1}$ are pairwise different

$N(\partial^{* h_n}) \geq \left| \left\{ W^{(n)}_0, \dots, W^{(n)}_{q_n-1} \right\} \right| = q_n$

$x_{n+1} \in [0,\dots,q_{n+1}-1] \longmapsto \sigma^{(n)}_{x_{n+1}} \in \left\{ \sigma \in \mathrm{Sym}[0,\dots,q_n-1],\ \sigma(0)=0,\ \sigma(q_n-1)=q_n-1 \right\}$

is a bijection

$\dfrac{\log N(\partial^{* h_n})}{h_n} \geq \dfrac{\log q_n}{h_n} =: v_n$

Lemma :
- (v_n) is decreasing
- $v_0 - 6 \leq v_n \leq v_0$
- $\dfrac{1}{\log^{(0m)}(q_n)}$ is summable

Proof : • $\boxed{q_{n+1} = (q_n - 2)!} \le q_n! \le q_n^{q_n}$

$$v_{n+1} \le \frac{q_n \log q_n}{h_{n+1}} = v_n \qquad h_{n+1} = q_n h_n$$

• $\log(k!) \ge k \log(k) - k$

$$\log q_{n+1} = \log(q_n!) - \log(q_n - 1) - \log(q_n)$$
$$\ge q_n \log(q_n) - 3 q_n$$

$$v_{n+1} \ge v_n - \frac{3}{h_{n-1}} \qquad v_n \ge v_0 - \underbrace{\Sigma}_{\le 6}$$

• N s.t $(v_0 - 6) h_n \ge 1 \quad \forall n \ge N$

$$\log q_{n+1} = h_{n+1} v_{n+1} \ge q_n \underbrace{(v_0 - 6) h_n}_{\ge 1} \ge q_n$$

$$\log(\underbrace{\log q_{n+2}}_{\ge q_{n+1}}) \ge \log q_{n+1} \ge q_n$$

$$\vdots$$

$$\forall n \ge N, \ \log^{(m)}(q_{n+m}) \ge q_n$$

Proof of th B : • $q_0 > e^6 \quad (v_0 - 6 > 0)$

$$q_{n+1} = (q_n - 2)!$$

$$x_{n+1} \mapsto \tau^{(n)}_{x_{n+1}} \text{ bijection}$$

$$h_{top}(T) \ge h_{top}(T, \mathcal{P}^*) = \lim \frac{\log N(\mathcal{P}^{*h_n})}{h_n} \ge \lim v_n \ge v_0 - 6 > 0$$

• check (C1) to get φ-int $\varphi = \frac{\log}{\log^{(om)}}$

$$\frac{\varphi(h_{n+1})}{h_n} = \frac{1}{\underbrace{\log^{om}(h_n)}_{\geq q_{n-1}}}\left(\underbrace{\frac{\log(h_n)}{h_n}}_{\leq 1} + \underbrace{\frac{\log q_n}{h_n}}_{v_n \leq v_0}\right)$$

$$\leq \frac{1}{\log^{(om)}(q_{n-1})}(1+v_0)$$

$\rightsquigarrow$ summable

so $\sum \frac{\varphi(h_{n+1})}{h_n}$ converges.

4) Comments

Minimal Cantor homeomorphism

$\rightsquigarrow$ dimension group

[Giordano, Putnam, Skau] : dim group is a complete invariant of strong OE

Th (Boyle, Handelman 94): $\alpha > 0$ or $+\infty$, S dyadic odometer

$\exists T$ minimal Cantor homeo s.t.

- $h_{top}(T) = \alpha$
- S, T are strongly OE

I proved (Th B') the same with • a φ-int
$\varphi = \frac{\log}{\log^{\circ m}}$
• a large class of odometers

B-H : - OE established in an abstract way
- they actually built a Bratteli diagram of an ... odomutant!

me : - explicit OE
- other formalism to define T
+ find properties