A quasi-isometric classification of permutational wreath products

Starting point: a finitely group $G = \langle S \rangle$ is a metric space, so which other metric space is quasi-isometric to G?
Or given G, H, is there a Q.I. $G \to H$?

Definition: A map $f : (X, d_X) \longrightarrow (Y, d_Y)$ is a quasi-isometry if there are $C \geq 1$, $K \geq 0$ such that:

(i) $\frac{1}{C} d_X(x,y) - K \leq d_Y(f(x), f(y))$
$\leq C\, d_X(x,y) + K \quad \forall x,y \in X.$

(ii) $d_Y(y, f(X)) \leq K \quad \forall y \in Y.$

Equivalently, f satisfies (i) and $\exists g : Y \to X$ such that $d(\underbrace{g\circ f, Id_X}), d(f\circ g, Id_Y) \leq K$

↳ where $d(f,g) := \sup_{x \in X} d_Y(f(x), g(x))$

Remark: Being quasi-isometric is an equivalence relation between metric spaces.

What is known about the Q.I. rigidity of amenable groups?

Definition: • G is Q.I. rigid if any group Q.I. to G is isomorphic to a finite index subgroup of G, or to a quotient of G by a finite normal subgroup (or the other way around.)

• A class $\mathcal{C}$ of groups is Q.I. rigid if any group Q.I. to a group $G \in \mathcal{C}$ is isom. to a f.i. subgroup or to a quotient by a finite subgroup of some $G' \in \mathcal{C}$.

Theorem: $\forall n \geq 1$, $\mathbb{Z}^n$ is Q.I. rigid.

Theorem (Farb-Mosher 1999, 2000): $\forall n \geq 1$, the Baumslag-Solitar group

$$BS(1,n) = \langle a, t \mid tat^{-1} = a^n \rangle$$

is Q.I. rigid.

Moreover, $BS(1,n)$ and $BS(1,m)$ are Q.I. iff n, m are powers of a common number.

Open questions :

- Is the class of finitely presented solvable groups Q.I. rigid?
- Is the class of f.p. metabelian groups Q.I. rigid?
- Is it true that being virtually polycyclic is a Q.I. invariant?

Wreath products

Given groups G, H, define

$$G \wr H := \left(\bigoplus_H G \right) \rtimes H$$

where H acts on $\bigoplus_H G$ by permuting the coordinates.

In addition, if $G = \langle S_G \rangle$ and $H = \langle S_H \rangle$ then $G \wr H$ is generated by

$$\{ \delta_a : a \in S_G \} \cup S_H$$

where $\delta_a : H \to G$, $h \mapsto \begin{cases} a & \text{if } h = 1_H \\ 1_G & \text{otherwise.} \end{cases}$

Thus, if we think $(c,p) \in G \wr H$ as a colouring of (the Cayley graph of) H with colors in G, finitely supported, together with an arrow $p \in H$, then right multiplying (c,p) by a generator amounts

- either to keep the same colouring and moving the arrow to a neighbour $q = ps$ of p:

$$(c,p) \cdot (\mathbb{1}, s) = (c, ps), \quad s \in S_H.$$

- or to keep the arrow where it is (i.e. on $p \in H$) and changing the color of this vertex, from $c(p)$ to $c(p)a$:

composition law in $\bigoplus_H G$

$$(c,p)\cdot(\delta_a, 1_H) = (c + p\cdot\delta_a, p), \quad a \in S_G$$

More generally, given $n \geq 2$ and a bounded degree graph X, $\mathcal{L}_n(X)$ is the graph:

- whose vertices are pairs (c,p) with $c: V(X) \longrightarrow \mathbb{Z}_n$ is a colouring of the vertices of X, $\underbrace{\text{finitely}}_{\substack{\text{i.e. } |\mathrm{supp}(c)| < \infty \\ \text{where } \mathrm{supp}(c) := \{x \in V(X) : c(x) \neq 0\}}}$ supported

  and $p \in V(X)$;
- whose edges connect $(c,p), (c',p')$ if either $c = c'$ and $p \underset{X}{\sim} p'$, or if $p = p'$ and c, c' only differ on p: $\mathrm{supp}(c'-c) = \{p\}$.

When $X = \mathrm{Cay}(H,S)$, then $\mathcal{L}_n(X)$ is the Cayley graph of $\mathbb{Z}/n\mathbb{Z} \wr H$ with respect to the generating set $\mathbb{Z}/n\mathbb{Z} \cup S$.

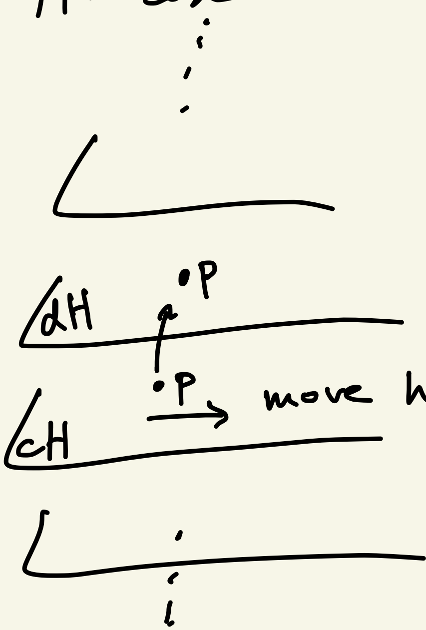
Given a colouring $c: V(X) \to \mathbb{Z}_n$, the set

$$L(c) = \{ (c,p) \in \mathcal{L}_n(X) : p \in V(X) \}$$

is called a leaf.

Remark: in $F \wr H$, leaves are just H-cosets: $L(c) = (c, 1_H) \cdot H$.

in $F \wr H$



move vertically: moves of the second kind.

move horizontally: moves of the first kind

Question: When are two wreath products $G_1 \wr H_1$, $G_2 \wr H_2$ quasi-isometric?

The answer $H_1 = H_2 = \mathbb{Z}$ is due to Eskin-Fisher-Whyte.

Theorem: $F_1 \wr \mathbb{Z}$ and $F_2 \wr \mathbb{Z}$ are Q.I. iff there exist $a, r, s \geq 1$ such that $|F_1| = a^r$, $|F_2| = a^s$.

The proof relies on a technique of "coarse differentiation", and is specific to two ended groups.

With completely different methods, Genevois and Tessera proved the following:

$$\left(G \wr H \overset{\text{“}\bigoplus_H G \rtimes H\text{''}}{} \text{ is amenable} \iff G, H \text{ are amenable} \right)$$

Theorem (2024):

Let F_1, F_2 be two non trivial finite groups. Let H_1, H_2 be finitely presented and one-ended groups. Then:

(i) If H_1 is not amenable, then $F_1 \wr H_1$ and $F_2 \wr H_2$ are Q.I. if and only if $|F_1|, |F_2|$ have the same prime divisors and H_1, H_2 are quasi-isometric.

(ii) If H_1 is amenable, then $F_1 \wr H_1$ and $F_2 \wr H_2$ are Q.I. if and only if there are $a, r, s \geq 1$ s.t. $|F_1| = a^r$, $|F_2| = a^s$ and there exist a quasi-$\frac{s}{r}$-to-one quasi-isometry $H_1 \longrightarrow H_2$.

Remark: More generally, they classify $\mathcal{L}_n(X)$ for some assumptions on X.

Scaling quasi-isometries

Notations: In a metric space X, given $A \subseteq X$ and $R \geq 0$, A^{+R} the R-neighborhood of A,

$$A^{+R} := \bigcup_{a \in A} B_X(a, R).$$

Note that $(A^{+R})^{+S} \subseteq A^{+(R+S)}$.

The Hausdorff distance between $A, B \subseteq X$ is

$$d_{Haus}(A, B) := \inf \{ R \geq 0 : A \subset B^{+R}, B \subset A^{+R} \}.$$

Lemma: Let $f : X \to Y$ be a (C, K)-Q.I. Let $A \subseteq X$. Then $f(A^{+R}) \subseteq f(A)^{+(CR+K)}$.

Definition: A Q.I. $f : X \to Y$ between two bonded degree graphs is quasi-k-to-one (for $k > 0$), if there is $C>0$ such that

$$\big|\, k|A| - |f^{-1}(A)| \,\big| \leq C \cdot |\partial_Y A|,$$

for all finite subsets $A \subset Y$, where

$$\partial_Y A := \{ y \in Y \setminus A : \exists a \in A,\ y \underset{Y}{\sim} a \}.$$

Theorem (Whyte 1999): A Q.I. $f: X \to Y$ is quasi-one-to-one if and only if it lies at bounded distance from a bijection.

Remark: Between non amenable spaces, a Q.I. $f: X \to Y$ is quasi-k-to-one for any $k>0$. Indeed, as Y is not amenable, there is $\varepsilon > 0$ such that $|\partial A| > \varepsilon |A|$ for any $A \subset Y$ finite, and thus

$$\big|k|A| - |f^{-1}(A)|\big| \leq k|A| + |f^{-1}(A)|$$

$\leq (k+P)|A|$ (where $P \geq 1$ is a uniform bound on $|f^{-1}(\{y\})|, y \in Y$)

$$\leq \frac{k+P}{\varepsilon} |\partial A|.$$

Consequence: Any quasi-isometry between non amenable spaces is at bounded distance from a bijection.

Main example of a scaling Q.I: the inclusion $H \hookrightarrow G$ of a f.i. subgroup into G is quasi-$\frac{1}{[G:H]}$-to-one.

For amenable spaces, we have:

Lemma: Let X amenable. If $f: X \to Y$ is quasi-k-to-one and quasi-k'-to-one, then $k = k'$.

Proof: let $(F_n)_{n\in\mathbb{N}}$ be a Følner sequence of X, i.e. $\frac{|\partial F_n|}{|F_n|} \xrightarrow[n\to\infty]{} 0$.

f quasi-k-to-one $\Rightarrow \exists C>0$ s.t.

$$\forall n\in\mathbb{N},\ \big|k|F_n| - |f^{-1}(F_n)|\big| \leq C\cdot|\partial F_n|$$

i.e. $\forall n\in\mathbb{N},\ \left|k - \frac{|f^{-1}(F_n)|}{|F_n|}\right| \leq C\cdot\frac{|\partial F_n|}{|F_n|}$

$$\Rightarrow k = \lim_{n\to\infty}\frac{|f^{-1}(F_n)|}{|F_n|} = k'. \ \square$$

Stability properties

Theorem: let $f: X\to Y$, $g: Y\to Z$, $h: X\to Y$ be Q.I.

(i) If $f: X\to Y$ is quasi-k-to-one, and $d(h,f)<\infty$, then h is quasi-k-to-one.

(ii) If $f: X\to Y$ is quasi-k-to-one

and $g: Y \to Z$ is quasi-k'-to-one, then $g \circ f$ is quasi-kk'-to-one.

(iii) If $f: X \to Y$ is quasi-k-to-one, then any of its quasi-inverses is quasi-$\frac{1}{k}$-to-one.

All this shows that, when X is amenable, there is a well-defined group morphism

$$Sc: QI_{sc}(X) \big/ \text{bounded distance} \longrightarrow (\mathbb{R}_{>0}, \cdot)$$

$$f \text{ quasi-}k\text{-to-one} \longmapsto k$$

and we call $Sc(X) := Im(Sc)$ the scaling group of X.

Proposition (Genevois and Tessera):

- $Sc(\mathbb{Z}^n) = \mathbb{R}_{>0}$, and more generally $Sc(\Gamma) = \mathbb{R}_{>0}$ for any $\Gamma < G$

lattice in a Carnot group;

- $Sc(BS(1,n)) = \mathbb{R}_{>0}$ $\forall n \geq 1$.

Towards Genevois–Tessera's strategy: 3 main steps.

Step 1: Fix an (A,B) - Q.I.

$$q: F_1 \wr H_1 \longrightarrow F_2 \wr H_2 .$$

Then q lies at finite distance from a Q.I. $\tilde{q}$ that sends H_1-cosets into H_2-cosets, and has a quasi-inverse that does the same in the other direction. This follows from the following:

Theorem: Let G be a finitely presented group and let H a one ended group. Then, for any coarse embedding $\rho : G \hookrightarrow F \wr H$, there is $R \geq 0$ such that $\rho(G)$ lies into the R-neighborhood of an H-coset.

Remark: They proved the same statement for general lamplighter graphs.