A generalization of Dye's reconstruction theorem by Fremlin

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Abstract

In this note we provide context and give a demonstration of a generalization of Dye's reconstruction theorem [Dye63, Thm. 2], proved by Fremlin [Fre02, Thm. 384D] following Eigen's arguments from [Eig82]. The theorem is stated for the very general setting of Dedekind complete Boolean algebras and their automorphism groups, but we will specifically work with measure algebras of standard spaces.

Contents

Organization of the note

This note is organized in the following way: Section 1 contains both Dye and Fremlin's Reconstruction theorems, both stated in their (almost)-original setting, followed by the relevant definitions and elementary facts. Section 2 explains why Dye's setting satisfies the hypotheses of Fremlin's theorem, and in particular it establishes the link between the two different languages. The three subsections are specifically dedicated to order-completeness, the supports of Borel bijections, and to finding many involutions in our full groups, respectively. Finally, section 3 is dedicated to the proof of a version of the theorem specifically stated in the context of measure algebras, although the general proof is sensibly the same. The reader only interested in the proof can (quasi)-safely go to Theorem 2.6 and skip the rest of Sections 1 and 2, then go (pseudo)-directly to Section 3.

1 Statement and definitions

1.1 Dye's Reconstruction theorem

Let us start by recalling Dye's theorem, in its most commonly (in ergodic theory) found form. For its most general form, see Section 2.3.2.

Theorem 1.1 ([Dye63, Thm. 2], see also [Kec10, Sec. I.4])**.** *Let $\mathbb{G}$ and $\mathbb{H}$ be two ergodic full groups on a standard probability space (X,μ). Then any isomorphism between $\mathbb{G}$ and $\mathbb{H}$ is the conjugation by some measure-preserving bijection. In other words, for any group isomorphism $\psi : \mathbb{G} \to \mathbb{H}$, there exists S in $\mathrm{Aut}(X,\mu)$ such that for all $T \in \mathbb{G}$, we have*

$$\psi(T) = STS^{-1}.$$

Before giving Fremlin's version of the theorem, we recall the terminology and give basic definitions and facts.

Definition 1.2. A **Polish space** is a separable and completely metrizable topological space.

Remark 1.3. There always exists a bounded (often by 1) metric which is compatible with the topology of a Polish space. Indeed, if d is a compatible metric, then $\min(1,d)$ is suitable, as it is equivalent to d. Moreover $\min(1,d)$ has the same Cauchy sequences as d, in particular every Polish space admits a compatible complete bounded metric.

Definition 1.4. A **standard Borel space** X is an uncountable measure space with a σ-algebra $\mathcal{B}(X)$ of subsets that are Borel for some Polish topology on X. Let us now endow $(X,\mathcal{B}(X))$ with a measure, we are mainly interested in two different cases:

1. If μ is a nonatomic (or diffuse) probability measure defined on $\mathcal{B}(X)$, then $(X,\mathcal{B}(X),\mu)$ is a **standard probabilty space**.
2. If λ is a nonatomic σ-finite measure defined on $\mathcal{B}(X)$ such that $\lambda(X)$ is infinite, then $(X,\mathcal{B}(X),\lambda)$ is a **standard σ-finite space**.

In the rest of this note, the σ-algebra $\mathcal{B}(X)$ will usually be omitted it in the notations, and we will denote by (X,λ) (resp. (X,μ)) a standard σ-finite (resp. probability) space.

All standard Borel spaces are isomorphic (see [Kec95, Thm. 15.6]). Moreover, all standard probability spaces are isomorphic (see [Kec95, Thm. 17.41]), and by σ-finiteness this implies that all σ-finite spaces are isomorphic, justifying the terminologies.

Definition 1.5. The **measure algebra of a standard probability space** (X,μ) is the space of Borel subsets of X, where two such subsets are identified if the measure of their symmetric difference is equal to zero. We denote this algebra by $\mathrm{MAlg}(X,\mu)$. It is equipped with the metric $d_{X,\mu}$ defined by $d_{X,\mu}(A,B) := \mu(A\Delta B)$.

In the case of a standard σ-finite space (X,λ), we similarly denote by $\mathrm{MAlg}(X,\lambda)$ the space of Borel subsets of X, where once again two such subsets are identified if the measure of their symmetric difference is equal to zero. As the symmetric difference of two sets can have infinite measure, there is no well-defined analogous metric on $\mathrm{MAlg}(X,\lambda)$.

The following definitions also make sense in the setting of a standard σ-finite space (X,λ).

Definition 1.6. Let (X, μ) be a standard probability space. The group $\mathrm{Aut}(X, \mu)$ is defined as the group of measure-preserving bijections of (X, μ), identified up to measure zero. It naturally acts on $\mathrm{MAlg}(X, \mu)$.

Definition 1.7. Consider (T_n) a sequence of elements of $\mathrm{Aut}(X, \mu)$. An element T in $\mathrm{Aut}(X, \mu)$ is obtained by **cutting and pasting** (T_n) if there exists a countable partition (A_n) of X such that for all n in $\mathbb{N}$ we have

$$T_{\restriction A_n} = T_{n \restriction A_n}.$$

Definition 1.8. A subgroup $\mathbb{G}$ of $\mathrm{Aut}(X, \mu)$ is a **full group** if it is stable under the operation of cutting and pasting any sequence of elements of $\mathbb{G}$.

Definition 1.9. We say that a subgroup $\mathbb{G}$ of $\mathrm{Aut}(X, \mu)$ is **ergodic** if for every $A \subseteq X$ such that $\mu(T(A) \Delta A) = 0$ for every T in $\mathbb{G}$, we have that A is either null or conull.

1.2 Fremlin's theorem

We now state Fremlin's theorem in full generality, then we will give the corresponding definitions. Although the proof will be given specifically in the context of measure algebras, the level of generality described by Fremlin still provides some insight. The reader interested in having reformulations more adapted to the measurable context will find them in Section 2.

Theorem 1.10 ([Fre02, Thm. 384D]). *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two Dedekind complete Boolean algebras, $\mathbb{G}$ and $\mathbb{H}$ two subgroups of $\mathrm{Aut}(\mathfrak{A})$ and $\mathrm{Aut}(\mathfrak{B})$ respectively, both having many involutions. If $\psi : \mathbb{G} \to \mathbb{H}$ is a group isomorphism, there exists a unique Boolean isomorphism $S : \mathfrak{A} \to \mathfrak{B}$ such that for all $T \in \mathbb{G}$, we have*

$$\psi(T) = STS^{-1}.$$

Definition 1.11. A **Boolean algebra** is a ring $(\mathfrak{A}, \Delta, \cap)$ such that $A^2 = A \cap A = A$ for all A in $\mathfrak{A}$, and such that $\mathfrak{A}$ is unital with multiplicative identity $1_{\mathfrak{A}}$. A **boolean homomorphism** between two Boolean algebras $\mathfrak{A}$ and $\mathfrak{B}$ is a unital ring homomorphism between $\mathfrak{A}$ and $\mathfrak{B}$: for any A, B in $\mathfrak{A}$ a Boolean homomorphism ϕ satisfies $\phi(A \Delta B) = \phi(A) \Delta \phi(B)$, $\phi(A \cap B) = \phi(A) \cap \phi(B)$ and $\phi(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}$.

We denote by $\mathrm{Aut}(\mathfrak{A})$ the group of bijective boolean homomorphisms from $\mathfrak{A}$ to itself.

Remark 1.12. In particular, for any Boolean algebra $\mathfrak{A}$, for all A in $\mathfrak{A}$ we have $A \Delta A = 0$, in other words any element is equal to its additive inverse element. (Indeed we have $A \Delta A = (A \Delta A)^2 = (A \Delta A) \cap (A \Delta A) = A^2 \Delta A^2 \Delta A^2 \Delta A^2 = A \Delta A \Delta A \Delta A$.) In particular, a Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ maps $0_{\mathfrak{A}}$ to $0_{\mathfrak{B}}$.

Remark 1.13. The most natural example of a Boolean algebra is the algebra of subsets of a set, equipped with the operations of symmetric difference and intersection, and with the (multiplicative) identity being the whole set. In this case the complement of a set A corresponds to $1_{\mathfrak{A}} \setminus A = 1_{\mathfrak{A}} \Delta A$, and the union of two sets A and B is $A \cup B = 1_{\mathfrak{A}} \Delta ((1_{\mathfrak{A}} \Delta A) \cap (1_{\mathfrak{A}} \Delta B))$. We in fact have the following theorem, justifying the notations.

Theorem 1.14 ([Fre02, Thm. 311E]). ***Stone's theorem, weak version:*** *Let $\mathfrak{A}$ be any Boolean ring, and let Z be the set of ring homomorphisms from $\mathfrak{A}$ onto $\{0, 1\}$. Then we have an injective ring homomorphism*

$$\begin{array}{rcl} \mathfrak{A} & \longrightarrow & \mathcal{P}(Z) \\ a & \longmapsto & \widehat{a} := \{z \in Z \mid z(a) = 1\}. \end{array}$$

If $\mathfrak{A}$ is a Boolean algebra (a unital Boolean ring), then $\widehat{1_{\mathfrak{A}}} = Z$.

Remark 1.15. Stone's theorem states that it makes sense to consider that Boolean algebras are *fields of sets*, a set along with a family of subsets. We will freely use this identification in these notes. Stronger versions of the theorem give more topological information on these *Stone spaces*, or *Stone representations* of Boolean algebras, but they are unneeded for the scope of this note.

We will be needing the following equivalence about Boolean algebra homomorphisms, the proof is straightforward and relies solely on the previously defined boolean properties.

Proposition 1.16 ([Fre02, Prop. 312H])**.** *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two boolean algebras, and $f : \mathfrak{A} \to \mathfrak{B}$ be a function. Then the following are equivalent:*

(1) f is a boolean homomorphism,

(2) $\forall A, B \in \mathfrak{A}$, we have $f(A \cap B) = f(A) \cap f(B)$ and $f(1_{\mathfrak{A}} \setminus A) = 1_{\mathfrak{B}} \setminus f(A)$,

(3) $\forall A, B \in \mathfrak{A}$, we have $f(A \cup B) = f(A) \cup f(B)$ and $f(1_{\mathfrak{A}} \setminus A) = 1_{\mathfrak{B}} \setminus f(A)$,

(4) $f(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}$, and $\forall A, B \in \mathfrak{A}$ such that $A \cap B = 0_{\mathfrak{A}}$ we have $f(A \cup B) = f(A) \cup f(B)$ and $f(A) \cap f(B) = 0_{\mathfrak{B}}$.

Proof. (1) $\implies$ (4): Immediate.

(4) $\implies$ (3): Consider A, B in $\mathfrak{A}$. From *(4)* we get

$$f(A) = f(A \cap B) \cup f(A \setminus B),$$

and similarly for B, which yields

$$f(A \cup B) = f(A) \cup f(B \setminus A) = f(A \cap B) \cup f(A \setminus B) \cup f(B \setminus A) = f(A) \cup f(B).$$

In particular, for $B = 1_{\mathfrak{A}} \setminus A$ we get $f(A) \cup f(1_{\mathfrak{A}} \setminus A) = f(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}$, and $f(A) \cap f(1_{\mathfrak{A}} \setminus A) = 0_{\mathfrak{B}}$, so by intersecting with $1_{\mathfrak{B}} \setminus f(A)$, we get $f(1_{\mathfrak{A}} \setminus A) = 1_{\mathfrak{B}} \setminus f(A)$, hence *(3)*.

(3) $\implies$ (2): For any A, B in $\mathfrak{A}$ we have

$$f(A \cap B) = f(1_{\mathfrak{A}} \setminus ((1_{\mathfrak{A}} \setminus A) \cup (1_{\mathfrak{A}} \setminus B))) = 1_{\mathfrak{B}} \setminus ((1_{\mathfrak{B}} \setminus f(A)) \cup (1_{\mathfrak{B}} \setminus f(B))) = f(A) \cap f(B).$$

(2) $\implies$ (1): For any A, B in $\mathfrak{A}$ we have

$$\begin{aligned} f(A \Delta B) &= f((1_{\mathfrak{A}} \setminus ((1_{\mathfrak{A}} \setminus A) \cap (1_{\mathfrak{A}} \setminus B))) \cap (1_{\mathfrak{A}} \setminus (A \cap B))) \\ &= (1_{\mathfrak{B}} \setminus ((1_{\mathfrak{B}} \setminus f(A)) \cap (1_{\mathfrak{B}} \setminus f(B))) \cap (1_{\mathfrak{B}} \setminus (f(A) \cap f(B)))) = f(A) \Delta f(B), \end{aligned}$$

and finally we also have $f(1_{\mathfrak{A}}) = f(1_{\mathfrak{A}} \setminus 0_{\mathfrak{A}}) = 1_{\mathfrak{B}} \setminus f(0_{\mathfrak{A}}) = 1_{\mathfrak{B}} \setminus 0_{\mathfrak{B}} = 1_{\mathfrak{B}}$ (because any element is its own Δ-inverse), so f is a boolean homomorphism, hence the implication. □

A Boolean algebra (or a field of sets) comes naturally with an order structure. We define *order-completeness*, or *Dedekind-completeness*.

Definition 1.17. Seeing a Boolean algebra $\mathfrak{A}$ as a field of sets, we can define a natural order $\subseteq$ by setting $A \subseteq B$ iff $A \cap B = A$. This makes $(\mathfrak{A}, \subseteq)$ a partially ordered set (poset), with least element $0_{\mathfrak{A}}$ and greatest element $1_{\mathfrak{A}}$. Setting $\sup\{A, B\} := A \cup B$ and $\inf\{A, B\} := A \cap B$ makes it a lattice.

Remark 1.18. We have to take care when extending the notions of supremum and infimum to infinite families. For instance, in the measure algebra of a a standard probability space (in particular it is non-atomic!) the uncountable union of all points is equal to the whole space. In the measure algebra, each point corresponds to 0, but the whole space is 1. In other words, an uncountable union does not correspond to a satisfying notion of supremum. We will see how to correctly define those notions in Section 2.1.

Remark 1.19 ([Fre02, Prop. 313La])**.** Any element of $\mathrm{Aut}(\mathfrak{A})$ is order-preserving.

Definition 1.20. Any poset P is **Dedekind-complete** if any non-empty subset of P with an upper-bound admits a *least upper-bound* (a supremum).

Definition 1.21. Let $\mathfrak{A}$ be a Boolean algebra, and T an element of $\mathrm{Aut}(\mathfrak{A})$. We say that $A \in \mathfrak{A}$ **supports** T if $T(B) = B$ for any $B \subseteq 1_{\mathfrak{A}} \setminus A$. If $\operatorname{supp} T \coloneqq \inf \{A \in \mathfrak{A} \mid A \text{ supports } T\}$ is defined, we call it the **support of** T.

Proposition 1.22 ([Fre02, Cor. 381F])**.** *If $\mathfrak{A}$ is Dedekind-complete, every element of* $\mathrm{Aut}(\mathfrak{A})$ *has a support.*

Definition 1.23. A subgroup $\mathbb{G}$ of $\mathrm{Aut}(\mathfrak{A})$ has **many involutions** if for every non-zero $A \in \mathfrak{A}$ there exists an non-trivial involution V in $\mathbb{G}$ such that $\operatorname{supp} V \subseteq A$.

2 Why Fremlin's theorem generalizes Dye's theorem

We say that two measures μ and ν on a standard space X are equivalent if they are both absolutely continuous with regards to the other one (equivalently, they have the same null and conull sets), and denote by $[\mu]$ the **measure class** of μ, containing all measures equivalent to μ. Recall that if λ is a σ-finite measure on X, there always exists a probability measure $\mu \in [\lambda]$. The non-singular setting is the adequate one for stating Fremlin's theorem for measure algebras.

Definition 2.1. Let (X, μ) be a standard probability space. We call **non-singular**, or **measure class-preserving**, a bimeasurable bijection that preserves the measure class, and denote by $\mathrm{Aut}(X, [\mu])$ the group of all non-singular bijections from X to itself, identified up to measure zero. We define on $\mathrm{Aut}(X, [\mu])$ the weak topology τ_w in the following way: (T_n) τ_w-converges to T if and only if for all Borel subsets of X one has $\mu(T_n(A) \Delta T(A)) \to 0$ and

$$\left\| \frac{d(T_{n*}\mu)}{d\mu} - \frac{d(T_*\mu)}{d\mu} \right\|_1 \to 0.$$

Proposition 2.2 ([IT65])**.** *The group* $(\mathrm{Aut}(X, [\mu]), \tau_w)$ *is a Polish group.*

Remark 2.3. The definitions of full groups and of ergodicity given in Section 1 extend to subgroups of $\mathrm{Aut}(X, [\mu])$ without any issue.

The first important step is to establish the link between the conjugations obtained in the different statements of the theorem. As it is stated in Theorem 1.10, the isomorphism obtained is an isomorphism of Boolean algebras. As such, it preserves the neutral elements for both Δ and $\cap$. This is equivalent to saying (for measure algebras) that it is a non-singular bijection of the underlying measure spaces. In other words, for $\mathfrak{A} = \mathrm{MAlg}(X, \mu)$, we have

$$\mathrm{Aut}(\mathfrak{A}) = \mathrm{Aut}(X, [\mu]).$$

Indeed, the class of a non-singular bijection clearly yields an automorphism of the boolean algebra, and conversely a boolean isomorphism yields the class of non-singular bijection. If a genuine function acting on the points (up to null sets) is needed, we will need to use the separability of the measure algebra (see Section 2.1.2). This is the next proposition.

The notion of atom of a measure algebra is convenient, and as such it is recalled.

Definition 2.4. An **atom** in a boolean algebra $\mathfrak{A}$ is a non-zero element $A \in \mathfrak{A}$ such that the only elements $\subseteq$-smaller than A are $0_{\mathfrak{A}}$ and A itself. Do note that for measure algebras it corresponds to the usual measure-theoretic notion of atoms (see [Fre02, Thm. 322B]).

Proposition 2.5 ([LM14, Thm. A.14, Cor. A.15])**.** *Let (X, μ) and (Y, ν) be two standard probability spaces.*

- *Any boolean homomorphism $\Phi : \mathrm{MAlg}(Y, \nu) \to \mathrm{MAlg}(X, \mu)$ yields a Borel non-singular application $\varphi : (X, \mu) \to (Y, \nu)$ that is unique up to null sets.*
- *If $\Phi : \mathrm{MAlg}(X, \mu) \to \mathrm{MAlg}(Y, \nu)$ is an isomorphism, there exist two conull Borel subsets A and B of X and Y respectively and a non-singular bijection $\psi : A \to B$ that is unique up to null sets.*

Proof. We start by fixing d_Y a compatible complete bounded metric on Y that induces its Borel structure. We consider the space $\mathrm{L}^0(X, \mu, Y)$ of measurable functions from X to Y, up to equality on μ-null sets, equipped with the metric defined by

$$d_\infty(f, g) \coloneqq \operatorname*{ess\,sup}_{x \in X} d_Y(f(x), g(x)),$$

which is complete since d_Y is. Let now (A_n) be a sequence of reprensentatives of a dense sequence in $\mathrm{MAlg}(X, \mu)$, which is separable by Proposition 2.11. Since Y is separable in particular it is Lindelöf so we can consider a sequence (y_k) such that $Y = \cup_k B(y_k, 2^{-n})$ for each $n \in \mathbb{N}$. By induction we then define for any $n \in \mathbb{N}$:

$$P_n \coloneqq P_{n-1} \wedge (B(y_k, 2^{-n}))_{k \in \mathbb{N}} \wedge \{A_n\},$$

the algebra (not σ-algebra!) generated by P_{n-1}, $(B(y_k, 2^{-n}))_{k \in \mathbb{N}}$, $\{A_n\}$. The sequence (P_n) is an increasing sequence of countable and atomic sub-algebras of $\mathcal{B}(Y)$, such that each atom of P_n has diameter less than 2^{-n} and $A_n \in P_n$ for any n.

We can now define the desired function as a limit in $\mathrm{L}^0(X, \mu, Y)$. For each $n \in \mathbb{N}$ and each atom $A \in P_n$, we choose $y_A \in A$ and define $\varphi_n \in \mathrm{L}^0(X, \mu, Y)$ by setting $\varphi_n(x) = y_A$ if and only if $x \in \Phi(A)$. By construction, (φ_n) is Cauchy for d_∞, we denote by φ its limit. By density of the classes of the A_n in $\mathrm{MAlg}(Y, \nu)$, φ lifts to Φ, and any other such lift also has to be the limit of the φ_n, so it is equal to φ up to a null set.

We now prove the second part of the statement. We apply the previous point to Φ and Φ^{-1}, yielding two Borel non-singular applications $\varphi' : Y \to X$ and $\varphi : X \to Y$. By uniqueness up to null sets and since $\Phi\Phi^{-1} = \mathrm{id}_{\mathrm{MAlg}(Y,\nu)}$ and $\Phi^{-1}\Phi = \mathrm{id}_{\mathrm{MAlg}(X,\mu)}$, we have two conull Borel subsets A and B of X and Y respectively, such that $(\varphi \circ \varphi')_{\restriction B} = \mathrm{id}_B$ and $(\varphi' \circ \varphi)_{\restriction A} = \mathrm{id}_A$. Setting $\psi = \varphi$ concludes the proof, as the uniqueness is a direct consequence of the first part. □

The version of the theorem that we will actually prove can be seen as a corollary of the most general version of Fremlin's theorem. The statement is the following:

Theorem 2.6. *Let (X, μ) be a standard probability space and $\mathbb{G}$ and $\mathbb{H}$ be two subgroups of* $\mathrm{Aut}(X, [\mu])$ *with many involutions. Then any isomorphism between $\mathbb{G}$ and $\mathbb{H}$ is the conjugation by some non-singular bijection. In other words, for any group isomorphism $\psi : \mathbb{G} \to \mathbb{H}$, there exists S in* $\mathrm{Aut}(X, [\mu])$ *such that for all $T \in \mathbb{G}$, we have*

$$\psi(T) = STS^{-1}.$$

There are three things that we need to prove. Firstly, that measure algebras are Dedekind-complete (we provide two proofs), then that ergodic full groups have many involutions (we also provide two proofs, in different settings), and finally that for the case of a probability measure μ, the non-singular bijection S is in fact measure-preserving ([Fre02, Cor. 383K]).

The third verification is actually very easy (see e.g. [LM14, Rem. 1.28]), and we give a more general argument that is interesting in its own right, especially for anyone considering infinite measures.

Proposition 2.7. *Let (X, λ) be a standard σ-finite space. If $\mathbb{G}$ is an ergodic subgroup of* $\mathrm{Aut}(X, \lambda)$*, any non-singular bijection S of (X, λ) that verifies $S\mathbb{G}S^{-1} \leqslant \mathrm{Aut}(X, \lambda)$ preserves λ up to multiplication by a positive scalar.*

Proof. Start by noticing that (STS^{-1}) preserves $S_*\lambda$, for any T in $\mathbb{G}$. Indeed, as T preserves λ, we have

$$\begin{aligned}(STS^{-1})_* S_*\lambda &= S_* T_* {S^{-1}}_* S_*\lambda \\ &= S_* T_* \lambda \\ &= S_*\lambda.\end{aligned}$$

As S is non-singular, by Radon-Nikodym's theorem there exists a Borel function $f : X \mapsto \mathbb{R}_+$ such that for any Borel subset A we have

$$\lambda(S^{-1}(A)) = \int_A f d\lambda$$

We will show that f is actually essentially constant. Let $U = STS^{-1}$ be an element of $S\mathbb{G}S^{-1}$. For any Borel subset A of X we have

$$\begin{aligned}U_*(S_*\lambda)(A) &= \lambda(S^{-1}(U^{-1}A)) \\ &= \int_{U^{-1}A} f(x) d\lambda(x) \\ &= \int_A f(U^{-1}x) d\lambda(x),\end{aligned}$$

as $U = STS^{-1}$ preserves λ, because $U \in S\mathbb{G}S^{-1} \leqslant \mathrm{Aut}(X, \lambda)$ by assumption. This means that the Radon-Nikodym derivative of $U_*(S_*\lambda)$ is $f \circ U^{-1}$. We previously proved that this pushforward measure $U_*(S_*\lambda) = (STS^{-1})_* S_*\lambda$ is equal to $S_*\lambda$, so by uniqueness $f = f \circ U^{-1}$, up to a null set. As this is valid for any U in $S\mathbb{G}S^{-1}$, which is ergodic, then this means that f is essentially constant. Indeed, take a set of the form $A_q = \{x \in X \mid f(x) < q\}$. It is invariant by $S\mathbb{G}S^{-1}$, and is therefore null or conull since S is non-singular. The only possibility for f is to be constant, up to a null set. □

In particular, taking λ to be a finite measure implies that S is actually measure-preserving.

2.1 Dedekind-completeness of measure algebras

We saw in the definition of Boolean algebras that notions of infimum and supremum exist for finite families. As it is often the case with measure theory, it is easy to extend it to countable families (what is sometimes called *Dedekind σ-completeness*), but here the existence of infimum and supremum for arbitrary families is required.

In the first proof we also obtain Dedekind-completeness for $\mathrm{MAlg}(X, \lambda)$ in the standard σ-finite context. The second proof establishes and makes use of the separability and metric-completeness of $\mathrm{MAlg}(X, \mu)$ in the standard probability context, which is enough for our needs.

2.1.1 Proof by essential supremum

Definition 2.8 ([Fre03, Def. 211G]). We say that a measure space (X, Σ, μ) is **localizable** if it satisfies the following two properties.

- (X, Σ, μ) is **semi-finite**, *i.e.* for any $E \in \Sigma$ such that $\mu(E) = +\infty$, there exists $F \in \Sigma$ such that $F \subseteq E$ and $0 < \mu(F) < +\infty$;
- For any (non necessarily countable!) family $\mathcal{E} \subseteq \Sigma$ there exists an **essential supremum** H of $\mathcal{E}$, that is to say a measurable subset such that:

  (i) for any $E \in \mathcal{E}, \mu(E \setminus H) = 0$

  (ii) if $G \in \Sigma$ is such that $\mu(E \setminus G) = 0$ for any $E \in \mathcal{E}$ then $\mu(H \setminus G) = 0$.

Proposition 2.9 ([Fre03, Thm. 211Ld]). *A standard σ-finite space is localizable.*

Proof. Let $(X, \mathcal{B}(X), \lambda)$ be a standard σ-finite space. As it is standard, it is semi-finite.

By σ-finiteness, write $X = \bigsqcup_{k \in \mathbb{N}} X_k$, with $\lambda(X_k)$ finite for any k. Fix $\mathcal{E} \subseteq \mathcal{B}(X)$, and define

$$\mathcal{F} \coloneqq \{F \in \mathcal{B}(X) \mid \forall E \in \mathcal{E}, \lambda(F \cap E) = 0\}.$$

First note that $\mathcal{F}$ is stable by countable union. For any k in $\mathbb{N}$, define $\gamma_k \coloneqq \sup\{\lambda(F \cap X_k) \mid F \in \mathcal{F}\} \in [0, \lambda(X_k)]$, and choose a sequence $(F^k_n)_{n \in \mathbb{N}}$ in $\mathcal{F}$ such that $\lim_n \lambda(F^k_n \cap X_k) = \gamma_k$. We now define the following sets:

$$\begin{cases} F^k \coloneqq \bigcup_{n \in \mathbb{N}} F^k_n, \\ F \coloneqq \bigsqcup_{k \in \mathbb{N}} F^k \cap X_k, \\ H \coloneqq X \setminus F. \end{cases}$$

We have that $F \cap X_k = F^k \cap X_k$ for any k, and all the F^k and F are in $\mathcal{F}$ by stability. We have to prove that H is the essential supremum of $\mathcal{E}$.

(i) Fix $E \in \mathcal{E}$. We have $\lambda(E \setminus H) = \sum_k \lambda(E \cap (F \cap X_k)) = \sum_k \lambda(E \cap (F^k \cap X_k)) = 0$ by definition of $\mathcal{F}$ and the fact that $F_k \in \mathcal{F}$.

(ii) Fix $G \in \mathcal{B}(X)$ such that $\lambda(E \setminus G) = 0$ for any $E \in \mathcal{E}$. We have that $X \setminus G$ and $F' \coloneqq F \cup (X \setminus G)$ are both in $\mathcal{F}$. Notice already that $F \subseteq F'$ and $F' \setminus F = H \setminus G$. For any k we have the following:

$$\begin{cases} \lambda(F' \cap X_k) \leqslant \gamma_k \ \text{ by definition of } \gamma_k, \\ \lambda(F \cap X_k) = \lambda(F^k \cap X_k) \geqslant \lim_n \lambda(F^k_n \cap X_k) = \gamma_k \ \text{ by definition of } F^k. \end{cases}$$

This ensures that for any k, $\lambda(F \cap X_k) = \lambda(F' \cap X_k)$, and since $\lambda(X_k)$ is finite, this yields that $\lambda((F' \setminus F) \cap X_k) = 0$. By summing over k, we have $\lambda(F' \setminus F) = \lambda(H \setminus G) = 0$, which concludes the proof. □

Proposition 2.10 ([Fre02, Thm. 322Be]). *The measure algebra* $\mathrm{MAlg}(X, \mu)$ *of a localizable standard measure space* (X, Σ, μ) *is Dedekind-complete as a Boolean algebra.*

Proof. To differentiate the class (in $\mathfrak{A} = \mathrm{MAlg}(X,\mu)$) of a measurable subset from the subset itself, we will denote the class of A by $\widetilde{A}$. We fix $\mathcal{E} \subseteq \Sigma$ and remark the following:

$$\begin{aligned} &\mu(E \setminus F) = 0 \text{ (for any } E \text{ in } \mathcal{E}) \\ \iff &\widetilde{E} \setminus \widetilde{F} = 0_{\mathfrak{A}} \text{ (for any } E \text{ in } \mathcal{E}) \\ \iff &\widetilde{F} \text{ is an upper bound of } \widetilde{\mathcal{E}} := \left\{ \widetilde{E} \mid E \in \mathcal{E} \right\}. \end{aligned}$$

Denote by H the essential supremum of $\mathcal{E}$. We now prove that $\widetilde{H}$ is the supremum of $\widetilde{\mathcal{E}}$ in $\mathfrak{A}$. We set $\mathcal{F} := \{F \in \Sigma \mid \mu(E \setminus F) = 0 \ \forall E \in \mathcal{E}\}$, and notice that $\widetilde{\mathcal{F}}$ is the set of upper bounds of $\widetilde{\mathcal{E}}$. Therefore, (H is an essential supremum of $\mathcal{E}$) $\iff$ ($H \in \mathcal{F}$ and $\widetilde{H}$ is a lower bound of $\widetilde{\mathcal{F}}$) $\iff$ ($\widetilde{H} = \sup \widetilde{\mathcal{E}}$). □

2.1.2 Proof by completeness of $\mathrm{MAlg}(X,\mu)$

This proof is from Le Maître, and is available in their PhD manuscript [LM14, Annexe A].

Proposition 2.11 ([LM14, Prop. A.4(iii)] or [LM22, Lem. 2.1])**.** *Let (X,μ) be a standard probability space. The metric space $(\mathrm{MAlg}(X,\mu), d_{X,\mu})$ is complete and separable.*

Proof. Let's start with separability. As (X,μ) is standard it is isomorphic to $([0,1], \mathrm{Leb})$, and therefore finite unions of rational endpoints intervals are dense in $\mathrm{MAlg}(X,\mu)$.

Now for completeness, we let (A_n) be a $d_{X,\mu}$-Cauchy sequence of Borel subsets (identified up to null sets). Passing to a subsequence if necessary, we may assume that $d_{X,\mu}(A_n, A_{n+1}) = \mu(A_n \Delta A_{n+1}) < 2^{-n}$ for any n. The Borel-Cantelli Lemma ensures us that

$$B := \{x \in X \mid \exists N, \forall n \geqslant N : x \notin A_n \Delta A_{n+1}\}$$

is conull. Define then

$$A := \{x \in X \mid \exists N, \forall n \geqslant N : x \in A_n\}.$$

If $x \in B \setminus A$, then there exists N big enough such that for any $n \geqslant N : x \notin A_n$ (indeed, if x is not in A, there exists infinitely many A_n that do not contain x, but if there also existed infinitely many A_n containing x, that would contradict the fact that x is in B). This is enough to conclude, as we then have $A \Delta A_N \subseteq \bigcup_{n \geqslant N} A_n \Delta A_{n+1}$, and since the measure of $\bigcup_{n \geqslant N} A_n \Delta A_{n+1}$ tends to zero, $d_{X,\mu}(A, A_N)$ also tends to zero. □

We then have the following Proposition, ensuring the existence of supremums of arbitrary family of elements in a measure algebra.

Proposition 2.12 ([LM14, Lem. A.5, Prop. A.6])**.** *Consider (X,μ) a standard probability space, and $\mathrm{MAlg}(X,\mu)$ the associated measure algebra.*

(1) Any upwards directed family of elements of $\mathrm{MAlg}(X,\mu)$ admits a supremum, which is obtained as the limit of an increasing sequence of elements of the family.

(2) Any family of elements of $\mathrm{MAlg}(X,\mu)$ admits a supremum, which is obtained as the limit of an increasing sequence of finite reunions of elements of the family.

Proof. (1). Let us start by assuming that $\mathcal{F}$ is an upwards directed family of elements of $\mathrm{MAlg}(X,\mu)$, in other words for any A, B in $\mathcal{F}$ there exists C in $\mathcal{F}$ such that $A \subseteq C$ and $B \subseteq C$.

Let $M = \sup_{A \in \mathcal{F}} \mu(A)$ and fix (A_n) a sequence of elements of $\mathcal{F}$ with $\mu(A_n)$ converging to M. As $\mathcal{F}$ is upwards directed, by induction we can find $B_n \in \mathcal{F}$ such that $A_n \subseteq B_n$ and

$B_{n-1} \subseteq B_n$, which gives us an increasing sequence in $\mathcal{F}$ satisfying $\mu(B_n) \to M$. We prove that it is a Cauchy sequence.

Let $\varepsilon > 0$ and let $N \in \mathbb{N}$ be such that any $n \geqslant N$ satisfies $\mu(B_n) > M - \varepsilon$. Then for any $n > m \geqslant N$, $d_{X,\mu}(B_n, B_m) = \mu(B_n \Delta B_m) = \mu(B_n \setminus B_m) = \mu(B_n) - \mu(B_m) < \varepsilon$. Therefore (B_n) is a Cauchy sequence in $(\mathrm{MAlg}(X,\mu), d_{X,\mu})$ which is complete by the previous proposition, and its limit is $\sup \mathcal{F}$. In particular $\mu(\sup \mathcal{F}) = \sup_{A \in \mathcal{F}} \mu(A)$.

(2). We now assume that $\mathcal{F}$ is any family of elements of $\mathrm{MAlg}(X,\mu)$. Denote by $\mathcal{G}$ the family of finite reunions of elements of $\mathcal{F}$. It is upwards directed, and therefore by *(1)* it admits a supremum $\sup \mathcal{G}$, which is also the supremum of $\mathcal{F}$. □

Remark 2.13. If a family of elements of $\mathrm{MAlg}(X,\mu)$ is stable by countable union (in particular it is upwards directed), the supremum of the family is actually a maximum, which means that it is an element of the family.

Remark 2.14. Of course, by taking complements, everything we said in this section remains true for infimum and minimum, with intersections rather than reunions.

2.2 Supports and separators

Our first remark is that the notion of support defined in Section 1 is easier to manipulate when we consider non-singular bijections. We establish this, then we chose to write the rest of section in the setting of general Borel bijections when possible, without any measure involved. Indeed, the proofs are quite elegant and not much more difficult. It is indeed also possible to express all of the following in the language of Boolean algebras, and we refer the interested reader to [Fre02, 382].

Lemma 2.15. *Let T be a boolean automorphism of $\mathrm{MAlg}(X,\mu)$. Then $\operatorname{supp} T$ is equal to the class of $\{x \in X \mid T(x) \neq x\}$ in $\mathrm{MAlg}(X,\mu)$.*

Proof. In this proof we identify Borel subsets of X with their class in $\mathrm{MAlg}(X,\mu)$. We set $\mathcal{T} \coloneqq \{A \in \mathrm{MAlg}(X,\mu) \mid T(B) = B \text{ for any } B \subseteq X \setminus A\}$. By definition, we have $\operatorname{supp} T = \inf \mathcal{T}$, and $\{x \in X \mid T(x) \neq x\} \in \mathcal{T}$, so $\operatorname{supp} T \subseteq \{x \in X \mid T(x) \neq x\}$. For the converse inclusion, we notice that $\mathcal{T}$ is stable by countable intersection, so $\operatorname{supp} T \in \mathcal{T}$ by Remark 2.13, and since $\{x \in X \mid T(x) \neq x\} \subseteq A$ for any $A \in \mathcal{T}$, the proof is over. □

The previous lemma links the terminology used in the following definition, which corresponds to the "usual" notion of support, with the previously used one. We then give useful facts written in the language of Borel bijections, adding measure-theoretic "corollaries" when they play a crucial role and need to be stated explicitely.

Definition 2.16. For a Borel bijection T of a standard Borel space X, The **support of** T is defined as $\operatorname{supp} T \coloneqq \{x \in X \mid T(x) \neq x\}$. For a non-singular bijection of a standard measure space, the support of T is defined similarly, but up to measure zero. It is straightforward to see that for two bijections S and T, we have $\operatorname{supp}(STS^{-1}) = S(\operatorname{supp} T)$.

Lemma 2.17. *Let T be a Borel bijection of a standard Borel space X and consider a Borel subset $A \subseteq X$. The following are equivalent:*

(i) $\operatorname{supp} T \subseteq A$;

(ii) $B \cap T(B) = \emptyset \implies B \subseteq A$;

(iii) $\emptyset \neq B \subseteq X \setminus A \implies B \cap T(B) \neq \emptyset$.

In particular, if A does not support T, there exists a non-empty Borel subset $B \subseteq X \setminus A$ such that $B \cap T(B) = \emptyset$.

Proof. *(i)* $\implies$ *(ii)*: Let B be such that $B \cap T(B) = \emptyset$. We have $(B \setminus A) \cap B \subseteq T(B \setminus A) \cap B \subseteq T(B) \cap B = \emptyset$, so $B \subseteq A$.

(ii) $\implies$ *(iii)* is immediate.

(iii) $\implies$ *(i)*: Assume $\operatorname{supp} T \not\subseteq A$. This means that there exists $\emptyset \neq C \subseteq \operatorname{supp} T \setminus A$ such that $T(C) \neq C$. As T is a bijection, $\emptyset \neq B \coloneqq C \setminus T(C) \subseteq C \subseteq X \setminus A$ satisfies $T(B) \cap B \subseteq T(C) \setminus T(C) = \emptyset$, which concludes the proof. □

The following measure-theoretic reformulation of Lemma 2.17 is used widely.

Lemma 2.18. *let T be an element of $\mathrm{Aut}(X, [\lambda])$, and let $D \subseteq X$ be non-trivial. The following are equivalent:*

(i) $\operatorname{supp} T \not\subseteq D$;

(ii) $T_{\restriction X \setminus D} \neq \mathrm{id}_{X \setminus D}$;

(iii) There exists $C \subseteq X \setminus D$ such that $C \neq \emptyset$ and $C \cap T(C) = \emptyset$.

Lemma 2.19 ([EG16, Lem. 5.1])**.** *Let T be a Borel bijection of a standard Borel space X. Then there exists a Borel partition (A_k) of $\operatorname{supp} T$ such that for any k, A_k is disjoint from $T(A_k)$.*

Remark 2.20. Like the authors, we adopt here the convention that a partition can contain multiple times the empty set. We also thank Corentin Correia for simplifying the following proof.

Proof. Let $\mathcal{C}$ be a countable separating set, *i.e.* for any $x \neq y$ in X, there exists $C \in \mathcal{C}$ such that $x \in C$ but $y \notin C$ (on the real line, one can think of intervals with rational endpoints for example). Define now $\mathcal{B} \coloneqq \{C \cap T^{-1}(X \setminus C) \mid C \in \mathcal{C}\}$.

Fix now x in $\operatorname{supp} T$. As $\mathcal{C}$ separates points, there exists $C \in \mathcal{C}$ such that $x \in C$ but $T(x) \notin C$, which means that $x \in C \cap T^{-1}(X \setminus C)$. Therefore, there exists $B \in \mathcal{B}$ such that $x \in B$, and $\mathcal{B}$ covers $\operatorname{supp} T$.

Moreover if $B \in \mathcal{B}$, there exists $C \in \mathcal{C}$ such that $B = C \cap T^{-1}(X \setminus C) \subseteq C$, so we have $T(B) = T(C) \cap (X \setminus C) \subseteq X \setminus C$. In particular B is disjoint from $T(B)$.

We conclude by defining the desired partition by setting $A_k \coloneqq B_k \setminus \bigcup_{l<k} B_l$, where (B_n) is an enumeration of $\mathcal{B}$. □

Lemma 2.21. *Let T be a Borel bijection of a standard Borel space X. There exists a Borel subset A of $\operatorname{supp} T$, such that $\operatorname{supp} T = A \sqcup \big(T(A) \cup T^{-1}(A)\big)$.*

Proof. Using Lemma 2.19, we write $\operatorname{supp} T = \bigsqcup_{n \in \mathbb{N}} B_n$, where (B_n) is a partition of X, and for all $n \in \mathbb{N}$, $T(B_n)$ is disjoint from B_n.

We now inductively define a sequence (A_n) of Borel subsets of $\operatorname{supp} T$ as follows. Fix $A_0 = B_0$, and let A_{i+1} be defined by

$$A_{i+1} = B_{i+1} \setminus \left(\left(\bigsqcup_{j \leqslant i} T(A_j) \right) \bigcup \left(\bigsqcup_{j \leqslant i} T^{-1}(A_j) \right) \right).$$

We conclude the proof by noticing that $A = \bigsqcup_n A_n$ is suitable. Indeed, for any i we have $A_{i+1} \cap \left(\bigsqcup_{j \leqslant i} T(A_j) \right) = \emptyset$ and $A_{i+1} \cap \left(\bigsqcup_{j \leqslant i} T^{-1}(A_j) \right) = \emptyset$ so A is disjoint from both $T(A)$ and $T^{-1}(A)$. Finally we have $\operatorname{supp} T = A \cup T(A) \cup T^{-1}(A)$, as any $x \in \operatorname{supp} T \setminus A$ is either in $T(A)$ or in $T^{-1}(A)$. □

Remark 2.22. The set A in the statement of Lemma 2.21 is sometimes called a **separator** for the bijection T (see [Fre02, 382]).

The following definition is fundamental, and exchanging involutions play a crucial role in the proof, as they encapsulate most of "the information of an ergodic full group", in a way.

Definition 2.23. Let V be a Borel bijection of X and A, B be disjoint, such that $V(A) = B$. Denote by $V_{A,B}$ the Borel involution defined by

$$\begin{cases} V_{A,B}(C) = V(C) \text{ for any Borel subset } C \subseteq A, \\ V_{A,B}(C) = V^{-1}(C) \text{ for any Borel subset } C \subseteq B, \\ V_{A,B} = \mathrm{id}_X \text{ on } X \setminus (A \sqcup B). \end{cases}$$

We call $V_{A,B}$ the **(A, B)-exchanging involution associated with V and sending A to B.** In particular $\operatorname{supp} V_{A,B} = A \sqcup B$.

In fact, every non-singular bijection of a standard measure space is an exchanging involution:

Proposition 2.24 ([Fre02, Cor. 382F])**.** *Let V be an involution in $\mathrm{Aut}(X, [\lambda])$. Then there exists two disjoint Borel subsets A and B such that $V = V_{A,B}$ is an (A, B)-exchanging involution.*

Proof. Let V be an involution. By Lemma 2.21 there exists a Borel subset A such that $\operatorname{supp} V = A \sqcup (V(A) \cup V^{-1}(A))$. However V is an involution, therefore $A \sqcup (V(A) \cup V^{-1}(A)) = A \sqcup V(A)$. □

2.3 Ergodic full groups have many involutions

As stated before, having many involutions is a property for subgroups of $\mathrm{Aut}(X, [\mu])$, which is in particular satisfied by ergodic full groups. We give a direct proof of this fact in the measure-preserving setting, by actually proving something stronger. In a second time, we recall the original statement of Dye in the context of type II groups, and after giving as few definitions as possible, we explain why this version of the theorem implies the first version of Dye's, while still falling under Fremlin's.

2.3.1 Strong version of having many involutions

In this section, we actually prove something stronger than the fact that ergodic full groups have many involutions. The proof is from [KM04, Lem. 7.10] but stands only for countable groups. To extend it we use Proposition 2.2 and prove what we need for a dense countable subgroup.

Proposition 2.25. *Let $\mathbb{G} \leqslant \mathrm{Aut}(X, [\mu])$ be an ergodic subgroup, and let Γ be a countable τ_w-dense subgroup of $\mathbb{G}$. Then for any Borel subset A of X, $\mu(\gamma(A)\Delta A) = 0$ $(\forall \gamma \in \Gamma) \Rightarrow A$ is either null or conull.*

Proof. For any A in $\mathrm{MAlg}(X, \mu)$, the application $T \in \mathrm{Aut}(X, \lambda) \mapsto \mu(T(A)\Delta A)$ is continuous. Therefore if (γ_n) converges weakly to an element T in $\mathrm{Aut}(X, \lambda)$, for any A in $\mathrm{MAlg}(X, \mu)$ we have $\mu(\gamma_n(A)\Delta A) \to \mu(T(A)\Delta A)$. The result follows from ergodicity of $\mathbb{G}$. □

The notion of pseudo-full group is adequate and allows us to give a somewhat precise description of the involutions we define.

Definition 2.26. Let $\mathbb{G}$ be a subgroup of $\mathrm{Aut}(X, [\lambda])$.

(1) Let A and B be two Borel subsets of X. Saying that $\phi : A \to B$ is a **partial isomorphism** of $\mathbb{G}$ means that there exists a partition $(A_n)_{n\in\mathbb{N}}$ of A, a partition $(B_n)_{n\in\mathbb{N}}$ of B, and a sequence $(T_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{G}$ such that $T_n(A_n) = B_n$ and $T_{n\restriction A_n} = \phi_{\restriction A_n}$, for every n in $\mathbb{N}$. The domain of ϕ is $\mathrm{dom}(\phi) = A$ and its range is $\mathrm{rng}(\phi) = B$, up to null sets.

(2) The set of all partial isomorphisms of $\mathbb{G}$ is called the **pseudo-full group** of $\mathbb{G}$, and is denoted by $[[\mathbb{G}]]$.

Proposition 2.27. *Let $\mathbb{G}$ be an ergodic subgroup of $\mathrm{Aut}(X,[\mu])$. If there exists a σ-finite measure λ in $[\mu]$ which is preserved by $\mathbb{G}$, then for any two Borel subsets A and B of X such that $\lambda(A)=\lambda(B)$, there exists a partial automorphism ϕ in $[[\mathbb{G}]]$ such that $\mathrm{dom}(\phi)=A$ and $\mathrm{rng}(\phi)=B$ (up to null sets).*

Proof. The proof is roughly the same as the proof of Lemma 7.10 of [KM04], which is stated in the countable case, but easily adapts to our case by considering a countable dense subgroup. We fix a σ-finite infinite measure λ which is preserved by $\mathbb{G}$.

Since $\mathbb{G}$ is a subgroup of $\mathrm{Aut}(X,[\mu])$, it is naturally endowed with the weak topology, which is separable thanks to Proposition 2.2. As such, we can consider (T_n) a countable dense subset of $\mathbb{G}$ and $\Gamma=(\gamma_n)$ the countable subgroup generated by (T_n). Let A and B first be two Borel subsets of X, such that $0<\lambda(A)=\lambda(B)<+\infty$. We recursively define a countable family of pairwise disjoint subsets A_n of A as follows:

$$\begin{cases} A_0=(\gamma_0^{-1}B)\cap A \\ A_{n+1}=\left(\gamma_{n+1}^{-1}\left(B\setminus\bigcup_{m\leqslant n}\gamma_m A_m\right)\right)\bigcap\left(A\setminus\bigcup_{m\leqslant n}A_m\right). \end{cases}$$

The set A_n represents the elements sent by γ_n to B, after removing the elements previously sent. Now set $A'=\bigsqcup_{n\in\mathbb{N}}A_n$ and $B'=\bigsqcup_{n\in\mathbb{N}}\gamma_nA_n$, and let $\phi:A'\to B'$ be the Borel application that sends $x\in A_n$ to $\gamma_n(x)\in\gamma_nA_n$. By definition, ϕ is a partial isomorphism between A' and B'. In particular, $\lambda(\mathrm{dom}(\phi))=\lambda(\mathrm{rng}(\phi))$.

Let us now suppose that either $\lambda(A\setminus\mathrm{dom}(\phi))>0$ or $\lambda(B\setminus\mathrm{rng}(\phi))>0$. As A and B have finite measure, we have $\lambda(A\setminus\mathrm{dom}(\phi))=\lambda(B\setminus\mathrm{rng}(\phi))>0$. Define $\widetilde{B}=\bigcup_{n\in\mathbb{N}}\gamma_n(A\setminus\mathrm{dom}(\phi))$, and notice that $\widetilde{B}$ is non null and invariant under the action of Γ. Ergodicity and the fact that Γ is dense in $\mathbb{G}$ ensure that $\widetilde{B}$ is conull, by Proposition 2.25. This coupled with the fact that $\lambda(B\setminus\mathrm{rng}(\phi))>0$ implies that there exists an integer n such that

$$\lambda\left((B\setminus\mathrm{rng}(\phi))\bigcap\gamma_n(A\setminus\mathrm{dom}(\phi))\right)>0.$$

We define n_0 as the smallest such integer. As the action is measure-preserving, we then have

$$\lambda\left(\gamma_{n_0}^{-1}(B\setminus\mathrm{rng}(\phi))\bigcap(A\setminus\mathrm{dom}(\phi))\right)>0.$$

Notice now that $(B\setminus\mathrm{rng}(\phi))\subseteq\left(B\setminus\bigcup_{m\leqslant n_0-1}\gamma_mA_m\right)$ and $(A\setminus\mathrm{dom}(\phi))\subseteq\left(A\setminus\bigcup_{m\leqslant n_0-1}A_m\right)$, which means that $\left(\gamma_{n_0}^{-1}(B\setminus\mathrm{rng}(\phi))\bigcap(A\setminus\mathrm{dom}(\phi))\right)$ is contained in A_{n_0} by construction, and thus it is contained in $\mathrm{dom}(\phi)$. This is the contradiction we sought, as this set has positive measure and is contained both in $\mathrm{dom}(\phi)$ and in $A\setminus\mathrm{dom}(\phi)$.

Now if $\lambda(A)=\lambda(B)=+\infty$, observe that $(A,\lambda_{\restriction A})$ and $(B,\lambda_{\restriction B})$ are both standard σ-finite spaces. It is then possible to write $A=\bigsqcup A_i$ and $B=\bigsqcup B_i$, with $\lambda(A_i)=\lambda(B_i)<+\infty$ for every i in $\mathbb{N}$. The previous argument gives us a sequence (ϕ_i) of partial isomorphisms with domains (A_i) and ranges (B_i). The partial isomorphism defined on A by $\phi_{\restriction A_i}=\phi_i$ is in $[[\mathbb{G}]]$ by construction, and is suitable. $\square$

Corollary 2.28. *Let $\mathbb{G} \leqslant \mathrm{Aut}(X, [\mu])$ be an ergodic full group. If there exists a σ-finite measure λ in $[\mu]$ that is preserved by $\mathbb{G}$, then for any Borel subsets A and B of X such that $\lambda(A) = \lambda(B)$ and $\lambda(X \setminus A) = \lambda(X \setminus B)$, there exists an involution V in $\mathbb{G}$ such that $V(A) = B$, and such that V is the identity on $X \setminus (A \cup B)$. In particular, $\mathbb{G}$ has many involutions.*

Proof. Define A' by $A' = A$ if A and B are disjoint, and $A' = A \setminus B$ otherwise. Define B' in the same way, such that A' and B' are disjoint, and notice that A' and B' verify the same measure conditions as A and B.

Proposition 2.27 gives us a measure-preserving partial automorphism ϕ defined by the action of $\mathbb{G}$ such that $\phi(A') = B'$, up to a null set. The following involution, defined by

$$V_{A',B'} = \begin{cases} \phi & \text{on } A' \\ \phi^{-1} & \text{on } B' \\ \mathrm{id}_X & \text{elsewhere.} \end{cases}$$

is in $\mathbb{G}$ and is suitable. □

As stated earlier, we proved a "strong version of having many involutions", namely that ergodic full groups contain involutions with supports *equal* to any Borel subset, and not just included in it. Without any mention of ergodicity, Fremlin actually proved that for a full group, the weak and strong versions of having many involutions are equivalent. His statement is once again in the general language of Boolean algebras, but we give the non-singular version.

Theorem 2.29 ([Fre02, Thm. 382Q])**.** *Let $\mathbb{G} \leqslant \mathrm{Aut}(X, [\mu])$ be a full group with many involutions. Then for any subset $A \subseteq X$ of positive measure, there exists an involution $U \in \mathbb{G}$ with $\operatorname{supp} U = A$.*

2.3.2 Dye's original statement

The statement of Theorem 1.1 is given in its simplest form for people interested in ergodic theory, and more specifically in ergodic full groups. However, in [Dye63], the author works with *type II full groups*, which is a condition equivalent (for full groups!) to having many involutions. We give this general statement, with no mention of ergodicity (in particular, even working with full groups of measure-preserving bijections, the conjugation obtained is only non-singular, since Proposition 2.7 crucially uses ergodicity).

The adequate language to talk about type II full groups is the language of relative atoms in the measure algebras. We chose to remain concise on this subject matter, and to give only the necessary proofs for our needs with the reconstruction theorem(s). In particular, we do not define type I. We refer the interested reader to [LM14, Sec. 1.4].

Definition 2.30. Let N be a closed subalgebra of $M \coloneqq \mathrm{MAlg}(X, \mu)$. We say that:

- $\emptyset \neq A \in M$ is **an atom relatively to** N if for all $B \subseteq A$ there exists $C \in N$ such that $B = A \cap C$,
- N is of **type II** if M does not have any atoms relatively to N;
- a full group $\mathbb{G} \leqslant \mathrm{Aut}(X, [\mu])$ is of **type II** if the algebra $M_{\mathbb{G}}$ of $\mathbb{G}$-invariant elements of M is of type II.

Theorem 2.31 ([Dye63, Thm. 2])**.** *Let $\mathbb{G}$ and $\mathbb{H}$ be two full subgroups of* $\mathrm{Aut}(X, \mu)$ *of type* II *on a standard probability space (X, μ). Then any isomorphism between $\mathbb{G}$ and $\mathbb{H}$ is the conjugation by some non-singular bijection. In other words, for any group isomorphism $\psi : \mathbb{G} \to \mathbb{H}$, there exists S in* $\mathrm{Aut}(X, [\mu])$ *such that for all $T \in \mathbb{G}$, we have*

$$\psi(T) = STS^{-1}.$$

For an ergodic (full) group $\mathbb{G}$, the algebra $M_{\mathbb{G}}$ of $\mathbb{G}$-invariant elements of $\mathrm{MAlg}(X, \mu)$ is equal to $\{\emptyset, X\}$, so an atom relatively to $M_{\mathbb{G}}$ is just an atom. In particular, $M_{\mathbb{G}}$ is of type II, and thus $\mathbb{G}$ is also of type II. It remains to show that being of type II implies having many involutions. It happens to be an equivalence. The following proof uses some notions of countable Borel equivalence relations, we refer to [LM14] or to [KM04] for more on this subject.

Proposition 2.32. *Let (X, μ) be a standard probability space and $\mathbb{G} \leqslant \mathrm{Aut}(X, [\mu])$ be a full group. The following are equivalent:*

(1) $\mathbb{G}$ is of type II*;*

(2) $\mathbb{G}$ has many involutions;

(3) for any weakly-dense countable subgroup $\Gamma \leqslant \mathbb{G}$, for any Borel subset $A \subseteq X$, $\mathcal{R}_{\Gamma\restriction A}$ does not admit a Borel fundamental domain.

Proof. The first thing to take note of is that, since $\mathrm{Aut}(X, [\mu])$ is Polish (Proposition 2.2), we can consider a countable dense subgroup $\Gamma \leqslant \mathbb{G}$. We fix such a subgroup Γ, and immediately have $M_{\mathbb{G}} \subseteq M_{\Gamma}$. The converse inclusion directly follows from density, and thus $M_{\mathbb{G}} = M_{\Gamma}$.

We will not prove the implication *(3)* $\implies$ *(1)*. It is done in [LM14, Prop. 1.44] and stated in the probability-measure-preserving context, but generalises to the non-singular one.

(1) $\implies$ *(3)*: By contraposition, assume that there exists a non-trivial Borel subset $A \subseteq X$ such that $\mathcal{R}_{\Gamma\restriction A}$ admits a Borel fundamental domain D. Let C be a non-trivial Borel subset of D, we have $\Gamma\text{-sat}(C) \in M_{\Gamma}$, where $\Gamma\text{-sat}(C)$ is the Γ-saturation of C, and $\Gamma\text{-sat}(C)$ exists by Proposition 2.12 (for more on this see [LM14, Prop. 1.8]). Therefore, by virtue of D being a fundamental domain we have $C = \Gamma\text{-sat}(C) \cap D$, so D is an atom relatively to M_{Γ}. Since $M_{\mathbb{G}} = M_{\Gamma}$, $\mathbb{G}$ is not of type II.

(3) $\implies$ *(2)*: Consider a non-trivial $A \subseteq X$. We know that the restriction of $\mathcal{R}_{\Gamma}$ to any Borel subset does not admit a Borel fundamental domain, therefore $\{x \in A \mid \Gamma \cdot x \cap A \text{ is finite}\}$ is null. Let $\Gamma = (\gamma_n)$ be an enumeration of Γ. Denote by N the integer defined by

$$N := \min \{n \in \mathbb{N} \mid \mu(\{x \in A \mid \gamma_n(x) \neq x \text{ and } \gamma_n(x) \in A\}) > 0\},$$

which exists by the previous argument. This means that there exists a Borel subset $B \subseteq \operatorname{supp} \gamma_N \cap A \cap \gamma_N^{-1}(A)$ satisfying $\mu(B) > 0$ and $\mu(B \cap \gamma_N(B)) = 0$. We define a $(B, \gamma_N(B))$-exchanging involution V by

$$\begin{cases} V(C) = \gamma_N(C) \text{ for any } C \subseteq B \\ V(C) = \gamma_N^{-1}(C) \text{ for any } C \subseteq \gamma_N(B) \\ V(C) = C \text{ for any } C \subseteq X \setminus (B \cup \gamma_N(B)). \end{cases}$$

As $\mathbb{G}$ is a full group, the involution V is in $\mathbb{G}$, its support is contained in A, and since A is arbitrary $\mathbb{G}$ has many involutions.

(2) $\implies$ *(3)*: By contraposition again, assume that there exists a non-trivial Borel subset $A \subseteq X$ such that $\mathcal{R}_{\Gamma\restriction A}$ admits a Borel fundamental domain D. Assume that there exists a non-trivial involution $V \in \mathbb{G}$ with $\operatorname{supp} V = B \sqcup V(B) \subseteq D$ by Proposition 2.24. We have $B = \Gamma\text{-sat}(B) \cap D$, $\Gamma\text{-sat}(B) \in M_\Gamma = M_\mathbb{G}$, and $V(D) = D$, so

$$V(B) = V(\Gamma\text{-sat}(B)) \cap V(D) = \Gamma\text{-sat}(B) \cap D = B,$$

a contradiction. □

We then have the following chain of implications with the different versions of the reconstruction theorem, Fremlin's version being the "strongest", and the ergodic statement of Dye's theorem being the "weakest".

Theorem 1.1 ⇐ Theorem 2.31 ⇐ Theorem 2.6 ⇐ Theorem 1.10

3 A proof of Fremlin's reconstruction theorem

Before jumping to the proof, we give the following lemma, which is very useful when working with involutions.

Lemma 3.1 ([Fre02, Lem. 384.A]). *Let $\mathbb{G}$ be a subgroup of the group of Borel bijections of X with many involutions. For any non-trivial Borel subset B of X, there exists $T \in \mathbb{G}$ of order exactly 4 and such that $\operatorname{supp} T \subseteq B$.*

Proof. As $\mathbb{G}$ has many involutions, there exists an involution $U \in \mathbb{G}$ such that $\operatorname{supp} U \subseteq B$. By Proposition 2.24, there exists A such that $\operatorname{supp} U = A \sqcup U(A)$. Let V be an involution in $\mathbb{G}$ such that $\operatorname{supp} V \subseteq A$. Then, $UVU = UVU^{-1}$ in an involution with support equal to $U(\operatorname{supp} V)$, therefore it commutes with V. This means that $UVUV = VUVU$ is an involution. Consequently, $T = UV$ has order 4 (by looking at the supports we see that UV and $UVUV$ are not trivial, and so $UVUVUV$ isn't either). □

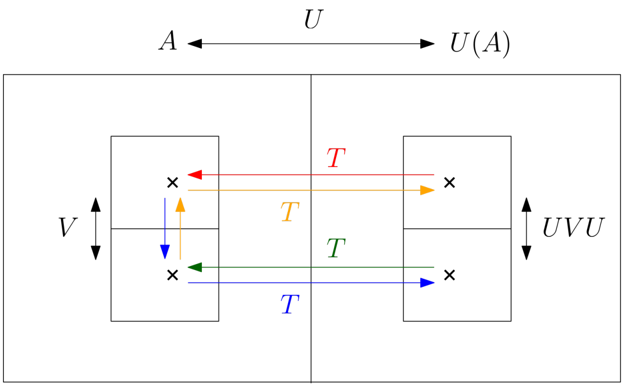

Figure 1: Construction of the bijection of order 4

We now fix all the necessary ingredients: (X, μ) is a standard probability space, $\mathbb{G}$ and $\mathbb{H}$ are two subgroups of $\operatorname{Aut}(X, [\mu])$ with many involutions, and $\psi : \mathbb{G} \to \mathbb{H}$ is a group isomorphism.

As previously stated, the proof is given in a measured context. As such, any equality between sets is to be understood as an equality in the measure algebra, that is to say an equality up to a null set. In particular, for two Borel subsets A and B of X, by $A \cap B = \emptyset$ we mean that $\mu(A \cap B) = 0$.

3.1 Linking measure-theoretic properties to algebraic properties

The goal of this section is to describe the support of a fixed involution purely in group-theoretic terms. Fix a non-trivial involution $V \in \mathbb{G}$. By Proposition 2.24, there exists two disjoint Borel subsets A and B such that $V = V_{A,B}$, which means that $\operatorname{supp} V = A \sqcup B = A \sqcup V(A)$. The involution V and these Borel subsets are fixed for the whole section. In the drawings, we shall imagine V to be the axial symmetry across the vertical separation of the support. We will be using the following notations:

$$
\begin{aligned}
\mathcal{C}_V &:= C(V) = \{T \in \mathbb{G} \mid TV = VT\} \\
\mathcal{D}_V &:= \{\text{involutions in } \mathcal{C}_V \text{ commuting with all their } \mathcal{C}_V\text{-conjugates}\} \\
&= \left\{T \in \mathcal{C}_V \mid T \text{ involution such that } \forall S \in \mathcal{C}_V : STS^{-1}T = TSTS^{-1}\right\} \\
\mathcal{E}_V &:= C(\mathcal{D}_V) = \{T \in \mathbb{G} \mid \forall S \in \mathcal{D}_V : ST = TS\} \\
\mathcal{F}_V &:= Sq(\mathcal{E}_V) = \left\{T^2 \mid T \in \mathcal{E}_V\right\} \\
\mathcal{G}_V &:= C(\mathcal{F}_V) = \{T \in \mathbb{G} \mid \forall S \in \mathcal{F}_V : ST = TS\}\,.
\end{aligned}
$$

Here C designates the centralizer in $\mathbb{G}$, and Sq designates the set of squared elements.

These sets already appear in the proof of Eigen's analogous theorem (see [Eig82]), however the distinction between the measure-theoretic properties and the group-theoretic properties was done by Fremlin.

We begin the study of the various "centralizer/squared"-sets we just defined. Our goal is Lemma 3.8, which will be the heart of the construction of the desired conjugation.

Lemma 3.2 (Property of $\mathcal{C}_V$). *For any T in $\mathcal{C}_V$, we have $T(\operatorname{supp} V) = \operatorname{supp} V$.*

Proof. Recall that in general we have $T(\operatorname{supp} V) = \operatorname{supp}(TVT^{-1})$, and $TVT^{-1} = V$ since T is in $\mathcal{C}_V$. □

Definition 3.3. For any Borel subset C of positive measure satisfying $V(C) = C$, we define the **induced exchanging-involution** V_C by

$$
\begin{cases} V_C(D) = V(D) \text{ for any } D \subseteq C \\ V_C(D) = D \text{ for any } D \subseteq X \setminus C. \end{cases}
$$

Notice that V_C is the $(C \cap A, C \cap B)$-exchanging involution associated with V.

Remark 3.4. If C and D have positive measure and are such that $V(C) = C$ and $V(D) = D$, then $V_C V_D = V_{C\Delta D} = V_D V_C$.

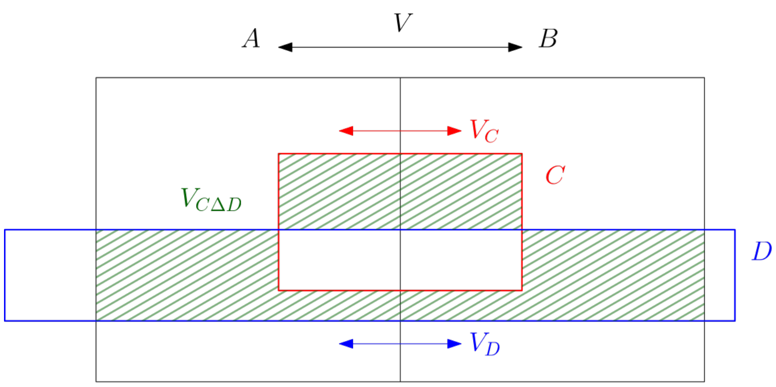

Figure 2: Commutativity of induced exchanging-involutions

Lemma 3.5 (Properties of $\mathcal{D}_V$). *We have the following:*

(i) For any T in $\mathcal{D}_V$, $\operatorname{supp} T \subseteq \operatorname{supp} V = A \sqcup B$;

(ii) For any Borel subset C of positive measure satisfying $V(C) = C$, V_C is in $\mathcal{D}_V$.

Proof. *(i)* By contraposition, assume that $T \in \mathcal{C}_V$ is such that $\operatorname{supp} T \nsubseteq \operatorname{supp} V$. By Lemma 2.18, there exists $C \subseteq X \setminus \operatorname{supp} V$ of positive measure such that $C \cap T(C) = \emptyset$. By Lemma 3.1, there exists S of order exactly 4 such that $\operatorname{supp} S \subseteq C$. The bijections S and V commute, as their supports are disjoint, so $S \in \mathcal{C}_V$. Moreover, $S \neq S^{-1}$ so there exists a non-trivial $D \subsetneq C$ such that $S(D) \neq S^{-1}(D)$.

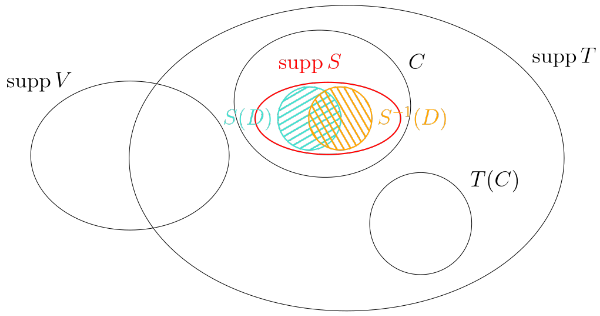

Figure 3: Visualisation of the sets in play

We have
$$C \cap T(D) = C \cap TS^{-1}(D) = \emptyset$$
therefore S and S^{-1} are trivial on $T(D)$ and on $TS^{-1}(D)$, and since T is an involution we have
$$STS^{-1}T(D) = ST^2(D) = S(D) \neq S^{-1}(D) = T^2S^{-1}(D) = TSTS^{-1}(D).$$
As T does not commute with its S-conjugate (with $S \in \mathcal{C}_V$), we have $T \notin \mathcal{D}_V$.

(ii) For any $S \in \operatorname{Aut}(X, [\lambda])$, it is easy to check that $SV_CS^{-1} = (SVS^{-1})_{S(C\cap A),S(C\cap B)}$. In particular, for $S = V$, we obtain $VV_CV^{-1} = V_C$, which means that V_C is in $\mathcal{C}_V$.

Now for any $S \in \mathcal{C}_V$ (in particular $VS(C) = S(C)$), we have
$$SV_CS^{-1} = (SVS^{-1})_{S(C\cap A),S(C\cap B)} = V_{S(C\cap A),S(C\cap B)} = V_{S(C)}.$$

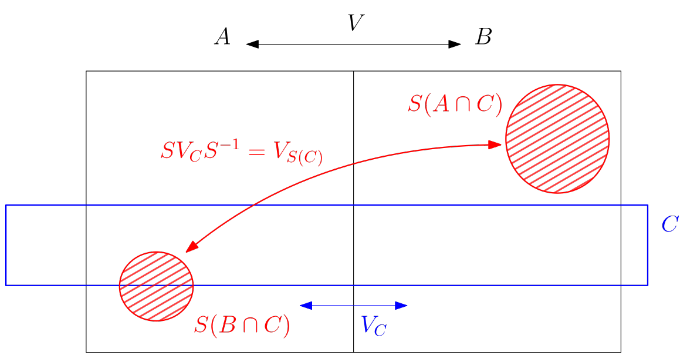

Figure 4: An example of a conjugate of an induced exchanging-involution

By Remark 3.4, $V_{S(C)}V_C = V_CV_{S(C)}$, and as S is arbitrary in $\mathcal{C}_V$, we proved that $V_C \in \mathcal{D}_V$. $\square$

Lemma 3.6 (Properties of $\mathcal{E}_V$)**.** *We have the following:*

(i) $\mathcal{E}_V \subseteq \mathcal{C}_V$*;*

(ii) Let T *be in* $\mathcal{E}_V$*. For any* $C \subseteq \operatorname{supp} V$ *we have* $T(C) \subseteq C \cup V(C)$*;*

(iii) Let T *be in* $\mathcal{E}_V$*. For any* $C \subseteq \operatorname{supp} V$ *we have* $T^2(C) = C$*;*

(iv) If $T \in \mathbb{G}$ *is such that* $\operatorname{supp} V \cap \operatorname{supp} T = \emptyset$*, then* $T \in \mathcal{E}_V$*.*

Proof. (i) The involution V is in $\mathcal{D}_V$, so the result follows.

(ii) Assume the contrary: let C be of positive measure, and such that $T(C)$ is not contained in $C_0 := C \cup V(C)$ (note that $V(C_0) = C_0$). In particular if $C_1 := T(C_0) \setminus C_0$, then $C_1 \supseteq T(C) \setminus C_0 \neq \emptyset$.

By *(i)* and Lemma 3.2, $C_1 \subseteq T(\operatorname{supp} V) = A \sqcup B$. We also have $VT = TV$ so in particular $VT(C_0) = TV(C_0) = T(C_0)$, and therefore $V(C_1) = VT(C_0) \setminus V(C_0) = T(C_0) \setminus C_0 = C_1$. Define then $D := C_1 \cap B$.

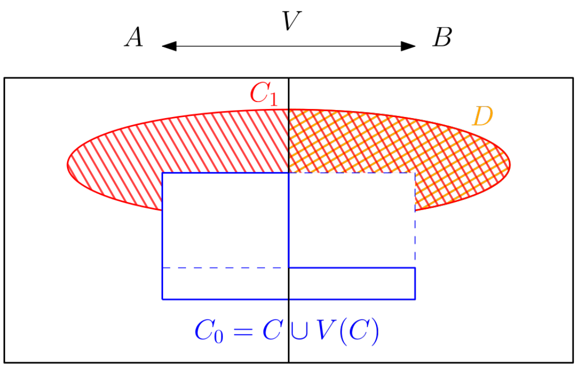

Figure 5: Visualisation of C_0, C_1 and D

Now we notice the following two facts:

• $C_1 \cap T(D) = \emptyset$. Indeed, $C_1 \subseteq T(C_0)$ so $C_1 \cap T(D) \subseteq T(C_0 \cap D) \subseteq T(C_0 \cap C_1) = \emptyset$.

• $C_1 = V(D) \sqcup D$. Indeed, $V(D) = V(C_1) \cap V(B) = C_1 \cap A$, so $C_1 = C_1 \cap (A \sqcup B) = (C_1 \cap A) \sqcup (C_1 \cap B) = V(D) \sqcup D$.

Therefore, we have

$$V_{C_1} T(D) = T(D) \neq TV(D) = TV_{C_1}(D).$$

Indeed, the first equality comes from the first point, the last equality is a direct consequence of the definitions, and the inequality comes from the second point. Since T is in $\mathcal{E}_V = C(\mathcal{D}_V)$ and $V_{C_1} \in \mathcal{D}_V$ by Lemma 3.5, this is a contradiction, as they do not commute.

(iii) By Lemma 2.18, $\operatorname{supp} T = \sup \{D \subseteq X \mid D \cap T(D) = \emptyset\}$ and so we have $C \cap \operatorname{supp} T = \sup \{D \subseteq C \mid D \cap T(D) = \emptyset\}$. Fix $D \subseteq C \cap \operatorname{supp} T$ such that $T(D) \cap D = \emptyset$. Since $D \subseteq C \subseteq \operatorname{supp} V$, by *(ii)* we have $T(D) \subseteq (D) \cup V(D)$, so in fact $T(D) \subseteq V(D)$ and thus by taking the supremum, $T(C \cap \operatorname{supp} T) \subseteq V(C \cap \operatorname{supp} T)$. By *(i)* we moreover know that $TV = VT$, so we have

$$T^2(C \cap \operatorname{supp} T) \subseteq TV(C \cap \operatorname{supp} T) = VT(C \cap \operatorname{supp} T) \subseteq V^2(C \cap \operatorname{supp} T) = C \cap \operatorname{supp} T.$$

Of course, the previous inclusion holds on $C \setminus \operatorname{supp} T$, so $T^2(C) \subseteq C$, and this holds for any $C \subseteq \operatorname{supp} V$. But by *(i)* and Lemma 3.2, we also have $\operatorname{supp} V = T(\operatorname{supp} V) = T^2(\operatorname{supp} V)$, so

$\operatorname{supp} V \setminus T^2(C) = T^2(\operatorname{supp} V \setminus C) \subseteq \operatorname{supp} V \setminus C$. We have proved both inclusions, so $T^2(C) = C$.

(iv) This is immediate from 3.5, as bijections with disjoint supports commute. □

Lemma 3.7 (Properties of $\mathcal{F}_V$). *If S is in $\mathcal{F}_V$, then $\operatorname{supp} V \cap \operatorname{supp} S = \emptyset$. Moreover, for any $\emptyset \neq C \subseteq X \setminus \operatorname{supp} V$, there exists a non-trivial involution S in $\mathcal{F}_V$, with $\operatorname{supp} S \subseteq C$.*

Proof. The first part of the statement is immediate from item *(iii)* of Lemma 3.6. For the second part, we know from 3.1 that there exists $T \in \mathbb{G}$ of order exactly 4 with $\operatorname{supp} T \subseteq C$, and from item *(iv)* of Lemma 3.6 we get that $T \in \mathcal{E}_V$, so $T^2 \in \mathcal{F}_V$ and is a non-trivial involution. □

Lemma 3.8 (Properties of $\mathcal{G}_V$). *The set $\mathcal{G}_V$ is comprised of all the elements in $\mathbb{G}$ that are supported by $\operatorname{supp} V$:*

$$\mathcal{G}_V = \{T \in \mathbb{G} \mid \operatorname{supp} T \subseteq \operatorname{supp} V\}.$$

Proof. ($\supseteq$) Let $T \in \mathbb{G}$ be such that $\operatorname{supp} T \subseteq \operatorname{supp} V$. For any $S \in \mathcal{F}_V$ we know from Lemma 3.7 that $\operatorname{supp} V \cap \operatorname{supp} S = \emptyset$, so T and S commute, which means that $T \in \mathcal{G}_V$.

($\subseteq$) Let $T \in \mathbb{G}$ be such that $\operatorname{supp} T \nsubseteq \operatorname{supp} V$. By Lemma 2.18 consider $C \subseteq X \setminus \operatorname{supp} V$ such that $C \cap T(C) = \emptyset$. By Lemma 3.7 we know that there exists an involution $S \in \mathcal{F}_V$ with $\operatorname{supp} S \subseteq C$. Therefore, for any $D \subseteq \operatorname{supp} S$ such that $S(D) \neq D$ we have

$$TS(D) \neq T(D) = ST(D),$$

which means that T does not commute with S, hence $T \notin \mathcal{G}_V$. □

3.2 Constructing the conjugation

Now armed with all the necessary properties of the previously defined sets, we can go back to the isomorphism $\psi : \mathbb{G} \to \mathbb{H}$ in order to construct the associated conjugation. It is immediate that $\psi(V)$ is an involution, and it is straightforward to check that the following hold:

$$\begin{aligned} \psi(\mathcal{C}_V) &= \mathcal{C}_{\psi(V)}, \\ \psi(\mathcal{D}_V) &= \mathcal{D}_{\psi(V)}, \\ \psi(\mathcal{E}_V) &= \mathcal{E}_{\psi(V)}, \\ \psi(\mathcal{F}_V) &= \mathcal{F}_{\psi(V)}, \\ \psi(\mathcal{G}_V) &= \mathcal{G}_{\psi(V)}. \end{aligned}$$

and in particular from Lemma 3.8, for any $T \in \mathbb{G}$ we see that

$$\operatorname{supp} T \subseteq \operatorname{supp} V \iff \operatorname{supp} \psi(T) \subseteq \operatorname{supp} \psi(V). \tag{$\star$}$$

We now define $S : \operatorname{MAlg}(X,\mu) \to \operatorname{MAlg}(X,\mu)$ and $S^* : \operatorname{MAlg}(X,\mu) \to \operatorname{MAlg}(X,\mu)$ to be such that for any $C \in \operatorname{MAlg}(X,\mu)$ we have

$$\begin{cases} S(C) := \sup \{\operatorname{supp} \psi(V) \mid V \in \mathbb{G} \text{ involution and } \operatorname{supp} V \subseteq C\}, \\ S^*(C) := \sup \left\{\operatorname{supp} \psi^{-1}(V) \mid V \in \mathbb{H} \text{ involution and } \operatorname{supp} V \subseteq C\right\}. \end{cases}$$

In particular S and S^* are ($\subseteq$)-order-preserving. We have to check that S and S^* are well-defined non-singular bijections of X (equivalently boolean automorphisms of $\operatorname{MAlg}(X,\mu)$), that $S^* = S^{-1}$, that S is the desired conjugation, and that it is unique in that regard.

By symmetry (as ψ is an isomorphism) it is enough to do the necessary verifications on S.

Lemma 3.9. *For any Borel subset $A \subseteq X$ and any involution $V \in \mathbb{G}$, if $\operatorname{supp} \psi(V) \subseteq S(A)$, then $\operatorname{supp} V \subseteq A$.*

Proof. We do a proof by contraposition. By Lemma 3.1 there exists an element $T \in \mathbb{G}$ of order exactly 4, with $\operatorname{supp} T \subseteq \operatorname{supp} V \setminus A$. Therefore, T^2 is a non-trivial involution with $\operatorname{supp} T^2 \subseteq \operatorname{supp} V$, so from $(\star)$ we get that $\operatorname{supp} \psi(T^2) \subseteq \operatorname{supp} \psi(V)$.

Now if $V' \in \mathbb{G}$ is an involution with $\operatorname{supp} V' \subseteq A$, then $T \in \mathcal{E}_{V'}$ by disjointness of the supports and Lemma 3.6, and therefore $T^2 \in \mathcal{F}_{V'}$. This yields that $\psi(T^2) \in \mathcal{F}_{\psi(V')}$, and by Lemma 3.7 $\operatorname{supp} \psi(T^2) \cap \operatorname{supp} \psi(V') = \emptyset$. As V' is arbitrary, $\operatorname{supp} \psi(T^2) \cap S(A) = \emptyset$, so

$$\emptyset \neq \operatorname{supp} \psi(T^2) \subseteq \operatorname{supp} \psi(V) \setminus S(A),$$

which concludes the proof. ◻

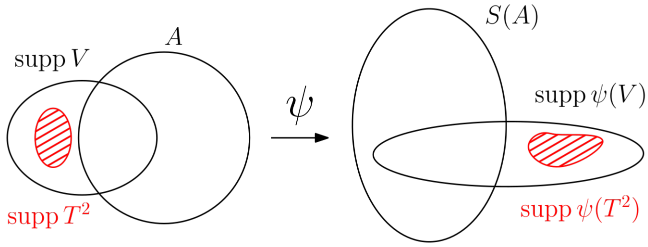

Figure 6: Visualisation of the proof of Lemma 3.9

Proposition 3.10. *For any $A \in \mathrm{MAlg}(X, \mu)$ we have $S^*S(A) = A$, and similarly, $SS^*(A) = A$.*

Proof. We assume that A is non-trivial.

($\supseteq$) Assume the contrary, consider $\emptyset \neq B \subseteq A \setminus S^*S(A)$, and an involution $V \in \mathbb{G}$ such that $\operatorname{supp} V \subseteq B$. We know that $\psi(V)$ is an involution in $\mathbb{H}$ and that $\operatorname{supp} \psi(V) \subseteq S(A)$, so

$$0 \neq \operatorname{supp} V = B \cap \operatorname{supp} \psi^{-1}\psi(V) \subseteq B \cap S^*S(A),$$

which is the contradiction.

($\subseteq$) Let now V be an involution in $\mathbb{H}$ such that $\operatorname{supp} V \subseteq S(A)$. Then $\psi^{-1}(V)$ is an involution in $\mathbb{G}$ with $\operatorname{supp} \psi\psi^{-1}(V) = \operatorname{supp} V \subseteq S(A)$. By Lemma 3.9, this means that $\operatorname{supp} \psi^{-1}(V) \subseteq A$. Again, as V is arbitrary, this means that $S^*S(A) \subseteq A$. ◻

Proposition 3.11. *The functions S and $S^* = S^{-1}$ are boolean automorphisms of $\mathrm{MAlg}(X, \mu)$.*

Proof. We start by noticing that since S and S^* are order-preserving, they send X to itself, and $\emptyset$ to itself. Indeed, $S(X) \subseteq X$, and $S^*(X) \subseteq X$, so by applying S, we get $X \subseteq S(X)$, thus $S(X) = X$. Similarly, $\emptyset \subseteq S(\emptyset)$, and $\emptyset \subseteq S^*(\emptyset)$, so by applying S, we get $S(\emptyset) \subseteq \emptyset$, thus $S(\emptyset) = \emptyset$.

We now prove that S is a Boolean automorphism by using characterization *(4)* of Proposition 1.16. To that end we fix A, B disjoint in $\mathrm{MAlg}(X, \mu)$, then we aim to establish that $S(A) \cap S(B) = \emptyset$ and $S(A \cup B) = S(A) \cup S(B)$.

We start with $S(A) \cap S(B)$. Let $C \subseteq S(A) \cap S(B)$, and let $V \in \mathbb{H}$ be an involution with $\operatorname{supp} V \subseteq C$. We have

$$\operatorname{supp} \psi\psi^{-1}(V) = \operatorname{supp} V \subseteq C \subseteq S(A) \cap S(B),$$

so by Lemma 3.9 applied to $S(A)$ and $S(B)$ separately, $\operatorname{supp} \psi^{-1}(V) \subseteq A \cap B = \emptyset$. This means that $\psi^{-1}(V)$ is the identity, so V is the identity, so $C = \emptyset$. Therefore $S(A) \cap S(B) = \emptyset$.

We finally have to take care of $S(A) \cup S(B)$. Since S is order-preserving we have $S(A) \cup S(B) \subseteq S(A \cup B)$. Fix then $C \subseteq S(A \cup B) \setminus (S(A) \cup S(B))$ and $D \subseteq A \cup B$ such that $S(D) = C$.

We also have $S(D \cap A) \subseteq S(D) \cap S(A) = C \cap S(A) = \emptyset$, and similarly for B. By applying S^*, which sends $\emptyset$ to itself, this means that $D \cap A = D \cap B = \emptyset$. This concludes the proof, as $C = S(D) = S((D \cap A) \cup (D \cap B)) = S(\emptyset) = \emptyset$, yielding that $S(A \cup B) \subseteq S(A) \cup S(B)$. □

We are now almost done. We only need to verify that ψ is the conjugation by S, and that it is the unique such automorphism. We prove the following final lemma, before finishing the proof with Proposition 3.13.

Lemma 3.12. *For any involutions $V \in \mathbb{G}$ and $V' \in \mathbb{H}$ we have $S(\operatorname{supp} V) = \operatorname{supp} \psi(V)$ and $S^{-1}(\operatorname{supp} V') = \operatorname{supp} \psi^{-1}(V')$.*

Proof. By definition of S we have $\operatorname{supp} \psi(V) \subseteq S(\operatorname{supp} V)$. For the converse inclusion, by Proposition 3.10 and the definition of S^{-1} we have

$$\operatorname{supp} \psi(V) = SS^{-1}(\operatorname{supp} \psi(V)) \supseteq S(\operatorname{supp} \psi^{-1}\psi(V)) = S(\operatorname{supp} V),$$

which concludes the proof. □

Proposition 3.13. *For any $T \in \mathbb{G}$, we have $\psi(T) = STS^{-1}$. Moreover, S is the only non-singular bijection of (X, μ) satisfying this.*

Proof. Assume the contrary: for a non-trivial $T \in \mathbb{G}$ the map $F \coloneqq \psi(T)^{-1}STS^{-1}$ is not id_X. By Lemma 2.18 this means that there exists $\emptyset \neq A \subseteq X$ such that $F(A) \cap A = \emptyset$. By applying $\psi(T)$ we get

$$STS^{-1}(A) \cap \psi(T)(A) = \emptyset. \tag{$*$}$$

Let now $V \in \mathbb{H}$ be a non-trivial involution with $B \coloneqq \operatorname{supp} V \subseteq A$. By Lemma 3.12, $\operatorname{supp} \psi^{-1}(V) = S^{-1}(B)$, and so by the general fact about the support of a conjugate we get that

$$\operatorname{supp}(T\psi^{-1}(V)T^{-1}) = T(\operatorname{supp} \psi^{-1}(V)) = TS^{-1}(B),$$

which yields (from Lemma 3.12 again):

$$\operatorname{supp}(\psi(T)V\psi(T)^{-1}) = \operatorname{supp}(\psi(T\psi^{-1}(V)T^{-1})) = S(\operatorname{supp}(T\psi^{-1}(V)T^{-1})) = STS^{-1}(B).$$

On the other hand, $\operatorname{supp}(\psi(T)V\psi(T)^{-1}) = \psi(T)(B)$. From the previous two equalities we get that $STS^{-1}(B) = \psi(T)(B)$, contradicting $(*)$.

We now turn to the uniqueness, and assume that S_1, S_2 are two distinct non-singular bijections. We have $S_2^{-1}S_1 \neq \mathrm{id}_X$, so by Lemma 2.18 again there exists $\emptyset \neq A \subseteq X$ such that $(S_2^{-1}S_1)(A) \cap A = \emptyset$. We then consider an involution $V \in \mathbb{G}$ with $\operatorname{supp} V \subseteq A$. We have $\operatorname{supp}((S_2^{-1}S_1)V(S_2^{-1}S_1)^{-1}) = (S_2^{-1}S_1)(\operatorname{supp} V) \subseteq (S_2^{-1}S_1)(A)$, so $(S_2^{-1}S_1)V(S_2^{-1}S_1)^{-1}$ cannot be equal to V. Therefore

$$S_2^{-1}S_1V \neq VS_2^{-1}S_1.$$

Multiplying by S_2 on the left and by S_1^{-1} on the right, we get that the conjugates of V by S_1 and S_2 are different. □

References

[Dye63] H. A. Dye. On groups of measure preserving transformations. ii. *American Journal of Mathematics*, 85(4):551–576, 1963. URL: http://www.jstor.org/stable/2373108. (document), 1.1, 2.3.2, 2.31

[EG16] Amichai Eisenmann and Yair Glasner. Generic IRS in free groups, after Bowen. *Proc. Am. Math. Soc.*, 144(10):4231–4246, 2016. doi:10.1090/proc/13020. 2.19

[Eig82] S. J. Eigen. The group of measure-preserving transformations of [0, 1] has no outer automorphisms. *Mathematische Annalen*, 259(2):259–270, June 1982. URL: https://doi.org/10.1007/BF01457312, doi:10.1007/BF01457312. (document), 3.1

[Fre02] D. H. Fremlin. *Measure theory 3, Measure algebras*. Measure theory. Torres Fremlin, 2002. (document), 1.10, 1.14, 1.16, 1.19, 1.22, 2.4, 2, 2.10, 2.2, 2.22, 2.24, 2.29, 3.1

[Fre03] D. H. Fremlin. *Measure theory. Vol. 2. Broad foundations*. Colchester: Torres Fremlin, corrected second printing of the 2001 original edition, 2003. 2.8, 2.9

[IT65] A. Ionescu Tulcea. On the category of certain classes of transformations in ergodic theory. *Transactions of the American Mathematical Society*, 114(1):261–279, 1965. URL: http://www.jstor.org/stable/1994001. 2.2

[Kec95] Alexander S. Kechris. *Classical Descriptive Set Theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. 1.1

[Kec10] Alexander S. Kechris. *Global aspects of ergodic group actions*. American Mathematical Society, 2010. 1.1

[KLM15] Adriane Kaïchouh and François Le Maître. Connected polish groups with ample generics. *Bulletin of the London Mathematical Society*, 47(6):996–1009, 2015. URL: https://londmathsoc.onlinelibrary.wiley.com/doi/abs/10.1112/blms/bdv078.

[KM04] Alexander S. Kechris and Benjamin D. Miller. *Topics in Orbit Equivalence*. Lecture Notes in Mathematics. Springer, 1 edition, 2004. 2.3.1, 2.3.1, 2.3.2

[LM14] François Le Maître. *Sur les groupes pleins préservant une mesure de probabilité*. Theses, Ecole normale supérieure de lyon - ENS LYON, May 2014. URL: https://theses.hal.science/tel-00992656. 2.5, 2, 2.1.2, 2.11, 2.12, 2.3.2, 2.3.2, 2.3.2

[LM22] François Le Maître. Polish topologies on groups of non-singular transformations. *J. Log. Anal.*, 14:15, 2022. Id/No 4. doi:10.4115/jla.2022.14.4. 2.11