Dye - Eigen - Fremlin Reconstruction theorem

I. Statements, context & Basic facts

Dye '63: $\mathbb{G}$, $\mathbb{H}$ two ergodic f.g. on a standard proba. space (X, μ)

then any $\psi : \mathbb{G} \to \mathbb{H}$ gp. isomorphism is the conjugation by some p.m.p. bijection.

i.e. $\exists S \in \mathrm{Aut}(X, \mu)$, $\forall T \in \mathbb{G}$: $\psi(T) = STS^{-1}$

$\mathrm{MAlg}(X, \mu)$

Fremlin' 2002: Let $\mathcal{A}$ and $\mathcal{B}$ be two Dedekind complete Boolean algebras

$\mathbb{G}$, $\mathbb{H}$ two subgroups of $\mathrm{Aut}(\mathcal{A})$ $\mathrm{Aut}(X, [\mu])$ and $\mathrm{Aut}(\mathcal{B})$ respectively, both with many involutions.

if $\psi : \mathbb{G} \to \mathbb{H}$ is a group isom., then

$\exists!$ Boolean isom. $S : \mathcal{A} \to \mathcal{B}$ s.t. $\forall T \in \mathbb{G}$: $\psi(T) = STS^{-1}$.

Definition: $\mathbb{G} \leq \mathrm{Aut}(X, [\mu])$ s.g. has many involutions,

if $\forall A \overset{\text{Borel}}{\subseteq} X$ of positive measure $\exists$ non-trivial involution $V \in \mathbb{G}$, $\mathrm{supp}\, V \subseteq A$

$$\mathrm{supp}\, V = \{x \in X \mid T(x) \neq x\}$$

Proposition: if G is an ergodic f.g. of (X,μ) then G has many involutions.

Proposition: ($MAlg(X,\mu)$ is Dedekind - complete)

phD manuscript of François. Any family of elements of $MAlg(X,\mu)$ admits a supremum which is obtained as the limit of an increasing sequence of finite reunions of elements of the family.

Disclaimer: We work "up to measure 0"

by $A \cap B = \emptyset$ I mean that $\mu(A \cap B) = 0$

Lemma: $T \in Aut(X,[\mu])$, $\emptyset \neq D \subseteq X$, TFAE:

(i) $supp\, T \nsubseteq D$

(ii) $T_{|X \setminus D} \neq id_{|X \setminus D}$

(iii) $\exists\, \emptyset \neq C \subseteq X \setminus D$ s.t. $C \cap T(C) = \emptyset$

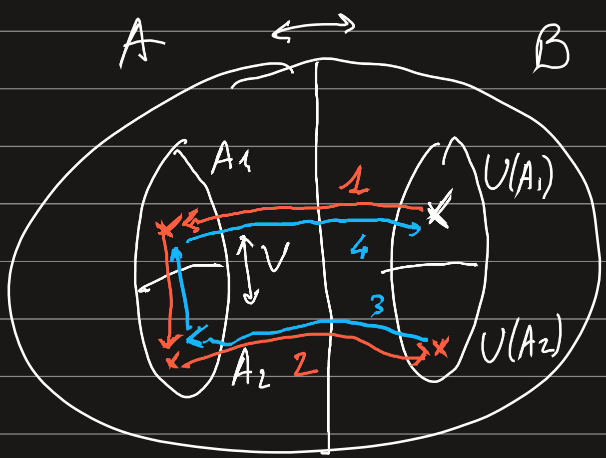
Moreover, $supp\, T = \sup \{ D \subseteq X \mid D \cap T(D) = \emptyset \}$

Proposition: V involut. $\in Aut(X,[\mu])$, then $\exists\, A, B \subseteq X$ disjoint, such that
$$\begin{cases} V(A) = B \\ supp\, V = A \sqcup B \end{cases}$$
we then say that $V = V_{A,B}$ is an (A,B) - exchanging involution.

II. Linking measure - theoretic properties & g. theoretic properties

Lemma 0: $G \leq \mathrm{Aut}(X,[\mu])$ w/ many involutions; $\forall\, \emptyset \neq C \subseteq X$, $\exists\, T \in G$ of order exactly 4 s.t. $\mathrm{supp}\, T \subseteq C$.

Proof: U (A,B)-exch. invol. in G, $A \sqcup B \subseteq C$
non-trivial



V (A_1, A_2)-exch. invo.
w/ $A_1 \sqcup A_2 \subseteq A$

$$\mathrm{Supp}(UVU) = U(\mathrm{Supp}\, V) \subseteq B$$

V and UVU commute
$UVUV$ is an involution

$$\Rightarrow \quad T := UV \text{ has order } 4. \ \blacksquare$$

For the rest of this section, we fix $G \leq \mathrm{Aut}(X,[\mu])$ w/ many invo. and V non trivial invo in G, w/ $\mathrm{supp}\, V = A \sqcup B$

Define the following:

$$\mathcal{G}_V := C(V) = \{T \in G \mid TV = VT\}$$

$$\mathcal{D}_V := \{\text{involutions in } \mathcal{G}_V \text{ commuting w/ all their } \mathcal{G}_V\text{-conjugates}\}$$
$$= \{T \in \mathcal{G}_V \mid T \text{ inv. and } \forall S \in \mathcal{G}_V : TSTS^{-1} = STS^{-1}T\}$$

$$\mathcal{E}_V := C(\mathcal{D}_V) = \{T \in G \mid \forall S \in \mathcal{D}_V : TS = ST\}$$

$$\mathcal{F}_V := \mathrm{Sq}(\mathcal{E}_V) = \{T^2 \mid T \in \mathcal{E}_V\}$$

$\setminus \mathcal{G}_V := C(\mathcal{F}_V) = \{ T \in \mathbb{G} \mid \forall S \in \mathcal{F}_V : ST = TS \}$

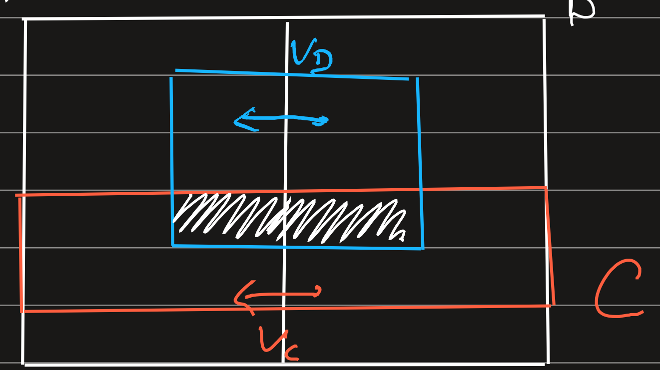
Lemma $\mathcal{G}$: $\forall T \in \mathcal{G}_V : \quad T(\operatorname{supp} V) = \operatorname{supp} V$

Proof: $T(\operatorname{supp} V) = \operatorname{Supp}(\underbrace{TVT^{-1}}) = \operatorname{supp}(V)$ ∎

Definition: $\forall \emptyset \neq C \subseteq X$ s.t. $V(C) = C$

the induced-exchanging-involution V_C is defined by

$$\begin{cases} V_C(D) = V(D) & \text{if } D \subseteq C \\ V_C(D) = D & \text{if } D \subseteq X \setminus C \end{cases}$$



Notice that:

- $V_C = V_{(A \cap C), (B \cap C)}$
- $\underline{V_C V_D} = V_{C \Delta D} = \underline{V_D V_C}$

Lemma $\mathcal{D}$: (1) $\forall T \in \mathcal{D}_V : \quad \operatorname{supp} T \subseteq \operatorname{supp} V$

(2) $\forall \emptyset \neq C \subseteq X$; s.t. $V(C) = C$
$V_C \in \mathcal{D}_V$.

Proof: (1) By contraposition.

Let $T \in \mathcal{G}_V$ (involution) s.t. $\operatorname{supp} T \not\subseteq \operatorname{Supp} V$

$\emptyset \neq C \subseteq \operatorname{Supp} T$ s.t.
$C \cap T(C) = \emptyset$

Lemma O $\Rightarrow \exists\, S \in \mathcal{G}$ of order 4, $\operatorname{supp} S \subseteq C$

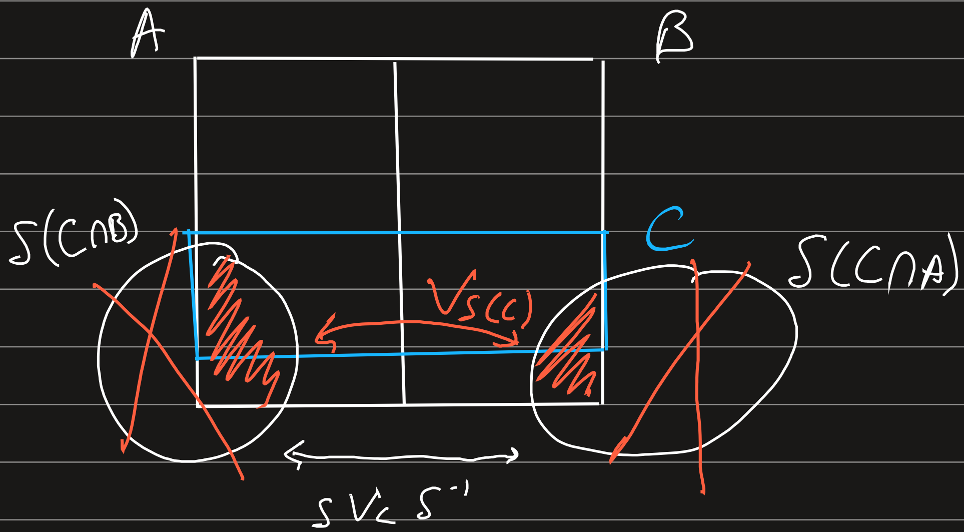
- $SV = VS \Rightarrow S \in \mathcal{G}_V$
- $S \neq S^{-1}$ so $\exists\, \emptyset\, D \subseteq C$, $S(D) \neq S^{-1}(D)$

we also have. $C \cap T(D) = \emptyset = C \cap TS^{-1}(D)$

$\rightarrow \quad ST\cancel{S^{-1}}T(D) = S\cancel{T}(D) \neq S^{-1}(D) = T\cancel{S}\underline{TS^{-1}(D)}$

so (D non-trivial) T and STS^{-1} do not commute.
$T \notin \mathcal{D}_V$.

(2) $\forall S \in \operatorname{Aut}(X, [\mu])$: $SV_CS^{-1} = (SVS^{-1})_{S(C\cap A),\, S(C\cap B)}$



In particular, for $S = V$, $VV_CV^{-1} = V_C$
$\Rightarrow V_C \in \mathcal{G}_V$.

Now for any $\underline{S \in \mathcal{G}_V}$ $\left(V\underline{S(C)} = SV(C) = \underline{S(C)}\right)$

$$\underline{SV_CS^{-1}} = (SVS^{-1})_{S(C\cap A),\, S(C\cap B)} = V_{S(C\cap A),\, S(C\cap B)} = V_{S(C)}$$

$V_{S(C)} V_C = V_C V_{S(C)}$ i.e. $V_C \in \mathcal{D}_V$. ∎

Lemma $\mathcal{E}$: (1) $\mathcal{E}_V \subseteq \mathcal{G}_V$
(2) $\forall T \in \mathcal{E}_V : \forall \emptyset \neq C \subseteq \operatorname{supp} V : T(C) \subseteq C \cup V(C)$

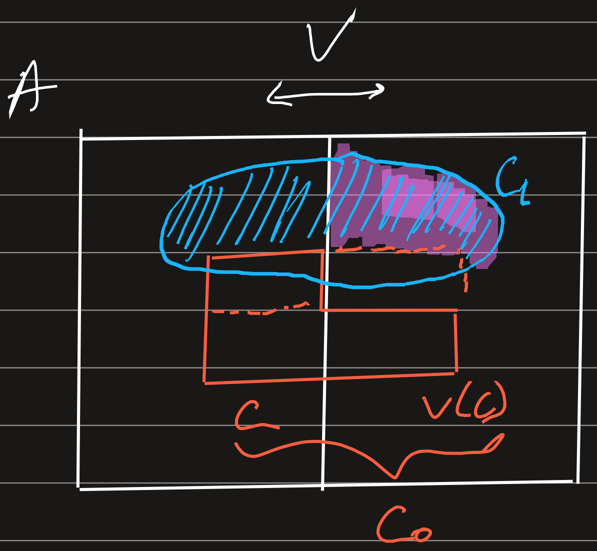
(3) $\forall T \in \mathcal{E}_V : \forall \emptyset \neq C \subseteq \operatorname{supp} V : T^2(C) = C$

(4) if $T \in \mathcal{G}$ is s.t. $\operatorname{supp} T \cap \operatorname{supp} V = \emptyset$, then $T \in \mathcal{E}_V$.

Proof: (1) $V \in \mathcal{D}_V$. ✓

(2) Assume the contrary: $\emptyset \neq C \subseteq \operatorname{supp} V$ s.t.

$$T(C) \not\subseteq \underbrace{C \cup V(C)}_{C_0}$$



$$V(C_0) = C_0$$

$$\emptyset \neq C_1 := T(C_0) \setminus C_0$$

(1) & Lemma 6

$$\Rightarrow C_1 \subseteq T(\underbrace{A \sqcup B}_{\operatorname{supp} V}) = A \sqcup B$$

$$VT(C_0) = TV(C_0) = T(C_0)$$

$$V(C_1) = VT(C_0) \setminus V(C_0) = T(C_0) \setminus C_0 = C_1$$

Define $D := B \cap C_1$ notice 2 things:

- $C_1 = V(D) \sqcup D$
- $C_1 \cap T(D) = \emptyset$: $C_1 \subseteq T(C_0)$ $\Rightarrow C_1 \cap T(D) \subseteq T$

Therefore, $V_{C_1} T(D) = T(D) \neq TV(D) = TV_{C_1}(D)$

But $V_{C_1} \in \mathcal{D}_V$ ✓

(3) $\quad \operatorname{supp} T = \sup \{ D \subseteq X \mid D \cap T(D) = \emptyset \}$

$D \subseteq C \cap \operatorname{supp} T = \sup \{ D \subseteq C \mid \underline{D \cap T(D) = \emptyset} \}$

s.t. $D \cap T(D) \neq \emptyset$

$C \cap \operatorname{supp} T \subseteq C \subseteq \operatorname{supp} V$ so by applying (2) to $C \cap \operatorname{supp} T$ we get

$\rightarrow T(D \cap \operatorname{supp} T) \subseteq (D \cap \operatorname{supp} T) \cup V(D \cap \operatorname{supp} T)$

in fact $T(D) \subseteq V(D)$

$\longrightarrow T(C \cap \operatorname{supp} T) \subseteq V(C \cap \operatorname{supp} T)$

by supremum.

We want $T^2(C) = C$

$\subseteq$ $\quad VT = TV$ so

$T^2(C \cap \operatorname{supp} T) \subseteq TV(C \cap \operatorname{supp} T)$

$\qquad = VT(C \cap \operatorname{supp} T) \subseteq V^2(C \cap \operatorname{supp} T)$

$\supseteq$ $\quad \operatorname{supp} V = T^2(\operatorname{supp} V) \qquad T \in \mathcal{E}_V \subseteq \mathcal{C}_V$

$\operatorname{supp} V \setminus T^2(C) = T^2(\operatorname{supp} V \setminus C) \subseteq \operatorname{supp} V \setminus C$

$\rightarrow T^2(C) \supseteq C$

(4) immediate from Lemma $\mathcal{D}$.(1) $\blacksquare$

Lemma $\mathcal{F}^1$: $\quad S \in \mathcal{F}_V \Rightarrow \operatorname{supp} S \cap \operatorname{supp} V = \emptyset$ (Lemma $\mathcal{E}$ (3))

Moreover, $\forall \emptyset \neq C \subseteq X \setminus \operatorname{supp} V$, $\exists S \in \mathcal{F}_V$, w/ $\operatorname{supp} S \subseteq C$

(involution non-trivial

Proof: Lemma Θ $\Rightarrow \exists T \in \mathbb{G}$, $\operatorname{supp} T \subseteq C$

Lemma $\mathcal{E}$ (4) $\Rightarrow T \in \mathcal{E}_V$. T^2 is as wanted. $\blacksquare$

Lemma $\mathcal{G}$: $\left(\mathcal{G}_V := \{ T \in \mathbb{G} \mid \operatorname{supp} T \subseteq \operatorname{supp} V \} \right)$

Proof: $\supseteq$ easy. fix $T \in \mathbb{G}$, $\underline{\operatorname{supp} T \subseteq \operatorname{supp} V}$

Lemma $\overline{\mathcal{F}}$ $\Rightarrow \forall S \in \widetilde{\mathcal{F}_V}$: $\operatorname{supp} S \cap \operatorname{supp} V = \emptyset$

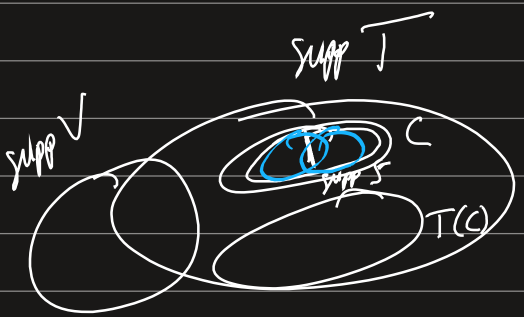
so $TS = ST$ ✓

$\subseteq$ By contraposition: fix $T \in \mathbb{G}$, $\operatorname{supp} T \not\subseteq \operatorname{supp} V$

fix $\emptyset \neq C \subseteq \operatorname{supp} T \setminus \operatorname{supp} V$ s.t. $C \cap T(C) = \emptyset$

Lemma $\overline{\mathcal{F}}$ $\Rightarrow \exists S$ non-trivial invo $\in \overline{\mathcal{F}_V}$

$\operatorname{supp} S \subseteq C$



$\overline{\mathcal{F}_V}$
$\mathbb{G}$

for any $D \subseteq \operatorname{supp} S$ s.t.

$D \neq S(D)$

$\underline{TS(D)} \neq T(D) = \underline{ST(D)}$

$\rightarrow T \notin \mathcal{G}_V$ $\blacksquare$

III. Constructing the conjugation

$\psi : \mathbb{G} \rightarrow \mathbb{H}$,

$$\psi(\mathcal{G}_V) = \mathcal{G}_{\psi(V)}$$

$$\forall T \in \mathbb{G}: \quad \operatorname{supp} T \subseteq \operatorname{supp} V \overset{(*)}{\iff} \operatorname{supp} \psi(T) \subseteq \operatorname{supp} \psi(V)$$

$$S, S^{*} \in (\operatorname{Aut}(X, [\mu]))$$

$$\begin{cases} S(C) = \sup\{\operatorname{supp} \psi(V) \mid V \in \mathbb{G} \text{ invo}, \text{ s.t. } \operatorname{supp} V \subseteq C\} \\ S^{*}(C) = \sup\{\operatorname{supp} \psi^{-1}(V) \mid V \in \mathbb{H} \text{ invo}, \text{ s.t. } \operatorname{supp} V \subseteq C\} \end{cases}$$

It remains to show:

easy
- S, S^{*} are well-defined.
- $S^{*} = S^{-1}$
- S is the desired conjugation.

easy
- unique in that regards.

Lemma:
from $(*)$

$$\begin{pmatrix} \forall A \subseteq X \\ \forall V \text{ invo} \in \mathbb{G} \end{pmatrix} : \quad \operatorname{supp} \psi(V) \subseteq S(A) \implies \operatorname{supp} V \subseteq A$$