Tautness and measure equivalence rigidity

Example:

$\Gamma, \Lambda \underset{\text{lattices}}{\leq} G$ lcsc grps

, act "similarly" on G $\qquad \Gamma \curvearrowright G \curvearrowleft \Lambda$

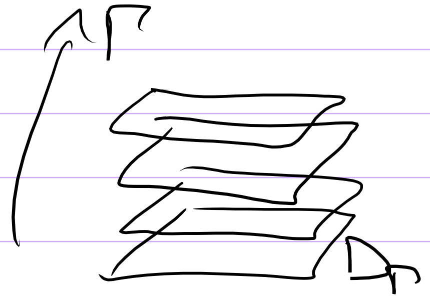
$\qquad\qquad\qquad\qquad \gamma \cdot \lambda^{-1}$

- G is large enough: $\Gamma \curvearrowright G$ free, $\Lambda \curvearrowright G$
- G is small enough: G/Γ G/Λ finite measure

Defn: Γ, Λ countable grps are measure equivalent (ME) when $\exists\, (\Sigma, m)$ standard Borel space with $\Gamma \times \Lambda \curvearrowright (\Sigma, m)$ measure preserving and

* $\Gamma \curvearrowright \Sigma$ free and admits a finite measure fund. domain D_Γ

$$\Sigma = \bigsqcup_{\gamma \in \Gamma} \gamma D_\Gamma \simeq \Gamma \times D_\Gamma$$



* $\Lambda \curvearrowright \Sigma$ —— " —— D_Λ

And (Σ, m) is a ME coupling

Facts: • ME with $D_\Gamma = D_\Lambda \iff$ OE (orbit equivalence)

• ME is an equiv relation on countable ∞ grps

Th (Ornstein–Weiss): "flexibility for ME"

$$\text{ME-class}(\mathbb{Z}) = \left\{ \begin{array}{c} \text{all amenable} \\ \text{infinite grps} \end{array} \right\}$$

Def°: Γ and Λ are almost isomorphic : Γ°_{a}

$$\left.\begin{array}{ll} \exists\, \Gamma^{\circ} \underset{f.i.}{\leq} \Gamma & \Lambda^{\circ} \underset{f.i.}{\leq} \Lambda \\ \exists \underset{\text{finite}}{M} \trianglelefteq \Gamma^{\circ} & \underset{\text{finite}}{N} \trianglelefteq \Lambda^{\circ} \end{array}\right\} \text{st } \Gamma^{\circ}/M \simeq \Lambda^{\circ}/N$$

remark: Almost iso $\Rightarrow$ ME

There exist grps for which $\Leftarrow$

examples of rigidity phenomena in ME:

1) Furman 1999: $\Gamma \underset{\text{lattice}}{\leq} G$ simple Lie grp higher rk

$\forall \Lambda$ arbitrary countable grp:

$\Lambda \overset{ME}{\sim} \Gamma \Rightarrow \Lambda \overset{\text{almost iso}}{\sim}$ a lattice in G

2) Bader - Furman - Sauer 2013: $\Gamma \underset{\text{lattice}}{\leq} G = \mathrm{Isom}(\mathbb{H}^n)$ $n \geq 3$

$\forall \Lambda$ arbitrary, $\Lambda \overset{L^1\text{-}ME}{\sim} \Gamma \Rightarrow \Lambda \overset{\text{almost iso}}{\sim}$ a lattice in G

3) Kida 2010: $\Gamma = MCG(\Sigma_g)$

Σ_g surface compact orientable of genus $g \geq 2$

$\Gamma = \text{diffeos}^{+} / \{\text{isotopy}\}$

$\forall \Lambda$ arbitrary countable grp: $\Lambda \overset{ME}{\sim} \Gamma \Leftrightarrow \Lambda \overset{\text{almost iso}}{\sim} \Gamma$

4) Guirardel - Horbez 2021: $\Gamma = \mathrm{Out}(\mathbb{F}_N)$ $N \geq 3$

$\forall \Lambda$, $\Gamma \overset{ME}{\sim} \Lambda \Leftrightarrow \Lambda \overset{\text{almost iso}}{\sim} \Gamma$

Def: Self ME coupling is ME coupling of Γ with itself

$$\Gamma_1 \times \Gamma_2 \curvearrowright \Sigma$$

ex: $\Gamma_1 \curvearrowright \Gamma \curvearrowleft \Gamma_2$ tautological self coupling

$\gamma_1 \cdot \gamma_2^{-1}$

but bigger: $G \underset{\text{f.i.}}{\geq} \Gamma$ $\Gamma_1 \curvearrowright G \curvearrowleft \Gamma_2$

Def: Γ is taut with respect to $G \geq \Gamma$

$\forall \Omega$ self coupling of Γ, $\Gamma \times \Gamma \curvearrowright \Omega$

$$\exists\, \Phi : {}^{\Gamma_1 \curvearrowright}\Omega^{\curvearrowleft \Gamma_2} \xrightarrow{\text{meas}} {}^{\Gamma_1 \curvearrowright}G^{\curvearrowleft \Gamma_2}$$

$$\left(\begin{array}{l} \Phi((\gamma_1,\gamma_2)\omega) = \gamma_1 \, \Phi(\omega) \, \gamma_2^{-1} \\ \forall \gamma_1,\gamma_2 \in \Gamma, \quad \forall^{*} \omega \in \Omega \end{array} \right.$$

Def: $\Gamma \hookrightarrow G$ is strongly ICC when the only proba measure on G invariant by Γ-conjugation is Dirac measure at 1.

ex: $\forall g \neq 1$, $\{\gamma g \gamma^{-1}\}_{\gamma \in \Gamma}$ is ∞ (G is discrete)

and Furman and BFS have strongly ICC

Morally: factorise the ME-rigidity proofs:

- $\Gamma \leq G$ Lie simple, lattice higher rk
- $\Gamma = MCG(\Sigma_g)$
- $\Gamma = Out(\mathbb{F}_N)$

$\longrightarrow$ Γ taut (and ICC) $\longrightarrow$ Γ is ME-rigid

Step 1: specific to geometry of the chosen Γ

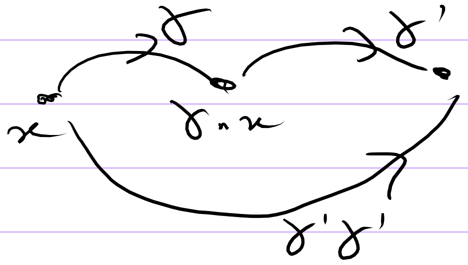
Step 2: general theorem on ME couplings

Tameness on the language of ME cocycles:

Def: $\Gamma \curvearrowright X$ p.m.p on a standard proba space X

A measurable cocycle is $c : \Gamma \times X \xrightarrow{\text{meas}} G$ (some other grp)

s.t.



$$c(\gamma'\gamma, x) = c(\gamma', \gamma \cdot x)\, c(\gamma, x)$$

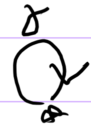
Two such cocycles c_1, c_2 are cohomologous when $\varphi : X \xrightarrow{\text{meas}} G$ and $\forall \gamma \in \Gamma$, $\forall^{*} x \in X$,

$$c_2(\gamma, x) = \varphi(\gamma \cdot x)\, c_1(\gamma, x)\, \varphi(x)^{-1}$$

remark: A meas cocycle is a measure theoretic generalisation of morphism of grps:

$$\underset{\substack{\uparrow \\ \text{meas}}}{C_x} : \Gamma \longrightarrow G$$

e.g. Γ

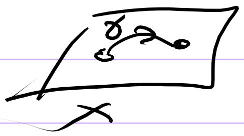


$X = \{*\}$

c is morph

Cohomology generalises conjugacy

$\varphi : X \to G$



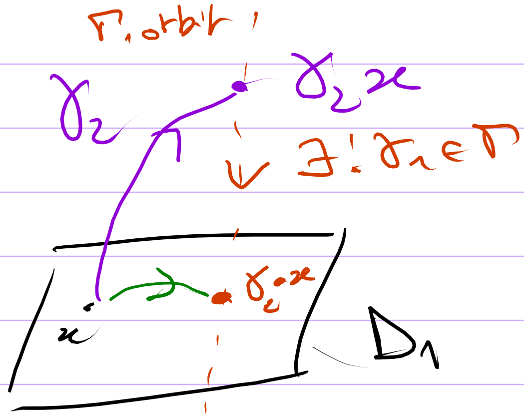
Any ME coupling of two grps $\Gamma_1 \times \Gamma_2 \curvearrowright \Omega$ fixed D_1, D_2 fund. domains. comes automatically with two measurable cocycles, called ME-cocycles:

Construct $\Gamma_2 \curvearrowright D_1$: $\gamma_2 \in \Gamma_2$, $x \in D_1$

$c_1 : \Gamma_2 \times D_1 \longrightarrow \Gamma_1$

$\gamma_2, x \longmapsto$ the γ_1 such that

$\gamma_1(\gamma_2 x) \in D_1$



Similarly $\Gamma_1 \curvearrowright D_2$

$c_2 : \Gamma_1 \times D_2 \longrightarrow \Gamma_2$

Now, in the case of self ME couplings:

$\Gamma_1 = \Gamma_2 = \Gamma$ $\qquad \Gamma_1 \times \Gamma_2 \curvearrowright \Omega$

Lemma [Kida] : If one choose D_1, D_2 such that $X = D_1 \cap D_2$ is of max measure,

Then, a cohom map $\varphi : X \to G \, (\geq \Gamma)$ between c_1 and c_2 gives rise to a tautening map

$$\Phi : \Omega \longrightarrow G$$

$\hookrightarrow$ idea : X is big enough $(\Gamma_1 \times \Gamma_2) X = \Omega$

Then, the fact that $c_2(\gamma, x) = \varphi(\gamma . x) \, c_1(\gamma, x) \, \varphi(x)^{-1}$

$$\varphi(\gamma . x) = \underbrace{c_2(\quad)}_{\gamma_2} \varphi(x) \underbrace{c_1(\quad)^{-1}}_{\gamma_1^{-1}}$$

$$\Phi((\gamma_1, \gamma_2) x) = \gamma_1 \varphi(x) \gamma_2^{-1} \in G$$

Examples:

*) **Furman 99** : "Step 1" is applying Zimmer's superrigidity of cocycles:

every ME cocycle $\Gamma \times X \to G$ is cohomologous to the identity $\Gamma \to G$

$\left[\Gamma \underset{\text{lattice}}{\leq} G \quad \begin{array}{l}\text{Lie simple} \\ \text{higher rk}\end{array} \right]$

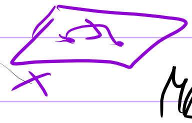
*) **Kida 2010** : Step 1 is a $\underbrace{\text{remarkable}}_{\substack{\text{step 1} \\ \text{Zimmer}}}$ $\underbrace{\text{generalisation}}_{\text{Step 2 !}}$ of Ivanov's theorem: "algebraic rigidity"

Ivanov 97 : Any $f : \Gamma \longrightarrow \Gamma = MCG(\Sigma_g)$ automorphism is a conjugation by some $g \in G = MCG^{\pm} = \text{Diffeos}^{\pm} / \text{isotopy}$

$\overset{\text{|| Ivanov}}{\text{Aut(curve graph)}}$

$$\left(f(\gamma) = g \gamma g^{-1} \right. \quad \gamma \in \Gamma$$

$\Gamma \ltimes X$



Kida generalises this: $f \text{ morph} \rightsquigarrow c$ ME-cocycle

Any ME-cocycle $c : \Gamma \times X \to \Gamma$. is cohomologous to the identity: $\varphi : X \to G$

$$c(\gamma, x) = \underbrace{\varphi(\gamma \cdot x)}_{\text{"}g\text{"}} \; \gamma \; \underbrace{\varphi(x)^{-1}}_{\text{"}g\text{"}}$$

Tautness and ICC implies ME-rigidity

What is "Step 2" ?

Th [Kida, Bader-Furman-Sauer]:

Γ countable grp $\leq G$.

Assume: $\begin{cases} * \ \Gamma \text{ tame wrt } G \\ * \ \Gamma \hookrightarrow G \text{ strongly ICC} \end{cases}$

Then

$\boxed{1}$ For any ME-coupling of Γ with some arbitrary Λ $\Gamma \times \Lambda \curvearrowright (\Sigma, m)$, there exist:

$$\begin{cases} \Psi : \Gamma \curvearrowright \Sigma \curvearrowleft \Lambda \xrightarrow{\text{meas}} \Gamma \curvearrowright G \curvearrowleft \rho(\Lambda) \\ \rho : \Lambda \xrightarrow{\text{morph}} G \end{cases}$$

$\forall \gamma \in \Gamma, \ \forall \lambda \in \Lambda, \quad \forall^a x \in \Sigma, \quad \Psi((\gamma,\lambda)x) = \gamma \Psi(x) \rho(\lambda)^{-1}$

$\boxed{2}$ (2a) If in addition, $G \underset{f.i.}{\geq} \Gamma$,

then $\begin{cases} \ker \rho < \infty \\ \operatorname{im} \rho \underset{f.i.}{\leq} G \end{cases}$ $\quad \Lambda \overset{ME}{\sim} \Gamma \Leftrightarrow \Lambda \overset{\text{almost iso}}{\sim} \Gamma$

In particular, $\Gamma \overset{\text{almost iso}}{\sim} \Lambda$

(2b) Γ lattice in G

$\begin{cases} \ker \rho < \infty \\ \operatorname{im} \rho \underset{\text{lattice}}{\leq} G \end{cases}$ $\quad \Lambda \overset{ME}{\sim} \Gamma \Leftrightarrow \Lambda \overset{\text{almost iso}}{\sim}$ a lattice in G

Proof of $\boxed{1}$: Fix an arbitrary $\Lambda \overset{ME}{\sim} \Gamma$

and fix coupling $\Gamma \times \Lambda \curvearrowright (\Sigma, m)$

Break the symmetry of $\Lambda \overset{ME}{\sim} \Gamma$: construct a self ME coupling of Γ: take two copies $\begin{cases} \Gamma_1 \curvearrowright \Sigma_1 \\ \Gamma_2 \curvearrowright \Sigma_2 \end{cases}$

$$\Omega_\Lambda := {}^{\Gamma_1 \curvearrowright}\Sigma_1 \times \Sigma_2{}^{\curvearrowleft \Gamma_2} \quad \underset{\Lambda}{\nwarrow \nearrow}$$

Λ diagonal action on $\Sigma \times \Sigma$:
$(\lambda x, \lambda y) \sim (x, y)$

$$= \left\{ \underset{\underset{[\lambda x, \lambda y]_\Lambda}{\parallel}}{[x,y]_\Lambda} \quad x, y \in \Sigma \right\}$$

Fix D fund. domain $\Lambda \curvearrowright \Sigma$, then:

$$\Omega_\Lambda \simeq {}^{\Gamma_1 \curvearrowright}\Sigma \times \overset{\curvearrowright 1}{D}$$

admits a fund domain of finite meas $\leftarrow \Sigma/\Lambda \times \Sigma/\Lambda$

$$\simeq D \times \Sigma^{\curvearrowleft \Gamma_2} \qquad \text{———} u \text{———}$$

So Ω_Λ is a self ME coupling of Γ.
By tameness, there exists

$$\Phi : \Omega_\Lambda \xrightarrow{\text{meas}} G$$

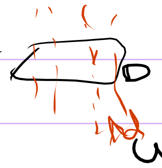
$$\Phi(\underset{\underset{[\gamma_1 x, \gamma_2 y]_\Lambda}{\parallel}}{(\gamma_1,\gamma_2)\,\omega}) = \gamma_1 \, \Phi[x,y]_\Lambda \, \gamma_2^{-1}$$

$\gamma_1, \gamma_2 \in \Gamma$ and $x, y \in \Sigma$

reminder: we want $\Psi : \Sigma \to G$
we have $\Phi : \Omega \to G$

Find Σ in Ω? "$\Psi = \Phi|_\Sigma$"

D is fund. domain for $\Lambda \curvearrowright \Sigma$:

For $d \in D$; define (in Σ )

$\Omega = \Sigma \times \Sigma / \Lambda$

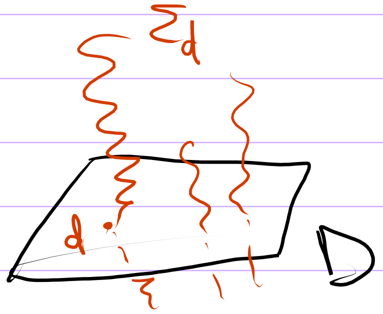
$$\Sigma_d := \{ [x,y]_\Lambda , \; y \in \Lambda d \}$$

$[x,y]_\Lambda \overset{\lambda^{-1}}{=} [\lambda^{-1} x , d]_\Lambda$

$\exists \; [x,y]_\Lambda = [\lambda x, \lambda y]_\Lambda$

$$\Sigma_d \simeq \Sigma \qquad \text{and} \qquad \Omega_\Lambda = \bigsqcup_{d \in D} \Sigma_d$$

$[x,d]_\Lambda \longmapsto x$



$$\begin{tikzcd} \text{in } \Omega_\Lambda : \arrow[d, "\Phi \text{ factoring}"] \\ G \end{tikzcd}$$

Define Ψ_d as the restriction of Φ to Σ_d

$$\Psi_d(\overset{\in \Sigma}{x}) := \Phi \underbrace{[x,d]_\Lambda}_{\in \Sigma_d}$$

* Γ-equivariance?

$$\psi_d(\gamma x) = \Phi[\gamma x, d]_\Lambda = \gamma \, \Phi[x,d]_\Lambda$$
$$= \gamma \, \psi_d(x)$$

* Λ-equivariance? we want

$$\rho : \Lambda \longrightarrow G$$

ICC

Claim: $\forall^* d \in D$, $\forall \lambda \in \Lambda$,

$$\left\{ \begin{array}{l} \Sigma \longrightarrow G \\ x \longmapsto \psi_d(\lambda x) \, \psi_d(x)^{-1} \end{array} \right.$$ is constant almost everywhere

For now, assume Claim holds. Fix $d \in D^*$.
Now, take $e(\lambda)$ the common value of the map in the claim.

ψ_d is equivariant!
e is morphism : $e : \Lambda \longrightarrow G$

So we have found "Furman's representation".

For $\boxed{1}$, it is left to prove the Claim.
Remark we have not yet used ICC asumptio

$$\left\{ \begin{array}{ll} (*) & \Phi[\underbrace{x, y]_\Lambda \Phi[y}_{(\ )^{-1}}, x]_\Lambda = 1 \quad \text{almost everywhere} \\ \\ (**) & \Phi[x, \underbrace{y] \; \Phi[y}_{\text{concatenatio}}, z] \; \Phi[z, x] = 1 \quad \text{a.e.} \end{array} \right.$$

We will use the "same ICC trick" twice:

For $(*)$: $F : \overset{\Gamma_1 2 \ \circlearrowleft \Gamma_2}{\Omega_\Lambda} \longrightarrow G$
$$[\underset{\gamma_1}{x}, \underset{\gamma_2}{y}]_\Lambda \longmapsto \underset{\gamma_1 \leftarrow}{\Phi[x}, \underbrace{y]_\Lambda \Phi[y}_{\gamma_2^{-1}\gamma_2}, \underset{\gamma_1^{-1}}{x]_\Lambda}$$

notice 1) F is Γ_2-invariant
2) equivariance $F(\gamma_1 x, y) = \gamma_1 F(x, y) \gamma_1^{-1}$
conjugatio

1) gives us $\bar{F} : \underbrace{\Omega / \Gamma_2}_{\text{finite measure } \mu} \longrightarrow G$

2) gives us that $\bar{F}_* \mu$ on G is finite and invariant by Γ_1-conjugatio: $\bar{F}_* \mu = \delta_{\{1\}}$ so $F \overset{a.e.}{=} 1$

for (*): $\Omega = \Sigma\times\Sigma/_{\Lambda} \longrightarrow G$

For (* *):

the new $F : \overset{\Gamma_1\ 2}{\Sigma} \times \overset{\Gamma_2\ 2}{\Sigma} \times \overset{\Gamma_3\ 1}{\Sigma} \Big/ \Lambda$ diagonal acts $\longrightarrow G$

$$\underset{\gamma_1}{[x},\ \underset{\gamma_2}{y},\ \underset{\gamma_3}{z}]_\Lambda \longmapsto \underset{\gamma_1}{\Phi[x,y)}\ \underset{\gamma_2^{-1}\gamma_2}{\Phi(y,z]}\ \underset{\gamma_3^{-1}\gamma_3}{\Phi[z,x]}\ \gamma_1^{-1}$$

F is $\Gamma_2\times\Gamma_3$ inv^t

F sends $\Gamma_1\ 2$ to Γ_1-conjugate

$\rightsquigarrow$ same trick $\quad F \overset{a.e.}{=} 1.$

Claim was $\forall^a x,y \in \Sigma$: $\forall \lambda\in\Lambda$
$\forall^a d\in D$

$$\underline{\Phi[\lambda x,d]}\ \Phi[x,d]^{-1} = \underline{\Phi[\lambda y,d]}\ \Phi[y,d]^{-1}$$

$$\Phi[\lambda x,d]\ \underbrace{\Phi[\lambda y,d]^{-1}}_{\text{inverse}} = \underbrace{\Phi[\lambda x,d]\ \Phi[d,\lambda y]}_{\text{concatenate}}$$

$$= \Phi[\lambda x,\lambda y]_\Lambda$$

$\downarrow$ the property of $[\cdot,\cdot]_\Lambda$

$$= \Phi[x,y]$$

$$= \Phi[x,d]\ \Phi[y,d]^{-1}$$

This yields the Claim and so we have $\boxed{1}$:

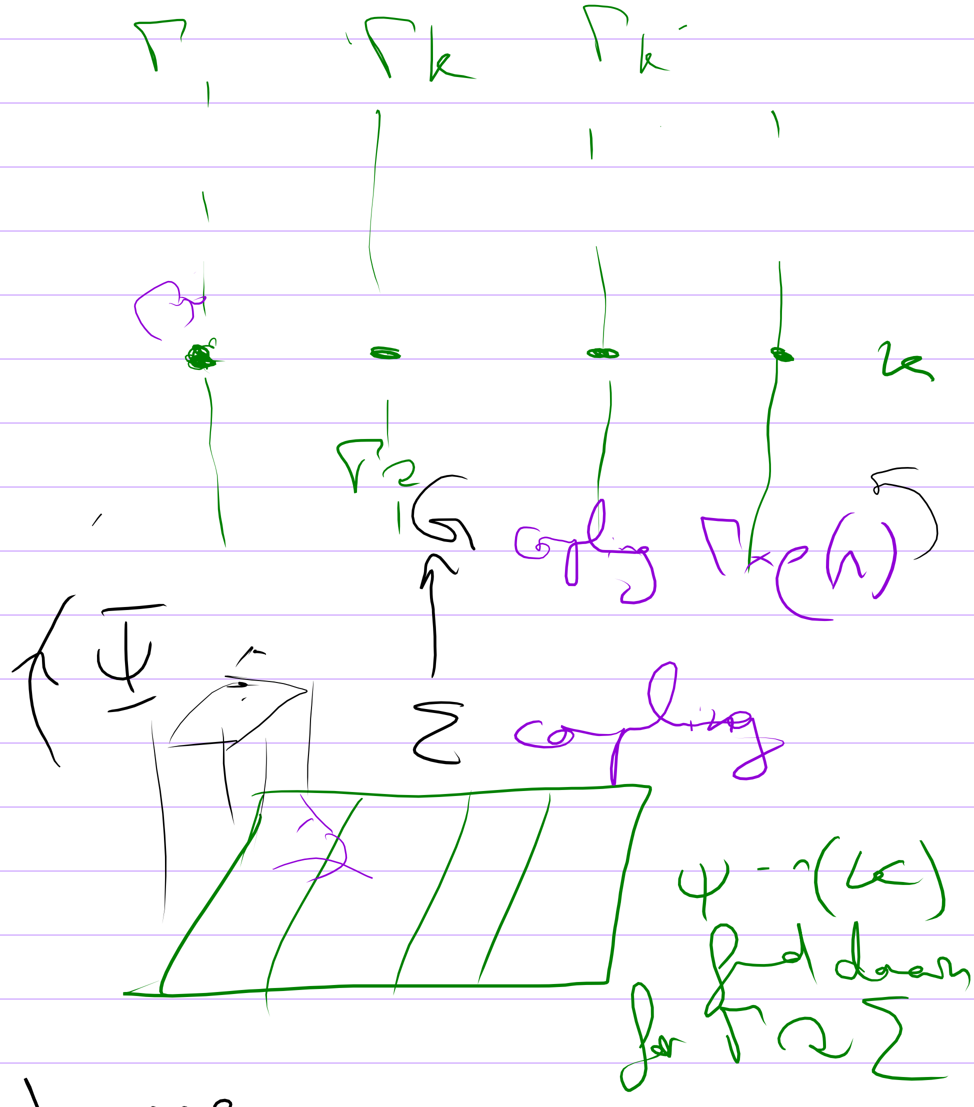
there exists $\begin{cases} \psi : \Sigma \longrightarrow G \\ e : \Lambda \longrightarrow G \end{cases}$

for $\boxed{2}$:

② $G \underset{f.i.}{\geq} \Gamma$ show $\begin{cases} \ker e < \infty \\ \operatorname{im} e \underset{f.i.}{\leq} G \end{cases}$

$\underbrace{}_{}$ K for G/Γ finite

in G



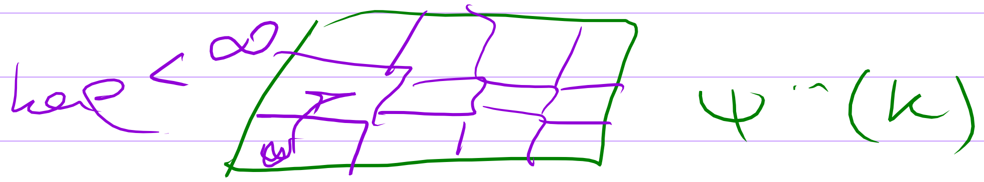
in Σ

coupling we have fund domains

<u>why $\ker e < \infty$</u>? $\Lambda \curvearrowright \Psi^{-1}(k)$

$\ker e$ preserves $\Psi^{-1}(k)$

$U = D_\Lambda \cap \Psi^{-1}(k) \underset{\text{measure}}{>} 0$



th [Bader Furman Sauer] :

th 2.6 "Integrable ME lattices"

th 1

the biggest generality of what we just proved

Γ, Λ ~~discrete~~ lcsc

$\boxed{1}$: G ⟶ Polish

$\Gamma \overset{\&}{\leq} G$

$\boxed{2}$ $\Gamma \xrightarrow{\pi} G$ (compact then closed image)

$\Downarrow$

same for $\rho : \Lambda \to G$

and ρ's such that

$\rho(\Lambda)\backslash G$ has finite measure

$$\mathrm{Isom}(H^n) = G$$