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Boundary representations and Wiener's Tauberian theorem for groups with a Gelfand pair

Joint with Jared White.

1) Notation

G locally compact group with left Haar measure μ.

$\Delta : G \longrightarrow \mathbb{R}_{>0}$ modular function: $\mu(Bg) = \Delta(g)\mu(B)$ $\forall B \subseteq G$ Borel, $\forall g \in G$.

$L^1(G) := \{ f : G \xrightarrow{\text{meas.}} \mathbb{C} : \int_G |f(x)| \, d\mu(x) < \infty \}$ is a Banach $*$-algebra:

$$\begin{cases} \|f\|_{L^1(G)} := \int_G |f(x)| \, d\mu(x) \\ (f * g)(x) := \int_G f(y) g(y^{-1}x) \, d\mu(y) \\ f^*(x) := \Delta(x^{-1}) \overline{f(x^{-1})}. \end{cases}$$

$\widehat{G}$ irreducible unitary reps of G upto equivalence called the unitary dual.

2) Fourier transform

Given $f \in L^1(G)$ and $\pi \in \widehat{G}$, define

$$\hat{f}(\pi) := \int_G f(x) \pi(x) \, d\mu(x) \in \mathcal{B}(\mathcal{H}_\pi).$$

$$\langle \hat{f}(\pi) v_1, v_2 \rangle := \int_G f(x) \langle \pi(x) v_1, v_2 \rangle \, d\mu(x)$$
$(v_1, v_2 \in \mathcal{H}_\pi)$.

Examples: $\widehat{\mathbb{R}} = \{ \chi_y(x) := e^{2\pi i x y} : y \in \mathbb{R} \} \cong \mathbb{R}$

$\widehat{\mathbb{Z}} = \{ \chi_\theta(n) := e^{2\pi i n \theta} : \theta \in [0,1) \} \cong \mathbb{T}$.

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3) Spectral synthesis

Given closed $I \subseteq L^1(G)$, define

$$h(I) := \{\pi \in \hat{G} : \hat{f}(\pi) = 0 \ \forall f \in I\} \qquad \text{(hull)}$$

Given $S \subseteq \hat{G}$ closed, define

$$k(S) := \{f \in L^1(G) : \hat{f}(\pi) = 0 \ \forall \pi \in S\}. \qquad \text{(kernel)}.$$

$k(S) \subseteq L^1(G)$ is the largest closed ideal in $L^1(G)$ with hull S.

$S \subseteq \hat{G}$ is a set of synthesis if $k(S)$ is the only closed ideal in $L^1(G)$ with hull S.

Some known results:

(i) If G is abelian, then every closed subset of $\hat{G}$ is a set of synthesis iff G is compact (Malliavin, 1959)

(ii) The unit sphere in $\mathbb{R}^d$ $(d \geq 3)$ is not a set of synthesis. [Schwartz, 1948]

(iii) If N is a 2-step connected nilpotent Lie group. points in $\hat{N}$ are sets of synthesis

(iv) $\emptyset \subseteq \mathbb{R}^d$ $(d \geq 1)$ is a set of synthesis. [Wiener, 1932]

Definition: A locally compact group G is called Wiener if $\emptyset \subseteq \hat{G}$ is a set of synthesis. Equivalently, for every proper closed $I \subsetneq L^1(G)$, $h(I)$ is non-empty.

Wiener	Not Wiener
Nilpotent	Non-compact semi-simple Lie
CGPG	Poguntke group: $H_3(\mathbb{R}) \rtimes \mathbb{R}$ $t \cdot \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & e^t x & z \\ & 1 & e^{-t} y \\ & & 1 \end{pmatrix}$
$[FC]^-$-groups	
Semi-direct of abe.	
Exp. Lie $\dim_{\mathbb{R}} \leq 3$	$Aut(T)$?
	Reductive p-adic?
Discrete?	Non-amenable?

Every connected locally compact group can be approximated by connected Lie groups. A connected locally compact group is Wiener iff its approximating Lie groups are.

Goal: understand Wiener TDLC groups.

4) TDLC groups.

Examples: (i) $\mathrm{Aut}(X)$, X connected locally finite graph.
Basis at id: $\mathrm{Aut}(X)_F := \{g \in \mathrm{Aut}(X) : gv = v \ \forall v \in F\}$
$F \subseteq VX$ finite

(ii) Linear algebraic groups over $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$.

Remark: every compactly generated tdlc group acts on a regular locally finite connected graph vertex transitively with compact ~~open vertex stab.~~ kernel.

5) New result

Theorem (C. - White, 2026)

(i) Let X be a connected locally finite graph with infinitely many ends. If G is non-compact tdlc acting on X, and the action on the ends is transitive, then G is not Wiener.

(ii) Any split non-abelian reductive group over a non-archimedean local field is not Wiener (e.g. GL_n, SO_n, Sp_{2n} ...).

6) Proof idea

Let $\pi \in \widehat{G}$. The rep π extends to a $*$-rep of $L^1(G)$:

$$\tilde{\pi} : L^1(G) \longrightarrow \mathcal{B}(\mathcal{H}_\pi)$$

defined by $\tilde{\pi}(f) := \hat{f}(\pi)$. The map $\pi \longmapsto \tilde{\pi}$ gives a bijection $\widehat{G} \longrightarrow \widehat{L^1(G)}$ preserving equivalence.

Then, G is Wiener iff for every proper closed ideal $I \subset L^1(G)$, $\exists \pi \in \widehat{G}$ such that $I \subseteq \ker_{L^1(G)}(\tilde{\pi})$.

Lemma: If G is Wiener, then every maximal closed ideal $I \subseteq L^1(G)$ is $*$-closed i.e. $I^* = I$.

Proof: G Wiener, $I \subset L^1(G)$ maximal closed. $\exists \pi \in \widehat{G}$ such that $I = \ker_{L^1(G)}(\tilde{\pi})$. This is $*$-closed since π is a $*$-rep.

Proof idea of theorem: construct maximal closed ideals in $L^1(G)$ which are not $*$-closed using reps of G on $L^p(\partial G)$.

7) Gelfand pairs

G locally compact group, $K \leq G$ compact subgroup. (G, K) is a <u>Gelfand pair</u> if

$$L^1(K \backslash G / K) = \{ f \in L^1(G) : f(k_1 g k_2) = f(g),\ \forall k_1, k_2 \in K,\ \forall g \in G \}$$

is commutative (under convolution).

In which case $L^1(K \backslash G / K)$ is a $*$-subalgebra of $L^1(G)$.

Theorem (Monod, 2020)
Let (G,K) be a Gelfand pair. There exists a maximal cocompact amenable subgroup $P \leq G$ such that $G = KP$ and $\partial G \cong G/P \cong K/K\cap P$.

Examples
(i) $G = \mathrm{Aut}(T)$, $K = \mathrm{Fix}(v)$ $v \in VT$, $P = \mathrm{Fix}(\omega)$ $\omega \in \partial T$.
(ii) $G = GL_n(\mathbb{Q}_p)$, $K = GL_n(\mathbb{Z}_p)$, $P =$ upper triangular $/\mathbb{Q}_p$.

8) Boundary representations.
(G,K) Gelfand pair and $P \leq G$ maximal cocompact amenable subgroup with $G = KP$ and $\partial G \cong G/P \cong K/K\cap P$.

Let μ be the pushforward of the Haar measure on K to G/P. The measure μ is G-quasi-invariant.

For $z \in \mathbb{C}$ and $p \in [1,\infty)$, define

$$\pi_{z,p} : G \longrightarrow \mathcal{B}(L^p(G/P))$$

$$\pi_{z,p}(g) f(xP) := \left(\frac{dg\mu}{d\mu}(xP)\right)^z f(g^{-1}xP)$$

The rep $\pi_{z,p}$ is strongly continuous if G is second countable i.e. $g \longmapsto \pi_{z,p}(g) f$ is continuous for all $f \in L^p(G/P)$.

Lemma: If $0 < \mathrm{Re}(z) < 1$ and $p = 1/\mathrm{Re}(z)$, then $\pi_{z,p}$ is isometric.

Proof: $$\|\pi_{z,p}(g) f\|^p_{L^p(G/P)} = \int_{G/P} \left|\left(\frac{dg\mu}{d\mu}(xP)\right)^z f(g^{-1}xP)\right|^p d\mu(xP)$$

$$= \int_{G/P} \left(\frac{dg\mu}{d\mu}(xP)\right)^{p\,\mathrm{Re}(z)} |f(g^{-1}xP)|^p \, d\mu(xP)$$

$$= \int_{G/P} \frac{dg\mu}{d\mu}(xP)\, |f(g^{-1}xP)|^p \, d\mu(xP)$$

$$= \int_{G/P} \frac{dg\mu}{d\mu}(xP)\, |f(g^{-1}xP)|^p \, \frac{d\mu}{dg\mu}(xP)\, d\mu(g^{-1}xP)$$

$$= \|f\|^p_{L^p(G/P)}. \qquad \blacksquare$$

These representations are also "admissible"

Let $\sigma \in \hat{K}$, $\chi_\sigma(k) := \mathrm{Tr}(\sigma(k))$, $p_\sigma := d_\sigma \overline{\chi_\sigma}$.

Note that $p_\sigma * p_\sigma = p_\sigma \in L^1(K)$.

Let $E = L^p(G/P)$. Then $P_\sigma := \int_K \pi_{z,p}(k) p_\sigma(k)\, dk$ is a projection of E onto a closed subspace $E(\sigma)$ called the σ-isotypic subspace of E.

[i.e. $E(\sigma)$ is the sum of subspaces of $\pi_{z,p}|_K$ which are isomorphic to σ].

Lemma: For arbitrary $z \in \mathbb{C}$ and $p \in [1,\infty)$, $E(\sigma)$ has finite dimension for all $\sigma \in \hat{K}$.

Theorem (Godement, 1952)

G locally compact, $K \leq G$ compact, (π, E) uniformly bounded top irr. rep on Banach space. If $\dim(E(\sigma)) < \infty$ for all $\sigma \in \hat{K}$, then $\ker_{L^1(G)}(\tilde{\pi})$ is maximal amongst closed ideals.

~~Goal now: For $0 < \mathrm{Re}(z) < 1$ and $p = 1/\mathrm{Re}(z)$, find z such that $\ker_{L^1(G)}(\tilde{\pi}_{z,p})$ is not $*$-closed and $\pi_{z,p}$ top. irr.~~

Goal now: Find $z \in \mathbb{C}$ such that $0 < \mathrm{Re}(z) < 1$ and for $p = 1/\mathrm{Re}(z)$, $\pi_{z,p}$ is top. irr. and $\ker_{L^1(G)}(\tilde{\pi}_{z,p})$ is not $*$-closed.

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Theorem (C. - White, 2026)

(G, K) Gelfand pair, G non-amenable. If $\exists z \in \mathbb{C}$ with $0 < \mathrm{Re}(z) < 1/2$ such that $\pi_{z,p}$ ($p = 1/\mathrm{Re}(z)$) is top. irr. and

$$\varphi_z(g) := \langle \pi_{z,2}(g) 1_{G/P}, 1_{G/P} \rangle_{L^2(G/P)}$$

is not positive definite, then G is not Wiener.

Comment on unitarizability

$\int_G \int_G f(x)\overline{f(y)}\varphi(y^{-1}x)\,dx\,dy \geq 0$ $\forall f \in C_c(G)$.

Proof idea: $\varphi_z(g)$ is a bounded spherical function on G i.e. K-bi-invariant and

$$\chi_{\varphi_z}(f) := \int_G f(x)\varphi_z(x)\,dx$$

is a character on $L^1(K \backslash G / K)$.

Since φ_z is not positive definite, $\ker_{L^1(K\backslash G/K)}(\chi_{\varphi_z})$ is not $*$-closed.

Then show that

$$\ker_{L^1(G)}(\tilde{\pi}_{z,p}) \cap L^1(K \backslash G / K) = \ker_{L^1(K\backslash G/K)}(\chi_{\varphi_z}).$$

9) Principal series representations G tdlc.

Let Δ_P be the modular function on P which is non-trivial if G is non-amenable.

For $z \in \mathbb{C}$, consider the smooth rep $\sigma_z := \mathrm{ind}_P^G(\Delta_P^z)$. The rep. space is

$$V_z := \left\{ f : G \to \mathbb{C} : \begin{array}{l} f(gp) = \Delta_P^z(p^{-1}) f(g) \ \forall p \in P \\ \exists U \in \mathrm{COS}(G) \text{ such that } f(ug) = f(g) \\ \forall u \in U, \forall g \in G. \end{array} \right\}.$$

Then σ_z acts on V_z by left translation.

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Restricting functions in V_z to K defines a linear map

$$R_K : V_z \longrightarrow C^\infty(K).$$

whose image consists of functions that are right translation ~~tra~~ invariant by $K \cap P$. In particular, it is a linear isomorphism

$$R_K : V_z \longrightarrow C^\infty(K/K\cap P)$$

Define a norm on V_z by

$$\|f\|_{K,p} := \left(\int_K |R_K(f)(k)|^p \, dk \right)^{1/p} \qquad p \in [1,\infty).$$

Denote the completion of V_z by this norm by $E_{z,p}$. The rep σ_z extends by continuity to a rep $\sigma_{z,p}$ on $E_{z,p}$. Also, R_K extends to $E_{z,p}$ by continuity and gives an iso

$$R_K : E_{z,p} \longrightarrow L^p(K/K\cap P).$$

Let $\varphi : K/K\cap P \longrightarrow G/P$, $k(K\cap P) \longmapsto kP$ denote the homeomorphism. Then

$$U_{z,p} : E_{z,p} \longrightarrow L^p(G/P),\ f \longmapsto R_K(f)\circ\varphi^{-1}$$

is an isometric isomorphism which intertwines $\sigma_{z,p}$ and $\pi_{z,p}$.

Theorem (C.-White, 2026)

(i) $\mathrm{ind}_P^G(\Delta_P^z)$ alg irr. iff $\pi_{z,p}$ top. irr.

(ii) $\mathrm{ind}_P^G(\Delta_P^z)$ unit. iff $\pi_{z,p}$ unit.

(iii) $\pi_{z,p}$ not unit. implies φ_z not positive def.

10) Open problems.

If G is non-amenable, ∂G is non-trivial and you can construct $\pi_{z,p}$ again. Under what assumptions can the reps $\pi_{z,p}$ give maximal closed ideals of $L^1(G)$ which are not $*$-closed?

Does there exist a non-amenable group G such that every maximal closed ideal of $L^1(G)$ is $*$-closed?