Compact Invariant Random Subgroups

j.w.
Tal Cohen
Helge Glöckner
Gil Goffer

Motivation

Γ countable group, $\Gamma \curvearrowright (X,\mu)$ p.m.p., ergodic

Claim: If μ-a.e. $x \in X$ has finite stabilizer, then $\exists$ finite normal subgroup containing a.e. stabilizer.

Proof: Γ has only countably many finite subgroups
$\Rightarrow$ (ergodicity) $\exists$ one conjugacy class $\{g\Delta g^{-1} \mid g \in \Gamma\}$ containing a.e. stab.
$\Rightarrow \forall g: \mu(\{x \in X \mid \mathrm{Stab}(x) = g\Delta g^{-1}\}) = \alpha$
$\Rightarrow \Gamma / N_\Gamma(\Delta)$ finite
$\Rightarrow \langle \mathrm{Stab} \rangle \subseteq \bigcup_{g \in G} \{g\Delta g^{-1} \mid g \in \Gamma\}$ finite union of finite sets

Conclude with 1970's

Thm (Ušakov – Wang): Let G be a loc cpct group. Let $C \subseteq G$ be a subset such that:

1. conj-inv: $gCg^{-1} = C \quad \forall g$
2. $\bar{C}$ compact
3. $\forall g \in C: \overline{\langle g \rangle}$ compact (i.e. g elliptic).

Then, $\overline{\langle C \rangle} \trianglelefteq G$ compact.

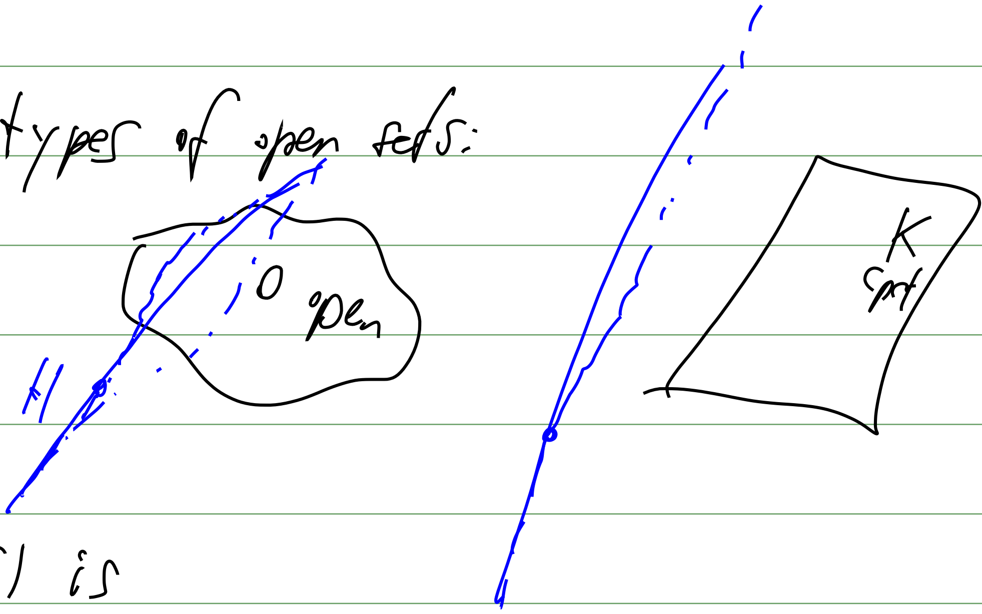
Qu: G loc cpct (second countable) $G \curvearrowright (X,\mu)$ p.m.p., erg.
Assume μ-a.e. stab is compact. Then what?

Use invariant random subgroups (IRS).

Chabauty spaces

Def: Let G be a loc cpct group. The Chabauty space of G is $\mathrm{Sub}(G) := \{H \subseteq G \mid H \text{ closed}\}$.

Topology: 2 types of open sets:



Properties: $\mathrm{Sub}(G)$ is

- compact
- metrizable (if G is second countable)

If G discrete: $\mathrm{Sub}(G) \subseteq \underline{\{0,1\}^G}$

Examples:

- $G \cong \mathbb{R}$: $\mathrm{Sub}(G) = \{x\mathbb{Z} \mid x \geq 0\} \cup \mathbb{R}$ $\cong [0,1]$

  $x_n\mathbb{Z} \to \{0\} \iff x_n \to \infty$

  $x_n\mathbb{Z} \to \mathbb{R} \iff x_n \to 0$

- $G = \mathbb{R}^2$: $\mathrm{Sub}(G) \cong S^4$

  Hubbard-Pizzeria '70's

- $G = \mathbb{R}^n$, $n \geq 3$?

Def: An <u>invariant random subgroup</u> on G (IRS) is a conj.-inv. Borel probability meas on $\mathrm{Sub}(G)$.
$G \curvearrowright \mathrm{Sub}(G)$ via conjugation

<u>Fact</u> (Varadarajan): $G \curvearrowright (X, \mu)$ p.m.p.,
$\mathrm{Stab}: X \to \mathrm{Sub}(G)$, $x \mapsto \mathrm{Stab}(x)$ is defined a.e.
$\mathrm{Stab}_*\mu$ is an IRS

<u>Prop:</u> every IRS arises this way.

<u>Examples:</u> •) δ_H is IRS $\Leftrightarrow$ $H \trianglelefteq G$
•) If $H \leq G$ finite index $\Rightarrow \sum_{g \in G} \frac{1}{[G : N_H(G)]} \delta_{gHg^{-1}}$
•) If $H \leq G$ is <u>cofinite</u>,
i.e. $\exists \nu$ prob meas on G/H s.t. $G \curvearrowright G/H$ is p.m.p.: $G/H \to \mathrm{Sub}(G)$
$gH \mapsto gHg^{-1}$
$\Rightarrow \nu_*$ is an IRS

Ex: $SL_n(\mathbb{Z}) < SL_n(\mathbb{R})$

We are interested in:

Compact IRS

IRS s.t. μ-a.e. $H \in Sub(G)$ is compact.

Example: $\Gamma = (V,E)$ a locally finite, connected graph

s.t. $\underset{= G}{Aut(\Gamma)} \curvearrowright V$ transitively

topology: $Stab(x)$ open $\forall x \in V$, then $Stab(x)$ is cpct, form a subbasis

$Aut(\Gamma) = \{ g: V \to V$ bij. $\mid x \circ\!\!-\!\!\circ y \iff gx \circ\!\!-\!\!\circ gy \}$

Consider $\{0,1\}^V$, measure: μ_p $0 < p < 1$

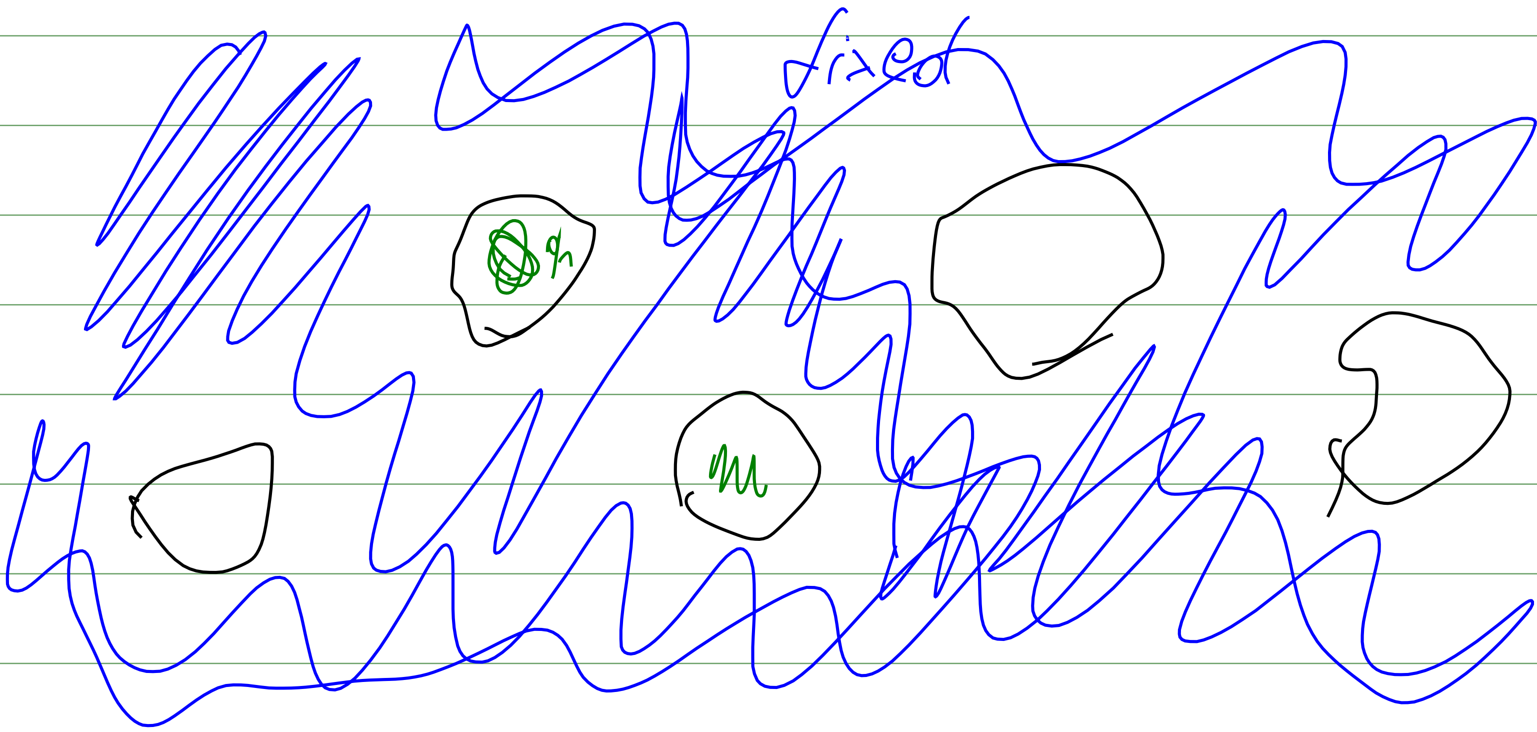
$$\nu_p = \bigotimes_{v \in V} p \cdot \delta_0 + (1-p) \cdot \delta_1$$

vertex x is fixed w. prob. p

$G \curvearrowright \{0,1\}^V \longrightarrow G \curvearrowright Sub(G)$, $A \mapsto pStab(A) = \{ g \mid \forall x \in A: gx = x \}$

$\mu_p = (\nu_p)_*$ is a cpct IRS

Assume p is big: $p > 1 - p_c$ critical vertex percolation



μ-a.e. $H \leq \underbrace{\{g \in G \mid g \text{ has finite support}\}}_{\text{compact (\& normal) (Trofimov)}}$

$g \in H$

$g = \lim g_n$

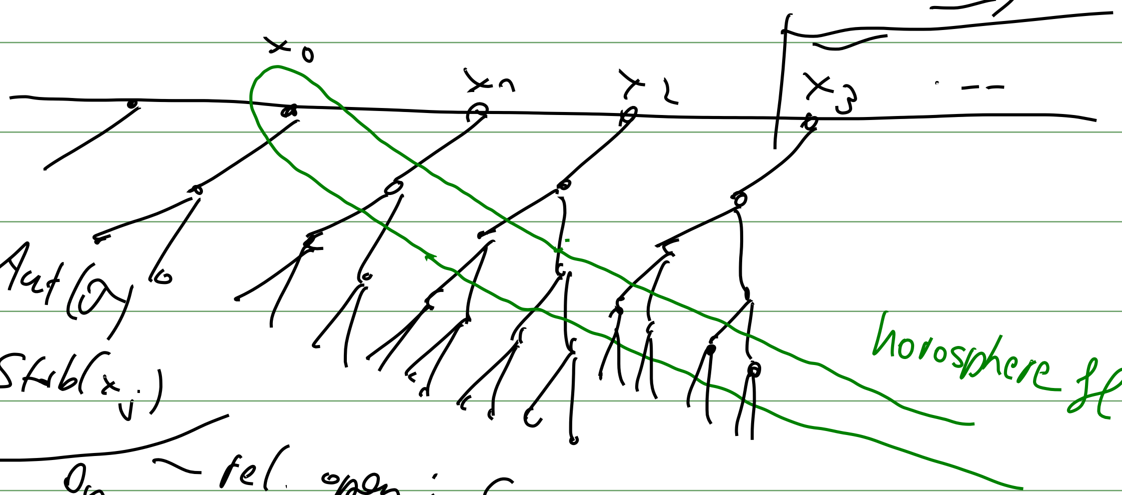
Conclusion: For big p,

$\exists\, N \trianglelefteq G$ cpct normal s.t.

μ-a.e. $H \leq N$.

Small p? Something weaker...

Example:



$G = \mathrm{Stab}(\mathcal{H}) \leq \mathrm{Aut}(T)$

$= \bigcup_i \underbrace{\bigcap_{j \geq i} \mathrm{Stab}(x_j)}_{O_n}$ ~ rel. open in G

Choose $A \subseteq \mathcal{H}$ unif at random w. prob p

$\{0,1\}^{\mathcal{H}} \to \mathrm{Sub}(G)$ ergodic IRS!

$\forall n \in \mathbb{N}$: $\mu(H \mid H \nsubseteq O_n) > 0$

$\Rightarrow \mu \not\leq$ cpct, normal subgroup

G $\boxed{\text{NOT}}$ cpctly gen.

Some structure theory

G lc group, some nice subgroups:

1. The <u>polycompact radical</u> of G is
$$W(G) := \bigcup_{\substack{N \trianglelefteq G \\ N \text{ cpct}}} N \quad \text{subgroup,}$$
not closed in general,
but (Trofimov): If G cpctly gen, then $W(G)$ is closed

2. The <u>locally elliptic radical</u> of G is
$$E(G) := \bigcup_{\substack{N \trianglelefteq G \\ N \text{ loc. ell.},\ N \text{ closed}}} N$$
$E(G) \geq W(G)$

locally elliptic: $\forall K \subseteq N$ cpct: $\overline{\langle K \rangle}$ is compact

Platonov, $E(G)$ is closed & locally elliptic

<u>Q (Caprace)</u>: μ cpct IRS $\overset{?}{\Longrightarrow}$ μ-a.e. $H \leq E(G)$?

We still don't know...

3. The <u>amenable radical</u> of G is
$$A(G) = \bigcup_{\substack{N \trianglelefteq G \text{ closed} \\ N \text{ amenable}}} N \qquad \text{closed, amenable}$$
$A(G) \geq E(G) \geq W(G)$

<u>Thm</u> (Bader–Duchesne–Lécureux):
μ amenable IRS $\Longrightarrow$ μ-a.e. $H \leq A(G)$

G totally disconnected, locally compact (tdlc).

Take $g \in G$. The <u>Levi subgroup</u> is

$lev(g) := \{h \in G \mid \{g^n h g^{-n} \mid n \in \mathbb{Z}\}$ has cpct closure$\}$

<u>closed</u>

$= \{h \in G \mid \{g^n h g^{-n} \mid n \in \mathbb{N}\}$ and $\{g^{-n} h g^n \mid n \in \mathbb{N}\}$ have accumulation points$\}$

Willis

The <u>approximate center</u> is $AC(G) := \bigcap_g lev(g)$

def by Willis

<u>Surprise thm:</u> G tdlc, μ cpct IRS.

Then, μ-a.e. $H \leq AC(G)$.

<u>Proof:</u> Define <u>ν</u> measure on G:

First choose a μ-random $H \leq G$

H cpct $\Rightarrow \exists\, m_H$ Haar prob. meas on H

Choose a m_H-random $h \in H$.

ν is conj.-inv. prob. meas on G, $\nu(\{g \mid g \text{ elliptic}\}) = 1$.

$G \curvearrowright (G, \nu)$ conj. p.m.p.,

look at $g \curvearrowright (G, \nu)$

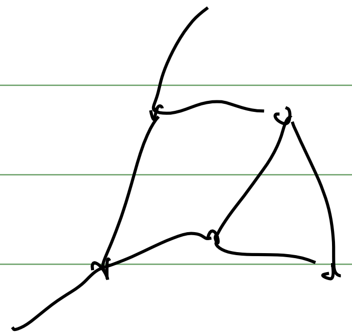
Poincaré recurrence $\Rightarrow \nu(\{h \mid \{g^n h g^{-n} \mid n \in \mathbb{N}\}$ has acc. pt$\}) = 1$

$\Rightarrow \nu(lev(g)) = 1$, i.e. $supp(\nu) \leq lev(g)$

$\Rightarrow \nu(AC(G)) = 1$

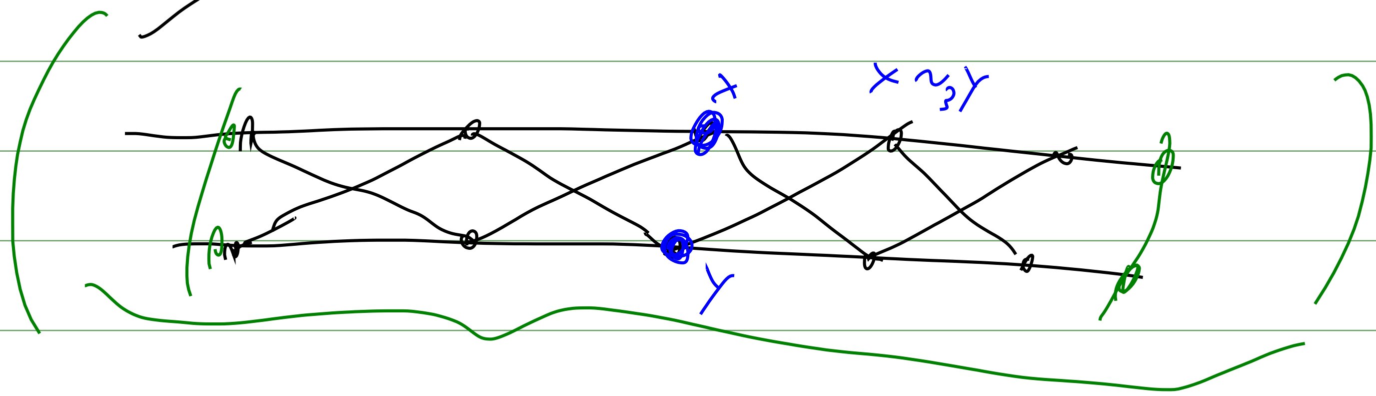
Back to the graph example...

$x \in V$ fixed w. prob $p < 1$



Def: $x \sim_n y$ if

$\forall m \geq n: \ B(x,m) = B(y,m)$



$\sim = \bigcup^{\nearrow} \sim_n$

Q: Find Γ s.t. $\sim$-classes are infinite.

Assume $x \not\sim y$ $\iff$ $\exists$ ∞-many z_i s.t. $d(x,z_i) \neq d(y,z_i)$

$\Rightarrow$ ν_p-a.e. subset contains some z_i

$\Rightarrow$ μ_p-a.e. subset: $Hx \neq Hy$

$\Rightarrow$ μ-a.e. $H \leq \bigcap_{x \in V} Stab([x])$

$= \bigcup^{\nearrow} Stab([x]_n)$ loc. ell. normal cpct

$gx \in [x]$

$\iff \exists n: gx \sim_n x \Rightarrow g[x]_n = [x]_n$

$\Rightarrow$ μ-a.e. $H \leq E(G)$.

$x \sim_n y$

$\Updownarrow$

$gx \sim_n gy$

Results for Lie groups

Thm G s.c. real Lie group, μ ergodic, cpct IRS $\Longrightarrow$ $\exists N \trianglelefteq G$ cpct, normal s.t. μ-a.e. $H \leq N$.

Proof idea: $\exists$ only countably many conj. classes of cpct subgroups

A amenable Lie group, $H \leq A$ cofinite $\Rightarrow$ A/H cpct

$\boxed{\#}$

Thm: G p-adic Lie group, μ ergodic, cpct IRS. If G is cptly gen, or algebraic, then $\exists$ cpct, normal N s.t. μ-a.e. $H \leq N$.

In general, $\mu \leq E(G)$.