Poisson boundaries in ergodic theory

I Definitions and cool properties

A Random walks

Setup G : discrete group (say fg), and $\mu \in \mathrm{Prob}(G)$ which is admissible : $\bigcup_{n \geq 0} (\mathrm{supp}\, \mu)^n = G$

$\mu \rightsquigarrow$ right random walk on G, defined by

$$Z_0 = e$$

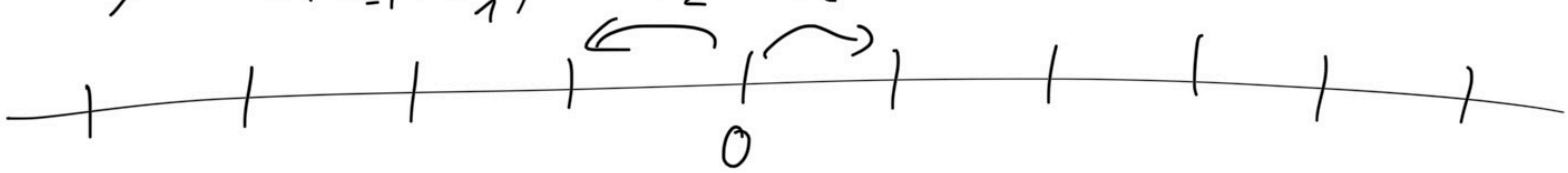
$$Z_n^{-1} Z_{n+1} \sim \mu \quad (\text{iid})$$

If we have variables $(g_i)_{i \in \mathbb{N}}$ iid of law μ,

$$Z_n = g_1 g_2 \cdots g_n$$

$$Z_{n+1} = Z_n g_{n+1}$$

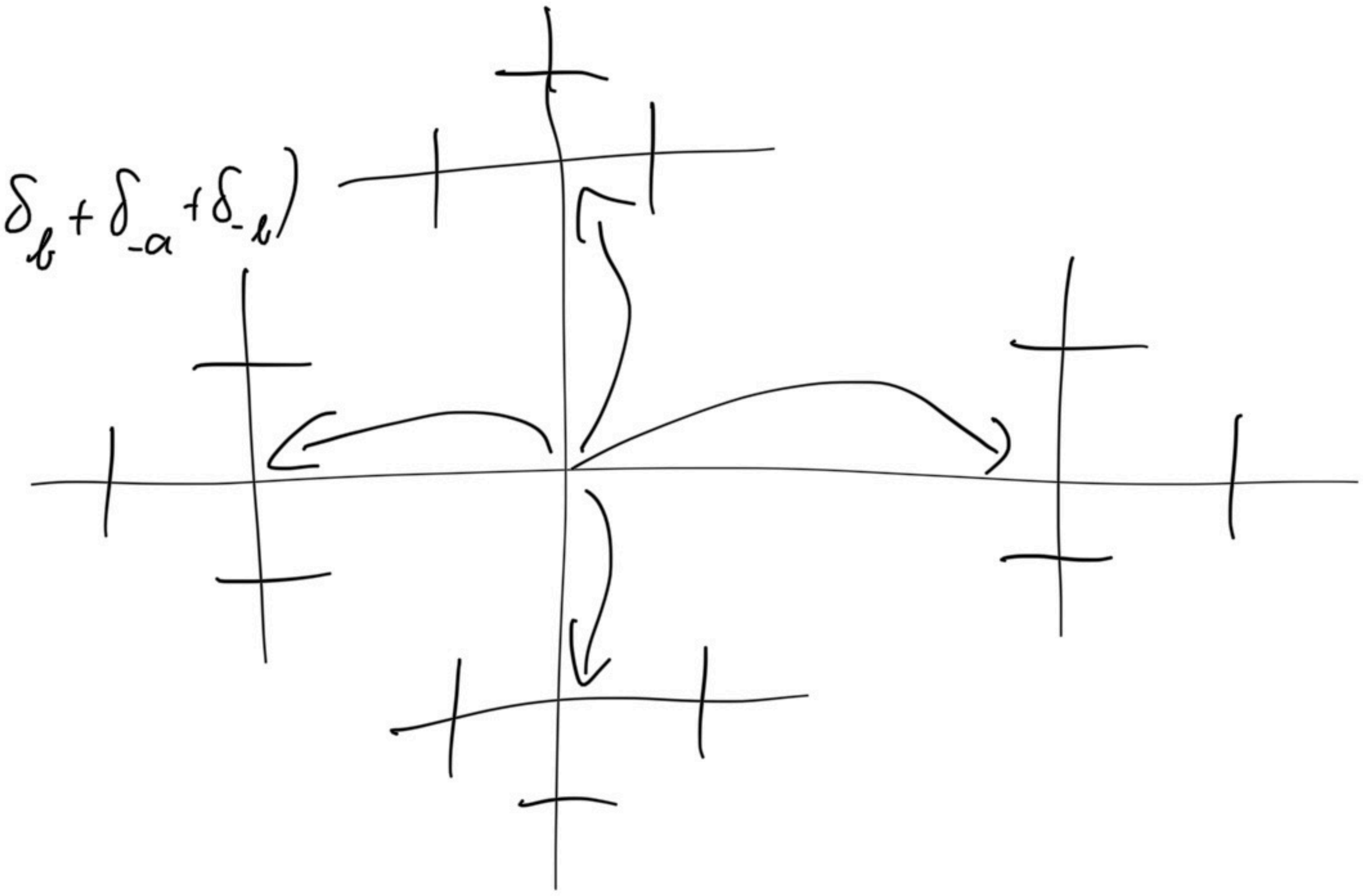
$G = \mathbb{Z}$ $\mu = 1/2(\delta_{-1} + \delta_1)$



$$Z_n = \sum_{i=1}^{n} X_i \quad , \quad X_i \sim \mu$$

$G = F_2$

$\mu = \frac{1}{4}(\delta_a + \delta_b + \delta_{-a} + \delta_{-b})$



$$\mathbb{E}\left(d(e, Z_{n+1}) - d(e, Z_n)\right) \geq \frac{1}{2}$$

" $Z_n \longrightarrow \partial F_2$ "

More generally, a nice RW is the simple RW on a Cayley graph of G (uniform probability among the neighbours).

Def Canonical state space

$$(G^{\mathbb{N}}, \mathbb{P})$$

$$\mathbb{P}(g_0, g_1, \dots, g_n, G^{\mathbb{N}_{>n}}) = \mathbb{1}_e(g_0) \prod_{i=0}^{n-1} \mu(g_i^{-1} g_{i+1})$$

$Z_n : G^{\mathbb{N}} \longrightarrow G$ is the n-th projection

(The law of Z_1 is μ, the law of Z_2 is $\mu^{*2}(g) = \int_G \mu(h^{-1}g)\, d\mu(h)$

Action of G on $G^{\mathbb{N}}$: $g \cdot (g_0, g_1, \dots) = (g g_0, g g_1, \dots)$

Let $g_* \mathbb{P} := \mathbb{P}_g$

$$\mathbb{P}_\nu := \int_G \mathbb{P}_g \, d\nu(g) \quad \text{for } \nu \in \mathrm{Prob}(G)$$

(Exercise: what is $\mathbb{P}_\mu$?)

$(X, \mathcal{E})$ Borel measure space, $x \mapsto \underbrace{\mu_x}_{\simeq x_* \mu} \in \mathrm{Prob}(X)$

B) Harmonic functions & the Poisson boundary

<u>Def</u> $f : G \to \mathbb{R}$ is (μ-)<u>harmonic</u> if $\forall g \in G$,

$$f(g) = \int_G f(gh) \, d\mu(h)$$ ie f has the mean value prop.

$$\left(\mathbb{E}_g(f(Z_0)) = \mathbb{E}_g(f(Z_1)) \right)$$

and in fact $f(Z_n) = \mathbb{E}(f(Z_{n+1}) \mid Z_n)$

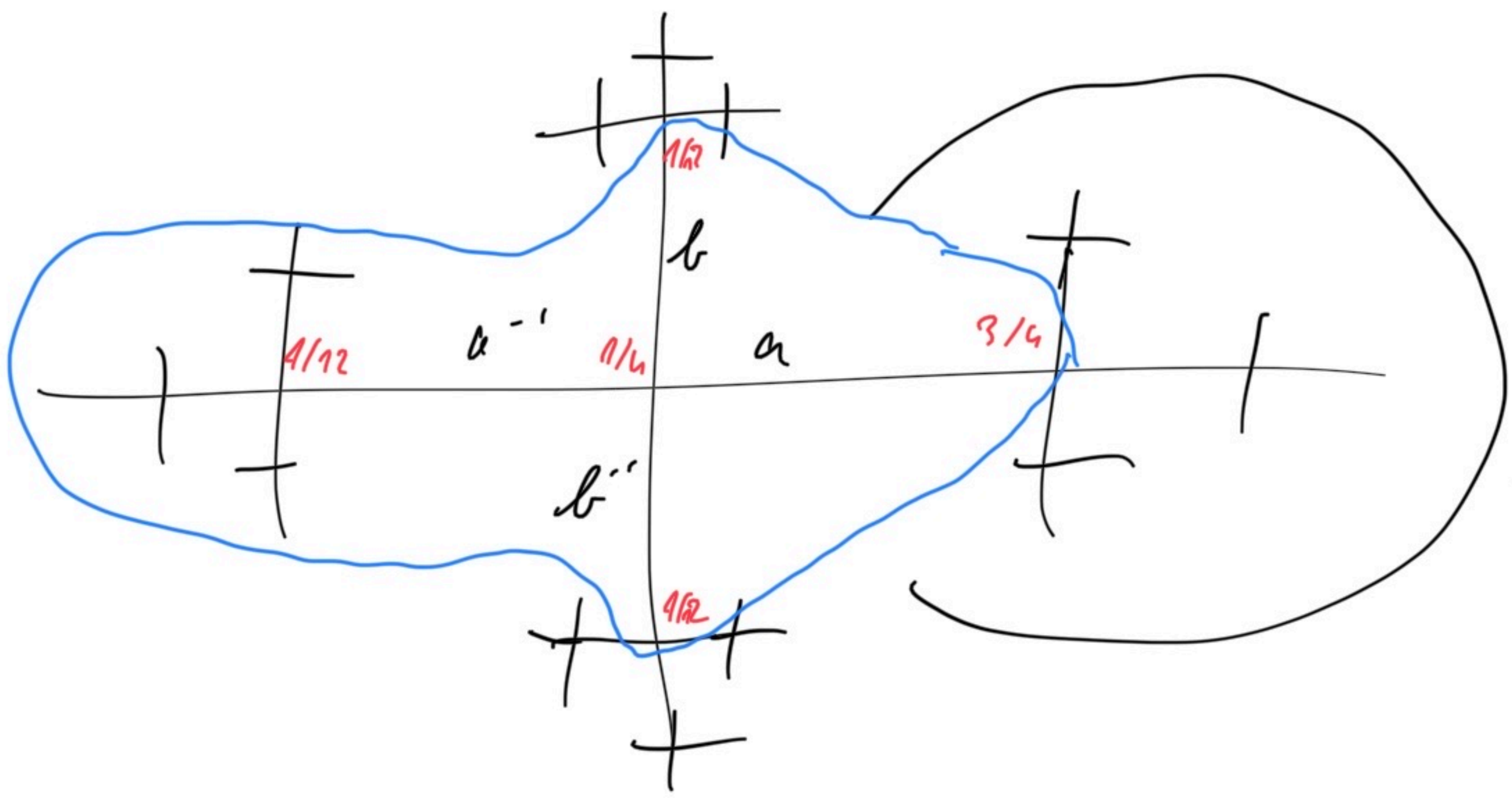
so $f(Z_n)$ is a martingale

Ex. $G = \mathbb{Z}$. f is harmonic iff $f(x) = \frac{1}{2}\big(f(x-1) + f(x+1)\big)$ $\forall x$

$$f(x) - f(x-1) = f(x+1) - f(x)$$

So $f(x) = ax + b$, where $a = f(x) - f(x-1)$ for any x

Every positive harmonic function is constant

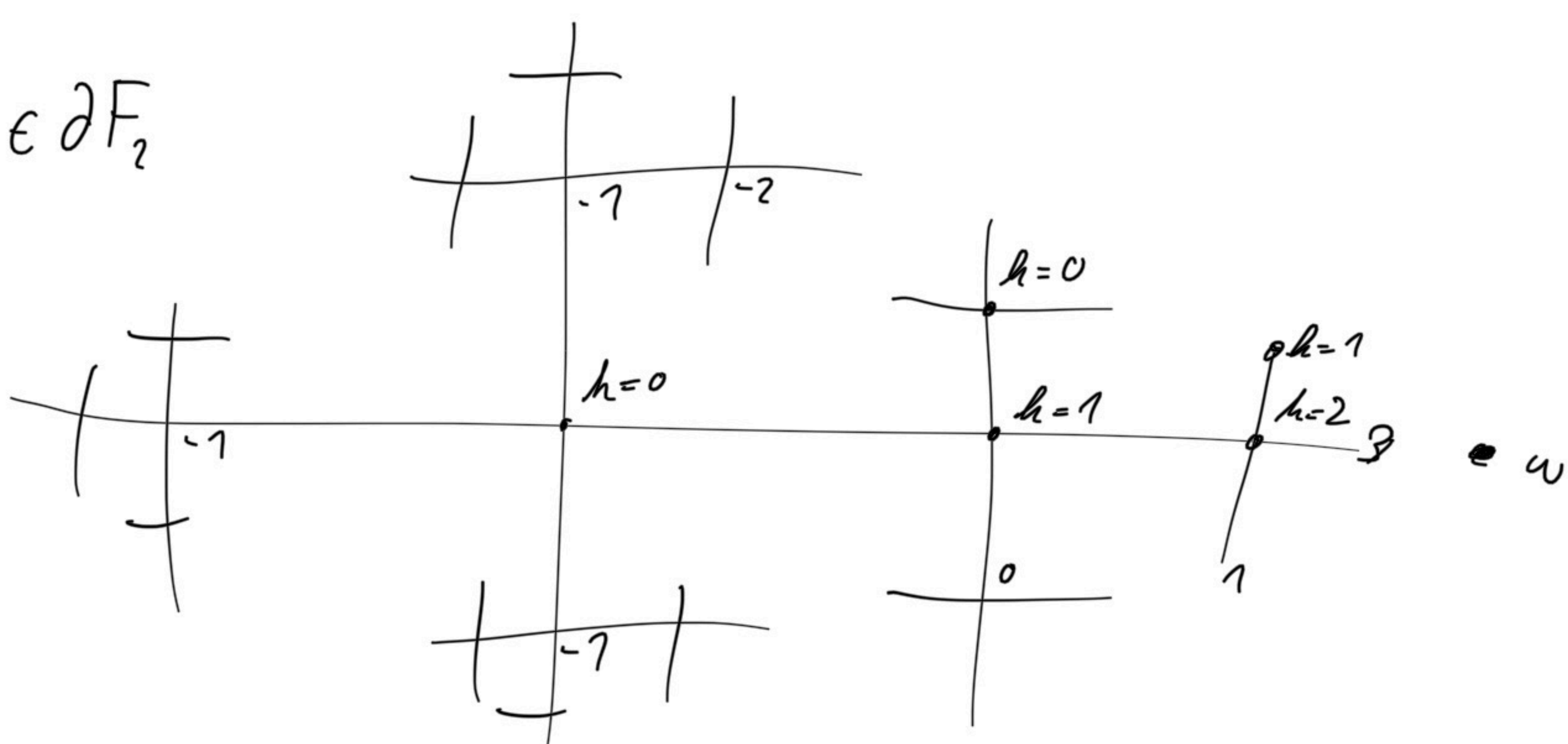


$A \subset \partial F_2$

$x \mapsto \mathbb{P}_x(Z_\infty \in A)$

$$\frac{1}{4}\left(\frac{3}{4} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12}\right) = \frac{1}{4}$$

Fix $\omega \in \partial F_2$



$x \mapsto 3^{h(x,\omega)}$ is

Let $T : G^{\mathbb{N}} \longrightarrow G^{\mathbb{N}}$ be the time shift, ie

$$T(g_0, g_1, g_2, \dots) = (g_1, g_2, \dots)$$

$$\mathbb{P}_\mu = T_\infty \mathbb{P} \quad , \quad T_\infty \mathbb{P}_\nu = \mathbb{P}_{\mu * \nu}$$

The Poisson boundary of (G, μ) is the space of ergodic components of T on $(G^{\mathbb{N}}, \mathbb{P})$

Let $L^\infty(G^{\mathbb{N}}, \mathbb{P})^T$ be the space of T-invariant maps.

Let $\pi : (G^{\mathbb{N}}, \mathbb{P}) \twoheadrightarrow (\Omega, \nu)$ be a projection such that

$$\pi^* L^\infty(\Omega, \nu) = L^\infty(G^{\mathbb{N}}, \mathbb{P})^T$$

Let $B_{PF}(G, \mu) := (\Omega, \nu)$

Note: $G \curvearrowright B_{PF}(G, \mu)$ in a measure-class preserving way.

The Poisson boundary parametrizes bounded harmonic functions

Let $h : G \to \mathbb{R}$ be bounded harmonic. Then $h(Z_n)$ is a martingale

for $\mathbb{P}$-a.e. $(x_n) \in G^{\mathbb{N}}$, the sequence $h(x_n)_n$ converges to some $\bar{h}(x)$

$$\bar{h} \in L^\infty(G^{\mathbb{N}}, \mathbb{P})^T = L^\infty(B_{PF}(G, \mu))$$

Now if $f \in L^\infty(B_{PF}(G,\mu))$,

$$h_f : g \longmapsto \int_\Omega f \, d g_* \nu \text{ is bounded harmonic}$$

These are inverse of each other

Ex : on $\mathbb{Z}$, $B_{PF}(\mathbb{Z},\mu) = \{*\}$

on F_2, $B_{PF}(F_2,\mu) = (\partial F_2, \nu)$

C- Properties of the Poisson boundary

If G is f.g. $\forall \mu \in \mathrm{Prob}(G)$, $B_{PF}(G,\mu) = \{*\} \Leftrightarrow G$ has poly. growth

(Choquet-Deny?)

$\exists \mu \in \mathrm{Prob}(G)$, $B_{PF}(G,\mu) = \{*\} \Leftrightarrow G$ is amenable

(Rosenblatt, Kaimanovich-Vershik)

Prop (Zimmer) $G \curvearrowright B_{PF}(G,\mu)$ is amenable

(it implies that for any $G \curvearrowright K$ compact metric,

there is a G-equivariant map $B_{PF}(G,\mu) \longrightarrow \mathrm{Prob}(K)$

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Proof. Let $C = \mathrm{Prob}(K)$, which is a convex compact G-space

Take $\varphi : \{e\} \times G^{\mathbb{N}^*} \to C$ any map, and extend φ by

$\varphi(g, g_i) = g\, \varphi(e, g^{-1} g_i)$ so that φ is G-equivariant.

Hence $\mathrm{Map}_G(G^{\mathbb{N}}, C) \neq \emptyset$, and $T \curvearrowright \mathrm{Map}_G(G^{\mathbb{N}}, C)$ by precomposition,

so $\mathrm{Map}_G(G^{\mathbb{N}}, C)^T \neq \emptyset$ by amenability of $\mathbb{N}$.

By definition of $B_{PF}(G,\mu)$, $\mathrm{Map}_G(B_{PF}(G,\mu), C) \neq \emptyset$

Prop $G \curvearrowright \underbrace{B_{PF}(G, \underbrace{\check{\mu}}_{\text{inverse } \mu})}_{\check{B}} \times \underbrace{B_{PF}(G,\mu)}_{B}$ is ergodic

Consider U the shift on $(G^{\mathbb{N}}, \mu^{\otimes \mathbb{N}})$

$$g_0, g_1, \dots \longmapsto e, g_0, g_0 g_1, g_0 g_1 g_2, \dots$$

Let $\tilde{U}$ the conjugate of U to $(G^{\mathbb{N}}, P)$.

$$\tilde{U} : (x_n)_n \longmapsto x_1^{-1} (x_{n+1})_n = x_1^{-1} T(x_n)$$

$$x_3 = g_1 g_2 g_3 \dashrightarrow g_2 g_3 = g_1^{-1} x_3 = x_1^{-1} x_3$$

$$(G^{\mathbb{Z}}, \mu^{\otimes \mathbb{Z}}) \simeq (G^{\mathbb{Z}_-}, \mu^{\mathbb{N}}) \times (G^{\mathbb{N}}, \mu^{\otimes \mathbb{N}})$$

$$x_n^{-1} x_{n+1} \sim \mu \iff x_{n+1}^{-1} x_n \sim \check{\mu}$$

$$G^{\mathbb{Z}}, \overline{\mathbb{P}} \simeq (G^{\mathbb{Z}_-}, \mu_* \mathbb{P}) \times (G^{\mathbb{N}}, \mathbb{P}) = (G^{\mathbb{N}}, \check{\mathbb{P}}) \times (G^{\mathbb{N}}, \mathbb{P})$$

$$\pi : (G^{\mathbb{Z}}, \overline{\mathbb{P}}) \twoheadrightarrow \underbrace{B_{PF}(G, \check{\mu})}_{(\check{B}, \check{\nu})} \times \underset{(B, \nu)}{B_{PF}(G, \mu)}$$

$$\tilde{\overline{U}} : G^{\mathbb{Z}} \to G^{\mathbb{Z}}$$
$$(x_n) \mapsto x_1^{-1} \overline{T}(x_n)$$

$$\overline{T}(x_n) = x_1 \overline{U}(x_n)$$
$$\pi(\tilde{U}(x_n)) = x_1^{-1} \pi((x_n))$$

Any G-invariant subset of $\check{B} \times B$ has a preimage by π which is $\tilde{\overline{U}}$-invariant, ergodicity follows from ergodicity of $\overline{U} \curvearrowright (G^{\mathbb{Z}}, \mu^{\otimes \mathbb{Z}})$.

Thm (slightly stronger ergodicity) (Bader-Furman)

For any $G \curvearrowright M$ metric, by isometries, separable, any G-map

$$\varphi : \check{B} \times B \longrightarrow M \quad \text{is essentially constant}$$

stronger than ergodicity ($G \curvearrowright \{0,1\}$ trivially), but it is weaker than ergodicity of $(\check{B} \times B)^2$ (nice exercise)

II Stationarity, entropy

A) (G, μ) spaces and boundaries

Def Let (X, ν) be a standard probability space, $G \curvearrowright X$.

ν is said to be μ-stationary if $\mu * \nu = \nu$

where $\mu * \nu = \int_G g_* \nu \, d\mu(g)$

(X, ν) is then said to be a (G, μ)-space.

If x is a random point of law ν, $Z_1 \cdot x$ has the same distribution

Props (i) If $G \curvearrowright (X, \nu)$ is pmp, it is a (G, μ)-space.

(ii) μ stationarity implies quasi invariance (admissibility)

(iii) The map $L^\infty(X, \nu) \longrightarrow L^\infty(G)$

$$f \longmapsto \left(h_f : g \mapsto \int_X f \, d g_* \nu\right)$$

has values in $H^\infty(G, \mu)$ (harmonic maps)

A common source is the following: if X is compact and $G \curvearrowright X$ by homeos there is a μ-stationary measure.

$$\phi : \underset{\text{Prob } X \circlearrowleft}{\nu \mapsto \mu * \nu} \quad \text{and amenability of } \mathbb{N}.$$

Prop If (X, ν) is a compact (G, μ) space, the limit

$$\nu_x := \lim_{n \to \infty} (x_n)_* \nu \text{ exists } \mathbb{P}\text{-a.s.}$$

This is T-invariant, so we have a map $B_{PF}(G, \mu) \to \text{Prob}(X)$
$$b \longmapsto \nu_b$$

Def-Prop TFAE: compact (X, ν) (G, μ) space

(i) ν_x is a Dirac mass $\mathbb{P}$-a.e.

(ii) The map $\Phi : L^\infty(X, \nu) \to L^\infty(B_{PF}(G, \mu))$
$$f \longmapsto \left(b \mapsto \int_X f \, d\nu_b\right) = \bar{h}_f$$
is an algebra morphism.

(iii) Φ is injective

If any of these hold, (X, ν) is called a (G, μ) boundary

Thm (Furstenberg) There exists a (unique) maximal (G, μ) boundary; all other boundaries are quotients of it