Poisson boundaries for the ergodic theorist II: [insert funny idea here]

G : group, discrete, $\mu \in \mathrm{Prob}(G)$ st $\bigcup_{n\geq 0} \operatorname{supp} \mu^{*n} = G$

If $G \curvearrowright (X, \nu)$ standard, (X, ν) is a (G, μ)-space if
$\nu = \mu * \nu = \int_G g_* \nu \, d\mu(g)$ (ν is μ-stationary)

$$Z_n(\omega)_* \nu \longrightarrow \nu_\omega \text{ for } \mathbb{P}\text{-ae } \omega.$$

If ν_ω is a point mass $\mathbb{P}$-a.e., we say that (X, ν) is (G, μ) boundary

Thm (Furstenberg) Every (G, μ) boundary is a quotient of the Poisson boundary $B_{PF}(G, \mu)$.

B - Entropy (es)

Def Let (X, ν) be a (G, μ) space. The Furstenberg entropy of (X, ν) is

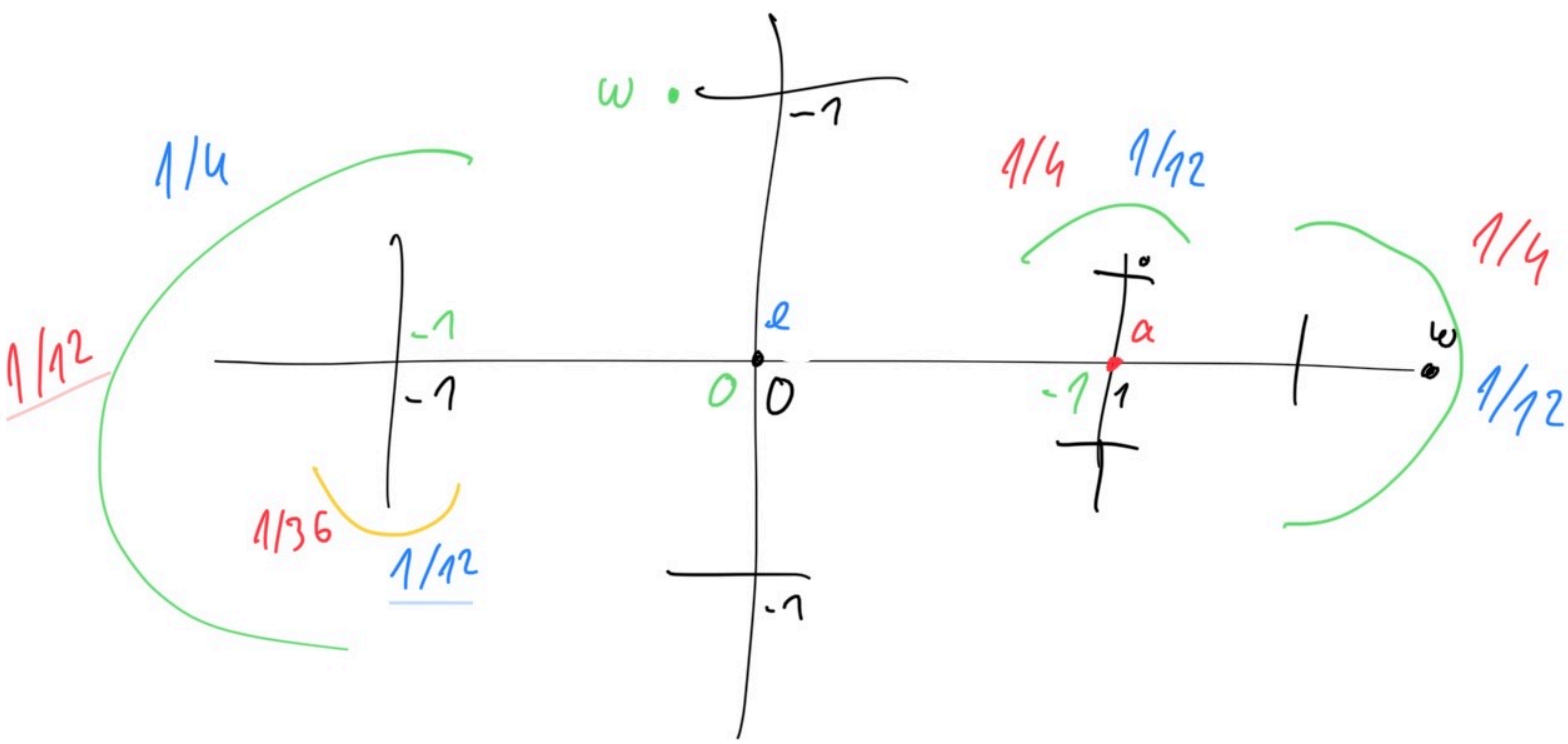
$$h_\mu(X, \nu) = \int_G \int_X -\log \frac{dg^{-1}\nu}{d\nu}(x) \, d\nu(x) \, d\mu(g)$$

$h_\mu(X,\nu) \geq 0$. The equality case is achieved exactly when $G \curvearrowright (X,\nu)$ is pmp.

Exercise: $h_{\mu^{*n}}(X,\nu) = n\, h_\mu(X,\nu)$ $\forall n \in \mathbb{N}$.

For a factor of (G,μ) spaces
$$\begin{array}{c}(X,\nu)\\ \downarrow \\ (X_0,\nu_0)\end{array}, \quad h_\mu(X,\nu) \geq h_\mu(X_0,\nu_0)$$

equality is achieved exactly when the factor map is relatively measure preserving (ie $g\nu_x = \nu_{gx}$, $(\nu_x)_x$ are the disintegration measures).



$$\frac{d g_* \nu}{d\nu}(\omega) = 3^{h(e,g;\omega)}$$

$$\int_{\partial F_2} -\log \frac{d(a^{-1})^{-1}_* \nu}{d\nu}(\omega)\, d\omega = \log 3 \underbrace{\int_{\partial F_2} -h(e,a;\omega)\, d\nu(\omega)}$$

-1 with measure $1/4$

1 with measure $3/4$

$$= \frac{1}{2}\log 3$$

$$h_\mu(X,\nu) = \frac{1}{2}\log 3$$

Def Let $\eta \in \mathrm{Prob}(G)$. The (Shannon) entropy of η is $H(\eta) := \sum_{g\in G} -\log \eta(g)\, \eta(g) \geq 0$

Exercise $H(\eta_1 * \eta_2) \leq H(\eta_1) + H(\eta_2)$

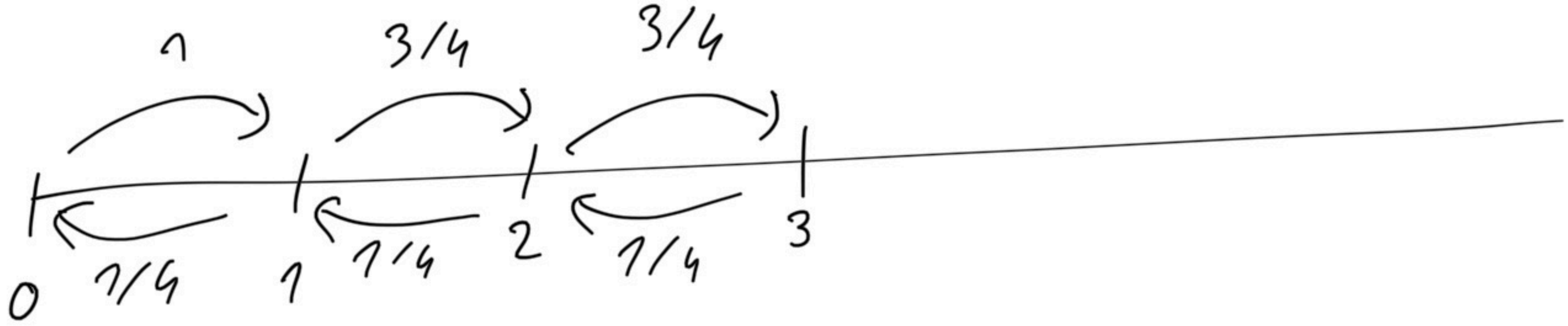
$$G\times G \to G$$

The sequence $H(\mu^{*n})$ is subadditive, so

$\frac{1}{n} H(\mu^{*n}) \longrightarrow h(G,\mu)$, the random walk entropy of (G,μ) (Avez)

Ex. $h(F_2,\mu)$ Let $S_k = \{g\in F_2,\ d(e,g)=k\}$, and

$p_n(k) = \mathbb{P}_e(Z_n \in S_k)$



$$\sum_{g\in F_2} -\log \mu^{*n}(g)\,\mu^{*n}(g) = \sum_{k\in\mathbb{N}}\ \sum_{g\in S_k} -\log\left(\frac{p_n(k)}{4\times 3^{k-1}}\right)\frac{p_n(k)}{4\times 3^{k-1}}$$

$$= \sum_{k\in\mathbb{N}} -\log\left(\frac{p_n(k)}{4\times 3^{k-1}}\right) p_n(k)$$

$$\frac{1}{n} H(\mu^{*n}) = \underbrace{\frac{1}{n} H(p_n)}_{\to 0} + \underbrace{\frac{1}{n}\log\left(\frac{4}{3}\right)}_{\to 0} + \frac{1}{n}\log 3 \underbrace{\sum_{k\in\mathbb{N}} k\, p_n(k)}_{\substack{\mathbb{E}_e(d(e,Z_n)) \\ \text{"drift"}}}$$

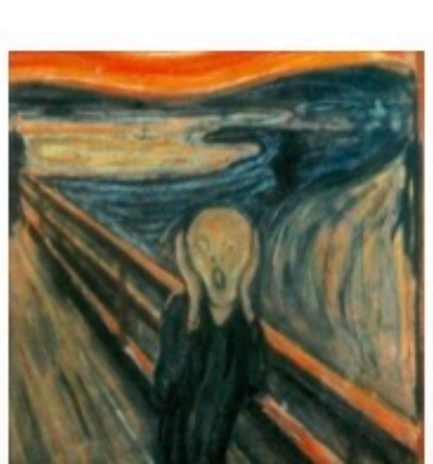
$$\frac{1}{n}\mathbb{E}_e(d(e,Z_n)) \longrightarrow \frac{1}{2}$$

$$\frac{1}{n} H(\mu^{*n}) \longrightarrow \frac{1}{2}\log(3)$$

Rem 1) Geometric quantities appeared in our computation

drift, growth (we always have entropy $\geq$ drift $\times$ growth)

2) Holy shit $h(G,\mu) = h_\mu(\partial F_2, \nu)$ $\left(= \frac{1}{2}\log 3\right)$



Thm Kaimanovich - Vershik '83 Let $\mu \in \mathrm{Prob}(G)$ be admissible and of finite entropy.

a) For any (G,μ) space (X,ν), $h_\mu(X,\nu) \leq h(G,\mu)$

b) If (X,ν) is a boundary, equality is achieved iff (X,ν) is the Poisson boundary of (G,μ).

c) The Poisson boundary is trivial iff $h(G,\mu)=0$

This theorem is veeeery useful when trying to prove that a candidate boundary is the Poisson boundary

Thm (Kaimanovich '00) If G is hyperbolic and μ has finite entropy and finite first log moment $\left(\mathbb{E}_e(\log(d(e,Z_1))) < \infty\right)$ then there is a unique μ-stationary measure ν on ∂G, and $(\partial G, \nu)$ is the Poisson boundary of (G,μ)

(The Poisson boundary: the identification problem)

III Rigidity using Poisson boundaries

A - Actions on trees

Assume that μ is symmetric

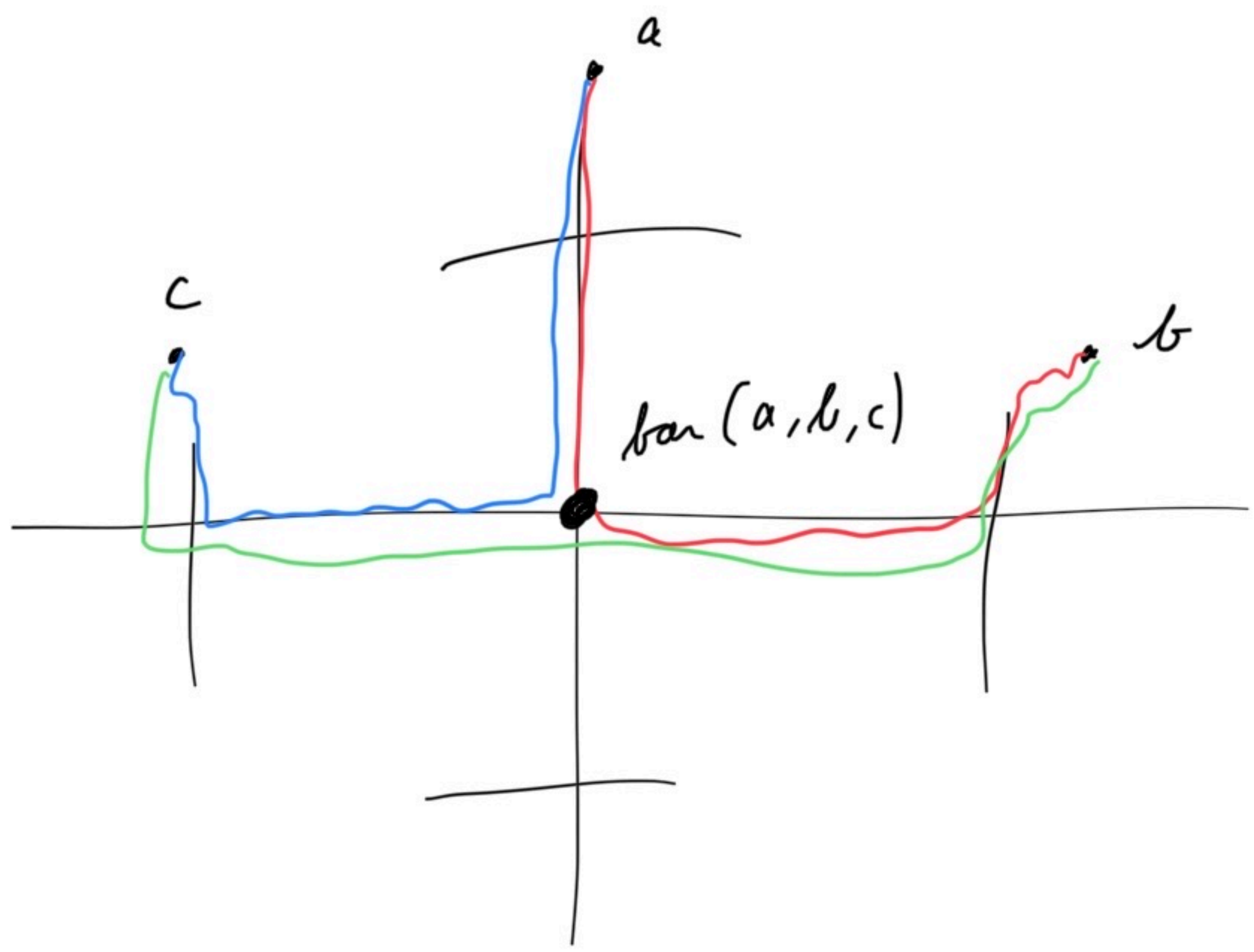
Theorem (à la Bader - Furman)

Given a morphism $G \to \mathrm{Aut}(T)$ which has non-elementary image, there is a unique G-map $\underset{= B_{PF}(G,\mu)}{B} \longrightarrow \partial T$, and every G-map $B \to \mathrm{Prob}(\partial T)$ factors through $\delta : \partial T \to \mathrm{Prob}(\partial T)$

$$\begin{matrix}\varphi \\ \psi\end{matrix} : B \to \partial T \qquad b \mapsto \frac{1}{2}\left(\delta_{\varphi(b)} + \delta_{\psi(b)}\right)$$

non elementary: $G \curvearrowright T$ does not fix a point in T, a point or a pair of points in ∂T

Take some G-map $B \to \mathrm{Prob}(\partial T)$
$$b \longmapsto \nu_b$$



Let $\phi : B \to \mathrm{Prob}(\partial T^3)$. If ν_b is non-atomic, $\phi(b)$
$$b \longmapsto \nu_b \times \nu_b \times \nu_b$$
has support in $\partial T^{(3)}$ (triples of distinct points)

If $|\mathrm{supp}\,\nu_b| \geq 3$ for ν-a.e. b, then I can condition $\phi(b)$ to $\partial T^{(3)}$

because $\partial T^{(3)}$ has positive $\phi(b)$ measure.

I can get $\Psi : B \to \mathrm{Prob}(\partial T^{(3)}) \xrightarrow{\text{bar}} \mathrm{Prob}(T)$

T is countable so $\mathrm{Prob}(T)$ is the 1-sphere in $\ell^1(T)$, on which G acts by isometries. By metric ergodicity of $G \curvearrowright B$, Ψ has to be essentially constant, but this essential value is a G-fixed point!

But those dont exist, so contradiction.

We have $|\mathrm{supp}\,\nu_b| \leq 2$ on a positive measure set. By ergodicity, $|\mathrm{supp}\,\nu_b| \leq 2$ a.e. (and $|\mathrm{supp}\,\nu_b|$ is essentially constant)

If $|\mathrm{supp}\,\nu_b| = 2$ a.e.,

$$\varphi : B \times B \to \mathrm{Prob}(\partial T) \quad \text{is a } G\text{-map, and if}$$
$$b, b' \mapsto \frac{1}{2}(\nu_b + \nu_{b'}) \quad |\mathrm{supp}\,\varphi(b,b')| = 2 \quad \nu\otimes\nu \text{ a.e.},$$
$$\mathrm{supp}\,\nu_b = \mathrm{supp}\,\nu_{b'} \quad \nu\otimes\nu\text{-a.e.}$$

But now $\mathrm{supp}\,\nu_b$ is essentially constant, it is a G-fixed pair in ∂T, contradiction

So $|\mathrm{supp}\,\varphi(b,b')| \geq 3$ $\nu\otimes\nu$ a.e. by ergodicity of $G \curvearrowright B \times B$

The contradiction is as previously.

Using the same kind of ideas, one gets

Theorem (Bader, Caprace, Furman, Sisto)

(B, ν), (G, μ) as before. Let (X, d) be a Gromov hyperbolic space, possibly non-proper, on which G acts by isometries, non-elementarily.

Then there is a G-map $\varphi : B \to \partial X$ such that

(i) $\varphi^2 : B^2 \to \partial X^2$ has essential image in $\partial X^{(2)}$

$b, b' \mapsto (\varphi(b), \varphi(b'))$

(ii) $\mathrm{Map}_G(B, \mathrm{Prob}(\partial X)) = \{\delta_\varphi\}$

(iii) $\mathrm{Map}_G(B \times B, \partial X) = \{\varphi \circ pr_1, \varphi \circ pr_2\}$

(iv) $\mathrm{Map}_G(B \times B, \partial X^{(2)}) = \{\varphi^2, \varphi^2 \circ \mathrm{flip}\}$

(If $G = (G_1 \times G_2)$, $B = B_1 \times B_2$ $\quad (b_1, b_2, b_1', b_2') \mapsto \begin{pmatrix} \varphi(b_1, b_2) \\ \varphi(b_1', b_2) \end{pmatrix}$

A (non-trivial) consequence of this is the following:

Theorem (BCFS, D.)

Let $G = \prod_{i=1}^{n} G_i$ where each G_i is either a simple Lie group or a lcsc group acting on a regular finite tree (building)

Let $\Gamma < G$ an irreducible lattice ($\overline{pr_i(\Gamma)} = G_i$), X hyperbolic, as above.

If $\Gamma \curvearrowright X$ is non elementary and coarsely minimal then $\Gamma \curvearrowright X$ is "equivalent" to the action of Γ on X_i (X_i is the symmetric space of G_i when G_i is rank one, or one of the trees)

$$\Gamma \hookrightarrow G \twoheadrightarrow G_i \longrightarrow \mathrm{Isom}(X_i)$$

Ex $\Gamma = SL_2(\mathbb{Z}[1/2]) \hookrightarrow SL_2(\mathbb{R}) \times SL_2(\mathbb{Q}_2)$, so the "only" actions of Γ on hyperbolic spaces are $\Gamma \curvearrowright \mathbb{H}^2$ or $\Gamma \curvearrowright T_3$

(Burger - Mozes groups)

Theorem (D.) Let G be a lcsc group, $\Gamma < G$ a lattice, μ some probability measure on G with [conditions] Then there is some $\nu \in \mathrm{Prob}(\Gamma)$ such that

$$B_{PF}(G, \mu) \underset{\Gamma}{\cong} B_{PF}(\Gamma, \nu)$$