Constructing the Haar measure on tdlc Polish groups

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October 20, 2014

Abstract

We give a short proof of the existence of the Haar measure for tdlc Polish groups.

The main tool that we are going to use is the Riesz representation theorem, which allows one to identify measures and positive linear functionals. A Borel probability measure $\mu : \mathcal{B}(X) \to [0,1]$ on a topological space X will be call regular if for all Borel set A, the measure of A is equal to the supremum of the measures of the compact subsets of A, and also equal to the infimum of the measures of the open sets containing A. Note that by [Kec95, Thm. 17.10], on a compact Polish space every probability measure is automatically regular.

Theorem 1 (Riesz representation theorem)**.** *Let X be a compact Polish space. Then for every normalized positive continuous linear functional $\varphi : \mathcal{C}(X) \to \mathbb{R}$, there exists a unique Borel probability measure $\mu : \mathcal{B}(X) \to [0,1]$ such that for all $f \in \mathcal{C}(X)$,*

$$\varphi(f) = \int_X f d\mu.$$

Moreover, such a measure μ is automatically regular.

The previous theorem uses in an essential way Carathéodory's extension theorem, which is itself a fundamental but difficult result. We will discuss a variant of the proof of the existence of the Haar measure in the last section, but it still require Carathéodory's theorem, which cannot be avoided when one wants to construct non atomic measures, to the author's knowledge. The theorem we are after is the following:

Theorem 2. *Let G be a Polish tdlc group. Then there exists a Borel left-invariant measure μ on G which is finite on compact sets and regular.*

We are going to take advantage of the fact that we are in the tdlc case through the use of van Dantzig's theorem:

Theorem 3 (van Dantzig)**.** *Let G be a tdlc group. Then for every neighborhood of the identity U, there exists a compact open subgroup which is a subset of U.*

1 Haar measure for profinite groups

We start with a profinite Polish group G. Let $(U_n)_{n\in\mathbb{N}}$ a countable basis neighborhood of identity, then by the previous theorem we may find $V_n \leqslant G$ compact open such that $V_n \subseteq U_n$. We then define $G_n = \bigcap_{k=1}^n V_k$, and observe that (G_n) forms a decreasing family of compact open subgroups which is a basis of neighborhoods of the identity. By replacing G_n by the kernel of the left action of G on G/G_n (because G/G_n is a finite space, this kernel is the intersection of finitely many conjugates of G_n, hence compact open), we may assume that G_n is a normal subgroup of G.

Definition 4. For $n \in \mathbb{N}$, we let $\mathcal{F}_n$ denote the vector space

$$\mathcal{F}_n = \{f \in \mathcal{C}(G) : f(xg) = f(x) \forall g \in G_n\}.$$

In other words, $\mathcal{F}_n$ is the vector space of all continuous functions which are right G_n-invariant.

It is an exercise to check that $\mathcal{F}_n$ is canonically identified to the set of functions on the finite space G/G_n. We define μ_n to be the counting measure on G/G_n normalized by

$$\mu_n(\{G_n\}) = \frac{1}{[G : G_n]},$$

and by Riesz's theorem [1] we may associate to it a continuous functional φ_n on $\mathcal{C}(G/G_n)$.

Remark 5. μ_n is the unique G-invariant measure on G/G_n which assign mass $1/[G : G_n]$ to the singleton $\{G_n\}$.

Lemma 6. *For every $n \leqslant m \in \mathbb{N}$, the vector space $\mathcal{F}_n$ is a subspace of $\mathcal{F}_m$, and φ_n is the restriction on φ_m to $\mathcal{F}_m$.*

Proof. The space $\mathcal{F}_n$ is a subspace of $\mathcal{F}_m$ because $G_n \geqslant G_m$ and $\mathcal{F}_n$ is the space of G_n-right-invariant functions. By the previous remark and Riesz representation theorem[2], we then only need to check that μ_m assigns mass $1/[G : G_n]$ to G_n. But this is a consequence of the formula

$$[G : G_n][G_m : G_n] = [G : G_n],$$

writing G_n as the reunion of $[G_m : G_n]$ right G_m-cosets and using the left G-invariance of our measures. $\square$

Lemma 7. *The reunion $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ is dense in $\mathcal{C}(G)$. Moreover, $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n^+$ is dense in $\mathcal{C}(G)^+$*

Proof. Let $\epsilon > 0$, and let $f \in \mathcal{C}(G)$. By the fact that f is continuous and van Dantzig's theorem, we find for every $x \in G$ a compact open subgroup G_x such that $f(xG_x)$ has diameter less than ϵ. By compactness, we find finitely many $x_1, ..., x_p$

[1] Here we only use the finite version, which is not hard to check!

[2] Again, rather the finite easy version of Riesz's representation theorem!

and indices $n_1, ..., n_p$ such that $(x_i G_{x_i})_{i=1}^p$ is a covering of G. Because (G_n) is a base of neighborhoods of the identity, we may fix $n \in \mathbb{N}$ such that

$$G_n \leqslant \bigcap_{i=1}^p G_{x_i}.$$

Now, we note that $\bigcup_{i=1}^p x_i G_{x_i}$ is a finite disjoint union of right G_n-cosets, so that for all $x \in G$, xG_n is a subset of some $x_i G_{x_i}$, and so $\mathrm{diam}(f(xG_n)) \leqslant \mathrm{diam}(f(x_i G_{x_i})) < \epsilon$. For each right G_n-coset xG_n, we pick $r_{xG_n} \in f(xG_n)$. We then define $\tilde{f} \in \mathcal{F}_n$ to be constant equal to r_{xG_n} on each right coset xG_n, and it is then straightforward to check that $\left\| f - \tilde{f} \right\| < \epsilon$.

The second part follows from the fact that if f positive, we could have picked $r_{xG_n} \geqslant 0$ for each coset $xG_n \in G/G_n$. □

To finish the proof, we note that by lemma 6, we may define a linear function φ on $\bigcup_{n \in \mathcal{F}_n}$ by putting $\varphi(f) = \varphi_n(f)$ for all $f \in \mathcal{F}_n$ and all $n \in \mathbb{N}$. Such a linear functional is of course still positive and continuous (it has norm one). We may thus extend it uniquely to $\overline{\bigcup_{n \in \mathbb{N}} \mathcal{F}_n} = \mathcal{C}(G)$. We now only need to check that the extension is positive. So let $f \in \mathcal{C}(G)$ be a positive function, by the previous lemma there is a sequence of positive functions $f_n \in \mathcal{F}_n$ which converges to f, and so

$$\varphi(f) = \lim_n \varphi(f_n) = \lim_n \varphi_n(f_n) \geqslant 0.$$

By Riesz's representation theorem, φ is continuous positive, and so it defines a unique Borel probability measure μ on G. This measure is both left and right invariant, for each μ_n is. We have proved

Theorem 8. *Let G be a Polish profinite group. Then there exists a regular probability measure μ on G which is both left and right-invariant.*

2 Haar measure for tdlc groups

Let G be a tdlc Polish group. We fix a compact open subgroup K. Such a compact subgroup is profinite, so by the previous theorem there is a probability measure ν on K which is both left and right invariant. We may as well assume that K has countable infinite index, because else G is already a profinite group, so that the previous theorem applies to it.

Write $G = \bigsqcup_{i \in \mathbb{N}} g_i K$. Equip each $g_i K$ with the pushforward measure $\mu_i = g_{i*}\nu$.

We then define a σ-finite measure μ on G by gluing together the μ_i's: for $A \subseteq G$ Borel, we let

$$\mu(A) = \sum_{i \in \mathbb{N}} \mu_i(A \cap g_i K) = \sum_{i \in \mathbb{N}} \mu(g_i^{-1}(A \cap g_i K)) = \sum_{i \in \mathbb{N}} \mu(g_i^{-1} A \cap K).$$

It is an exercise to check that μ is still regular. We now need to prove that μ is left-invariant.

Let $g \in G$. Then g permutes the partition $(g_i K)$ and so there is some $\sigma \in \mathfrak{S}(\mathbb{N})$ such that for all i, $gg_iK = g_{\sigma(i)}K$. For all Borel $A \subseteq X$,

$$\begin{aligned}\mu(g^{-1}A) &= \sum_{i\in\mathbb{N}} \mu((gg_i)^{-1}A \cap K) \\ &= \sum_{i\in\mathbb{N}} \mu(g_{\sigma(i)}^{-1} gg_i (gg_i)^{-1}A \cap K)\end{aligned}$$

by left K-invariance of μ, because $g_{\sigma(i)}^{-1}gg_i \in K$. But now we can simplify this expression and we get

$$\mu(g^{-1}A) = \sum_{i\in\mathbb{N}} \mu(g_{\sigma(i)}^{-1}A \cap K) = \sum_{i\in\mathbb{N}} \mu(g_i^{-1}A \cap K) = \mu(A).$$

So theorem 2 is proved, in other words we just built a Haar measure on any tdlcsc group. Note that we only used the left-invariance of the Haar measure we built for profinite groups.

3 A variant

We could also build the Haar measure directly on G without using the Haar measure on profinite groups, but the technical problem we run into is that on a non compact locally compact space, regular measures correspond to positive linear functionals on $\mathcal{C}_c(G)$ which are not continuous anymore. But because G is a countable increasing union of compact open sets, this does not matter too much and we now sketch how to do it.

This time we start with (G_n) a decreasing chain of compact open subgroups of G which is a base of neighborhoods for the identity. Again we let $\mathcal{F}_n$ be the space of right G_n-invariant compactly supported continuous functions on G. We may define a counting measure μ_n on G/G_n the same way as in the profinite case, and normalise it so that $\mu_n(G_1G_n) = 1$.

This gives a positive linear functional φ_n on $\mathcal{F}_n$. Now write G as a countable increasing union of compact open sets K_m, and let $\mathcal{G}_m$ denote the functions whose support is contained in K_m. It is straightforward to check that $\mathcal{C}_c(G) = \bigcup_m \mathcal{G}_m$. Now each φ_n restricts to a continuous linear form on $\mathcal{G}_m$, and we have $\mathcal{G}_m = \overline{\bigcup_{n\in\mathbb{N}}(\mathcal{F}_n \cap \mathcal{G}_m)}$ (same proof as lemma 7), so that by continuity and the compatibiliy of the φ_n, we may build a continuous invariant positive linear form ψ_m on $\mathcal{G}_m$ out of the restrictions of the φ_n's. It is then easy to check that ψ_{m+1} extends ψ_n, so $\psi = \bigcup \psi_m$ is a positive linear functional on G which yields a left Haar measure on G. The details are left as an exercise!

References

[Kec95] Alexander S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.