On the irreducibility of the Koopman representation of $\mathrm{Aut}(X, \mu)$

François Le Maître

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Abstract

In this note, we provide an elementary proof of the fact, due to Furstenberg-Glasner-Weiss [Gla03] and independently Bekka-de la Harpe (unpublished), that the Koopman representation of $\mathrm{Aut}(X, \mu)$ (or of any ergodic full group) is irreducible. The idea of our proof is to use von Neumann's ergodic theorem.

Theorem 1 (Glasner-Furstenberg-Weiss, Bekka-de la Harpe)**.** *Let G be an ergodic full group on a standard probability space (X, μ). Then its Koopman representation $\kappa : G \to \mathcal{U}(\mathrm{L}^2_0(X, \mu))$ is irreducible.*

Proof. Let $f \in \mathrm{L}^2_0(X, \mu)$ be a non null function, and let $\mathcal{K}$ be the the Hilbert space spanned by $\mathrm{Aut}(X, \mu) \cdot f \oplus \mathbb{C}1$. We have to show that $\mathcal{K} = \mathrm{L}^2(X, \mu)$. Because $\int_X f = 0$, we have $\mu(\{x \in X : f(x) > 0\}) > 0$, and we let $\alpha = \mu(\{x \in X : f(x) > 0\})$. By density of step functions in $\mathcal{H}$, it suffices to show that $\mathcal{K}$ contains all the characteristic functions of Borel subsets of X. Because G acts ergodically, and because the sum of the characteristic functions of two disjoint sets is the characteristic function of their union, it suffices to show that for every $\epsilon \in (0, \alpha)$, there exists $A \subseteq X$ such that $\mu(A) = \epsilon$ and $\mathcal{K}$ contains the characteristic function of A.

To this end, we fix $\epsilon \in (0, \alpha)$ and $A \subseteq \{x \in X : f(x) > 0\}$ of measure ϵ. Since G is ergodic, we may find $T \in G$ whose ergodic components are A and $X \setminus A$. We then apply von Neumann's ergodic theorem to T and f, which yields that the function

$$\tilde{f} = \frac{\int_A f}{\mu(A)} \chi_A + \frac{\int_{X \setminus A} f}{1 - \mu(A)} \chi_{X \setminus A}$$

arises as a limit of Cesaro averages, and thus belongs to $\mathcal{K}$. By substracting $\frac{\int_{X \setminus A} f}{1 - \mu(A)} \cdot 1$ to it and renormalising, we find that χ_A belongs to $\mathcal{K}$, which ends the proof (note that $\int_{X \setminus A} f < 0$, which guarantees that $\tilde{f}$ takes two distinct values). □

One may wonder which countable groups admit an irreducible Koopman representation. The group of dyadic permutations, being dense in $\mathrm{Aut}(X, \mu)$, is an example of an amenable group with an irreducible Koopman representation. Also note that on the other hand, no abelian group can have an irreducible Koopman representation. One may also construct groups with a free action generating an irreducible Koopman representation: just take a generic representation of $\mathbb{F}_2$ in $\mathrm{Aut}(X, \mu)$, by [Pra81] it generates a dense subgroup of $\mathrm{Aut}(X, \mu)$ acting freely (see [Kec10, Thm. 10.8]).

References

[Gla03] Eli Glasner. *Ergodic theory via joinings*, volume 101 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.

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[Pra81] V. S. Prasad. Generating dense subgroups of measure preserving transformations. *Proc. Amer. Math. Soc.*, 83(2):286–288, 1981.