L^1 full groups

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Standing asumptions

Throughout the talk,

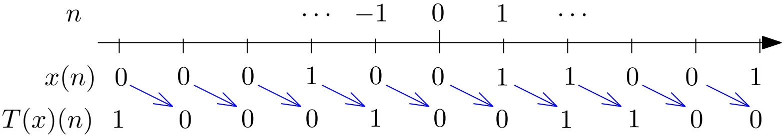
- (X, μ) denotes a standard atomless probability space, so it is isomorphic to $[0, 1]$ equipped with the Lebesgue measure.
- We will ignore null sets.
- All the sets we consider are Borel sets.

Definition

A Borel bijection $T : X \to X$ is a **measure-preserving transformation** if for all $A \subseteq X$ one has $\mu(T(A)) = \mu(A)$.

Two fundamental examples:

- Irrational rotations: $X = [0, 1[$ equipped with the Lebesgue measure, take $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and let $T_\alpha(x) = x + \alpha \mod 1$.
- Bernoulli shifts: (Y, ν) is a standard probability space possibly with atoms (e.g. Y={0,1} and $\nu = 1/2(\delta_0 + \delta_1)$). Let $X = (Y, \nu)^{\mathbb{Z}}$, $\mu = \nu^{\otimes \mathbb{Z}}$, and let $T(x) = n \mapsto x(n-1)$.



(Flip-)conjugacy of measure-preserving transformations

Definition

Two measure-preserving transformations T and T' are **conjugate** if there is a third measure-preserving transformation S such that

$$T = ST'S^{-1}.$$

Examples:

- Irrational rotations: note that $S(x) = -x \mod 1$ conjugates T_α and $T_{-\alpha} = T^{-1}$. Actually T_α is conjugate to T_β iff $\alpha = \pm\beta \mod 1$.
- An irrational rotation is never conjugate to a Bernoulli shift.
- A Bernoulli shift is conjugate to its inverse via $S(x)(n) = x(-n)$.

Definition

Two measure-preserving transformations T and T' are **flip-conjugate** if there is a third measure-preserving transformation S such that

$$T = ST'S^{-1} \text{ or } T = ST'^{-1}S^{-1}.$$

An obvious invariant of flip-conjugacy is the following.

Definition

A measure-preserving transformation T is **ergodic** if for every Borel set A, $T(A) = A$ implies $\mu(A) = 0$ or 1.

Irrational rotations and Bernoulli shifts are ergodic.

Here is another invariant of flip conjugacy.

Definition

Let T be a measure-preserving transformation. Its **full group** $[T]$ is defined as the group of all measure-preserving transformations U such that for all $x \in X$

$$U(x) \in \mathcal{O}_T(x),$$

where $\mathcal{O}_T(x) = \{T^n(x) : n \in \mathbb{Z}\}$ is the T-orbit of x.

Theorem (Dye 59)

Let T and T' be two ergodic measure-preserving transformations. Then there is a third measure-preserving transformation S such that

$$[T] = S[T']S^{-1}.$$

Suppose T is an **aperiodic** m.p.t., i.e. that all its orbits are infinite. Then $U \in [T]$ is completely determined by the **cocycle map** $c_U : X \to \mathbb{Z}$ defined by the equation

$$U(x) = T^{c_U(x)}(x).$$

We have the cocycle identity $c_{UU'}(x) = c_{U'}(x) + c_U(U'(x))$.

Definition

Let T be a measure-preserving transformation. Its L^1 **full group** $[T]_1$ is defined as the group of all $U \in [T]$ whose cocycle c_U satifies

$$\int_X |c_U(x)|\, d\mu(x) < +\infty.$$

- $[T]_1$ is a group !
- We have a natural metric defined by

$$d_1(U, U') = \int_X |c_U(x) - c_{U'}(x)|\, d\mu(x)$$

This metric is separable, **right invariant** and complete.

- So $[T]_1$ is a cli Polish group.

Proposition (LM)

Let T and T' be two ergodic measure-preserving transformations. TFAE:

- *T is flip conjugate to T',*
- *$[T]_1$ is topologically isomorphic to $[T']_1$,*
- *$[T]_1$ is abstractly isomorphic to $[T']_1$.*

This is a straightforward consequence of the following two results.

- A reconstruction result à la Dye: any abstract isomorphism between L^1 full groups is the conjugacy by a measure-preserving transformation.
- A rigidity result from Belinskaya (1969): if $[T]_1 = [T']_1$ then T and T' are flip-conjugate.

Topological rank and L^1 full groups

Definition

The **topological rank** $t(G)$ of a separable topological group G is the minimum $n \in \mathbb{N} \cup \{\infty\}$ such that there are $g_1, \ldots, g_n \in G$ generating a dense subgroup of G.

Question

What is the topological rank of $[T]_1$?

Theorem (LM)

Let T be an ergodic m.p.t. then TFAE:

- *$[T]_1$ has finite topological rank;*
- *T has finite entropy.*

Definition

An **observable** is a Borel map $\varphi : X \to I$ where I is a countable set.

Example: on $X = \{0,1\}^{\mathbb{Z}}$, let $\varphi(x) = \begin{cases} 0 & \text{if } x(0) = 0 \\ (1, x(1)) & \text{if } x(0) = 1 \end{cases}$.

Given $i \in I$, the amount of information provided by knowing $\varphi(x) = i$ is equal to $-\ln(\mu(\varphi^{-1}(\{i\})))$.

Definition

The **entropy** of an observable φ is the mean amount of information it provides:

$$H(\varphi) := -\sum_{i \in I} \mu(\varphi^{-1}(\{i\})) \ln(\mu(\varphi^{-1}\{i\}))$$

If $\varphi : X \to I$ and $\psi : X \to J$, we get a new observable $(\varphi, \psi) : X \to I \times J$.

Proposition (Subadditivity of entropy)

$H(\varphi, \psi) \leq H(\varphi) + H(\psi)$.

(Rokhlin) entropy of a m.p.t.

Definition

Let T be a m.p.t., an observable φ is **dynamically generating** if for all $x \neq y \in X$ there is $n \in \mathbb{Z}$ such that $\varphi(T^n(x)) \neq \varphi(T^n(y))$.

Definition

The **entropy** $h(T)$ of a m.p.t. T is the infimum of the entropies of its dynamically generating observables.

Theorem (Kolmogorov-Sinai, Rokhlin)

Let (Y, ν) be an countable atomic probability space and let $(X, \mu) = (Y^{\mathbb{Z}}, \nu^{\otimes \mathbb{Z}})$. Consider the observable $\varphi(x) = x(0)$. Then the entropy of the Bernoulli shift T on X is

$$h(T) = H(\varphi).$$

Proof that $t([T]_1) < \infty \Rightarrow h(T) < +\infty$.

Let $U_1, \ldots, U_n \in [T]_1$ generate a dense subgroup. It is a well-known fact that every integrable observable has finite entropy, so by subadditivity

$$H(c_{U_1}, \ldots, c_{U_n}) < +\infty.$$

One then has to check that the observable $(c_{U_1}, \ldots, c_{U_n})$ is dynamically generating, which follows from two facts:

- The elements of the closed group generated by $(U_1, \ldots, U_n)$ have a cocycle belonging to the T-invariant σ-algebra generated by $c_{U_1}, \ldots, c_{U_n}$.
- There are sufficiently many elements in $[T]_1$: as we will see shortly, for every $A \subseteq X$ we can find $U \in [T]_1$ such that $A = \{x \in X : c_U(x) \neq 0\}$.

So T has a dynamically generating observable of finite entropy, hence by definition T has finite entropy.

Proving that $h(T) < +\infty \Rightarrow t([T]_1)$

To prove the reverse implication, we will need to understand better L^1 full groups. I will go over several basic properties of independent interest.

Definition

Let T be an aperiodic m.p.t. and let $U \in [T]_1$. Its **index** is

$$I(U) = \int_X c_U(x) d\mu(x).$$

Example: $I(T^n) = n$.

By the cocycle identity, the index a group homomorphism $[T]_1 \to \mathbb{R}$. We will see that when T is ergodic it takes values into $\mathbb{Z}$.

Let T be a m.p.t. and $A \subseteq X$. Poincaré's recurrence theorem states for almost all $x \in A$ there is $n \in \mathbb{N}^{>0}$ such that $T^n(x) \in A$. Let $n_A(x) \in \mathbb{N}^{>0}$ be the smallest such integer, and put $n_A(x) = 0$ for $x \notin A$.

Theorem (Kac's return time theorem)

We have $\int_X n_A(x) = \mu(\bigcup_{n\in\mathbb{Z}} T^n(A))$.

Definition

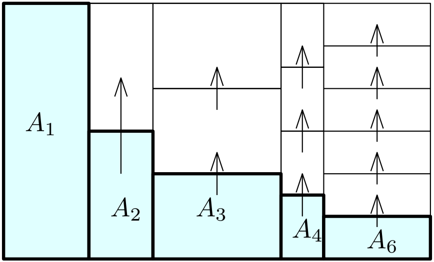
Let T be a m.p.t. and let $A \subseteq X$ non null. The transformation T_A induced by T on A is defined by

$$T_A(x) = T^{n_A(x)}(x)$$

Kac's result yields $T_A \in [T]_1$.

Proof of Kac's return time theorem

For $n \in \mathbb{N}^{>0}$ we let $A_n := \{x \in A : n_A(x) = n\}$ and $Y = \bigcup_{n\in\mathbb{Z}} T^n(A)$. Then $(A_n)_{n>0}$ is a partition of A and we have a **Kakutani-Rohlin partition** of Y:



Thus $\sum_{n\in\mathbb{N}} n\mu(A_n) = \mu(Y)$, i.e. $\int_X n_A(x) = \mu(Y)$.

Proposition (LM)

Let T be an aperiodic m.p.t., then for every $U \in [T]_1$ and $A \subseteq X$, we have $U_A \in [T]_1$ and if we let $Y = \bigcup_{n\in\mathbb{Z}} U^n(A)$ then $I(U_A) = \int_Y c_U(x) d\mu(x)$.

The **support** of a m.p.t. T is the set $\operatorname{supp} T := \{x \in X : T(x) \neq x\}$.

Proposition (LM)

Let T be an aperiodic m.p.t. and $\epsilon > 0$. The L^1 full group of T is generated by elements whose support has measure less than ϵ.

Proof.

Take $A \subseteq X$ intersecting every T-orbit with $\mu(A) < \epsilon$. Then UU_A^{-1} is **periodic** (has only finite orbits) hence has a diffuse ergodic decomposition. So one can write it as a product of elements whose support has measure less than ϵ. Since $\operatorname{supp}(U_A) \subseteq A$ and $U = (UU_A^{-1})U_A$, we are done. □

Definition

Let T be an aperiodic m.p.t.. The **derived L^1 full group** of T is the closure of the subgroup generated by commutators. It is denoted by $[T]_1'$.

Using the commutator trick via a refinement of the previous proposition + additional work, we show:

Theorem (LM)

Let T be an aperiodic m.p.t., we have the following.

- $[T]_1'$ *is topologically generated by involutions.*
- $[T]_1' = \ker I$.
- $[T]_1'$ *is the connected component of the identity.*
- $[T]_1'$ *is generated by periodic elements.*
- $[T]_1'$ *is topologically simple iff T is ergodic.*
- *If T is ergodic then I takes values in $\mathbb{Z}$ so $[T]_1 = [T]_1' \rtimes_T \mathbb{Z}$.*

Generating the L^1 full group (3)

So $[T]_1$ is topologically generated by involutions along with T itself. The easiest involutions to work with arise as follows: let $A \subseteq X$ such that A and $T(A)$ are disjoint, and define

$$I_{T,A}(x) = \begin{cases} T(x) & \text{if } x \in A, \\ T^{-1}(x), & \text{if } x \in T(A), \\ x & \text{else.} \end{cases}$$

Proposition (LM, following work of Kittrell-Tsankov for full groups)

Let T be an aperiodic m.p.t., then $[T]_1'$ is topologically generated by $\{I_{T,A} : A \cap T(A) = \emptyset\}$.

Corollary (LM, following work of Marks for full groups)

Let T_α be an irrational rotation, then $t([T_\alpha]_1) = 2$.

Note that irrational rotations have entropy zero...

Let X be the Cantor space ($X = \{0,1\}^{\mathbb{N}}$), and let T be a homeomorphism of X. Define the **full group** of T to be the group $[T]$ of homeomorphisms U of X such that for every $x \in X$,

$$U(x) \in \mathcal{O}_T(x).$$

Again when T is aperiodic we have a well-defined cocycle map c_U given by $U(x) = T^{c_U(x)}(x)$.

Definition (Giordano-Putnam-Skau)

Let T be an aperiodic homeomorphism of X. The **topological full group** $[T]_c$ of T is the group of T is the group of $U \in [T]$ such that c_U is *continuous*.

This characterization along with the fact that topological full groups are invariants of flip conjugacy (in the topological context) provide inspiration and motivation for the study of L^1 full groups as measurable analogues of topological full groups.

Link with L^1 full groups

Proposition (LM)

Let T be an aperiodic homeomorphism of X, let μ be a T-invariant probability measure. Then the involutions of $[T]_c$ generate a dense subgroup of $[T]_1'$.

Proof.

Approximate $I_{T,A}$ by $I_{T,U}$ where U is clopen. □

Corollary (LM)

Let T be an aperiodic homeomorphism of X, let μ be a T-invariant ergodic probability measure. Then $[T]_c$ is dense in $[T]_1$.

Proof.

Recall that $[T]_1$ is generated by $[T]_1'$ along with T. But $T \in [T]_c$! □

Proof that $h(T) < +\infty \Rightarrow t([T]_1) < +\infty$

- By Krieger's generator theorem, we may assume that T is a minimal subshift of finite type.
- By Matui's theorem we then have that $[T]_c'$ is finitely generated.
- So by the result from the previous slide $[T]_1'$ is topologically finitely generated.
- Since $[T]_1 = [T]_1' \rtimes_T \mathbb{Z}$, we conclude that $t([T]_1) < +\infty$.

An obvious question

Theorem

Let T be an ergodic m.p.t. then TFAE:

- $[T]_1$ *has finite topological rank;*
- $[T]_1'$ *has finite topological rank;*
- T *has finite entropy.*

Question

Is there a formula relating $t([T]_1)$ to $h(T)$?

Beyond the $\mathbb{Z}$ case

- Given a measure-preserving graphing Φ, define its L^1 full group by

$$[\Phi]_1 = \{U \in [\mathcal{R}_\Phi] : \int_X d_\Phi(x, U(x)) d\mu(x) < +\infty\}.$$

- When the graphing comes from a finite generating set of a countable group Γ acting by m.p.t., the L^1 full group and its topology do not depend on the choice of the finite generating set: we have a well-defined Polish cli group $[\Gamma \curvearrowright X]_1$.
- The result on topological finite generation generalizes thanks to work of Seward and Nekrashevych, but only to the derived group.

Beyond the $\mathbb{Z}$ case (2)

Theorem (Carderi,LM,Matte-Bon,Tsankov)

Let Γ be a finitely generated group acting freely and ergodically on (X,μ) by m.p.t.. TFAE:

- $[\Gamma \curvearrowright X]_1'$ *has finite topological rank;*
- $\Gamma \curvearrowright X$ *has finite Rohlin entropy.*

Question

What is $[\Gamma]_1/[\Gamma]_1'$?