Classification of non-free p.m.p. boolean actions of ergodic full groups and applications

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Abstract

We extend Dye's reconstruction theorem, which classifies isomorphisms between full groups, to a classification of homomorphisms between full groups. For full groups of ergodic p.m.p. equivalence relations, our result roughly says that such homomorphisms come only from actions of the equivalence relation, or of one of its symmetric powers. This has several rigidity consequences for homomorphisms between full groups. Our main application is a characterization of property (T) for ergodic p.m.p. equivalence relations purely in full group terms, without using their topology: an ergodic p.m.p. equivalence relation has (T) iff all non-free ergodic boolean actions of its full group are strongly ergodic.

1 Introduction

1.1 General context

1.1.1 Dynamical systems and their orbits

Given a bijection on a set, the associated partition into orbits characterizes the bijection up to conjugacy. Indeed the conjugacy class of the bijection is completely determined by the number of orbits of size $k \geqslant 1$ and the number of infinite orbits. The picture gets much more intricate when the set X is endowed with additional structure, for instance a topology or a measure. In this paper, we will work in a purely measurable context, but our results are inspired by results of Matte-Bon in the topological context [MB18] as we will explain later on.

We begin with a standard probability space (X, μ), i.e. a probability space isomorphic to the $[0, 1]$ interval endowed with the Lebesgue measure. The bijections that we consider are those which preserve this structure, called p.m.p. (probability measure-preserving) bijections, and we will restrict to the *aperiodic* ones, i.e. those all whose orbits are infinite. A good example to have in mind is the following: consider $X = [0, 1)$ endowed with the Lebesgue measure, let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then the irrational rotation T_α is the p.m.p. bijection of (X, μ) defined by: for all $x \in X$,

$$T_\alpha(x) = x + 1 \mod 1.$$

Contrarily to bijections on sets without additional structure, p.m.p. bijections are in general hard to classify up to conjugacy. In the measurable context, the appropriate definition of conjugacy is the following: two p.m.p. bijections T_1 and T_2 of a standard probability space (X, μ) are conjugate if there is another p.m.p. bijection S such that for *μ-almost all $x \in X$*,

$$ST_1 x = T_2 S x.$$

Note that this can be rewritten as the following equality of bijections up to measure zero: $S \circ T_1 \circ S^{-1} = T_2$, hence the name conjugacy.

In the context of irrational rotations, a good example of a conjugacy is any map $S : [0,1) \to [0,1)$ such that for almost all $x \in [0,1)$, $S(x) = 1 - x$: such a map conjugates T_α to $T_{-\alpha}$. This remark actually says it all for irrational rotations: the Halmos-von Neumann theorem implies that T_α is conjugate to T_β iff $\alpha = \pm\beta \mod 1$ [Hv42]. Another fundamental class of transformations which are completely characterized up to conjugacy are *Bernoulli shifts* [Kol58, Orn70], which act by shifting coordinates on an infinite product of a fixed (possibly finite) probability space, the product being indexed by $\mathbb{Z}$. In full generality, the problem of deciding whether two p.m.p. transformations are conjugate or not is intractable in a sense that can be made precise, see for instance [Hjo01, FRW11].

A basic invariant of conjugacy is ergodicity, which says that every measurable set which is a reunion of orbits must have measure 0 or 1. By design, such a property is remembered by the partition of the space into orbits. The irrational rotations are ergodic, and one can build aperiodic non-ergodic examples easily. For instance, noting the $[0,1)$ interval is equal to the disjoint union $[0,1/2) \sqcup [1/2,1)$ one can define a p.m.p. bijection which acts as an irrational rotation on each of these intervals. A finer invariant is then the *ergodic decomposition* of a p.m.p. bijection, which roughly tells us the sizes of the ergodic parts, plus the size of a diffuse part where ergodic components have measure zero (think of $T_\alpha \times \mathrm{id}_{[0,1]}$). Here is the fundamental result that started the study of dynamical systems through their orbits in the measurable context.

Theorem 1.1 ([Dye59, Thm. 5])**.** *Two aperiodic p.m.p. bijections of a standard probability space can be conjugated so as to have the same orbits iff they have the same ergodic decomposition.*

When a p.m.p. bijection S establishes a bijection bewteen T_1-orbits and T_2-orbits, let us say that S is an orbit equivalence. Dye's theorem tells us that there is an orbit equivalence between two p.m.p. bijections T_1 and T_2 iff they have the same ergodic decomposition. In particular, *any two ergodic p.m.p. bijections admit an orbit equivalence between them.*

1.1.2 Full groups

Dye's result was actually stated in a more abstract framework which is central to us. In order to define it properly, let us first denote by $\mathrm{Aut}(X, \mu)$ the group of all p.m.p. bijections of a standard probability space (X, μ), where we identity two such bijections if they coincide on a conull set. Now Dye defines a subgroup $\mathbb{G}$ of $\mathrm{Aut}(X, \mu)$ to be **full** if it is stable under *countable cutting and pasting*: whenever (A_n) is a partition of X, and (T_n) is a sequence of elements of $\mathbb{G}$ such that $(T_n(A_n))$ is also a partition of X, the p.m.p. bijection $T : X \to X$ defined by

$$T(x) = T_n(x) \text{ for all } x \in A_n \text{ and all } n \in \mathbb{N}$$

belongs to $\mathbb{G}$. The first example of full groups we want to give is that of the full group associated to a p.m.p. bijection $T \in \mathrm{Aut}(X, \mu)$, defined as

$$[T] \coloneqq \{U \in \mathrm{Aut}(X, \mu) \colon \forall x \in X,\ (x, U(x)) \in \mathrm{Orb}_T(x)\},$$

where $\mathrm{Orb}_T(x) = \{T^n(x) \colon n \in \mathbb{Z}\}$ is the T-orbit of X. It is a good exercise to check that $[T]$ is the smallest full group containing T, so we might call such full groups *singly generated*.

Moreover, it it another good exercise to check that $S \in \mathrm{Aut}(X, \mu)$ is an orbit equivalence between $T_1, T_2 \in \mathrm{Aut}(X, \mu)$ iff $S[T_1]S^{-1} = [T_2]$. So orbit equivalence is the study of full groups up to conjugacy, which was Dye's original viewpoint. Note however that there are many more full groups, e.g. $\mathrm{Aut}(X, \mu)$ itself is a full group. More importantly, replacing p.m.p. bijections by *countable groups of p.m.p. bijections*, the situation can get much richer: there are many non conjugate ergodic countably generated full groups. We won't say more on this in this introductory section, but the reader can consult [Gab11] for a broad overview of orbit equivalence, which has become a field in its own right.

Since conjugacy of full groups implies that they are isomorphic as abstract groups, it is natural to wonder whether the converse holds. As it turns out, Dye was also able to answer this question positively in the ergodic case (see Sec. 3 for the precise definition of ergodicity for full groups), obtaining the following result, also known as Dye's reconstruction theorem .

Theorem 1.2 ([Dye63, Thm. 2])**.** *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be two ergodic full groups. Then for every group isomorphism $\Psi : \mathbb{G}_1 \to \mathbb{G}_2$, there is a unique $S \in \mathrm{Aut}(X, \mu)$ such that for all $T \in \mathbb{G}_1$,*

$$\Psi(T) = S\mathbb{G}_1 S^{-1}.$$

The most general version of Dye's reconstruction theorem does not need ergodicity, but then S is only non-singular since Ψ might stretch ergodic components.

The main purpose of the present paper is to extend Dye's result by looking at group *homomorphisms* between ergodic full groups, rather than isomorphisms. The fact that such a classification might be possible was hinted to us by Matte Bon's remarkable work in the topological setup: he was able to classify actions of any minimal topological full group over a Cantor space X on another Cantor space Y, which in particular allows him to describe homomorphism between topological full groups [MB18]. He proved that either the action has some free behaviour, or that it has to come from the standard action of the full group on a symmetric power of X. We exhibit a similar classification for full groups, and obtain applications which show how full groups remember partitions into orbits as abstract groups.

1.2 Results

We will now state our main result after introducing some more terminology.

A p.m.p. **boolean action** of a full group $\mathbb{G}$ is simply a group homomorphism $\mathbb{G} \to \mathrm{Aut}(Y, \nu)$, where (Y, ν) is another standard probability space. Although full groups do not admit non-trivial measure-preserving actions on standard probability space because all their boolean actions are *whirly* (see [GW05] and [GP07, Sec. 5]), they do admit boolean actions such as the one provided by their inclusion into $\mathrm{Aut}(X, \mu)$. Other examples are provided by (symmetric) diagonal actions, which we now define along with some useful auxiliary notation.

Given a standard Borel space X and $n \in \mathbb{N}$, we have a natural action of the symmetric group $\mathfrak{S}_n$ on X^n by permuting coordinates and the quotient $X^n/\mathfrak{S}_n$ is still a standard Borel space which we denote by $X^{\odot n}$. Moreover, if (X, μ) is a standard probability space, we can endow the space $X^{\odot n}$ with the pushforward measure via the quotient map $\pi_n : (X^n, \mu^{\otimes n}) \to X^{\odot n}$, and we denote by $\mu^{\odot n} := \pi_{n*}\mu^{\otimes n}$ this measure. Given $x_1, \dots, x_n \in X$ we will use the notation $[x_1, \dots, x_n] := \pi_n(x_1, \dots, x_n) \in X^{\otimes n}$. We have a group homomorphism $\iota^{\odot n} : \mathrm{Aut}(X, \mu) \to \mathrm{Aut}(X^{\odot n}, \mu^{\odot n})$ defined by $\iota^{\odot n}(T)[x_1, \dots, x_n] = [Tx_1, \dots, Tx_n]$, in particular every full group admits countably many boolean actions obtained by restricting $\iota^{\odot n}$ to the full group in question. It is not hard to see that these boolean actions are not conjugate as soon as the full group is not the trivial group.

We can now state our measurable version of Matte Bon's theorem, which states roughly that every boolean action decomposes as a free part plus a countable union of boolean actions factoring onto the above defined $\iota^{\odot n}$ in a measure-preserving (and "support-preserving") manner.

Theorem A. *Let $\mathbb{G}$ be an ergodic full group over the standard probability space (X, μ) and let $\rho\colon \mathbb{G} \to \mathrm{Aut}(Y, \nu)$ be a boolean action on another standard probability space (Y, ν). Then there is a unique measurable $\rho(\mathbb{G})$-invariant partition*[1] *$\{Y_n\}_{n=0,\dots,\infty}$ of Y such that the boolean actions ρ_n induced on $(Y_n, \frac{\nu_{\restriction Y_n}}{\nu(Y_n)})$ are subject to the following conditions*

1. *ρ_0 is the trivial action (it maps every element of $\mathbb{G}$ to the identity map on Y_0);*
2. *for all $n \in \mathbb{N}^*$, there is a (unique) measure preserving map $\pi_n : Y_n \to X^{\odot n}$ satisfying the following two properties*
   (a) *for all $g \in \mathbb{G}$ and almost all $y \in Y_n$ we have $\pi_n(\rho_n(g)y) = \iota^{\odot n}(g)\pi_n(y)$;*
   (b) *for all $g \in \mathbb{G}$ and almost all $y \in Y_n$, if $\rho_n(g)(y) \neq y$ then $\iota^{\odot n}(g)\pi_n(y) \neq \pi_n(y)$;*
3. *ρ_∞ is free: for every $g \in \mathbb{G} \setminus \{\mathrm{id}\}$ we have that $\{y \in Y_\infty\colon \rho_\infty(g)y = y\}$ is a null-set.*

Let us briefly describe our approach to the above result. An important distinction with the case of topological full groups is that full groups of measure-preserving actions are uncountable. This difficulty is counter-balanced by the presence of the so-called *uniform metric*, which turns them into topological groups endowed with a complete biinvariant metric, not separable in general. Although our main result does not mention this group topology, it relies crucially on it, as our first observation towards it is an automatic continuity result which actually makes our life much easier than in the topological context (see Cor. 3.10).

We also make a crucial use of a classification of the invariant random subgroups of the group of dyadic permutations due to Thomas and Tucker-Drob, while an important part of Matte Bon's proof is to first obtain an analogous classification in the topological context, and to somehow extend it to the full group using the topology. All in all, although our result is heavily inspired by Matte Bon's, the proof is different and much easier in our setup. We also use the following result which might be of independent interest. Recall that a group topology is called **SIN** (Small Invariant Neighborhoods) when there is a basis of neighborhoods of the identity consisting of conjugacy invariant neighborhoods.

Theorem B. *Let $\mathbb{G}$ be an ergodic full group, let τ be a SIN group topology on $\mathbb{G}$. Then τ is either the discrete or the uniform topology.*

Note that the SIN hypothesis above is important, otherwise one can always endow $\mathbb{G}$ with the weak topology induced by $\mathrm{Aut}(X, \mu)$.

[1] We allow partitions to contain empty sets throughout the whole paper.

Our main application of the classification theorem for boolean action is about full groups which are *separable* for the uniform topology, or equivalently which are countably generated. Such full groups are better-known as full groups of *p.m.p. equivalence relations* and are thus closely related to the well-developed field of orbit equivalence of p.m.p. actions. This connection is particularly apparent in Dye's reconstruction theorem, which can be restated as the fact that any abstract group isomorphism between full groups of p.m.p. aperiodic equivalence relations must come from an isomorphism between the equivalence relations themselves. One thus expects group properties of full groups to reflect properties of the equivalence relation. To our knowledge, only three such examples were known so far for p.m.p. equivalence relations:

- A non-trivial p.m.p. equivalence relation is ergodic if and only if its full group is simple [Eig81].
- A non-trivial p.m.p. equivalence relation is aperiodic if and only if its full group has no index two normal subgroup [LM16, Thm. 1.14].
- An ergodic p.m.p. equivalence relation is amenable if and only if every action by homeomorphism of its full group on a compact metrizable space admits a fixed point [KT10, Cor. 1.4].

We should also mention that the two last properties have natural continuous counterparts which are crucial in establishing the above algebraic properties using automatic continuity. In particular, the last item above uses Giordano and Pestov's beautiful results on extreme amenability for full groups [GP07]. Another topological property which reflects properties of the p.m.p. equivalence relation is the topological rank [LM14a], but it has no natural purely algebraic counterpart.

Let us now explain how our main result allows us to obtain one more theorem of the above kind. First note that every p.m.p. action of an equivalence relation on a probability space (Y, ν) induces a boolean action of its associated full group. The main result of the present paper has the following consequence: *every non-free ergodic boolean action of the full group of a p.m.p. equivalence relation comes from a measure-preserving action of the equivalence relation itself or one of its symmetric powers* (see Corollary 5.2).

We can then obtain a dynamical characterization of property (T) for p.m.p. equivalence relations, purely in terms of their full group, thus adding a fourth item to the above list.

Theorem C. *Let $\mathcal{R}$ be a p.m.p. ergodic equivalence relation. Then $\mathcal{R}$ has (T) if and only if all the non-free ergodic boolean actions of its full group on standard probability spaces are strongly ergodic.*

Let us highlight the key ingredients of the above result, apart from Theorem A. The direct implication (shown in Section 7.4) relies on the fact that every ergodic p.m.p. action of a p.m.p. equivalence relation with property (T) is strongly ergodic, and that if $\mathcal{R}$ has (T) then so do all its symmetric powers. On the other hand, the converse relies on a natural generalization to p.m.p. equivalence relations of the Connes-Weiss result [CW80] which states that if all p.m.p. ergodic actions of a countable group are strongly ergodic then the group has property (T) (see Theorem 8.13). This result was also obtained independently by Grabowski, Jardón-Sánchez and Mellick in a work in preparation.

While the above results are to be expected, we take the time to prove them in details in the last two sections. We also give a purely quantitative version of property (T) of an ergodic p.m.p. equivalence relations as a kind of strong property (T) of the full group, but with respect to the restricted class of *full* unitary representations as defined in Definition 6.11) (see Section 7.2). Since we are far from understanding all unitary representations of full groups, the following question remains open:

Question 1.3. Let $\mathcal{R}$ be a p.m.p. ergodic equivalence relation with property (T). Does its full group $[\mathcal{R}]$ have strong property (T) ?

Other applications of our main result provide restrictions on homomorphisms between full groups of p.m.p. equivalence relations, and we refer the reader to Section 5 for details. It seems natural to expect that such restrictions also hold in the more general setup of *orbit full groups of p.m.p. actions of locally compact groups* (see [CLM18]). It would also be nice to remove the ergodicity hypothesis in our main result, but it is not known whether aperiodic full groups satisfy the automatic continuity property, and Theorem B is not true anymore (see [LM16, Sec. 7] for a finer SIN topology which however does satisfy the automatic continuity property).

Another interesting question is to understand non p.m.p. boolean actions of ergodic full groups, and also non p.m.p. boolean actions of non p.m.p. ergodic full groups. This would probably require an understanding of quasi-invariant random subgroups of $\mathfrak{S}_{2^\infty}$.

Finally, we point out that our main results implies Dye's reconstruction theorem for ergodic full groups in a rather straightforward manner (see Section 4.6). However thanks to the work of Fremlin [Fre02, 384D], Dye's reconstruction theorem is now known to hold for a much wider class of groups of measure-preserving transformations, e.g. the so-called L^1 full groups considered in [LM18]. It would be interesting to know if Theorem A can be extended to such groups, possibly at the cost of requiring continuity of the boolean action.

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Contents

2 Boolean actions

In this work we will denote by (X, μ) or (Y, ν) atomless standard probability spaces, while (Z, λ) will usually be a standard probability space possibly with atoms. We denote by $\mathrm{MAlg}(X, \mu)$ the *measure algebra* of (X, μ), that is the boolean algebra consisting of measurable subsets of X up to measure zero. This algebra is complete, which means that any family $(A_i)_{i\in I}$ of elements has a supremum and an infimum. We will denote these by $\bigvee_{i\in I} A_i$ and $\bigwedge_{i\in I} A_i$ respectively. Whenever I is countable, they coincide with set-theoretic union and intersection and we will also use the notations $\bigcup_{i\in I} A_i$ and $\bigcap_{i\in I} A_i$. For notational convenience, we will use the topology induced by the metric d_μ on $\mathrm{MAlg}(X, \mu)$ given by $d_\mu(A, B) = \mu(A \triangle B)$ (this metric is actually complete and separable, but we won't make direct use of this fact). We will implicitly use the fact that all the boolean operations are uniformly continuous with respect to d_μ. The topology induced by d_μ on $\mathrm{MAlg}(X, \mu)$ will be the only topology that we consider there, in particular we may omit to mention this topology explicitly in proofs.

All the subsets of X that we will consider are seen as elements of $\mathrm{MAlg}(X, \mu)$. In particular they are measurable and we will neglect what happens on sets of measure zero. We now give the pointwise definition of the group $\mathrm{Aut}(X, \mu)$.

Definition 2.1. A *measure-preserving bijection* of (X, μ) is a Borel bijection $T : X \to X$ such that $\mu(T^{-1}(A)) = \mu(A)$ for all Borel $A \subseteq X$. Then $\mathrm{Aut}(X, \mu)$ is the group of all measure-preserving bijections up to measure zero: we identify T_1 and T_2 as soon as $\mu(\{x \in X \colon T_1(x) \neq T_2(x)\}) = 0$.

Every measure-preserving bijection T defines naturally an automorphism of the measure algebra $A \mapsto T(A)$, which gives rise to a faithful action by automorphisms of $\mathrm{Aut}(X, \mu)$ on $\mathrm{MAlg}(X, \mu)$. Moreover, every measure-preserving automorphism of $\mathrm{MAlg}(X, \mu)$ arises in this manner (see e.g. [Kec10, Sec. 1]), so one can also view $\mathrm{Aut}(X, \mu)$ as the group of all measure-preserving automorphisms of the measure algebra $\mathrm{MAlg}(X, \mu)$. While this second viewpoint is conceptually more satisfying, we will sometimes work with the first which is more concrete, especially when dealing when a single element of $\mathrm{Aut}(X, \mu)$.

Given $T \in \mathrm{Aut}(X, \mu)$, we put $\mathrm{Fix}(T) := \{x \in X : \ Tx = x\}$ and we denote by $\mathrm{supp}(T)$ its complement, called the **support** of T. The latter can be defined purely in measure algebra terms as $\mathrm{supp}\, T = \bigvee\{A \in \mathrm{MAlg}(X, \mu) \colon T(A) \cap A = \emptyset\}$.

The group $\mathrm{Aut}(X, \mu)$ can be equipped with two metrizable group topologies which will play an important role in our arguments:

- the **weak topology** for which a sequence $(T_n)_n$ converges to T if for every $A \subset X$, we have that $(\mu(T_n A \Delta A))_n$ converges to 0 as n tends to infinity;
- the **uniform topology** which is the topology induced by the *uniform metric* d_u defined by

$$d_u(S, T) \coloneqq \mu(\{x \in X \colon S(x) \neq T(x)\}) = \mu(\mathrm{supp}(ST^{-1})).$$

The weak topology is a Polish topology, which means that it is separable and admits a compatible complete metric. The uniform topology refines the weak topology and it is not separable. The metric d_u is complete and biinvariant: $d_u(T_1 S T_2, T_1 T T_2) = d_u(S, T)$ for all $S, T, T_1, T_2 \in \mathrm{Aut}(X, \mu)$.

Definition 2.2. A (p.m.p.) **boolean action** of a group G on a standard probability space (X, μ) is a homomorphism $\rho \colon G \to \mathrm{Aut}(X, \mu)$. If G is a topological group, we say that a boolean action is **continuous** if it is continuous when we endow $\mathrm{Aut}(X, \mu)$ with the weak topology.

Remark 2.3. Actions which are actually continuous when we endow $\mathrm{Aut}(X, \mu)$ with the stronger uniform topology will play an important role in our work, but since the adjective *uniformly continuous* could be confusing, we will not use it, and will explicitly state the topologies that we are considering instead.

Whenever a group G acts on a standard probability space (X, μ) by measure-preserving bijections, we automatically obtain a boolean action on (X, μ). Moreover, when the acting group is Polish and the action is Borel, the boolean action is continuous. Given a Polish group G, not every continuous boolean G-action on (X, μ) lifts to a Borel measure-preserving G-action in general, e.g. the natural (continuous) boolean $\mathrm{Aut}(X, \mu)$-action on (X, μ) does not have a so-called pointwise realization, see [GTW05]. However when G is locally compact second-countable, every continuous boolean G-action on (X, μ) lifts to a genuine Borel action on (X, μ) by measure-preserving bijections. We will often use without mention the much easier fact that when Γ is a countable discrete group, every boolean Γ-action lifts to a Γ-action on (X, μ) by measure-preserving bijections.

Given $H \leqslant G$ and a boolean action $\rho : G \to \mathrm{Aut}(X, \mu)$, we will denote the **restriction** of ρ to H by $\rho_{\restriction H} : H \to \mathrm{Aut}(X, \mu)$, which is a boolean H-action. Moreover, if $A \in \mathrm{MAlg}(X, \mu)$ is non zero and ρ (resp. $\rho_{\restriction H}$)-invariant, we call $\rho^A : G \to \mathrm{Aut}(A, \mu_A)$ (resp. $\rho^A_{\restriction H} : H \to \mathrm{Aut}(A, \mu_A)$) the **induced** boolean action on (A, μ_A), where μ_A is the probability measure on A obtained by renormalizing μ to a probability measure.

Definition 2.4. A boolean action $\rho \colon G \to \mathrm{Aut}(X, \mu)$ is **free** if the support of any element has full measure. It is **nowhere free** if its restriction to every invariant positive measure set is not free.

Any boolean action $\rho \colon G \to \mathrm{Aut}(X, \mu)$ splits as a free and a nowhere free boolean action. Indeed if we set

$$X_{\mathrm{free}} \coloneqq \bigwedge_{g \in G \setminus \{\mathrm{id}\}} \mathrm{supp}(\rho(g))$$

then it is clear that X_{free} is ρ-invariant, the boolean G-action induced by ρ on X_{free} is free and the boolean G-action induced by ρ on $X \setminus X_{\mathrm{free}}$ is nowhere free.

Definition 2.5. Given two boolean actions $\rho_X : G \to \mathrm{Aut}(X, \mu)$ and $\rho_Y : G \to \mathrm{Aut}(Y, \nu)$, we say that a measure-preserving map $\pi : (Y, \nu) \to (X, \mu)$ is a **factor map** if for every $g \in G$, and every $A \in \mathrm{MAlg}(X, \mu)$ we have that

$$\rho_Y(g)\pi^{-1}(A) = \pi^{-1}(\rho_X(g)A).$$

From the pointwise viewpoint, a measure-preserving map π is a factor map if and only if for every $g \in G$ and almost every $y \in Y$,

$$\pi(\rho_Y(g)y) = \rho_X(g)\pi(y).$$

It follows that for any factor map π we have that $\pi^{-1}(\mathrm{supp}(\rho_X(g)) \subseteq \mathrm{supp}(\rho_Y(g))$. Factor maps where the reverse inclusion holds play a key role in our work, so let us give them a name.

Definition 2.6. Given two boolean actions $\rho_X : G \to \mathrm{Aut}(X, \mu)$ and $\rho_Y : G \to \mathrm{Aut}(Y, \nu)$, we say that a factor map $\pi : (Y, \nu) \to (X, \mu)$ is **support-preserving** when $\pi^{-1}(\mathrm{supp}(\rho_X(g)) = \mathrm{supp}(\rho_Y(g))$ for all $g \in G$.

Remark 2.7. Let Γ be a countable group. Consider two boolean actions $\rho_X : \Gamma \to \mathrm{Aut}(X, \mu)$ and $\rho_Y : \Gamma \to \mathrm{Aut}(Y, \nu)$ viewed as Γ-actions by measure-preserving bijections, and let $\pi : (Y, \nu) \to (X, \mu)$ be a factor map. Straightforward considerations on actions on sets show that the following are equivalent:

- π is support-preserving.
- for almost every $y \in Y$, π induces a bijection from the orbit $\rho_Y(\Gamma)y$ to the orbit $\rho_X(\Gamma)\pi(y)$.
- for almost every $y \in Y$, the ρ_Y-stabilizer of y is equal to the ρ_X-stabilizer of $\pi(y)$.

2.1 The symmetric-diagonal action

Given a standard probability space (X,μ) and a natural number $n \geqslant 1$, we have a natural action of the symmetric group $\mathfrak{S}_n$ on X^n by permuting coordinates. We denote the quotient by $X^{\odot n} := X^n/\mathfrak{S}_n$ and the quotient map by $\pi_n : X^n \to X^{\odot n}$. The standard Borel space $X^{\odot n}$ can be endowed with the pushforward probability measure $\mu^{\odot n} := \pi_{n*}\mu^{\otimes n}$. The standard probability space $(X^{\odot n}, \mu^{\odot n})$ is called the *n-symmetric power* of (X,μ). We will also use the notation $(X^{\odot 0}, \mu^{\odot 0})$ for the trivial measure space, that is $X^{\odot 0}$ is a point. Finally, given a sequence $(\alpha_i)_{0\leqslant i<\infty}$, we will set

$$(X^{\odot(\alpha_i)_i}, \mu^{\odot(\alpha_i)_i}) := \left(\bigsqcup_i X^{\odot i}, \sum_i \alpha_i \mu^{\odot i}\right).$$

Given $A_1,\dots,A_n \subseteq X$ measurable we will use the notation

$$[A_1,\dots,A_n] := \pi_n(A_1\times\ldots\times A_n) \subseteq X^{\odot n}.$$

Clearly $[A_1,\dots,A_n] = [A_{\sigma(1)},\dots,A_{\sigma(n)}]$ for every $\sigma\in\mathfrak{S}_n$.

Lemma 2.8. *The measure algebra* $\mathrm{MAlg}(X^{\odot n},\mu^{\odot n})$ *is generated by elements of the form* $[A_1,\dots,A_n]$ *where* $A_1,...,A_n$ *is a* partition *of* X*.*

Proof. Let $X^{(n)} \subseteq X^n$ be the set of pairwise distinct n-uples in X. Then $\mu^n(X^{(n)}) = 1$ and it is no hard to check that the σ-algebra of $X^{(n)}$ is generated by products of the form $A_1\times\cdots\times A_n$ where $(A_1,...,A_n)$ is a measurable partition of X. The result follows by pushing forward to $X^{\odot n}$. ∎

We have a boolean action $\iota^{\odot n} : \mathrm{Aut}(X,\mu) \to \mathrm{Aut}(X^{\odot n},\mu^{\odot n})$ uniquely defined by

$$\iota^{\odot n}(T)[A_1,\dots,A_n] = [TA_1,\dots,TA_n],$$

From the pointwise viewpoint it is given by $\iota^{\odot n}(T)\pi_n(x_1,\dots,x_n) = \pi_n(Tx_1,\dots,Tx_n)$. For $n=0$, we let $\iota^{\odot 0}$ be the unique homomorphism from $\mathrm{Aut}(X,\mu)$ to the trivial group $\mathrm{Aut}(X^{\odot 0},\mu^{\odot 0})$, which is the trivial homomorphism. Note that for all $n\geqslant 1$ we have and all $T\in\mathrm{Aut}(X,\mu)$ we have

$$\begin{aligned}\mathrm{Fix}(\iota^{\odot n}(T)) =& [\mathrm{Fix}(T),\dots,\mathrm{Fix}(T)]\\ \mathrm{supp}(\iota^{\odot n}(T)) =& [\mathrm{supp}(T),X,\dots,X],\end{aligned}$$

hence $\mu^{\odot n}(\mathrm{Fix}(\iota^{\odot n}(T))) = \mu(\mathrm{Fix}(T))^n$. In particular, the map $\pi_n : X^n \to X^{\odot n}$ is a support-preserving factor map from ι^n to $\iota^{\odot n}$, where $\iota^n : \mathrm{Aut}(X,\mu)\to\mathrm{Aut}(X^n,\mu^{\otimes n})$ is the diagonal boolean action.

Gluing the boolean actions $\iota^{\odot n}$ together, for every sequence of non-negative reals $(\alpha_i)_{i<\infty}$, we obtain a boolean action $\iota^{\odot(\alpha_i)_i}$ as the diagonal embedding

$$\iota^{\odot(\alpha_i)_i} : \mathrm{Aut}(X,\mu) \to \prod_{i\geqslant 0}\mathrm{Aut}(X^{\odot i},\alpha_i\mu^{\odot i}) \leqslant \mathrm{Aut}(X^{\odot(\alpha_i)_i},\mu^{\odot(\alpha_i)_i}),$$
$$\text{where } \iota^{\odot(\alpha_i)_i}(T)(A) := \iota^{\odot i}(T)A \text{ if } A\subseteq X^{\odot i}.$$

Definition 2.9. Let $G\leqslant\mathrm{Aut}(X,\mu)$. Given a sequence of non-negative reals $(\alpha_i)_{i<\infty}$ which sums to 1, the **symmetric diagonal sum** boolean action $\iota^{\odot(\alpha_i)_i}$ is the boolean action $\iota^{\odot(\alpha_i)_i} : G\to\mathrm{Aut}(X^{\odot(\alpha_i)_i},\mu^{\odot(\alpha_i)_i})$ obtained by restricting the above defined map to G.

Observe that symmetric diagonal sums are uniform-to-uniform continuous. We will see that for ergodic full groups, they arise naturally from the non-free part of any boolean action.

Remark 2.10. Our work provides a reasonable classification for nowhere free boolean actions of $\mathrm{Aut}(X, \mu)$, and more generally of its full subgroups. Let us point out that there are however a priori many free boolean actions of $\mathrm{Aut}(X, \mu)$ (and hence of full groups). Indeed, given any sequence of non negative integers $(m_n)_{n\geqslant 1}$, where infinitely many m_n's are non zero, the natural boolean diagonal action $\prod_n (\iota^{\odot n})^{m_n}$ of $\mathrm{Aut}(X, \mu)$ on the infinite direct product

$$\prod_{n\geqslant 1} (X^{\odot n}, \mu^{\odot n})^{m_n}$$

is free. We wonder how many such actions there are up to conjugacy.

2.2 Diagonally support-preserving factorizable actions

We will be interested in boolean actions which factorize over a symmetric diagonal sum action in a support-preserving manner, which we will abbreviate as follows.

Definition 2.11. Let $G \leqslant \mathrm{Aut}(X, \mu)$ be a subgroup. A boolean action $\rho\colon G \to \mathrm{Aut}(Y, \nu)$ is **diagonally support-preserving factorizable** if there is $Y_\infty \subseteq Y$ such that

- Y_∞ is G-invariant and the action induced by ρ on Y_∞ is a free boolean action,
- the action induced by ρ on $Y \setminus Y_\infty$ factorizes over a symmetric diagonal sum in a support-preserving way.

The point of the above definition is to give a compact way of stating Theorem A. Note that it does not cover the uniqueness in it, but we will now see why uniqueness is automatic for the groups that we are considering.

Here is the general condition on the inclusion $G \leqslant \mathrm{Aut}(X, \mu)$ which will imply the uniqueness of such factorizations. This condition is a natural generalization of *absolute non freeness*, considered in [DG18, Def. 11].

Definition 2.12. A subgroup $G \leqslant \mathrm{Aut}(X, \mu)$ is called **highly absolutely non free** if for every measurable partition of X into n pieces $A_1, \ldots, A_n$ and every $\epsilon > 0$, there are $T_1, \ldots, T_n \in G$ with disjoint support such that for all $i \in \{1, \ldots, n\}$ we have

$$\mu(A_i \bigtriangleup \operatorname{supp} T_i) < \epsilon.$$

This condition is met by $\mathrm{Aut}(X, \mu)$, more generally ergodic full groups as well as some natural countable subgroups such as the group of dyadic permutations, see Lemma 2.17.

The next section will be devoted to the proof of the following proposition, which shows the uniqueness of the symmetric diagonal sum factor appearing in Theorem A.

Proposition 2.13. *Let $G \leqslant \mathrm{Aut}(X, \mu)$ be a highly absolutely non free subgroup and consider a diagonally support-preserving factorizable boolean action $\rho\colon G \to \mathrm{Aut}(Y, \nu)$. Then there is a unique partition up to null-sets $(Y_i)_{i=0,\ldots,\infty}$ of Y which is ρ-invariant and such that*

- *the action of G on Y_∞ is free;*
- *for every $i < \infty$, the action of G on Y_i factorizes onto the action $\iota^{\odot i}$ on $X^{\odot i}$ in a support-preserving manner, and the corresponding support-preserving factor map is unique.*

2.3 Proof of the key proposition for uniqueness in Theorem A

Before proving Proposition 2.13, we need a notion of independent interest.

Let $G \leqslant \mathrm{Aut}(X,\mu)$ be a subgroup. The associated isotropy subalgebra $\mathrm{Iso}_G(X,\mu)$ is the measure algebra generated by the supports (or equivalently the set of fixed points) of all the elements of G, that is $\mathrm{Iso}_G(X,\mu) \coloneqq \langle \operatorname{supp} g : g \in G\rangle \subseteq \mathrm{MAlg}(X,\mu)$.

Definition 2.14 ([Ver12])**.** A subgroup $G \leqslant \mathrm{Aut}(X,\mu)$ is said to be **totally non free** (TNF) if $\mathrm{Iso}_G(X,\mu) = \mathrm{MAlg}(X,\mu)$.

Clearly highly absolutely non free actions are totally non free. The following lemma is a simple observation which will yield Proposition 2.13.

Lemma 2.15. *Let $\rho_1 : G \to \mathrm{Aut}(Y_1,\nu_1)$ and $\rho_2 : G \to \mathrm{Aut}(Y_2,\nu_2)$ be two boolean actions of a group G. Suppose $\rho_2(G)$ is totally non free, then there is at most one support-preserving factor map $\phi : (Y_1,\nu_1) \to (Y_2,\nu_2)$.*

Proof. Suppose φ is a support-preserving factor map. By total non freeness the measure algebra generated by $\{\operatorname{supp}\rho_2(g) : g \in G\}$ is equal to $\mathrm{MAlg}(Y_2,\nu_2)$, so φ^{-1} is completely determined by the values it takes on $\{\operatorname{supp}\rho_2(g) : g \in G\}$. Since φ is support-preserving, for all $g \in G$ we have $\varphi^{-1}(\operatorname{supp}\rho_2(g)) = \operatorname{supp}\rho_1(g)$, which proves uniqueness. □

Proposition 2.16. *Let $G \leqslant \mathrm{Aut}(X,\mu)$ be a highly absolutely non free subgroup. Let $(\alpha_i)_{0\leqslant i<\infty}$ be a sequence of non-negative reals such that $\sum_i \alpha_i = 1$, and let $\iota^{\odot(\alpha_i)_i}$ be the associated symmetric diagonal sum boolean action. Then $\iota^{\odot(\alpha_i)_i}(G) \leqslant \mathrm{Aut}(X^{\odot(\alpha_i)_i},\mu^{\odot(\alpha_i)_i})$ is not discrete for the uniform topology and totally non free.*

Proof. Observe that by high absolute non freeness we have a sequence of non-trivial elements $T_n \in G$ such that $\mu(\operatorname{supp} T_n) \to 0$. Since $\sum_i \alpha_i = 1$, we can exchange the limit and obtain

$$\lim_n \mu^{\odot(\alpha_i)_i}(\operatorname{supp}\iota^{\odot(\alpha_i)_i}(T_n)) = \lim_n \sum_i \alpha_i(1-\mu(\mathrm{Fix}(T_n)^i) = \sum_i \alpha_i \lim_n (1-\mu(\mathrm{Fix}(T_n)^i) = 0$$

so that $\rho(G)$ is not discrete as claimed.

Let us now assume $\alpha_0 = 0$ so that the trivial part vanishes. For any $0 < i < \infty$ and $T_1,\dots,T_k \in G$ with pairwise disjoint supports, we have that

$$\bigcap_{j\leqslant k} \operatorname{supp}\iota^{\odot i}(g_j) = \begin{cases} \emptyset & \text{if } k > i;\\ [\operatorname{supp}(T_1),\dots,\operatorname{supp}(T_i)] & \text{if } k = i. \end{cases}$$

Observe that since G is highly absolutely non free, Lemma 2.8 implies that sets of the form $[\operatorname{supp}(T_1),\dots,\operatorname{supp}(T_i)]$ generate $\mathrm{MAlg}(X^{\odot i},\mu^{\odot i})$, where $T_1,\dots,T_k \in G$ have pairwise disjoint supports. Finally remark that

$$\bigvee_{\operatorname{supp}T_1,\dots,\operatorname{supp}T_n \text{ pairwise disjoint}} \left(\bigwedge_{j=1}^{n} \operatorname{supp}\iota^{\odot(\alpha_i)_i}(T_j)\right) = \bigvee_{i\geqslant n} X^{\odot i} \subseteq X^{\odot(\alpha_i)_i} \qquad (1)$$

which implies that G is totally non-free as claimed. □

The proof of Proposition 2.13 follows a very similar scheme.

Proof of Proposition 2.13. Set $Y_\infty \coloneqq \bigwedge_{g\in G\setminus\{\mathrm{id}\}} \operatorname{supp}(\rho(T))$. Then the boolean action induced by ρ on Y_∞ is free. Let us now set $Y_{<\infty} \coloneqq Y \setminus Y_\infty$. By assumption, there is a sequence $(\alpha_i)_{0\leqslant i<\infty}$ of non-negative reals such that $\sum_{i\geqslant 0}\alpha_i = 1-\mu(Y_\infty)$ and a support-preserving factor map $\pi : (Y_{<\infty},\nu) \to (X^{\odot(\alpha_i)_i}, \mu^{\odot(\alpha_i)_i})$. Set $Y_i \coloneqq \pi^{-1}(X^{\odot i})$, which has measure α_i. Observe that since π is support-preserving, Equation (1) implies that for every $n \geqslant 0$,

$$\bigvee_{i\geqslant n} Y_i = \bigvee_{\operatorname{supp} T_1,\dots,\operatorname{supp} T_n \text{ pairwise disjoint}} \left(\bigwedge_{i=1}^{n} \operatorname{supp}\rho(T_i)\right).$$

Therefore the Y_i's (and hence the α_i's) are uniquely defined. Finally since the action $\iota^{\odot(\alpha_i)_i}$ is totally non-free, Lemma 2.15 implies that the factor map π is unique. ☐

2.4 Diagonal factorization for the dyadic symmetric group

Fix the standard Cantor space $(X,\mu) \coloneqq (\{0,1\}^{\mathbb{N}}, \mathcal{B}(1/2)^{\otimes\mathbb{N}})$, which we equip with the product of 1/2 Bernoulli measures $\mathcal{B}(1/2) \coloneqq \frac{1}{2}(\delta_0+\delta_1)$. We view the symmetric groups $\mathfrak{S}_{2^n} \coloneqq \mathfrak{S}(\{0,1\}^n)$ as a subgroup of $\operatorname{Aut}(X,\mu)$ as follows: for each $(x_k)_k \in X$ and each $\sigma \in \mathfrak{S}_{2^n}$, we set

$$(\sigma\cdot(x_k)_k)_i \coloneqq \begin{cases} (\sigma((x_k)_{k\leqslant n}))_i & \text{if } i\leqslant n,\\ x_i & \text{if } i>n;\end{cases}$$

that is σ acts as the concatenation of the finite permutation in $\mathfrak{S}_{2^n}$ and the identity. Note that with this identification in mind, for each $n\in\mathbb{N}$ we have $\mathfrak{S}_{2^n} \leqslant \mathfrak{S}_{2^{n+1}} \leqslant \operatorname{Aut}(X,\mu)$. We then define the **dyadic symmetric group** $\mathfrak{S}_{2^\infty}$ as the union of all these finite groups:

$$\mathfrak{S}_{2^\infty} \coloneqq \bigcup_{n\in\mathbb{N}} \mathfrak{S}_{2^n} \leqslant \operatorname{Aut}(X,\mu).$$

Lemma 2.17. *The dyadic symmetric group $\mathfrak{S}_{2^\infty} \leqslant \operatorname{Aut}(X,\mu)$ is highly absolutely non free.*

Proof. This follows from the fact that any cylinder subset of X is the support of some element of $\mathfrak{S}_{2^\infty}$. ☐

Proposition 2.18. *Every measure-preserving action $\rho : \mathfrak{S}_{2^\infty} \to \operatorname{Aut}(Y,\nu)$ is diagonally support-preserving factorizable.*

The above proposition will be deduced from a theorem of Thomas and Tucker-Drob [TTD14]. This theorem is stated in the context of invariant random subgroups (IRS) and we will need to introduce a little bit of terminology before applying it.

For a countable group Γ, we denote by $\operatorname{Sub}(\Gamma) \subseteq \{0,1\}^\Gamma$ the Borel set of subgroups of Γ. The group Γ acts on $\operatorname{Sub}(\Gamma)$ by conjugation and an IRS of Γ is by definition a conjugacy invariant probability measure on $\operatorname{Sub}(\Gamma)$. Note that the stabilizer of any $\Lambda\in\operatorname{Sub}(\Gamma)$ for the Γ-action by conjugacy is equal to its normalizer.

Definition 2.19. An IRS $\zeta \in \operatorname{Prob}(\operatorname{Sub}(\Gamma))$ is **self-normalizing** if for ζ-almost every $\Lambda \in \operatorname{Sub}(\Gamma)$ we have that the normalizer of Λ (defined as $N_\Gamma(\Lambda) \coloneqq \{\gamma\in\Gamma\colon \gamma\Lambda\gamma^{-1} = \Lambda\}$) is equal to Λ.

Let us fix a p.m.p. action of the countable group Γ on (X,μ). We define the measurable stabilizer map $\operatorname{Stab}_\Gamma : X \to \operatorname{Sub}(\Gamma)$ which maps $x\in X$ to its stabilizer $\operatorname{Stab}_\Gamma(x) \coloneqq \{\gamma\in\Gamma : \gamma x = x\}$. The pushforward measure $(\operatorname{Stab}_\Gamma)_*(\mu)$ is an IRS, which we call the **IRS of the action**.

Lemma 2.20 ([Ver12, Prop. 4])**.** *The IRS of any totally non free action is self-normalizing.*

Here is a useful observation.

Lemma 2.21. *Consider a p.m.p. action $\alpha : \Gamma \to \mathrm{Aut}(X, \mu)$ of a countable group Γ, and denote by $\beta : \Gamma \to \mathrm{Aut}(\mathrm{Sub}(\Gamma), (\mathrm{Stab}_\Gamma)_*(\mu))$ the action associated to the IRS of α. Assume that the IRS $(\mathrm{Stab}_\Gamma)_*(\mu)$ is self-normalizing. Then the stabilizer map is support-preserving: for every $\gamma \in \Gamma$*

$$\mathrm{Stab}_\Gamma^{-1}(\operatorname{supp} \beta(\gamma)) = \operatorname{supp} \alpha(\gamma).$$

Proof. Since β is a factor of α via the stabilizer map, we have $\mathrm{Stab}^{-1}(\operatorname{supp} \beta(\gamma)) \subseteq \operatorname{supp} \alpha(\gamma)$ for every $\gamma \in \Gamma$. For the converse, the hypothesis that IRS is self-normalizing yields

$$\operatorname{supp} \beta(\gamma) = \{\Lambda \leqslant \Gamma : \gamma \notin N_\Gamma(\Lambda)\} = \{\Lambda \leqslant \Gamma : \ \gamma \notin \Lambda\}.$$

Using the definition of the stabilizer map, we finally have

$$\mu(\mathrm{Stab}^{-1}(\operatorname{supp} \beta(\gamma))) = \mu(\{x \in X : \alpha(\gamma)x \neq x\}) = \mu(\operatorname{supp} \alpha(\gamma))$$

so the inclusion $\mathrm{Stab}^{-1}(\operatorname{supp} \beta(\gamma)) \subseteq \operatorname{supp} \alpha(\gamma)$ must be an equality and we are done. □

Let us go back to the proof of Proposition 2.18. Assume that $(X, \mu) := (\{0,1\}^{\mathbb{N}}, \mathcal{B}(1/2)^{\otimes \mathbb{N}})$ and consider $\mathfrak{S}_{2^\infty} \leqslant \mathrm{Aut}(X, \mu)$. For every $i \geqslant 1$, we denote by ζ_i the IRS associated to the symmetric diagonal action of $\mathfrak{S}_{2^\infty}$ on $(X^{\odot i}, \mu^{\odot i})$.

Note that for every non negative sequence $(\alpha_i)_{i \geqslant 0}$ such that $\sum_{i \geqslant 0} \alpha_i = 1$, the IRS $\mathfrak{S}_{2^\infty} \curvearrowright (\mathrm{Sub}(\mathfrak{S}_{2^\infty}), \sum_{i \geqslant 0} \alpha_i \zeta_i)$ is conjugate to the diagonal sum action $\iota^{(\alpha_i)_i}$ of $\mathfrak{S}_{2^\infty}$ as a consequence of Lemma 2.17 and Proposition 2.16 .

Finally we denote by ζ_∞ the Dirac measure on the identity, which is the IRS associated to any free action and by ζ_0 the Dirac measure on $\mathfrak{S}_{2^\infty}$, which is the IRS associated to any trivial action. We now state Thomas and Tucker-Drob's main result from [TTD14].

Theorem 2.22. *Every IRS of $\mathfrak{S}_{2^\infty}$ can be written uniquely as an infinite convex combination of elements of the family $(\zeta_i)_{i=0,\dots,\infty}$.*

Note that the theorem can be restated by saying that the IRS associated to any nowhere free action of $\mathfrak{S}_{2^\infty}$ is a diagonal sum action. We are now ready to give the proof of Proposition 2.18.

Proof of Proposition 2.18. Consider a p.m.p. action ρ of $\mathfrak{S}_{2^\infty}$ on (Y, ν) and assume that the action is nowhere free. Denote by $\zeta := (\mathrm{Stab}_\Gamma)_*(\nu)$ the IRS associated to the action, which we view as a p.m.p. action using the same letter ζ. Denote by $(\alpha_i)_{0 \leqslant i \leqslant \infty}$ the coefficients provided by Theorem 2.22, since the action ρ is nowhere free $\alpha_\infty = 0$. As remarked above, it is a consequence of Lemma 2.17 and Proposition 2.16 that ζ is a diagonal sum action, and ρ factorizes onto ζ by construction. The only thing we have to show is that the factor map is support preserving.

Proposition 2.16 and Lemma 2.17 imply that ζ is totally non-free. Lemma 2.20 tells us that totally non free IRS are self normalizing, in particular so is ζ. Therefore we can use Lemma 2.21 to obtain that the factor map from ρ to ζ is support-preserving, which finishes the proof. □

3 Ergodic full groups

As in the last section, we will denote by (X, μ) a standard probability space without atoms. Let us start by recalling Dye's definition of full groups.

Definition 3.1. A **full group** is a subgroup $\mathbb{G}$ of $\mathrm{Aut}(X, \mu)$ which is stable under *cutting and pasting*, that is: whenever $(A_n)_n$ is a measurable partition of X and $(T_n)_n$ is a sequence of elements of $\mathbb{G}$ such that $(T_n(A_n))_n$ is also a partition of X, then the element $T \in \mathrm{Aut}(X, \mu)$ defined by $T(x) \coloneqq T_n(x)$ for $x \in A_n$ belongs to $\mathbb{G}$.

Given $G \leqslant \mathrm{Aut}(X, \mu)$, the smallest full group containing G is denoted by $[G]$ and called the full group generated by G; it can be constructed by cutting and pasting the elements of G. A subgroup $G \leqslant \mathrm{Aut}(X, \mu)$ is **ergodic** if the only G-invariant elements of $\mathrm{MAlg}(X, \mu)$ are X and $\emptyset$. Recall that ergodicity can also be detected at the level of the L^2 space:

Proposition 3.2 (see e.g. [KL16, Prop. 2.7])**.** *A subgroup $G \leqslant \mathrm{Aut}(X, \mu)$ is ergodic iff the only functions $f \in \mathrm{L}^2(X, \mu)$ such that $f \circ T = f$ for all $T \in G$ are the constant functions.*

The following theorem can be proven exactly as in [Fat78] (see [LM14b, Thm. 3.11] for the detailed proof in our setup).

Theorem 3.3. *A full group is ergodic if and only if it is simple.*

We will use the following important property of ergodic full groups.

Proposition 3.4 ([Dye59, Lem. 3.2])**.** *Let $\mathbb{G} \leqslant \mathrm{Aut}(X, \mu)$ be an ergodic full group. Then for every $A, B \in \mathrm{MAlg}(X, \mu)$ such that $\mu(A) = \mu(B)$, there is an involution $T \in \mathbb{G}$ such that $T(A) = B$ and $\mathrm{supp}(T) = A \cup B$.*

Since (X, μ) is atomless, every measurable set can be written as the disjoint union of two measurable sets of equal measure. The previous proposition thus implies the following.

Corollary 3.5. *Let $\mathbb{G} \leqslant \mathrm{Aut}(X, \mu)$ be an ergodic full group and let $A \in \mathrm{MAlg}(X, \mu)$. Then there is an involution $U \in \mathbb{G}$ whose support is equal to A.*

Corollary 3.5 clearly implies that the inclusion $\mathbb{G} \leqslant \mathrm{Aut}(X, \mu)$ is a highly absolutely non free and hence totally non free.

We will often use the following other well-known consequence of Proposition 3.4.

Proposition 3.6. *Let $\mathbb{G} \leqslant \mathrm{Aut}(X, \mu)$ be an ergodic full group. Then two involutions in $\mathbb{G}$ are conjugate if and only if their supports have the same measure.*

Proof. Since all the elements of $\mathbb{G}$ preserve the measure, if two involutions are conjugate then their supports must have the same measure.

Conversely, let $U, V \in \mathbb{G}$ be two involutions whose supports have the same measure. Since all standard Borel spaces are isomorphic, we can assume that $X = [0, 1]$ and in particular that we have a Borel linear order $<$ on X. Set $A \coloneqq \{x \in X : U(x) > x\}$ and $B \coloneqq \{x \in X : V(x) > x\}$, then it is straightforward to check that $\mathrm{supp}\, U = A \sqcup U(A)$ and $\mathrm{supp}\, V = B \sqcup V(B)$. Since U and V preserve the measure and $\mu(\mathrm{supp}\, U) = \mu(\mathrm{supp}\, V)$, we deduce that $\mu(A) = \mu(B)$. Applying Proposition 3.4, we find $T \in \mathbb{G}$ such that $T(A) = B$. Define a partial bijective measure-preserving map W by

$$W(x) \coloneqq \begin{cases} T(x) & \text{if } x \in A \\ VTU(x) & \text{if } x \in U(A). \end{cases}$$

The element W is only partially defined, but since its domain and range have equal measure, using Proposition 3.4 one can easily extend W to an element $\tilde{W}$ of $\mathbb{G}$. Then $\tilde{W}U\tilde{W}^{-1} = V$. □

The following result is key to all automatic continuity results on full groups. It is due to Ryzhikov [Ryz85] (see also the very neat proof in Miller's thesis [Mil04]).

Theorem 3.7 (Ryzhikov)**.** *Let $T \in \mathrm{Aut}(X,\mu)$. Then T can be written as the product of three involutions which belong to the full group generated by T.*

Finally, we note that the notion of support-preserving factor map as defined in Definition 2.6 behaves well with respect to full groups. In what follows, given a subgroup $G \leqslant \mathrm{Aut}(X,\mu)$, we denote by ι_G its inclusion in $\mathrm{Aut}(X,\mu)$, which is a boolean action.

Proposition 3.8. *Let $G \leqslant \mathrm{Aut}(X,\mu)$ and $\rho_Y : G \to \mathrm{Aut}(Y,\nu)$ be boolean action, suppose $\pi : (Y,\nu) \to (X,\mu)$ is a support-preserving factor map from ρ_Y to ι_G. Then there is a unique extension $[\rho_Y] : [G] \to \mathrm{Aut}(Y,\nu)$ of ρ_Y such that π is still support-preserving from $[\rho_Y]$ to $\iota_{[G]}$.*

Proof. Since π is support-preserving, if $U_1, U_2 \in G$ satisfy that $U_{1\restriction A} = U_{2\restriction A}$ for some measurable $A \subseteq X$, we have $\rho_Y(U_1)_{\restriction \pi^{-1}(A)} = \rho_Y(U_2)_{\restriction \pi^{-1}(A)}$. Now take $T \in [G]$, let (A_n) be measurable partition of X and (T_n) a sequence of elements of G such that $T_{\restriction A_n} = T_{n\restriction A_n}$ for every $n \in \mathbb{N}$, we are forced to define $[\rho_Y](T)$ by

$$[\rho_Y](T)_{\restriction \pi^{-1}(A_n)} = \rho(T_n)_{\restriction \pi^{-1}(A_n)}$$

so that π is still support-preserving. □

3.1 Topologies, automatic continuity and proof of Theorem B

Any full group $\mathbb{G} \leqslant \mathrm{Aut}(X,\mu)$ can be equipped with the uniform topology coming from $\mathrm{Aut}(X,\mu)$, that is the topology induced by the *uniform metric* d_u defined by

$$d_u(S,T) = \mu(\{x \in X \colon S(x) \neq T(x)\}) = \mu(\mathrm{supp}(ST^{-1})).$$

Full groups are always closed for the uniform topology, so the uniform metric restricts to a complete metric on them [Dye59, Lem. 5.4]. The full groups which are moreover separable for the uniform topology are exactly the full groups of *countable p.m.p. equivalence relations* [CLM16, Prop. 3.8]. The following result was proved by Kittrell and Tsankov for such full groups [KT10], but their proof extends verbatim to the general case, as was already observed by Ben Yaacov, Berenstein and Melleray in the case $\mathbb{G} = \mathrm{Aut}(X,\mu)$ [BYBM13] (see [LM14b, Thm. 3.18] for an explicit general statement).

Theorem 3.9. *Let $\mathbb{G} \leqslant \mathrm{Aut}(X,\mu)$ be an ergodic full group. Then every group homomorphism $\mathbb{G} \to H$, where H is a Polish group, has to be continuous with respect to the uniform topology on $\mathbb{G}$.*

We will only need the following corollary.

Corollary 3.10. *Let $\mathbb{G}$ be an ergodic full group, suppose that we have a boolean action $\rho : \mathbb{G} \to \mathrm{Aut}(Y,\nu)$. Then ρ is uniform to weak continuous.*

Proof. The corollary follows from the above theorem and the fact that $\mathrm{Aut}(X,\mu)$ equipped with the weak topology is a Polish group. $\square$

The uniform metric on $\mathrm{Aut}(X,\mu)$ is bi-invariant, which implies that the uniform topology is SIN (the identity element admits a basis of conjugacy invariant neighborhoods). We now prove that the discrete topology is the only other SIN topology on any ergodic full group.

Theorem 3.11. *Let $\mathbb{G} \leqslant \mathrm{Aut}(X,\mu)$ be an ergodic full group, let τ be a SIN group topology on $\mathbb{G}$. Then τ is either the discrete or the uniform topology.*

We will use the fact that the uniform topology can be interpreted as a topology of uniform convergence:

Proposition 3.12 ([Dye59, Lem. 5.4])**.** *The metric δ_u on $\mathrm{Aut}(X,\mu)$ defined by*

$$\delta_u(T,U) \coloneqq \sup_{A\in\mathrm{MAlg}(X,\mu)} \mu(T(A) \bigtriangleup U(A)) = 2 \sup_{A\in\mathrm{MAlg}(X,\mu)} \mu(T(A) \setminus U(A))$$

is equivalent to the uniform metric d_u.

In the proof of the theorem we will need the following lemma.

Lemma 3.13. *Let $\mathbb{G}$ be an ergodic full group and let $\mathcal{V} \subseteq \mathbb{G}$ be a conjugacy-invariant symmetric subset. Let $T \in \mathcal{V}$ and $A \subseteq X$ measurable. Then $\mathcal{V}^2$ contains every involution whose support has measure at most*

$$2\mu(TA \setminus A).$$

Proof. Let $\varepsilon \in [0, \mu(TA \setminus A)]$. We have to show that $\mathcal{V}^2$ contains all involutions whose support has measure 2ε.

Set $B \coloneqq A \setminus TA$. Let U be an involution whose support has measure ε and is contained in B. Clearly TUT^{-1} is an involution whose support is contained in TB. Since $B \cap TB = \emptyset$, $U(TUT^{-1})$ is an involution and its support has measure $2\mu(\mathrm{supp}(U)) = 2\varepsilon$. Since $\mathcal{V}^2$ is conjugacy invariant $UTUT^{-1} \in \mathcal{V}T\mathcal{V}T^{-1} = \mathcal{V}^2$. Therefore $\mathcal{V}^2$ contains an involution whose support has measure 2ε. Finally since $\mathcal{V}^2$ is conjugacy invariant, Proposition 3.6 implies that $\mathcal{V}^2$ must contain all involution whose support has measure 2ε as wanted. $\square$

Proof of Theorem 3.11. We start by proving τ has to refine the uniform topology. By Proposition 3.12, a basis of neighborhoods of the identity for the uniform topology is given by sets of the form

$$\mathcal{V}_\varepsilon \coloneqq \{T \in \mathbb{G} \colon \mu(T(A) \setminus A) < \varepsilon \text{ for all } A \subseteq X \text{ measurable}\}.$$

Therefore let us fix $\varepsilon \in (0, \frac{1}{2})$. We have to exhibit a τ-neighborhood of the identity $\mathcal{W}$ such that $\mathcal{W} \subseteq \mathcal{V}_\varepsilon$.

Take an involution U whose whose support has measure 2ε. Since the topology is Hausdorff and SIN, we find a conjugacy-invariant τ-neighborhood of the identity $\mathcal{W}'$ which does not contain U. Choose a conjugacy-invariant and symmetric τ-neighborhood of the identity $\mathcal{W}$ such that $\mathcal{W}^2 \subseteq \mathcal{W}'$. Then Lemma 3.13 and the fact that $\mathcal{W}^2$ does not contain U imply that $\mu(TA \setminus A) < \varepsilon$ for every $T \in \mathcal{W}$, that is $\mathcal{W} \subseteq \mathcal{V}_\varepsilon$. So τ does refine the uniform topology.

Let us now show that if τ is not discrete, it has to be equal to the uniform topology. So given a τ-neighborhood of the identity $\mathcal{W}$, we have to show that there is $\varepsilon > 0$ such that $\mathcal{W} \supseteq \{T \in \mathbb{G} :\ d_u(T, \mathrm{id}) \leqslant \varepsilon\}$.

Since τ is not discrete, there there exists a conjugacy-invariant, symmetric τ-neighborhood of the identity $\mathcal{V} \subseteq \mathcal{W}$ such that $\mathcal{V} \neq \{\mathrm{id}\}$ and $\mathcal{V}^6 \subset \mathcal{W}$. Let $T \in \mathcal{V} \setminus \{\mathrm{id}\}$ and let $A \subseteq X$ such that $\mu(TA \setminus A) > 0$. Then Lemma 3.13, implies that $\mathcal{V}^2$ contains every involution whose support has measure at most $\epsilon := 2\mu(TA \setminus A)$. By Ryzhikov theorem, Theorem 3.7, every element of $\mathbb{G}$ whose support has measure at most ϵ is the product of 3 involution whose support also has measure at most ϵ. So $\mathcal{W} \supseteq \mathcal{V}^6$ contains every element whose support has measure at most than ϵ as claimed. □

We will only use the following corollary when the target group G is $\mathrm{Aut}(Y, \nu)$ equipped with the uniform topology.

Corollary 3.14. *Let $\mathbb{G}$ be an ergodic full group and let G be a group with a SIN group topology τ. Assume that we have a homomorphism $\rho : \mathbb{G} \to G$. If $\rho(\mathbb{G})$ is not discrete, then ρ is uniform-to-τ continuous.*

Proof. We may as well assume there is some $T \in \mathbb{G}$ such that $\rho(T) \neq 1$ because otherwise ρ is clearly continuous. It follows that ρ is injective since $\mathbb{G}$ is simple by Theorem 3.3. Consider the pullback topology $\rho^*(\tau)$ on $\mathbb{G}$, that is the topology generated by the pre-images of open sets in G. Since ρ is injective and τ is Hausdorff, $\rho^*(\tau)$ is Hausdorff. Moreover τ is SIN so $\rho^*(\tau)$ is a SIN group topology. So by the above theorem, if the image of $\mathbb{G}$ is not discrete, $\rho^*(\tau)$ has to be the uniform topology and hence ρ is continuous (actually an embedding) of topological groups. □

We conclude this section by recalling a fundamental tool in the study of full groups, namely induced bijections. Given $T \in \mathrm{Aut}(X, \mu)$ and $A \subseteq X$ measurable, for almost all $x \in A$ there is $n \geqslant 1$ such that $T^n(x) \in A$. If we define $\tau_A(x)$ as the smallest such n, and let $\tau_A(x) = 0$ whenever $x \notin A$, the bijection T_A **induced** by T on A is by definition given by: for all $x \in X$,

$$T_A(x) = T^{\tau_A(x)}(x).$$

Observe that T_A always belongs to the full group $[T]$ generated by T. The following well-known continuity property follows from [Kea70, Lem. 3].

Proposition 3.15. *Let $T \in \mathrm{Aut}(X, \mu)$. Endow the measure algebra $\mathrm{MAlg}(X, \mu)$ with the metric $d_\mu(A, B) := \mu(A \triangle B)$ and the full group of T with the uniform topology. Then the map*

$$\begin{array}{rcl} \mathrm{MAlg}(X, \mu) & \to & [T] \\ A & \mapsto & T_A \end{array}$$

is continuous.

3.2 P.m.p. equivalence relations and their full groups

Let us start by recalling some terminology; the unfamiliar reader is refered to [KM04]. Let X be a standard Borel space. A **Borel equivalence relation** $\mathcal{R}$ on X is an equivalence relation which is Borel when one see it as a subset of the product space $\mathcal{R} \subseteq X \times X$ equipped with the product σ-algebra. We will be mainly interested in equivalence relation with *countable classes*, also called countable equivalence relations. The prototypical example of Borel countable equivalence relations is obtained as follows: given a countable group Γ acting on X by Borel bijections, we get a Borel equivalence relation $\mathcal{R}_\Gamma$ on X whose classes are the Γ-orbits, defined by

$$(x, y) \in \mathcal{R}_\Gamma \text{ if } y \in \Gamma \cdot x.$$

The Feldman-Moore theorem ensures us that conversely every countable Borel equivalence relation comes from a Borel action of some countable group Γ on X.

Define the Borel full group of a countable Borel equivalence relation $\mathcal{R}$ as the group of all Borel bijections T of X such that $(x, T(x)) \in \mathcal{R}$ for every $x \in X$. Note that if $\mathcal{R}$ is a countable Borel equivalence relation and $T : X \to X$ is a Borel function such that T restricts to a bijection on each $\mathcal{R}$-class, then T belongs to the Borel full group of $\mathcal{R}$. Also observe that if $\mathcal{R} = \mathcal{R}_\Gamma$, then $T : X \to X$ is in the Borel full group of $\mathcal{R}$ if and only if there is a partition of X into Borel subsets $(A_\gamma)_{\gamma\in\Gamma}$ such that $(\gamma A_\gamma)_{\gamma\in\Gamma}$ is also a partition of X and for all $\gamma \in \Gamma$ and all $x \in A_\gamma$, $T(x) = \gamma \cdot x$.

Now suppose that (X, μ) be a standard probability space, i.e. X is a standard Borel space equipped with a Borel probability measure. It follows from the previous discussion that if we now have a Borel action by measure-preserving bijections of a countable group Γ on (X, μ), then every element of the Borel full group of $\mathcal{R}_\Gamma$ preserves the measure μ. This means that $\mathcal{R}_\Gamma$ is a p.m.p. equivalence relation as defined below.

Definition 3.16. A countable Borel equivalence relation $\mathcal{R}$ on a standard probability space (X, μ) is **measure-preserving** when the elements of its Borel full group preserve the measure μ. We will simply call **p.m.p.** equivalence relations such equivalence relations.

Definition 3.17. The **full group** of a Borel equivalence relation $\mathcal{R}$ is the group

$$[\mathcal{R}] \coloneqq \{T \in \mathrm{Aut}(X, \mu) \colon (T(x), x) \in \mathcal{R} \text{ for all } x \in X\}.$$

Observe that when $\mathcal{R}$ is p.m.p., the group $[\mathcal{R}]$ is the quotient of the Borel full group of $\mathcal{R}$ by the normal subgroup of elements whose support has measure zero. It is not hard to check that $[\mathcal{R}]$ is always a full group on the sense of Definition 3.1. As recalled before, the restriction of the uniform distance d_u on $[\mathcal{R}]$ is complete separable and thus defines a Polish group topology on $[\mathcal{R}]$.

Remark 3.18. As discovered in [CLM16], there are actually many full groups of uncountable Borel equivalence relations whose full group is a Polish group.

We say that a p.m.p. equivalence relation $\mathcal{R}$ is **ergodic** if every measurable $A \subseteq X$ which is a union of $\mathcal{R}$-classes has measure either 0 or 1. When $\mathcal{R} = \mathcal{R}_\Gamma$, this is equivalent to every Γ-invariant set having measure 0 or 1, which is equivalent to every almost everywhere invariant set having measure 0 or 1. It follows that a p.m.p. equivalence relation is ergodic if and only if its full group is.

Definition 3.19. Say that a p.m.p. equivalence relation $\mathcal{R}$ is **finite** when it has only finite classes, and **hyperfinite** when it can be written as an increasing union of p.m.p. finite subequivalence relations.

Here is the prototypical of an ergodic hyperfinite equivalence relation. Let us work in the standard Cantor space $(X, \mu) \coloneqq (\{0, 1\}^{\mathbb{N}}, \mathcal{B}(1/2)^{\otimes\mathbb{N}})$, equipped with the standard Bernoulli product measure.

Definition 3.20. Denote by $\mathcal{R}_0$ the equivalence relation on $\{0, 1\}^{\mathbb{N}}$ defined by $((x_n), (y_n)) \in \mathcal{R}_0$ if and only if there is $N \in \mathbb{N}$ such that $x_n = y_n$ for all $n \geqslant N$.

Recall from Section 2.4 that we have a natural inclusion of the dyadic symmetric group $\mathfrak{S}_{2^\infty}$ into $\mathrm{Aut}(X,\mu)$. Observe that the equivalence relation $\mathcal{R}_0$ is the equivalence relation associated to this p.m.p. action, in particular $\mathcal{R}_0$ is a p.m.p. equivalence relation. Note that $\mathcal{R}_0$ can be written as the increasing union of the finite p.m.p. equivalence relations $\mathcal{S}_k$ where $\mathcal{S}_k$ is given by $((x_n),(y_n)) \in \mathcal{S}_k$ if and only if $x_n = y_n$ for all $n \geqslant k$. We thus see that $\mathcal{R}_0$ is hyperfinite, and it is not hard to check that it is ergodic.

We finally define the important notion of orbit equivalence between p.m.p. equivalence relations.

Definition 3.21. Let $\mathcal{R}$ and $\mathcal{S}$ be two p.m.p. equivalence relations on (X,μ) and (Y,ν) respectively. A measure-preserving bijection $\varphi : (X,\mu) \to (Y,\nu)$ is called an **orbit equivalence** between $\mathcal{R}$ and $\mathcal{S}$ if up to restricting $\mathcal{R}$ and $\mathcal{S}$ to full measure subsets, we have $\varphi \times \varphi(\mathcal{R}) = \mathcal{S}$. When there is such an orbit equivalence between $\mathcal{R}$ and $\mathcal{S}$ we say that $\mathcal{R}$ and $\mathcal{S}$ are orbit equivalent.

By a theorem of Dye, the equivalence relation $\mathcal{R}_0$ is the unique hyperfinite ergodic equivalence relation up to orbit equivalence ([Dye59, Thm. 3], see also [KM04, Sec. 7] for a statement and proof in a setup closer to ours. Note that Dye only consider full groups and that the notion of hyperfiniteness is then stated as *approximate finiteness* of the full group; see [LM14b, Prop. 1.57] for the equivalence between the two notions).

It is well-known that $\varphi : (X,\mu) \to (Y,\nu)$ is an orbit equivalence between p.m.p. equivalence relations $\mathcal{R}$ on (X,μ) and $\mathcal{S}$ on (Y,ν) if and only if $\varphi[\mathcal{R}]\varphi^{-1} = [\mathcal{S}]$ (see for instance [LM14b, Prop. 1.30]). Moreover, Dye's reconstruction theorem [Dye63, Thm. 2] yields that any abstract isomorphism between ergodic full groups must come from an isomorphism of the underlying measured spaces so that full groups of ergodic p.m.p. equivalence relations completely remember the equivalence relation up to orbit equivalence. Our main result allows us to recover Dye's reconstruction theorem with a completely different proof (see Section 4.6).

3.3 Products and symmetric products of p.m.p. equivalence relations

Definition 3.22. Let $\mathcal{R}$ be a countable p.m.p. equivalence relation on (X,μ) and let $n \in \mathbb{N}^*$ a positive integer. Then we define the **product equivalence relation** $\mathcal{R}^n$ on (X^n, μ^n) by declaring

$$((x_i)_{i=1}^n, (y_i)_{i=1}^n) \in \mathcal{R}^n \text{ if for every } i \in \{1,\dots,n\},\ (x_i,y_i) \in \mathcal{R}.$$

Recall the definition of the symmetric n-power $(X^{\odot n}, \mu^{\odot n})$ from Section 2.1. We then define the **symmetric product equivalence relation** $\mathcal{R}^{\odot n}$ on $(X^{\odot n}, \mu^{\odot n})$ by declaring

$$([x_i]_{i=1}^n, [y_i]_{i=1}^n) \in \mathcal{R}^{\odot n} \text{ if there is } \sigma \in \mathfrak{S}_n \text{ such that for all } i \in \{1,\dots,n\},\ (x_i, y_{\sigma(i)}) \in \mathcal{R}.$$

One can easily check that both $\mathcal{R}^n$ and $\mathcal{R}^{\odot n}$ are countable p.m.p. equivalence relations. Here is one way of seeing this.

Assume that the equivalence relation $\mathcal{R}$ is generated by an action α of the countable group Γ on the probability space (X,μ). Then the action α^n of Γ^n on X^n defined by

$$\alpha^n(\gamma_1,\dots,\gamma_n)(x_1,\dots,x_n) = (\alpha(\gamma_1)x_1,\dots,\alpha(\gamma_n)x_n)$$

generates the equivalence relation $\mathcal{R}^n$. Now consider the group $\Gamma^n \rtimes \mathfrak{S}_n$ where $\mathfrak{S}_n$ acts on Γ^n by permuting the copies. Then the Γ^n-action α^n on X^n naturally extends to a $\Gamma^n \rtimes \mathfrak{S}_n$-action by declaring

$$\alpha^n((\gamma_1,\dots,\gamma_n)\sigma)(x_1,\dots,x_n) \coloneqq \left(\alpha(\gamma_1)x_{\sigma^{-1}(1)},\dots\alpha(\gamma_n)x_{\sigma^{-1}(n)}\right).$$

Let $D \subseteq X^n$ be a measurable fundamental domain of the action of $\mathfrak{S}_n$ on the full measure subset $X^{(n)} = \{(x_1, \dots, x_n) \colon x_i \neq x_j \text{ for all } i \neq j\}$ (e.g. fix a Borel linear order $<$ on X and let $D \coloneqq \{(x_1, \dots, x_n) \in X^n \colon x_1 < x_2 < \dots < x_n\}$). The projection $\pi_n \colon X^{(n)} \to X^{\odot n}$ restrict to an isomorphism $\pi_n \colon D \to X^{\odot n}$ which scales the measure by $n!$. Then, it is easy to show that π_n defines an *orbit equivalence*[2] between the restriction of the orbit equivalence relation of $\alpha^n(\Gamma^n \rtimes \mathfrak{S}_n)$ on D and $\mathcal{R}^{\odot n}$, that is given $x, y \in D$

$$\text{there is } g \in \Gamma^n \rtimes \mathfrak{S}_n \text{ such that } y = \alpha^n(g)x \text{ if and only if } (\pi_n(x), \pi_(y)) \in \mathcal{R}^{\odot n}.$$

This actually shows the following lemma.

Lemma 3.23. *The p.m.p. equivalence relation $\mathcal{R}^{\odot n}$ is stably orbit equivalent*[3] *to the equivalence relation*

$$\mathcal{R}^n \rtimes \mathfrak{S}_n \coloneqq \{((x_i)_{i=1}^n, (y_i)_{i=1}^n) \colon \exists \sigma \in \mathfrak{S}_n \forall i \in \{1, \dots, n\}, (x_i, y_{\sigma(i)}) \in \mathcal{R}\}. \quad \square$$

The construction of $\mathcal{R}^{\odot n}$ is naturally compatible with the symmetric diagonal action as follows.

Lemma 3.24. *Let $\mathcal{R}$ be an ergodic countable p.m.p. equivalence relation and let $n \in \mathbb{N}^*$ be a positive integer. Consider the symmetric diagonal boolean action $\iota^{\odot n} \colon [\mathcal{R}] \to \mathrm{Aut}(X^{\odot n}, \mu^{\odot n})$. Then $\iota^{\odot n}([\mathcal{R}]) \leqslant [\mathcal{R}^{\odot n}]$ and the full group generated by $\iota^{\odot n}([\mathcal{R}])$ is equal to $[\mathcal{R}^{\odot n}]$.*

Proof. By the Feldman-Moore theorem, we may and do fix a measure-preserving action α of a countable group Γ such that $\mathcal{R} = \mathcal{R}_\Gamma$. We thus get an action α^n of $\Gamma^n \rtimes \mathfrak{S}_n$ on X^n as explained above.

Let again $D \subseteq X^n$ be a measurable fundamental domain of the action of $\mathfrak{S}_n$ on X^n. By the discussion which precedes the lemma, for every $T \in [\mathcal{R}^{\odot n}]$ there is a measurable partition $\{D^g\}_{g \in \Gamma^n \rtimes \mathfrak{S}_n}$ of D such that

$$T\pi_n(x_1, \dots, x_n) = \pi_n(\alpha^n(g)(x_1, \dots, x_n))$$

for every $(x_1, \dots, x_n) \in D^g$. Up to discarding the null set of non-injective n-uples in X^n, we have a countable cover

$$\{A_1^i \times \dots \times A_n^i\}_{i \in \mathbb{N}}$$

of X^n such that for every $i \in \mathbb{N}$, the sets $A_1^i, \dots, A_n^i$ are pairwise disjoint.

Let us now fix $g = (\gamma_1, \dots, \gamma_n)\sigma \in \Gamma^{\odot n}$. Observe that for all $A_1, \dots, A_n \subseteq X$ measurable we have

$$\alpha^n(g)(A_1 \times \dots \times A_n) = \alpha(\gamma_1)A_{\sigma^{-1}(1)} \times \dots \times \alpha(\gamma_n)A_{\sigma^{-1}(n)}.$$

Now for every $i, j \in \mathbb{N}$ and $k \in \{1, \dots, n\}$, we set

$$C_k^{g,i,j} \coloneqq A_k^i \cap \alpha(\gamma_k)^{-1} A^j_{\sigma^{-1}(k)}.$$

[2]Two p.m.p. equivalence relations $\mathcal{R}$ and $\mathcal{S}$ are orbit equivalent if there is a p.m.p. map which induces an isomorphism between $\mathcal{R}$ and $\mathcal{S}$ up to a null set.

[3]Two p.m.p. equivalence relations $\mathcal{R}$ and $\mathcal{S}$ on (X, μ) are stably orbit equivalent if there are positive measure sets A and B such that their respective restrictions to A and B (equipped with appropriate renormalized measures) are orbit equivalent.

Since $\{A_1^i \times \ldots \times A_n^i\}_{i\in\mathbb{N}}$ is a cover of X^n, we first have that

$$\left\{\alpha^n(\gamma_1, ..., \gamma_n)^{-1}(A^j_{\sigma^{-1}(1)} \times \ldots \times A^j_{\sigma^{-1}(n)})\right\}_{j\in\mathbb{N}}$$

is a cover of X^n. We deduce that $\{C_1^{g,i,j} \times \ldots \times C_n^{g,i,j}\}_{i,j\in\mathbb{N}}$ forms a cover of X^n and hence of D^g. Let us now fix $i, j \in \mathbb{N}$. By construction for all $k \in \{1, \ldots, n\}$ we have $\alpha(\gamma_k)C_k^{g,i,j} \subseteq A^j_{\sigma^{-1}(k)}$, so the sets

$$\alpha(\gamma_1)C_1^{g,i,j}, \ldots, \alpha(\gamma_n)C_n^{g,i,j}$$

are pairwise disjoint. Therefore using Proposition 3.4 we can find $U \in [\mathcal{R}]$ such that for all $k \in \{1, \ldots, n\}$,

$$U(x) = \alpha(\gamma_k)x \text{ for every } x \in C_k^{g,i,j}.$$

Finally remark that by construction, for every $(x_1, \ldots, x_n) \in D^g \cap C_1^{g,i,j} \times \ldots \times C_n^{g,i,j}$ we have

$$\begin{aligned}
\iota^{\odot n}(U)\pi_n(x_1, \ldots, x_n) &= \pi_n(\alpha(\gamma_1)x_1, \ldots, \alpha(\gamma_n)x_n) \\
&= \pi_n(\alpha^n(g)(x_1, \ldots, x_n)) \\
&= T\pi_n(x_1, \ldots, x_n).
\end{aligned}$$

Therefore T can be obtained by cutting and pasting elements of $\iota^{\odot n}([\mathcal{R}])$ and hence the proof is completed. □

3.4 Boolean actions of full groups coming from actions of the equivalence relations

Let $\mathcal{R}$ be a p.m.p. equivalence relation on the probability space (X, μ). Let $\mathcal{S}$ be a p.m.p. equivalence relation on (Y, ν) and let $\pi : Y \to X$ a measure-preserving map.

- We say that π is a **factor map** if for every $(y_1, y_2) \in \mathcal{S}$, one has that $(\pi(y_1), \pi(y_2)) \in \mathcal{R}$.

Clearly π is a factor map if and only if the map $\pi \times \pi : Y \times Y \to X \times X$ restrict to a map from $\mathcal{S}$ to $\mathcal{R}$. Therefore we will also use the convention to denote a factor map as a map $\pi\colon \mathcal{S} \to \mathcal{R}$. Observe also that Y can be seen inside $\mathcal{S}$ as the set $\Delta_Y \coloneqq \{(y, y)\colon y \in Y\}$ and π is compatible with this restriction.

- We say that a factor map $\pi\colon \mathcal{S} \to \mathcal{R}$ is **class-bijective**, if there is $Y_0 \subseteq Y$ of full measure such that for all $y_1, y_2 \in Y_0$, if $(y_1, y_2) \in \mathcal{S}$ and $y_1 \neq y_2$, then $\pi(y_1) \neq \pi(y_2)$.

If $\pi\colon \mathcal{S} \to \mathcal{R}$ is a class-bijective factor map, then $\mathcal{S}$ is often called a **class-bijective extension** of $\mathcal{R}$ and we say that π is a **measure-preserving action** of $\mathcal{R}$.

Example 3.25. Let $\mathcal{R}$ be a p.m.p. equivalence relation on (X, μ), let $n \geqslant 1$. Then $\mathcal{R}^n$ is a class-bijective extension of $\mathcal{R}^{\odot n}$ via $\pi : X^n \to X^{\odot n}$. Indeed for almost every $(x_1, \ldots, x_n)$, we have that the x_i's belong to distinct $\mathcal{R}$-classes. This yields that the restriction of π to the $\mathcal{R}^n$ equivalence class of $(x_1, \ldots, x_n)$ (which is equal to $[x_1]_{\mathcal{R}} \times \cdots \times [x_n]_{\mathcal{R}}$) is injective as wanted.

Measure-preserving actions of equivalence relations give rise to boolean actions of the associated full groups.

Lemma 3.26. *Let $\mathcal{R}$ be a p.m.p. equivalence relation on (X,μ) and $\mathcal{S}$ be a p.m.p. equivalence relation on (Y,ν). Let π be a class-bijective factor map from $\mathcal{S}$ to $\mathcal{R}$ and for $T\in[\mathcal{R}]$ and $y\in Y$ define*

$$\rho_\pi(T)y \text{ as the unique } y'\in Y \text{ such that } \pi(x)=T\pi(y) \text{ and } (y,y')\in\mathcal{S}.$$

Then $\rho_\pi\colon[\mathcal{R}]\to[\mathcal{S}]$ is a group homomorphism. Moreover ρ_π is support-preserving: for all $T\in[\mathcal{R}]$ we have $\operatorname{supp}(\rho_\pi(T))=\pi^{-1}(\operatorname{supp}(T))$, and the full group generated by $\rho_\pi([\mathcal{R}])$ is equal to $[\mathcal{S}]$.

Proof. Fix $T\in[\mathcal{R}]$, then remark that

$$\pi^{-1}(\{(x,Tx)\in\mathcal{R}\colon x\in X\})=\{(y,\rho_\pi(T)y)\in\mathcal{S}\colon y\in Y\}$$

so the graph of $\rho_\pi(T)$ is Borel, which implies that $\rho_\pi(T)$ is a Borel map (see [Kec95, Thm. 14.12]). By construction $\rho_\pi(T)$ preserves $\mathcal{S}$-classes, and since π is class-bijective, $\rho_\pi(T)$ induces a bijection on each $\mathcal{S}$-class. It follows that $\rho_\pi(T)$ is in the full group of $\mathcal{S}$. To see that ρ_π is a group homomorphism, observe that

$$\pi(\rho_\pi(T_1)\rho_\pi(T_2)y)=T_1\pi(\rho_\pi(T_2)y))=T_1T_2\pi(y).$$

We have by definition $\rho_\pi(T)(y)=y$ if and only if $T(\pi(y))=\pi(y)$, so ρ_π is support-preserving.

Finally, to see that the full group generated by $\rho_\pi([\mathcal{R}])$ is equal to $[\mathcal{S}]$, let $\Gamma\leqslant[\mathcal{R}]$ be a countable group such that $\mathcal{R}=\mathcal{R}_\Gamma$, then since π is class-bijective we have that $\mathcal{S}$ is generated by the action ρ_π of Γ. In particular the full group of $\mathcal{S}$ is contained in the full group generated by $\rho_\pi(\Gamma)$. Since the full group generated by $\rho_\pi(\Gamma)$ is contained in the full group generated $\rho_\pi([\mathcal{R}])$, which is itself contained in $[\mathcal{S}]$ by construction, the conclusion follows. $\square$

The above lemma has a converse that will play an important role.

Lemma 3.27. *Let $\mathcal{R}$ be a p.m.p. equivalence relation on (X,μ) and consider a boolean action $\rho:[\mathcal{R}]\to\operatorname{Aut}(Y,\nu)$ such that ρ factors in a support-preserving way on the inclusion $\iota:[\mathcal{R}]\to\operatorname{Aut}(X,\mu)$. Then there is a p.m.p. equivalence relation $\mathcal{S}$ on Y and a class-bijective factor map $\pi:\mathcal{S}\to\mathcal{R}$ such that $\rho=\rho_\pi$.*

Proof. Let Γ be a countable group acting on (X,μ) so that $\mathcal{R}=\mathcal{R}_\Gamma$. Denote by $\pi:Y\to X$ the support-preserving factor map. Let $\mathcal{S}$ be the orbit equivalence relation of the action ρ of Γ. The map π intertwines by definition the two actions of Γ, that is $\pi\colon\mathcal{S}\to\mathcal{R}$ is a factor map. Moreover the hypothesis that π is support-preserving implies thatπ is class-bijective (see Remark 2.7).

Finally observe that $\rho_\pi(\gamma)=\rho(\gamma)$ for every $\gamma\in\Gamma$. Since ρ and ρ_π are both support-preserving, they agree on the full group generated by $\alpha(\Gamma)$, which finishes the proof. $\square$

Example 3.28. Let $\mathcal{R}$ be a p.m.p. equivalence relation on (X,μ), let $n\geqslant 1$. The diagonal action $\iota^n:[\mathcal{R}]\to[\mathcal{R}^n]$ factors onto $\iota^{\odot n}:[\mathcal{R}]\to[\mathcal{R}^{\odot n}]$ via the quotient map $\pi:X^n\to X^{\odot n}$, and this factor map is support-preserving since

$$\operatorname{Fix}(\iota^{\odot n}(T))=[\operatorname{Fix}T,\dots,\operatorname{Fix}T]=\pi^{-1}(\operatorname{Fix}T\times\cdots\times\operatorname{Fix}T)=\pi^{-1}\operatorname{Fix}(\iota^n(T)).$$

The full group generated by $\iota^{\odot n}([\mathcal{R}])$ is equal to $[\mathcal{R}^{\odot n}]$ so by Proposition 3.8 ι^n extends in a support preserving manner to $\widetilde{\iota^n} : [\mathcal{R}^{\odot n}] \to [\mathcal{R}^n]$, which can be rewritten as $\widetilde{\iota^n} = \rho_\pi$ when viewing π as a class-bijective factor map $\mathcal{R}^n \to \mathcal{R}^{\odot n}$.

We refer the reader to Section 6 for the notion of boolean action of a p.m.p. equivalence relation, which as we prove there is essentially the same as a p.m.p. action (this is needed for our characterization of property (T) in terms of full groups).

4 Diagonal support-preserving factorization for boolean actions

The aim of this section is to prove Theorem A which we restate here.

Theorem 4.1. *Let $\mathbb{G}$ be an ergodic full group over the standard probability space (X,μ) and let $\rho\colon \mathbb{G} \to \mathrm{Aut}(Y,\nu)$ be a boolean action on another standard probability space (Y,ν). Then there is a unique measurable $\rho(\mathbb{G})$-invariant partition $\{Y_n\}_{n=0,\dots,\infty}$ of Y such that the boolean actions $\rho_n = \rho_{\restriction\left(Y_n, \frac{\nu_{\restriction Y_n}}{\nu(Y_n)}\right)}$ are subject to the following conditions*

1. *the map ρ_0 maps every element to the identity on Y_0;*
2. *for all $n \in \mathbb{N}^*$, there is a (unique) measure-preserving map $\pi_n\colon Y_n \to X^{\odot n}$ which is a support-preserving factor map from the boolean action ρ_n to $\iota^{\odot n}\colon \mathbb{G} \hookrightarrow \mathrm{Aut}(X^{\odot n}, \mu^{\odot n})$;*
3. *for every $T \in \mathbb{G} \setminus \{\mathrm{id}\}$ we have that $\{y \in Y_\infty :\ \rho_\infty(T)y = y\}$ is a null-set.*

Let $\mathbb{G}$ be an ergodic full group. We first observe that Corollary 3.5 yields that the inclusion $\mathbb{G} \leqslant \mathrm{Aut}(X,\mu)$ is highly absolutely non-free. Therefore we can apply Proposition 2.13: Theorem 4.1 is implied by the following theorem.

Theorem 4.2. *Let $\mathbb{G} \leqslant \mathrm{Aut}(X,\mu)$ be an ergodic full group and let $\rho\colon \mathbb{G} \to \mathrm{Aut}(Y,\nu)$ be a boolean action. Then ρ is diagonally support-preserving factorizable.*

The rest of this section is devoted to the proof of Theorem 4.2. We will first prove it in the hyperfinite case although it is not needed for the general case. Indeed, the proof in the hyperfinite case is much easier and we find it of independent interest. Nevertheless, the hasty reader should jump to Section 4.2.

4.1 Proof in the hyperfinite case

First recall that by Dye's theorem, there is up to isomorphism only one full group of ergodic p.m.p. hyperfinite equivalence relation, namely the full group $\mathbb{G} = [\mathcal{R}_0]$ (see Definition 3.20 and the paragraphs below). We will thus prove Theorem 4.2 when $\mathbb{G} = [\mathcal{R}_0]$. This will rely on the following fact several times.

Proposition 4.3 (see [Kec10, Prop. 2.8])**.** *The closure of $\mathfrak{S}_{2^\infty} \leqslant \mathrm{Aut}(X,\mu)$ in the uniform topology is the full group $[\mathcal{R}_0]$.*

Let us fix a boolean action $\rho\colon [\mathcal{R}_0] \to \mathrm{Aut}(Y,\nu)$. By automatic continuity (see Theorem 3.9) we obtain that ρ is uniform-to-weak continuous. We then have the following straightforward consequence.

Lemma 4.4. *Let $A \in \mathrm{MAlg}(Y, \nu)$ such that $\rho(\sigma)A = A$ for every $\sigma \in \mathfrak{S}_{2^\infty}$. Then $\rho(T)A = A$ for every $T \in [\mathcal{R}_0]$.*

Proof. By uniform-to-weak continuity, the set of all $T \in [\mathcal{R}_0]$ such that $\rho(T)A = A$ is closed in the uniform topology, and by assumption it contains the dense subgroup $\mathfrak{S}_{2^\infty}$ (see Proposition 4.3), so it has to be equal to $[\mathcal{R}_0]$. $\square$

Lemma 4.5. *It the boolean action $\rho : [\mathcal{R}_0] \to \mathrm{Aut}(Y, \nu)$ is not free, then its restriction to the subgroup $\mathfrak{S}_{2^\infty}$ is not free either.*

Proof. We prove the contrapositive; assume that the restriction of ρ to $\mathfrak{S}_{2^\infty}$ is free. Since $\mathcal{R}_0$ is ergodic, all involutions in $[\mathcal{R}_0]$ with support of the same measure are conjugate, see Proposition 3.6. Remark that for every $n \geqslant 1$, there is an involution in $\mathfrak{S}_{2^\infty}$ whose support in $\mathrm{Aut}(X, \mu)$ has measure 2^{-n}. Since the action of $\mathfrak{S}_{2^\infty}$ is free and all involutions of support of measure 2^{-n} are conjugate in $[\mathcal{R}_0]$, we get that $\mathrm{supp}\, \rho(U)$ has full measure for every involution $U \in [\mathcal{R}_0]$ whose support has measure 2^{-n}.

Take $T \in [\mathcal{R}_0]$ different from the identity and consider a subset of positive measure $A \subset X$ such that $TA \cap A = \emptyset$. For any large enough $n \geqslant 1$, consider an involution U_n with support of measure 2^{-n-1} and contained in A. Remark that TU_nT^{-1} is an involution with support contained in TA and in particular U_n and TU_nT^{-1} are involutions with disjoint supports. So $U_nTU_nT^{-1}$ is an involution with support of measure 2^{-n} and hence $\mathrm{supp}\, \rho(U_nTU_nT^{-1})$ has full measure. On the other hand U_n tends to the identity in $[\mathcal{R}_0]$. By uniform-to-weak continuity, this implies that $\rho(U_n)\, \mathrm{supp}\, \rho(T)$ tends to $\mathrm{supp}\, \rho(T)$ and hence

$$Y = \mathrm{supp}\, \rho(U_nTU_nT^{-1}) \subseteq \rho(U_n)\, \mathrm{supp}\, \rho(T) \cup \mathrm{supp}\, \rho(T) \to \mathrm{supp}\, \rho(T)$$

which implies that $\mathrm{supp}\, T = Y$. The boolean action ρ is thus free. $\square$

Recall now that by Proposition 2.18 the action $\rho : \mathfrak{S}_{2^\infty} \to \mathrm{Aut}(Y, \nu)$ is diagonally support-preserving factorizable. We thus have a unique sequence $(Y_i)_{i=0,\ldots,\infty}$ of measurable subsets of Y such that the action restricted to Y_∞ is free and the action restricted to Y_i for $i < \infty$ factorizes onto the action $\iota^{\odot i}$ on $X^{\odot i}$ in a support preserving manner, see Proposition 2.13. For $i < \infty$, we denote by $\pi_i \colon Y_i \to X^{\odot i}$ the factor maps.

Remark now that Lemma 4.4 implies that Y_i is $\rho([\mathcal{R}_0])$-invariant for every $i = 0, \ldots, \infty$. Moreover Lemma 4.5 implies that the restriction of ρ on Y_∞ is free, seen as a boolean action of $[\mathcal{R}_0]$. Let us first show that π_i induces a factor map for the entire group $[\mathcal{R}_0]$.

Lemma 4.6. *For every $[B_1, \ldots, B_n] \subset X^{\odot i}$ and $T \in [\mathcal{R}_0]$ we have that*

$$\rho(T)\pi_i^{-1}([B_1, \ldots, B_n]) = \pi_i([TB_1, \ldots, TB_i]).$$

Proof. Fix a sequence $(\sigma_n)_{n\geqslant 0}$ of elements of $\mathfrak{S}_{2^\infty}$ converging to T. Then for every n, we have that

$$\rho(\sigma_n)\pi_i^{-1}([B_1, \ldots, B_n]) = \pi_i([\sigma_n B_1, \ldots, \sigma_n B_i]).$$

The lemma now follows from the continuity of ρ. $\square$

The following lemma is well-known.

Lemma 4.7. *Let $(T_n)_{n\geqslant 0}$ be a sequence of elements of $\mathrm{Aut}(X, \mu)$ which converges weakly to $T \in \mathrm{Aut}(X, \mu)$. Assume moreover that there is $B \subseteq X$ such that $\lim_n \mu(B \Delta \,\mathrm{supp}\, T_n) = 0$. Then $\mathrm{supp}\, T \subseteq B$.*

Proof. Recall that $\operatorname{supp} T$ is the supremum of the family of $A \in \operatorname{MAlg}(X,\mu)$ such that $TA \cap A = \emptyset$, so it suffices to show that every such A is contained in B.

So take $A \subseteq X$ such that $TA \cap A = \emptyset$. Since T_n tends weakly to T, we have that $\lim_n \mu(T_n A \triangle TA) = 0$. Since A is disjoint from TA we obtain $\lim_n \mu(T_n A \cap A) = 0$, and hence $\lim_n \mu(A \setminus \operatorname{supp} T_n) = 0$. Since $\lim_n \mu(B \triangle \operatorname{supp} T_n) = 0$, we conclude that $\mu(A \setminus B) = 0$, so $A \subseteq B$ as wanted. $\square$

We can now finish the proof by showing that the factor map is support preserving.

Lemma 4.8. *For every $T \in [\mathcal{R}_0]$, we have that*

$$Y_i \cap \operatorname{supp} \rho(T) = \pi_i^{-1}([\operatorname{supp} T, \dots, \operatorname{supp} T]).$$

Proof. Fix a sequence $(\sigma_n)_{n\geqslant 0}$ of elements of $\mathfrak{S}_{2^\infty}$ converging to T. By Lemma 4.6 we have that

$$\operatorname{supp} \rho(T) \supseteq \pi_i^{-1}([\operatorname{supp} T, \dots, \operatorname{supp} T]).$$

Also observe that since σ_n converges to T,

$$\pi_i^{-1}([\operatorname{supp} T, \dots, \operatorname{supp} T]) = \lim_n \pi_i^{-1}([\operatorname{supp} \sigma_n, \dots, \operatorname{supp} \sigma_n]) = \lim_n \operatorname{supp} \rho(\sigma_n).$$

Finally since ρ is continuous, we can use Lemma 4.7 to obtain the following diagram of inclusions and equalities:

$$\begin{tikzcd}
\lim_n \operatorname{supp} \rho(\sigma_n) \cap Y_i \arrow[r, phantom, "\supseteq"] \arrow[d, equal] & \operatorname{supp} \rho(T) \cap Y_i \arrow[d, phantom, sloped, "\supseteq"] \\
\lim_n \pi_i^{-1}([\operatorname{supp} \sigma_n, \dots, \operatorname{supp} \sigma_n]) \arrow[r, equal] & \pi_i^{-1}([\operatorname{supp} T, \dots, \operatorname{supp} T]).
\end{tikzcd}$$

So the four terms above are equal, in particular $\operatorname{supp} \rho(T) \cap Y_i = \pi_i^{-1}([\operatorname{supp} T, \dots, \operatorname{supp} T])$ as wanted. $\square$

Therefore the proof of Theorem 4.2 for the case $\mathbb{G} = [\mathcal{R}_0]$ is concluded. We now proceed to the general case. Recall that the general case does not require what we have done in the present section.

4.2 Setup in the general case

Let $\mathbb{G}$ be an ergodic full group over the standard probability space (X,μ). Since all the standard probability spaces are isomorphic, we can assume that $X = \{0,1\}^{\mathbb{N}}$ and $\mu = \mathcal{B}(1/2)^{\otimes \mathbb{N}}$. Moreover we can always assume that $\mathbb{G} \leqslant \operatorname{Aut}(X,\mu)$ contains the standard copy of $\mathfrak{S}_{2^\infty}$ acting on (X,μ). Indeed by different results of Dye, every ergodic full group contains the full group of a hyperfinite p.m.p. ergodic equivalence relation [Dye59, Thm. 4], all these full groups are conjugated in $\operatorname{Aut}(X,\mu)$ [Dye59, Thm. 3] and we already observed that $\mathfrak{S}_{2^\infty}$ is contained the full group of the hyperfinite p.m.p. ergodic equivalence relation $\mathcal{R}_0$. Therefore from now on, we will always assume that $\mathfrak{S}_{2^\infty} \leqslant \mathbb{G}$.

Fix a boolean action $\rho : \mathbb{G} \to \operatorname{Aut}(Y,\nu)$. Let $A_\infty := \bigcap_{\sigma \in \mathfrak{S}_{2^\infty} \setminus \{\mathrm{id}\}} \operatorname{supp}(\rho(\sigma))$ be the free part of the restriction of ρ to $\mathfrak{S}_\infty$. Proposition 2.18 applied to $\rho_{\restriction \mathfrak{S}_{2^\infty}}$ yields that the action induced by $\rho_{\restriction \mathfrak{S}_{2^\infty}}$ is diagonally support-preserving factorizable. Let $(\alpha_i)_{i\geqslant 0}$ such that the action $\rho^{Y \setminus A_\infty}_{\restriction \mathfrak{S}_{2^\infty}}$ induced on $Y \setminus A_\infty$ by the restriction of ρ to $\mathfrak{S}_{2^\infty}$ factorises in a support-preserving manner on the diagonal sum action $\iota^{\odot(\alpha_i)_i}$ (see Definition 2.9; the α_i's are well defined thanks to Proposition 2.13).

We now consider the associated character $\chi\colon \mathbb{G} \to \mathbb{R}$, defined by $\chi(T) \coloneqq \nu(\mathrm{Fix}(\rho(T)))$. The definition of the diagonal sum action $\iota^{\odot(\alpha_i)_i}$ and the fact that the factor map onto it is support-preserving imply that

$$\chi(T) = \sum_{i\geqslant 0} \alpha_i \mu(\mathrm{Fix}(T))^i \tag{2}$$

for every $T \in \mathfrak{S}_{2^\infty}$. We will show in Proposition 4.13 that the equation holds for every $T \in \mathbb{G}$.

Remark 4.9. We do not know at this moment whether the character χ is continuous, and hence we cannot apply the classification of characters on full groups [CLM16, Prop. 5.3]. Indeed ρ is a priori not uniform-to-uniform continuous and the map $T \mapsto \mu(\mathrm{Fix}(T))$ is not continuous for the weak topology.

We will first understand the free part of the action.

4.3 The free part is the same as the free part of the restriction to $\mathfrak{S}_{2^\infty}$

Our aim now is to prove the following.

Proposition 4.10. *Given a boolean action $\rho : \mathbb{G} \to \mathrm{Aut}(Y, \nu)$, we have that*

$$\bigcap_{\sigma\in\mathfrak{S}_{2^\infty}\setminus\{\mathrm{id}\}} \mathrm{supp}(\rho(\sigma)) = \bigwedge_{T\in\mathbb{G}\setminus\{\mathrm{id}\}} \mathrm{supp}(\rho(T));$$

that is, the free parts of ρ and of the restriction of ρ to $\mathfrak{S}_{2^\infty}$ coincide.

We first show an intermediate result. Recall that in the previous section we defined the free part of the restriction of ρ to $\mathfrak{S}_{2^\infty}$ as $A_\infty = \bigcap_{\sigma\in\mathfrak{S}_{2^\infty}\setminus\{\mathrm{id}\}} \mathrm{supp}(\rho(\sigma))$, which is clearly $\rho(\mathfrak{S}_{2^\infty})$-invariant.

Lemma 4.11. *The subset A_∞ is $\rho(\mathbb{G})$-invariant.*

Proof. Suppose by contradiction that A_∞ is not $\rho(\mathbb{G})$-invariant and let $T \in \mathbb{G}$ such that $\nu(\rho(T)A_\infty \setminus A_\infty) > \varepsilon$ for some $\varepsilon > 0$.

Let (B_n) a sequence of cylinder sets such that $\mu(B_n) \to 0$. By automatic continuity, Corollary 3.10, we have that ρ is uniform to weak continuous. Since taking induced bijections is continuous (see Lemma 3.15), for n large enough we have that the induced bijection $T_{X\setminus B_n}$ also satisfies $\nu(\rho(T_{X\setminus B_n})A \setminus A) > \varepsilon$. So if we take a sufficiently large n and put $U \coloneqq T_{X\setminus B_n}$, we have found $U \in \mathbb{G}$ and a cylinder subset $B \subset X$ such that $\mathrm{supp}(U) \cap B = \emptyset$ and $\nu(\rho(U)A_\infty \setminus A_\infty) > \varepsilon$.

Equation (2) implies that whenever $(\sigma_n)_n$ is a sequence of elements of $\mathfrak{S}_{2^\infty}$ which tends to the identity, $\nu(\mathrm{supp}\,\sigma_n)$ converges to $\nu(A_\infty)$. Therefore there exists an element $\sigma \in \mathfrak{S}_{2^\infty}$ with support contained in B such that $\nu(\mathrm{supp}(\rho(\sigma))) < \nu(A) + \varepsilon$.

On the other hand observe that σ and U commute, whence

$$\mathrm{supp}(\rho(\sigma)) = \mathrm{supp}(\rho(U\sigma U^{-1})) = \rho(U)\,\mathrm{supp}(\rho(\sigma)) \supseteq \rho(U)A_\infty.$$

This implies that

$$\nu(\mathrm{supp}(\rho(\sigma))) \geqslant \nu(A_\infty) + \nu(\rho(U)A_\infty \setminus A_\infty)) > \nu(A_\infty) + \varepsilon,$$

a contradiction. $\square$

Proof of Proposition 4.10. We have to show

$$\bigcap_{\sigma \in \mathfrak{S}_{2^\infty} \setminus \{\mathrm{id}\}} \mathrm{supp}(\rho(\sigma)) = \bigwedge_{T \in \mathbb{G} \setminus \{\mathrm{id}\}} \mathrm{supp}(\rho(T));$$

The left hand side is equal to A_∞ and clearly contains the right hand side. By Lemma 4.11, the set A_∞ is $\mathbb{G}$-invariant. Therefore we can assume that $Y = A_\infty$ and need to prove that for every $T \in \mathbb{G}$ the support of $\rho(T)$ is the entire space X, whenever $T \neq \mathrm{id}$.

Proposition 3.6 tells us that every involution $U \in \mathbb{G}$ whose support has measure $1/2^n$ for some $n \in \mathbb{N}$ is conjugate to an element of $\mathfrak{S}_{2^\infty}$, and hence $\rho(U)$ has support of full measure. We now argue as in Lemma 4.5: let $T \in \mathbb{G}$ be a nontrivial element, we can find $B \subseteq X$ measurable non null such that $TB \cap B = \emptyset$. For large enough n, we then find an involution U_n supported on B such that its support has measure $1/2^n$. The commutator $[T, U_n]$ is an involution whose support has measure $2/2^n$, and so $\rho([T, U_n])$ has full measure support.

But the support of $\rho([T, U_n])$ is contained in $\mathrm{supp}\,\rho(T) \cup \rho(U_n)(\mathrm{supp}\,\rho(T))$. By automatic continuity, Corollary 3.10, the sequence $(\rho(U_n))_n$ tends weakly to the identity, therefore $\rho(U_n)\,\mathrm{supp}\,T$ tends to $\mathrm{supp}\,T$. This implies that $Y = \mathrm{supp}\,\rho([T, U_n]) \subseteq \mathrm{supp}\,\rho(T)$ as wanted. □

4.4 Continuity and support dependency on the non-free part

We will now show that $\mathrm{supp}\,\rho(T)$ only depends on $\mathrm{supp}\,T$. To this end, we first need to know that ρ is uniform-to-uniform continuous, which is an easy consequence of what we have done so far.

Proposition 4.12. *Given a boolean action $\rho : \mathbb{G} \to \mathrm{Aut}(Y, \nu)$, we have that the action induced by ρ on its non-free part is uniform-to-uniform continuous.*

Proof. By Proposition 2.18, the restriction $\rho_{\restriction \mathfrak{S}_{2^\infty}}$ factors in a support preserving manner on a symmetric diagonal sum which by Proposition 2.16 is not discrete in the uniform topology. Since the factor map is support-preserving, this implies ρ has non-discrete image. We can now apply Corollary 3.14 to obtain that ρ is indeed uniform-to-uniform countionus. □

To see that $\mathrm{supp}\,\rho(T)$ only depends on $\mathrm{supp}\,T$, we will use again the character associated to our boolean action. Since we now know that ρ is uniform-to-uniform continuous, [CLM16, Prop. 5.3] implies that Equation (2) holds for every $T \in \mathbb{G}$. However, there is a direct proof of this fact in the same line of that of [CLM16, Prop. 5.3], and we now present it for the convenience of the reader.

Proposition 4.13. *Let $\chi : \mathbb{G} \to \mathbb{R}$ be defined by $\chi(T) = \nu(\mathrm{Fix}\,\rho(T))$. Then there is a unique sequence $(\alpha_i)_{0 \leqslant i < \infty}$ such that for every $g \in \mathbb{G}$ we have that*

$$\chi(T) = \sum_{i \geqslant 0} \alpha_i \mu(\mathrm{Fix}\,T)^i.$$

Proof. First observe that the uniqueness of (α_i) is a consequence of the uniqueness of the coefficients of the power series $\sum_{i \geqslant 0} \alpha_i x^i$ whose radius of convergence is at least 1 and of the fact that $\mu(\mathrm{Fix}(T))$ can take any value in $[0, 1]$.

Without loss of generality, we assume that the free part of ρ is trivial and by Proposition 4.10, the free part of the restriction of ρ to $\mathfrak{S}_{2^\infty}$ is also trivial. By Equation (2) the desired result holds for elements of $\mathfrak{S}_{2^\infty}$ (see also Prop. 2.18).

Fix $T \in \mathbb{G}$. By Rokhlin's lemma and ergodicity, T is approximately conjugate to an element of the full group $[\mathcal{R}_0]$ and Proposition 4.3 tells us that $[\mathcal{R}_0] = \overline{\mathfrak{S}_{2^\infty}}$. So there are sequences $U_n \in \mathbb{G}$ and $\sigma_n \in \mathfrak{S}_{2^\infty}$ such that $U_n \sigma_n U_n^{-1}$ converges to T. By uniform-to-uniform continuity (Proposition 4.12), $\rho(U_n\sigma_nU_n^{-1})$ converges to $\rho(T)$ in the uniform topology and hence

$$\nu(\operatorname{Fix}\rho(U_n\sigma_nU_n^{-1}) \triangle \operatorname{Fix}\rho(T)) \to 0.$$

In particular, $\nu(\operatorname{Fix}\rho(U_n\sigma_nU_n^{-1})) \to \nu(\operatorname{Fix}\rho(T))$. Now observe that

$$\nu(\operatorname{Fix}\rho(U_n\sigma_nU_n^{-1})) = \nu(\operatorname{Fix}\rho(\sigma_n)) = \sum_{i\geqslant 0} \alpha_i \mu(\operatorname{Fix}\sigma_n)^i = \sum_{i\geqslant 0} \alpha_i \mu(\operatorname{Fix} U_n\sigma_nU_n^{-1})^i.$$

Clearly $\mu(\operatorname{Fix} U_n\sigma_nU_n^{-1})^i \leqslant 1$, so using $\mu(\operatorname{Fix}\rho(U_n\sigma_nU_n^{-1})) \to \mu(\operatorname{Fix}\rho(T))$ we obtain that the right-hand term converges to $\sum_i \alpha_i \mu(\operatorname{Fix}(\rho(T)))^i$. Since the left-hand term converges to $\nu(\operatorname{Fix}\rho(T)) = \chi(T)$, the conclusion follows. $\square$

The exact formula in the previous proposition is not really important, all that matters is that we now know that $\nu(\operatorname{supp}\rho(T))$ depends only on $\mu(\operatorname{supp} T)$. Moreover this dependency is continuous.

Corollary 4.14. *If T is aperiodic on its support, then so is $\rho(T)$.*

Proof. Observe that a measure-preserving bijection T is aperiodic on its support if and only if for every n, $\mu(\operatorname{Fix}(T^n)) = \mu(\operatorname{Fix}(T))$. Now if T is aperiodic on its support, then for every n we have $\nu(\operatorname{Fix}(\rho(T)^n)) = \chi(T^n) = \chi(T) = \nu(\operatorname{Fix}(\rho(T))$ so $\rho(T)$ is aperiodic on its support. $\square$

Here is another useful observation. Note that $\operatorname{supp} UT^n \subseteq \operatorname{supp} T \cup \operatorname{supp} U$.

Lemma 4.15. *Let T and U be measure-preserving bijections, with T aperiodic when restricted to its support. Then* $\lim_n (\operatorname{supp}(UT^n)\Delta(\operatorname{supp} T \cup \operatorname{supp} U)) = 0$.

Proof. Observe that if $x \in \operatorname{supp} U \setminus \operatorname{supp} T$, then $UT^n(x) \neq x$. For each $n \in \mathbb{N}$, let $A_n := \{x \in \operatorname{supp} T : UT^n(x) = x\}$. Then $x \in A_n$ implies $T^n(x) = U^{-1}(x)$ and hence the aperiodicity of T implies that the measurable subsets $\{A_n\}_n$ are pairwise disjoint. Hence $\lim_n \mu(A_n) = 0$ and the lemma is proved. $\square$

We can now prove that $\operatorname{supp}\rho(T)$ only depends on $\operatorname{supp} T$.

Proposition 4.16. *Let $T, U \in \mathbb{G}$ have the same support, then $\rho(T)$ and $\rho(U)$ also have the same support.*

Proof. As a first step, let us assume that T is aperiodic on its support. By Lemma 4.15, we have

$$\lim_n \mu(\operatorname{supp}(UT^n)) = \mu(\operatorname{supp} T \cup \operatorname{supp} U) = \mu(\operatorname{supp} T).$$

Proposition 4.13 then implies $\lim_n \nu(\operatorname{supp}(\rho(UT^n))) = \nu(\operatorname{supp}\rho(T))$. Corollary 4.14 tells us that $\rho(T)$ is aperiodic on its support. We can therefore use a second time Lemma 4.15 to get

$$\lim_n \nu(\operatorname{supp}\rho(UT^n)) = \nu(\operatorname{supp}\rho(T) \cup \operatorname{supp}\rho(U)).$$

Hence $\nu(\operatorname{supp}\rho(U) \setminus \operatorname{supp}\rho(T)) = 0$. On the other hand, using Proposition 4.13 again we have $\nu(\operatorname{supp}\rho(U)) = \nu(\operatorname{supp}\rho(T))$ and hence $\operatorname{supp}\rho(U) = \operatorname{supp}\rho(T)$.

Now if T is not aperiodic on its support, we can find an element $T' \in \mathbb{G}$ which is aperiodic on its support and has same support as T and U. The above argument then yields $\operatorname{supp}\rho(T) = \operatorname{supp}\rho(T') = \operatorname{supp}\rho(U)$ as wanted. □

4.5 End of the proof of Theorem A

As we already mentioned at the beginning of Section 4, the uniqueness part in Theorem A is a direct consequence of Proposition 2.13 so in order to finish the proof of Theorem A, we have to show that Theorem 4.2 holds, namely that every boolean $\mathbb{G}$-action is diagonally support-preserving factorizable.

Let $\mathbb{G} \leqslant \operatorname{Aut}(X,\mu)$ be an ergodic full group and fix a boolean action $\rho\colon \mathbb{G} \to \operatorname{Aut}(Y,\nu)$. We can assume that the free part of the action of $\mathbb{G}$ on (Y,ν) is trivial and by Proposition 4.10 this implies that the restriction of ρ to $\mathfrak{S}_{2^\infty}$ also has trivial free part. Proposition 2.18 provides a symmetric diagonal sum boolean action $\iota^{\odot(\alpha_i)_i}$ of $\mathfrak{S}_{2^\infty}$ on $(X^{\odot(\alpha_i)_i}, \mu^{\odot(\alpha_i)_i})$, and a support-preserving factor map $\pi\colon Y \to Z$ of $\rho_{\restriction \mathfrak{S}_{2^\infty}}$ onto $\iota^{\odot(\alpha_i)_i}$.

We will show that π is still a measure preserving support preserving factor map of ρ onto the same diagonal sum action $\iota^{\odot(\alpha_i)_i}$, now seen as a boolean action of $\mathbb{G}$.

Given $\sigma \in \mathfrak{S}_{2^\infty}$, we already know that $\pi^{-1}(\operatorname{supp}\iota^{\odot(\alpha_i)_i}(\sigma)) = \operatorname{supp}\rho(\sigma)$ and by uniform-to-uniform continuity (see Proposition 4.12), the same is true for elements of $[\mathcal{R}_0] = \overline{\mathfrak{S}_{2^\infty}}$, cf. Proposition 4.3. Fix $T \in \mathbb{G}$. By Corollary 3.5, there exists $T' \in [\mathcal{R}_0]$ such that $\operatorname{supp}(T') = \operatorname{supp}(T)$ and hence by Proposition 4.16 we have

$$\pi^{-1}(\operatorname{supp}\iota^{\odot(\alpha_i)_i}(T)) = \pi^{-1}(\operatorname{supp}\iota^{\odot(\alpha_i)_i}(T')) = \operatorname{supp}\rho(T') = \operatorname{supp}\rho(T).$$

We conclude that π is indeed support-preserving. Let us now show it is a factor map. Take $T, U \in \mathbb{G}$ then we have

$$\begin{aligned}\pi^{-1}(\iota^{\odot(\alpha_i)_i}(T)\operatorname{supp}\iota^{\odot(\alpha_i)_i}(U)) &= \pi^{-1}(\operatorname{supp}\iota^{\odot(\alpha_i)_i}(TUT^{-1})) \\ &= \operatorname{supp}\rho(TUT^{-1}) \\ &= \rho(T)\operatorname{supp}\rho(U),\end{aligned}$$

so π is equivariant on elements of the form $\operatorname{supp}\iota^{\odot(\alpha_i)_i}(U)$. The action $\iota^{\odot(\alpha_i)_i}$ of $\mathbb{G}$ is totally non free, Proposition 2.16, so by definition these sets generate the measure algebra of $(X^{\odot(\alpha_i)_i}, \mu^{\odot(\alpha_i)_i})$. Therefore π is equivariant and the proof of Theorem 4.2 is concluded.

4.6 Recovering Dye's reconstruction theorem

We can now recover Dye's reconstruction theorem in the ergodic case [Dye63, Thm. 2].

Theorem 4.17. *Let $\mathbb{G}$ and $\mathbb{H}$ be two ergodic full groups on (X,μ), let $\varphi : \mathbb{G} \to \mathbb{H}$ be a group isomorphism. Then there is a unique $S \in \operatorname{Aut}(X,\mu)$ such that for all $T \in \mathbb{G}$,*

$$\varphi(T) = STS^{-1}.$$

Proof. Let us start by proving uniqueness: suppose that $S \in \operatorname{Aut}(X,\mu)$ satisfies that for all $T \in \mathbb{G}$, $\varphi(T) = STS^{-1}$. Let $A \in \operatorname{MAlg}(X,\mu)$, then by Corollary 3.5 there is $T \in \mathbb{G}$ whose support is equal to A, and then $\operatorname{supp}\varphi(T) = \operatorname{supp} STS^{-1} = S(\operatorname{supp} T) = S(A)$ so we see that $S(A)$ is completely determined by φ, which implies the uniqueness of S.

We now prove the existence. Let $X_0, \dots, X_\infty$ be the sets arising from the decomposition of φ as a diagonal support-preserving factorizable boolean action from Theorem 4.2. Since $\mathbb{H}$ is ergodic, φ is an ergodic boolean action, so one of these sets is equal to X. Since $\mathbb{H}$ contains elements of arbitrary support and φ is an isomorphism, both X_0 and X_∞ must be empty. Also, if $X_n = X$ for some $n \geqslant 2$, we see that φ factors onto a nontrivial symmetric diagonal action of $\mathbb{G}$ in a support-preserving manner, but then $\varphi(\mathbb{G})$ is not a full group, a contradiction. We conclude that $X_1 = X$, which means that we have a measure-preserving map $\pi : (X,\mu) \to (X,\mu)$ which factors φ onto ι in a support preserving manner, where ι is the inclusion map of $\mathbb{G}$ in $\mathrm{Aut}(X,\mu)$.

If π is not almost surely injective, then sets of the form $\pi^{-1}(A)$ where $A \in \mathrm{MAlg}(X,\mu)$ form a proper subalgebra of $\mathrm{MAlg}(X,\mu)$, and since π is a support-preserving factor map of φ onto ι, we see that all the $\varphi(T)$ have support in this proper subalgebra, contradicting the fact that every element of $\mathrm{MAlg}(X,\mu)$ is the support of some element of $\mathbb{H}$. We conclude that $\pi \in \mathrm{Aut}(X,\mu)$, in other words π is an isomorphism between φ and the inclusion of $\mathbb{G}$ in $\mathrm{Aut}(X,\mu)$, which means that for all $T \in \mathbb{G}$ and almost all $x \in X$ $\pi\varphi(T)x = \iota(T)\pi(x)$. If we let $S = \pi^{-1}$, we finally get $\varphi(T) = STS^{-1}$. □

5 Boolean actions of full groups of p.m.p. equivalence relations

We will now explain how Theorem 4.1 implies that for every measure-preserving boolean action of a full group $[\mathcal{R}]$, the non-free part of the action comes from measure-preserving actions of the equivalence relation and its symmetric tensor powers, yielding a statement more similar to the main result of Matte Bon [MB18]. The rest of the section is then devoted to applications of this result.

Theorem 5.1. *Let $\mathcal{R}$ be a p.m.p. ergodic equivalence relation on (X,μ). Let $\rho\colon [\mathcal{R}] \to \mathrm{Aut}(Y,\nu)$ be a boolean action. Then there is a unique partition $Y = \bigsqcup_{n\in\mathbb{N}} Y_n^\rho \sqcup Y_\infty^\rho$ into (possibly null) $\rho([\mathcal{R}])$-invariant sets such that:*

- *the restriction of ρ to Y_0^ρ is the trivial action;*
- *for all $n \geqslant 1$ such that $\nu(Y_n^\rho) > 0$, there is a p.m.p. equivalence relation $\mathcal{S}_n$ on Y_n and a class-bijective factor $\pi\colon \mathcal{S}_n \to \mathcal{R}^{\odot n}$ such that*

$$\rho_{\restriction Y_n^\rho} = \rho_\pi \circ \iota_n,$$

  where $\iota_n : [\mathcal{R}] \to [\mathcal{R}^{\odot n}]$ is the diagonal embedding and ρ_π is obtained π as defined in Lemma 3.26;
- *the restriction of ρ to Y_∞^ρ is free.*

Proof. We know from Theorem 4.1 that ρ is diagonally support-preserving factorizable, which yields the unique partition $Y = \bigsqcup_{n\in\mathbb{N}} Y_n^\rho \sqcup Y_\infty^\rho$ that we seek. The two first items are satisfied by definition, let us see why the third holds. We have that ρ restricted to Y_n^ρ factors onto $\iota^{\odot n} : [\mathcal{R}] \to [\mathcal{R}^{\odot n}]$ in a support-preserving manner.

Let $n \geqslant 1$, assume that $\nu(Y_n^\rho) > 0$ and denote by $\nu_{Y_n^\rho}$ the probability measure on Y_n^ρ obtained by renormalizing the restriction of ν to Y_n^ρ. Let us define $\rho_n : \iota^{\odot n}([\mathcal{R}]) \to \mathrm{Aut}(Y_n^\rho, \nu_{Y_n^\rho})$ by $\rho_n(\iota^{\odot n}(T)) = \rho(T)$. By Lemma 3.24 the full group generated by $\iota^{\odot n}([\mathcal{R}])$

is equal to $[\mathcal{R}^{\odot n}]$ so by Proposition 3.8 we may extend ρ' to a support-preserving boolean action $\rho' : [\mathcal{R}^{\odot n}] \to \mathrm{Aut}(Y_n^\rho, \nu_{Y_n^\rho})$. It now follows from Lemma 3.27 that there is a measure-preserving action $\pi : \mathcal{S}_n \to \mathcal{R}^{\odot n}$ of $\mathcal{R}^{\odot n}$ on (Y, ν) for some measure-preserving equivalence relation $\mathcal{S}_n$ on (Y, ν) such that $\rho' = \rho_\pi$. Since $\rho'(\iota^{\odot n}(T)) = \rho_{\restriction Y_n}(T)$ for all $T \in [\mathcal{R}]$, this finishes the proof. $\square$

Let $\mathcal{R}$ be a p.m.p. ergodic equivalence relation on (X, μ) and let $\rho\colon [\mathcal{R}] \to \mathrm{Aut}(Y, \nu)$ be a boolean action. Denote by ρ_∞ the restriction of ρ to the measurable subset Y_∞^ρ given by Theorem 5.1. Then $\rho_\infty([\mathcal{R}]) \leqslant \mathrm{Aut}(Y_\infty^\rho, \nu_{Y_\infty^\rho})$ is discrete. In particular, if $\rho([\mathcal{R}])$ is separable with respect to the uniform topology, we must have that Y_∞^ρ has measure zero. We thus get the following corollary.

Corollary 5.2. *Let $\mathcal{R}$ and $\mathcal{S}$ be p.m.p. ergodic equivalence relations on (X, μ) and (Y, ν) respectively. Let $\rho\colon [\mathcal{R}] \to [\mathcal{S}]$ be a boolean action. Then there is a unique partition $Y = \bigsqcup_{n \in \mathbb{N}} Y_n^\rho$ into $\rho([\mathcal{R}])$-invariant sets such that:*

- *the restriction of ρ to Y_0^ρ is the trivial action;*
- *for all $n \geqslant 1$, there is a p.m.p. equivalence relation $\mathcal{S}_n \subset \mathcal{S}$ on Y_n^ρ and a class-bijective factor $\pi\colon \mathcal{S}_n \to \mathcal{R}^{\odot n}$ such that*

$$\rho_{\restriction Y_n} = \rho_\pi \circ \iota_n,$$

  where $\iota_n : [\mathcal{R}] \to [\mathcal{R}^{\odot n}]$ is the diagonal embedding and ρ_π is obtained π as defined in Lemma 3.26. $\square$

Definition 5.3. Let $\mathcal{R}$ and $\mathcal{S}$ be p.m.p. ergodic equivalence relations on (X, μ). Let $\rho\colon [\mathcal{R}] \to [\mathcal{S}]$ be a boolean action. We say that ρ is **extension-like** when the set $Y_0^\rho \sqcup Y_1^\rho$ from Corollary 5.2 has full measure.

In other words, a boolean action $\rho : [\mathcal{R}] \to [\mathcal{S}]$ is extension-like if once we remove its trivial part Y_0^ρ, it comes from an extension $\pi : Y \setminus Y_0^\rho \to X$ of $\mathcal{R}$ by subequivalence relation $\mathcal{S}'$ of the restriction of $\mathcal{S}$ to $Y \setminus Y_0^\rho$.

5.1 Amenability

A p.m.p. equivalence relation $\mathcal{R}$ on (X, μ) is called **amenable** when up to restricting it to a full measure subset, we can find a sequence of Borel maps $f^n : \mathcal{R} \to \mathbb{R}^{\geqslant 0}$ such that for all $x \in X$, $\sum_{x' \in [x]_\mathcal{R}} f_n(x, x') = 1$, and if we denote by $\|\cdot\|_1$ the ℓ^1 norm on $\ell^1([x]_\mathcal{R})$, then for all $y, z \in [x]_\mathcal{R}$, we have

$$\|f_n(y, \cdot) - f_n(z, \cdot)\|_1 \to 0.$$

In other words, we have a measurable way to assign to every $x \in X$ a sequence of probability measures on $[x]_\mathcal{R}$ so that the probability measures assigned to points y and z in the same equivalence class are asymptotically the same. Such a sequence (f_n) is called a **Reiter sequence** for $\mathcal{R}$.

Our main result (Corollary 5.2) interacts with amenability as follows.

Proposition 5.4. *Let $\mathcal{R}$ and $\mathcal{S}$ be p.m.p. equivalence relations on (X, μ) and (Y, ν) respectively with $\mathcal{R}$ ergodic. Let us assume that there is a non-trivial group homomorphism $\rho\colon [\mathcal{R}] \to [\mathcal{S}]$. If $\mathcal{S}$ is amenable, then $\mathcal{R}$ is.*

In order to prove this proposition, we need a few well-known lemmas.

Lemma 5.5. *Every subequivalence relation of an amenable p.m.p. equivalence relation is amenable.*

Sketch of proof. Since every subequivalence relation of a hyperfinite equivalence relation has to be hyperfinite, a quick way to see this is to use the Connes-Feldman-Weiss theorem [CFW81] and the fact that hyperfinite equivalence relations are amenable. See [KM04, Sec. 6 and 10] for details. □

The following lemma is the p.m.p. equivalence relation version of the fact that if a countable group Γ is acting freely p.m.p. on (X,μ), then the associated equivalence relation $\mathcal{R}_\Gamma$ is amenable iff Γ is amenable.

Lemma 5.6. *Let $\mathcal{R}$ and $\mathcal{S}$ be p.m.p. equivalence relations on (X,μ) and (Y,ν) respectively. Let $\pi\colon Y\to X$ be a class-bijective factor. Then $\mathcal{S}$ is amenable if and only if $\mathcal{R}$ is.*

Proof. If $\mathcal{R}$ is amenable, let (f_n) be a Reiter sequence witnessing it. We build a Reiter sequence (g_n) for $\mathcal{S}$ by letting $g_n(y,y')=f_n(\pi(y),\pi(y'))$. Since for every $y\in Y$, π induces a bijection between $[y]_\mathcal{S}$ and $[\pi(y)]_\mathcal{R}$, it is not hard to check that (g_n) is indeed a Reiter sequence.

Conversely, let (g_n) be a Reiter sequence for $\mathcal{S}$. For all $(x,x')\in\mathcal{R}$ and $y\in Y$ such that $\pi(y)=x$, let $\pi_x^{-1}(x')$ denote the unique $y'\in Y$ such that $(y,y')\in\mathcal{S}$ and $\pi(y')=x'$. Let $(\nu_x)_{x\in X}$ be the disintegration of ν w.r.t. π. We then let $f_n(x,x')=\int_{\pi^{-1}(x)}g_n(y,\pi_x^{-1}(x'))d\nu_x(y)$. Observe that

$$\|f_n(x,\cdot)-f_n(x',\cdot)\|_1\leqslant\int_{\pi^{-1}(x)}\left\|g_n(y,\cdot)-g_n(\pi_x^{-1}(x'),\cdot)\right\|_1 d\nu_x(y).$$

Since (g_n) is Reiter, the integrand converges to zero pointwise, and it is bounded by 2 so by the Lebesgue dominated convergence we conclude that (f_n) is Reiter as wanted. □

Lemma 5.7. *Let $\mathcal{R}$ be a p.m.p. ergodic equivalence relations on (X,μ). Then the following are equivalent:*

(i) $\mathcal{R}$ is amenable;

(ii) $\mathcal{R}^{\odot n}$ is amenable for all $n\geqslant 1$;

(iii) $\mathcal{R}^{\odot n}$ is amenable for some $n\geqslant 1$.

Proof. Since $\mathcal{R}^n$ is a class-bijective extension of $\mathcal{R}^{\odot n}$, by Lemma 5.6 it suffices to show the same equivalences as above where $\mathcal{R}^{\odot n}$ is replaced by $\mathcal{R}^n$. Also note that the implication (ii)⇒(iii) is clear.

Let us show (i)⇒(iii). Assume that $\mathcal{R}$ is amenable and let (f_k) be a Reiter sequence, then

$$((x_1,...,x_n),(y_1,...,y_n))\mapsto f_k(x_1,y_1)\cdots f_k(x_n,y_n)$$

is easily seen to define a Reiter sequence for $\mathcal{R}^n$.

We finally show (iii)⇒(i). Suppose $\mathcal{R}^n$ is amenable. Observe that the projection on the first coordinate $\pi: X^n\to X$ is a class-bijective factor map from the subequivalence $\mathcal{S}$ of $\mathcal{R}^n$ given by $(((x_1,...,x_n),(y_1,...,y_n))\in\mathcal{S}$ iff $(x_1,y_1)\in\mathcal{R}$ and $x_i=y_i$ for all $i\in\{2,\dots,n\}$. Since every subequivalence relation of an amenable equivalence relation is amenable, $\mathcal{S}$ is amenable, so one more application of Lemma 5.6 yields that $\mathcal{R}$ is amenable as wanted. □

Proof of Proposition 5.4. Since ρ is non-trivial, Corollary 5.2 yields the existence of $n \geqslant 1$ such that some subequivalence relation of $\mathcal{S}$ is a class-bijective extension of $\mathcal{R}^{\odot n}$. Since $\mathcal{S}$ is amenable, we get that $\mathcal{R}^{\odot n}$ is amenable and hence by Lemma 5.7, $\mathcal{R}$ is amenable as well. $\square$

Remark 5.8. One can restate Proposition 5.4 as the fact that as soon as $\mathcal{R}$ is ergodic nonamenable and $\mathcal{S}$ is amenable, there are no non-trivial group homomorphisms $[\mathcal{R}] \to [\mathcal{S}]$. One can probably also rule out homomorphisms $[\mathcal{R}] \to [\mathcal{S}]$ when $\mathcal{R}$ has (T) and $\mathcal{S}$ has the Haagerup property. However proving this would require to develop the theory of property (T) and the Haagerup property for p.m.p. equivalence relations in the non-ergodic context, which is beyond the scope of this paper.

5.2 Cost and treeability

Given a p.m.p. equivalence relation $\mathcal{R}$, a **graphing** of $\mathcal{R}$ is a Borel subset $\mathcal{G} \subseteq \mathcal{R}$ which is symmetric: for all $(x, y) \in \mathcal{G}$, we have $(y, x) \in \mathcal{G}$. Given such a graphing, the **$\mathcal{G}$-degree** of $x \in X$ is $\deg_{\mathcal{G}}(x) := |\{y \in X : (x, y) \in \mathcal{G}\}|$. The **cost** of a graphing $\mathcal{G}$ is

$$\mathrm{Cost}(\mathcal{G}) := \frac{1}{2} \int_X \deg_{\mathcal{G}}(x).$$

A graphing of $\mathcal{R}$ **generates** $\mathcal{R}$ when $\mathcal{R}$ is the smallest equivalence relation containing $\mathcal{G}$, and the **cost** of $\mathcal{R}$ is the infimum of the costs of the graphings which generate it.

A **treeing** of $\mathcal{R}$ is a graphing of $\mathcal{R}$ whose connected components are trees (have no circuits). A p.m.p. equivalence relation $\mathcal{R}$ is **treeable** when it can be generated by a treeing. A fundamental theorem of Gaboriau shows that the cost of $\mathcal{R}$ has then to be equal to the cost of such a treeing [Gab00, Thm. 1].

We now explain how Theorem 5.1 yields restrictions for homomorphisms between full groups of p.m.p. ergodic equivalence relations related to the above definitions. Here is the result.

Proposition 5.9. *Let $\mathcal{R}$ and $\mathcal{S}$ be non-amenable p.m.p. equivalence relations on (X, μ) and (Y, ν) respectively, with $\mathcal{R}$ ergodic. Let us assume that there is a non-trivial group homomorphism $\rho \colon [\mathcal{R}] \to [\mathcal{S}]$. If $\mathcal{S}$ is treeable, then the cost of $\mathcal{R}$ is strictly greater than 1 and ρ is extension-like.*

Lemma 5.10. *Let $\mathcal{R}$ be a p.m.p. aperiodic equivalence relation on (X, μ). Then for every $n \geqslant 2$, the cost of $\mathcal{R}^{\odot n}$ is 1.*

Proof. Since $n \geqslant 2$, the equivalence relation $\mathcal{R}^n$ has cost 1 by [KM04, Thm. 24.9]. Moreover $\mathcal{R}^n$ has index $n!$ in

$$\mathcal{R}^n \rtimes \mathfrak{S}_n := \{((x_i)_{i=1}^n, (y_i)_{i=1}^n) \colon \exists \sigma \in \mathfrak{S}_n \forall i \in \{1, \dots, n\}, (x_i, y_{\sigma(i)}) \in \mathcal{R}\},$$

so we deduce from Gaboriau's result on cost for direct products [Gab00, Prop. V.1] (see also [KM04, Prop. 25.1]) that $\mathcal{R}^n \rtimes \mathfrak{S}_n$ has cost at most 1. But $\mathcal{R}^n \rtimes \mathfrak{S}_n$ has cost at least 1 by aperiodicity (this dates back to Levitt's paper where he introduces cost [Lev95, Thm. 2], see also [KM04, Cor. 21.3]) so it has cost 1. Since $\mathcal{R}^{\odot n}$ and $\mathcal{R}^n \rtimes \mathfrak{S}_n$ are stably orbit equivalent, Gaboriau's induction formula for stable orbit equivalence [Gab00, Invariance II.11.] (see also [KM04, Thm. 21.1]) yield that $\mathcal{R}^{\odot n}$ has cost 1 as wanted. $\square$

Proof of Proposition 5.9. Suppose by contradiction that the boolean action $\rho : [\mathcal{R}] \to [\mathcal{S}]$ is not extension-like, then since it is not trivial we must have that Y_n^ρ has positive measure for some finite $n \geqslant 2$ as per Corollary 5.2. In particular, the restriction of $\mathcal{S}$ to Y_n^ρ contains a class-bijective extension $\mathcal{S}_n$ of $\mathcal{R}^{\odot n}$. Since $\mathcal{S}$ is treeable, so is its restriction to Y_n^ρ by [Gab00, Prop. II.10.].

Moreover, $\mathcal{S}_n$ must have cost at most 1 since it is a class-bijective extension of a cost 1 p.m.p. equivalence relation (see the second remark after Proposition 10.14 in [Kec10]), so it has cost 1 by aperiodicity of $\mathcal{S}_n$ (which itself follows from the aperiodicity of its factor $\mathcal{R}^{\odot n}$). Also note that $\mathcal{S}_n$ is treeable by [Gab00, Thm. IV.4], and not amenable since it is a class-bijective extension of a the non-amenable p.m.p. equivalence relation $\mathcal{R}^{\odot n}$. We arrive at the desired contradiction by [Gab00, Cor. IV.2].

So our action is extension-like, and we thus get that the restriction of $\mathcal{S}$ to Y_1^ρ is a class-bijective extension of $\mathcal{R}$. Again if the cost of $\mathcal{R}$ were equal to 1, we would get that the treeable non-amenable equivalence relation $\mathcal{S} \cap (Y_1^\rho \times Y_1^\rho)$ has cost 1, a contradiction. So $\mathcal{R}$ has cost strictly greater than 1 as wanted. □

Remark 5.11. In the context of the above proposition, we do not know whether if $\mathcal{S}$ is treeable, then $\mathcal{R}$ has to be treeable as well. Indeed it is unknown whether there are non-treeable equivalence relations with treeable actions. Our result only exploits the fact that such actions do not exist when the cost of the acting equivalence relation is 1.

5.3 Solidity

Definition 5.12. A p.m.p. equivalence relation $\mathcal{R}$ is called **solid** if for any subequivalence relation $\mathcal{R}' \subseteq \mathcal{R}$, there is a partition $X = \bigsqcup_{n \geqslant 0} X_n$ of X into $\mathcal{R}'$-invariant sets such that the restriction of $\mathcal{R}'$ to X_0 is amenable, and for every $n \geqslant 1$ the restriction of $\mathcal{R}'$ to X_n is ergodic and non-amenable.

Note that by [CI10, Prop. 6], one could equivalently ask that for every $n \geqslant 1$ the restriction of $\mathcal{R}'$ to X_n is strongly ergodic.

Proposition 5.13. *A p.m.p. equivalence relation $\mathcal{R}$ is solid iff for every non-amenable subequivalence relation $\mathcal{R}'$, there is an $\mathcal{R}'$-invariant set $A \subseteq X$ of positive measure such that the restriction of $\mathcal{R}'$ to A is ergodic.*

Proof. The direct implication is clear. Conversely, let $\mathcal{R}'$ be any subequivalence relation of $\mathcal{R}$, let $X_0 \subseteq X$ measurable and maximal among measurable sets Y such that the restriction of $\mathcal{R}'$ to Y is amenable. Observe that X_0 is $\mathcal{R}'$-invariant. By maximality the restriction of $\mathcal{R}'$ to $X \setminus X_0$ is everywhere non-amenable, which by hypothesis yields that its ergodic decomposition is discrete as wanted. □

Our main result implies the following rigidity result for homomorphisms taking values into full groups of solid equivalence relations.

Proposition 5.14. *Let $\mathcal{R}$ and $\mathcal{S}$ be non-amenable p.m.p. equivalence relations on (X, μ) and (Y, ν) respectively, with $\mathcal{R}$ ergodic. Let us assume that there is a non-trivial group homomorphism $\rho \colon [\mathcal{R}] \to [\mathcal{S}]$. If $\mathcal{S}$ is solid, then $\mathcal{R}$ is solid as well and ρ is extension-like.*

Lemma 5.15. *Let $\mathcal{R}$ and $\mathcal{S}$ be non-amenable p.m.p. ergodic equivalence relations on (X, μ) and (Y, ν) respectively. Let $\pi \colon Y \to X$ be a class-bijective factor. If $\mathcal{S}$ is solid, then $\mathcal{R}$ is.*

Proof. We use the characterization provided by Proposition 5.13. Consider a non-amenable sub-equivalence relation $\mathcal{R}' \subseteq \mathcal{R}$. It yields a sub-equivalence relation $\mathcal{S}' \subseteq \mathcal{S}$ given by $(x, y) \in \mathcal{S}'$ if $(x, y) \in \mathcal{S}$ and $(\pi(x), \pi(y)) \in \mathcal{R}'$.

By construction π is a class-bijective factor map from $\mathcal{S}'$ to $\mathcal{R}'$. Lemma 5.6 yields that $\mathcal{S}'$ is not amenable. Since $\mathcal{S}$ is solid, there is a measurable $\mathcal{S}'$-invariant subset $A \subseteq Y$ such that $\mathcal{S}'$ restricted to A is ergodic. Let $B \subseteq X$ be the essential image of A, i.e. the unique up to measure zero measurable set such that

1. $\nu(\pi^{-1}(B) \cap A) = \nu(A)$;
2. for every positive measure subset $B' \subseteq B$ one has that $\nu(\pi^{-1}(B') \cap A) > 0$.

Then $\mathcal{R}'$ restricted to B is ergodic: if $B' \subseteq B$ is $\mathcal{R}'$-invariant, then $\pi^{-1}(B') \cap A$ is $\mathcal{S}'$-invariant, so it must be either empty or contain A, which implies that $B' = \emptyset$ or $B' = B$ as wanted. So we have shown that $\mathcal{R}'$ can be restricted to an ergodic equivalence relation, which by Proposition 5.14 shows that $\mathcal{R}$ is solid as wanted. $\square$

Proof of Proposition 5.14. We begin by showing that for all $n \geqslant 2$, the p.m.p. equivalence relation $\mathcal{R}^{\odot n}$ is not solid. Indeed $\mathcal{R}^n$ is not solid because it contains the non-amenable equivalence relation $\mathrm{id}_X \times \mathcal{R}^{n-1}$ which has diffuse ergodic decomposition. Since $\mathcal{R}^n$ is a class-bijective extension of $\mathcal{R}^{\odot n}$, we conclude by the previous lemma that $\mathcal{R}^{\odot n}$ is not solid.

Now, let $\rho : [\mathcal{R}] \to [\mathcal{S}]$ be a non-trivial group homomorphism. Since $\mathcal{S}$ is solid, all its restrictions are solid as well, and combining the non-solidity of $\mathcal{R}^{\odot n}$ for $n \geqslant 2$ and Corollary 5.2, we see that ρ must be extension-like. Since ρ is not trivial and any restriction of $\mathcal{S}$ has to be solid as well, it then follows from Lemma 5.15 that $\mathcal{R}$ is solid as wanted. $\square$

5.4 Conclusions

Let us recap the restrictions for non-trivial homomorphisms $\rho : [\mathcal{R}] \to [\mathcal{S}]$ that we have obtained when $\mathcal{R}$ is ergodic:

- If $\mathcal{S}$ is amenable, $\mathcal{R}$ must be amenable as well;
- If both $\mathcal{R}$ and $\mathcal{S}$ are non-amenable and $\mathcal{S}$ is treeable, then $\mathcal{R}$ has cost > 1 and ρ is extension-like;
- If both $\mathcal{R}$ and $\mathcal{S}$ are non-amenable and $\mathcal{S}$ is solid, then $\mathcal{R}$ is solid as well and ρ is extension-like.

As mentioned in Remark 5.8, a further natural result would be to rule out non-trivial homomorphisms when $\mathcal{R}$ has (T) and $\mathcal{S}$ has the Haagerup property, but seems to require to develop these properties in the non-ergodic case, which we leave as a question.

Question 5.16. Let $\mathcal{R}$ be a p.m.p. equivalence relation with property (T), is it true that if $\mathcal{S}$ is a class-bijective extension of $\mathcal{R}$ with the Haagerup property, then both $\mathcal{R}$ and $\mathcal{S}$ are periodic (have only finite classes)?

It would be interesting to investigate non-singular boolean actions of full groups, both of p.m.p. equivalence relations, but also of non-singular equivalence relations. When $\mathcal{R} = \mathcal{R}_\Gamma$ comes from a free p.m.p. Γ-action, one can use a non-singular Γ-action to build a non-singular extensions of $\mathcal{R}$ which is not p.m.p., so full groups of p.m.p. equivalence relations seem to often admit non p.m.p. boolean actions. If one wants to follow a similar approach to ours, here is a first important question.

Question 5.17. What are the quasi-invariant random subgroups of $\mathfrak{S}_{2^\infty}$, i.e. the probability measures on $\mathrm{Sub}(\mathfrak{S}_{2^\infty})$ which are quasi-preserved by the conjugacy action?

6 Homomorphisms and actions of p.m.p. equivalence relations

Any equivalence relation $\mathcal{R}$ on a set X can be seen as a groupoid where the product is given by $(x_1, x_2)(x_2, x_3) = (x_1, x_3)$ whenever (x_1, x_2) and (x_2, x_3) belong to $\mathcal{R}$. This allow one to make sense of actions of $\mathcal{R}$ on various structures. The reader acquainted with groupoids will note that our viewpoint amounts to working only with actions on constant bundles, thus making our life easier. Note that these are not the right kinds of actions when working with non-ergodic groupoids, but we do not consider them here.

Definition 6.1. Let G be a Polish group and let $\mathcal{R}$ be a p.m.p. equivalence relation. A **homomorphism**, or **cocycle** is a Borel map $c : \mathcal{R} \to G$ such that for all $(x_1, x_2), (x_2, x_3) \in \mathcal{R}$,

$$c(x_1, x_3) = c(x_1, x_2)c(x_2, x_3).$$

We will be interested in the cases where G is either the automorphism group of a probability space $\mathrm{Aut}(Z, \eta)$, the unitary group of a Hilbert space $\mathcal{U}(\mathcal{H})$ or the group of affine isometries of a real Hilbert space $\mathrm{Iso}(\mathcal{H})$. As we will see, there is a deep connection between these three kinds of actions stemming from natural embeddings between these three Polish groups. For now, let us stick to $G = \mathrm{Aut}(Z, \eta)$.

6.1 Boolean actions of p.m.p. equivalence relations

Definition 6.2. A (p.m.p.) **boolean action** of a p.m.p. equivalence relation $\mathcal{R}$ on (X, μ) is a homomorphism $c : \mathcal{R} \to \mathrm{Aut}(Z, \eta)$, where (Z, η) is a standard probability space, possibly with atoms.

Our aim is now to show that for ergodic p.m.p. equivalence relations, boolean actions are the same thing as p.m.p. actions as defined in Section 3.4.

Let $\mathcal{R}$ be a p.m.p. equivalence relation on the standard probability (X, μ). Consider another standard probability space (Z, η) and let $c\colon \mathcal{R} \to \mathrm{Aut}(Z, \eta)$ be a boolean action. Let $\Gamma \leqslant [\mathcal{R}]$ such that $\mathcal{R}$ is the equivalence relation generated by Γ. Denote by α the associated action of Γ on (X, μ). Then we would like to obtain an action β of Γ on $X \times Z$ via

$$\beta(\gamma)(x, z) = (\gamma x, c(\gamma x, x)z).$$

However, $c(\gamma x, x)$ is only defined up to null-sets, therefore we have to make sure that such a β would be measurable and an action.

The upcoming Proposition 6.4 will yield that, but first we need a lemma. Recall that given a standard probability space (X, μ) and a Polish space M, $\mathrm{L}^0(X, \mu, M)$ denotes the space of measurable maps $X \to M$ up to measure zero. The latter is a Polish space for the topology of convergence in measure, see e.g. [Kec10, Sec. 19].

Lemma 6.3. *Let (X, μ) and (Z, η) be standard probability spaces, possibly with atoms. Then we have a natural identification between* $\mathrm{MAlg}(X\times Z, \mu\otimes\eta)$ *and* $\mathrm{L}^0(X, \mu, \mathrm{MAlg}(Z, \eta))$.

Proof. In this proof, it is convenient for clarity to denote by $[A]$ the equivalence class of a Borel set in the corresponding measure algebra. For $A \subseteq X \times Z$ and $x \in X$, let $A_x \coloneqq \{z \in Z \colon (x, z) \in A\}$. Given $A \subseteq X \times Z$ Borel, the map $x \mapsto \eta(A_x)$ is Borel, and so for $B \subseteq Z$ Borel the map $x \mapsto \mu(A_x \triangle B)$ is Borel. This means that every $A \subseteq X \times Z$ Borel defines a Borel map $F_A : x \mapsto [A_x]$ from X to $\mathrm{MAlg}(Z, \eta)$. Denote by $[F_A]$ the equivalence class of F_A in $\mathrm{L}^0(X, \mu, \mathrm{MAlg}(Z, \eta))$.

Observe that if $[A] = [A']$ then by Fubini's theorem $[F_A] = [F_{A'}]$ In particular, the map $A \mapsto [F_A]$ quotients down to an injective map $\Phi : \mathrm{MAlg}(X \times Z, \mu \otimes \eta) \to \mathrm{L}^0(X, \mu, \mathrm{MAlg}(Z, \eta))$, and we now only have to show it is surjective.

Endow $\mathrm{L}^0(X, \mu, \mathrm{MAlg}(Z, \eta)$ with the complete metric

$$d(F, G) = \int_X \mu(F(x) \triangle G(x))d\mu(x).$$

Applying Fubini's theorem once more, we see that Φ is isometric. To see that Φ is surjective, first observe that the space of maps taking countably many values is dense in $\mathrm{L}^0(X, \mu, \mathrm{MAlg}(Z, \eta))$. Given such a map F taking countably many values $[B_0], [B_1], \dots,$ let $A_n = F^{-1}[B_n]$ (which is well-defined up to a null set). Then the set $A_f \coloneqq \bigcup_n A_n \times B_n$ satisfies $\Phi(A_F) = F$. So Φ has dense image, and since both $d_{\mu\otimes\eta}$ and d are complete we conclude that Φ is a surjective isometry as wanted. $\square$

Proposition 6.4. *Let (X, μ) and (Y, ν) be standard Borel spaces. Then for every Borel function $F : X \to \mathrm{Aut}(Z, \eta)$, there is an element $\tilde{F} \in \mathrm{Aut}(X \times Z, \mu \times \eta)$ such that for almost every x, we have that $\tilde{F}(x, z) = (x, F(x)z)$ for almost all $z \in Z$.*

Proof. Denote by $\cdot$ the natural action of $\mathrm{Aut}(Z, \eta)$ on $\mathrm{MAlg}(Z, \eta)$. Then for every $f \in \mathrm{L}^0(X, \mu, \mathrm{MAlg}(Z, \eta))$, we define $F(f) \in \mathrm{L}^0(X, \mu, \mathrm{MAlg}(Z, \eta))$ by

$$F(f)(x) = F(x) \cdot f(x).$$

Through the identification between $\mathrm{MAlg}(X\times Z, \mu\otimes\eta$ and $\mathrm{L}^0(X, \mu, \mathrm{MAlg}(Z, \eta))$ provided by the previous lemma, F thus defines an automorphism T_F of $\mathrm{MAlg}(X \times Z, \mu \otimes \eta)$.

Let us denote by $\tilde{F}$ an arbitrary lift of T_F to a measure-preserving bijection of $(X \times Z, \mu \otimes \eta)$, which we write as $T_F(x, z) = (\alpha(x, z), \beta(x, z))$ By construction T_F preserves elements of $\mathrm{MAlg}(X \times Z, \mu \otimes \eta)$ of the form $A \times Z$ where $A \in \mathrm{MAlg}(X, \mu)$, so T_F almost preserves Borel sets of the form $A \times Z$. If (C_n) is a separating sequence of Borel subsets of X, we conclude that after throwing away a null set, $\tilde{F}$ preserves every set of the form $C_n \times Z$, which implies that $\alpha(x, z) = x$ for almost all $x \in X$ and almost all $(x, z) \in X \times Z$.

We finally conclude that $\beta(x, \cdot) = F(x)$ for almost every x from the fact that up to measure zero, $\beta(x, \cdot)$ has to be by construction a measure-preserving bijection of Z which is equal to $F(x)$ in $\mathrm{Aut}(Z, \eta)$. $\square$

We can finally check that boolean actions yield p.m.p. actions.

Proposition 6.5. *Let $\mathcal{R}$ be a p.m.p. equivalence relation on the standard probability (X, μ) and let $c \colon \mathcal{R} \to \mathrm{Aut}(Z, \eta)$ be a Borel cocycle. Then there is an essentially unique p.m.p. equivalence relation $\mathcal{S}$ on $(X \times Z, \mu \otimes \eta)$ such that for almost every $((x, z), (x', z')) \in \mathcal{S}$ we have that $z' = c(x, x')z$. Moreover the projection $\pi : X \times Y \to X$ is a class-bijective factor map from $\mathcal{S}$ to $\mathcal{R}$.*

Proof. Let $\alpha : \Gamma \curvearrowright (X, \mu)$ generate the equivalence relation $\mathcal{R}$. We define a new Γ-action β on $(X \times Z, \mu \otimes \eta)$ by $\beta(\gamma)(x, z) = (\alpha(\gamma)x, c(\alpha(\gamma)x, x)z)$. Proposition 6.4 guarantees us that after possibly modifying it on a null set, β is a well-defined p.m.p. action such that the projection $\pi : X \times Z \to X$ is Γ-equivariant.

Denote by $\mathcal{S}$ the equivalence relation generated by β. Since π is Γ-equivariant, the map π is factor map from $\mathcal{S}$. To see that it is class-bijective, we need to show that it is injective when restricted to a β-orbit. By Remark 2.7, it suffices to show that if $\alpha(\gamma)x = x$, then for all $z \in Z$ we have $\beta(\gamma)(x, z) = (x, z)$, which is clear since $c(\alpha(\gamma)x, x) = c(x, x) = \mathrm{id}_Z$. Therefore the projection is a class-bijective factor map $\mathcal{S} \to \mathcal{R}$. □

Using Rokhlin's skew product theorem, we now show that conversely, every p.m.p. action of an ergodic p.m.p. equivalence relation is a boolean action.

Proposition 6.6. *Let $\mathcal{R}$ be a p.m.p. ergodic equivalence relation, let $\mathcal{S}$ be a p.m.p. equivalence relation on (Y, ν) which is a class-bijective extension of $\mathcal{R}$ via a map $\pi : Y \to X$. Then one can decompose (Y, ν) as a product space $Y = X \times Z$ endowed with the product measure $\mu \otimes \eta$, where η is a probability measure on the standard Borel space Z, possibly with atoms.*

Moreover in this decomposition π becomes the projection on the first coordinate and we get a cocycle $c : \mathcal{R} \to \mathrm{Aut}(Z, \eta)$ such that for all $z \in Z$ $c(x, x')z$ is the unique $z' \in Z$ such that $((x, z), (x', z')) \in \mathcal{S}$ and $\pi((x', z')) = x'$.

Proof. Let $\Gamma \leqslant [\mathcal{R}]$ such that $\mathcal{R} = \mathcal{R}_\Gamma$. Denote by β the Γ-action on (Y, ν) provided by the injection $\rho_\pi : [\mathcal{R}] \to [\mathcal{S}]$ as in Lemma 3.26, which is a class-bijective extension of the Γ-action on (X, μ) provided by the inclusion $\Gamma \leqslant [\mathcal{R}]$.

Since $\mathcal{R} = \mathcal{R}_\Gamma$ and $\mathcal{R}$ is ergodic, the Γ-action on (X, μ) provided by the inclusion $\Gamma \leqslant [\mathcal{R}]$ is ergodic. By Rokhlin's skew product theorem (see [Gla03, Thm. 3.18]), we get the desired decomposition of (Y, ν) as a product $(X \times Z, \mu \otimes \eta)$ and a cocycle $\tilde{c} : \Gamma \times X \to \mathrm{Aut}(Z, \nu)$ such that for all $(x, z) \in X \times Z$, $\beta(\gamma)(x, z) = (\gamma x, \tilde{c}(\gamma, x)z)$. Since π is class-bijective, we get a well-defined cocycle $c : \mathcal{R} \to \mathrm{Aut}(Z, \eta)$ by letting $c(x, \gamma x) = \tilde{c}(x, \gamma)$. □

We finally make the following convenient definition, which will be followed in the next section by its analogue for unitary representations.

Definition 6.7. Given a boolean action $\alpha : \mathcal{R} \to \mathrm{Aut}(Z, \eta)$ of a p.m.p. equivalence relation $\mathcal{R}$ on (X, μ), we denote by $[\alpha]$ the boolean measure-preserving action of $[\mathcal{R}]$ on $(X \times Z, \mu \otimes \eta)$ corresponding to the class-bijective extension of $\mathcal{R}$ given by the previous proposition. It is defined by: for all $T \in [\mathcal{R}]$,

$$[\alpha](T)(x, z) = (T(x), \alpha(x, T(x))z).$$

Remark 6.8. It follows from Lemma 3.27 and the correspondence between boolean and p.m.p. actions of equivalence relations that the boolean actions $[\alpha]$ of a full group $[\mathcal{R}]$ are also characterized as those for which factor onto the inclusion $\iota : [\mathcal{R}] \hookrightarrow \mathrm{Aut}(X, \mu)$ in a support-preserving manner.

6.2 Unitary representations of p.m.p. equivalence relations

Definition 6.9. Let $\mathcal{R}$ be an ergodic p.m.p. equivalence relation on a standard probability space (X, μ). A **unitary representation** of $\mathcal{R}$ is a homomorphism $\pi : \mathcal{R} \to \mathcal{U}(\mathcal{H})$, where $\mathcal{U}(\mathcal{H})$ is the unitary group of a separable Hilbert space $\mathcal{H}$.

Given unitary representation $\pi : \mathcal{R} \to \mathcal{U}(\mathcal{H})$ of a p.m.p. equivalence relation $\mathcal{R}$ on (X, μ), a **section** of the representation is a Borel map $\xi : X \to \mathcal{H}$. It is called **square-integrable** when $\int_X \|\xi(x)\|^2 d\mu(x)$ is finite. The space of square-integrable sections is naturally a Hilbert space for the scalar product $\langle \xi, \eta \rangle = \int_X \langle \xi(x), \eta(x) \rangle d\mu(x)$, and we denote this Hilbert space by $\mathrm{L}^2(X, \mu, \mathcal{H})$. Observe that $\mathrm{L}^2(X, \mu, \mathcal{H})$ has a natural $\mathrm{L}^\infty(X, \mu)$-module structure with $\mathrm{L}^\infty(X, \mu)$ acting by multiplication on sections. Moreover, one can associate to π a unitary representation $[\pi]$ of the full group of $\mathcal{R}$ on $\mathrm{L}^2(X, \mu, \mathcal{H})$: for all $T \in [\mathcal{R}]$ and all $\xi \in \mathrm{L}^2(X, \mu, \mathcal{H})$, we let

$$([\pi](T)\xi)(T(x)) = \pi(x, T(x))\xi(x).$$

We will now characterize algebraically unitary representations of the full group of the form $[\pi]$ in a manner analogous to Remark 6.8. We need the following additional structure.

Definition 6.10. An $\mathrm{L}^\infty(X, \mu)$**-module** is a Hilbert space $\mathcal{K}$ an a *faithful* normal $*$-representation of $\mathrm{L}^\infty(X, \mu)$ in $\mathcal{B}(\mathcal{K})$.

Given an $\mathrm{L}^\infty(X, \mu)$**-module** $\mathcal{K}$, we identify $\mathrm{L}^\infty(X, \mu)$ to its image in $\mathcal{B}(\mathcal{K})$. For all $A \subseteq X$, we denote by $\mathcal{K}_A$ the subspace $\chi_A \mathcal{K}$.

Our motivating example of $\mathrm{L}^\infty(X, \mu)$-modules is provided by unitary representations of p.m.p. equivalence relations: indeed the space of square-integrable sections is easily checked to be an $\mathrm{L}^\infty(X, \mu)$-module.

Here are the abstract properties of $[\pi]$ which will allow us to reconstruct π.

Definition 6.11. Let $\mathcal{R}$ be a p.m.p. equivalence relation on (X, μ), let $\rho : [\mathcal{R}] \to \mathcal{U}(\mathcal{K})$ be a unitary representation on a Hilbert space $\mathcal{K}$ equipped with an $\mathrm{L}^\infty(X, \mu)$-module structure. Say that ρ is **full** when it satisfies the following two conditions:

(i) For all $A \subseteq X$ Borel, $\rho(T)\chi_A = \chi_{T(A)}\rho(T)$;

(ii) For all $A \subseteq X$, and all $T, U \in [\mathcal{R}]$ such that $T_{\restriction A} = U_{\restriction A}$, we have $\rho(T)_{\restriction \mathcal{K}_A} = \rho(U)_{\restriction \mathcal{K}_A}$.

Note that the first condition yields that $\rho(T)\mathcal{K}_A = \mathcal{K}_{T(A)}$ for all $T \in [\mathcal{R}]$, while the second is equivalent to asking that $\rho(T)_{\restriction \mathcal{K}_{\mathrm{Fix}(T)}} = \mathrm{id}_{\mathcal{K}_{\mathrm{Fix}(T)}}$ for all $T \in [\mathcal{R}]$.

Theorem 6.12. *Let $\mathcal{R}$ be an ergodic p.m.p. equivalence relation on (X, μ). Every full unitary representation $\rho : [\mathcal{R}] \to \mathcal{U}(\mathcal{K})$ is unitarily equivalent to a unitary representation of the form $[\pi]$ for some unitary representation π of $\mathcal{R}$.*

Proof. By [KR97, Thm. 14.2.1], the $\mathrm{L}^\infty(X, \mu)$-module $\mathcal{K}$ can be decomposed as a direct integral of Hilbert spaces $(\mathcal{H}_x)_{x \in X}$. By condition (i) and ergodicity the dimension of $\mathcal{H}_x$ is almost surely constant. We can thus assume $\mathcal{H}_x$ is constant equal to a fixed Hilbert space $\mathcal{H}$ and so $\mathcal{K} = \mathrm{L}^2(X, \mu, \mathcal{H})$.

Denote by $\lambda : [\mathcal{R}] \to \mathrm{L}^2(X, \mu, \mathcal{H})$ be the unitary representation defined by $(\lambda(T)\xi)(T(x)) = \xi(x)$. Then for all $A \subseteq X$ Borel, we have $\lambda(T)\chi_A = \chi_{T(A)}\lambda(T)$. So by condition (i), for every $T \in [\mathcal{R}]$, the unitary $\lambda(T)^*\rho(T)$ commutes with $\mathrm{L}^\infty(X, \mu)$ and is thus a decomposable operator

$$\lambda(T)^*\rho(T) = \int_X c(T, x) d\mu(x),$$

where $c(T,x) \in \mathcal{U}(\mathcal{H})$ for almost all $x \in X$. Observe that by uniqueness of the decomposition of a decomposable operator, for any $U \in [T]$ we have $\lambda(U)\int_X c(T,x)d\mu(x)\lambda(U)^* = \int_X c(T,U^{-1}(x))d\mu(x)$. When $T_1, T_2 \in [\mathcal{R}]$, we thus have

$$\begin{aligned}
\lambda(T_1T_2)^*\rho(T_1T_2) &= \lambda(T_2)^*\lambda(T_1)^*\rho(T_1)\rho(T_2) \\
&= \lambda(T_2)^*\int_X c(T_1,x)d\mu(x)\rho(T_2) \\
&= \int_X c(T_1,T_2(x))d\mu(x)\lambda(T_2)^*\rho(T_2) \\
&= \int_X c(T_1,T_2(x))c(T_2,x)d\mu(x),
\end{aligned}$$

so for almost all $x \in X$, $c(T_1,T_2(x))c(T_2,x) = c(T_1T_2,x)$. We now pick a countable group Γ such that $\mathcal{R} = \mathcal{R}_\Gamma$. By restricting to a full measure set, we may assume that the above cocycle relation holds for all $\gamma \in \Gamma$ and all $x \in X$.

Furthermore by condition (ii), we have that $\rho(\gamma)$ is trivial when restricted to $\mathcal{K}_{\mathrm{Fix}(\gamma)}$, so up to restricting again we have $c(\gamma,x) = \mathrm{id}_\mathcal{H}$ for all $\gamma \in \Gamma$ and all $x \in \mathrm{Fix}(\gamma)$. It follows that we have a well defined unitary representation $\pi : \mathcal{R} \to \mathcal{U}(\mathcal{H})$ by letting $\pi(x,\gamma\cdot x) = c(\gamma,x)$. By construction we have $\rho(T) = \lambda(T)\int_X c(T,x)d\mu(x)$, so for all $\xi \in \mathrm{L}^2(X,\mu,\mathcal{H})$ and all $x \in X$ we have

$$\rho(T)(\xi)(T(x)) = \lambda(T)\int_X c(T,y)d\mu(y)\xi(T(x)) = c(T,x)\xi(x).$$

Now if we pick $\gamma \in \Gamma$ such that $\gamma \cdot x = T(x)$ we have by condition (ii)

$$c(T,x)\xi(x) = c(\gamma,x)\xi(x) = \pi(x,\gamma\cdot x)\xi(x) = \pi(x,T(x))\xi(x).$$

We thus have $\rho = [\pi]$ as wanted. □

7 Property (T)

7.1 First definition and strong ergodicity

Let $\mathcal{R}$ be a measure-preserving equivalence relation on (X,μ). Given a unitary representation $\pi : \mathcal{R} \to \mathcal{U}(\mathcal{H})$, a **unit section** is a section $\xi : X \to \mathcal{H}$ satisfying that for all $x \in X$, $\|\xi(x)\| = 1$. We say that a sequence (ξ_n) of unit sections is **almost invariant** when the sequence of functions $(x,y) \in \mathcal{R} \mapsto \|\pi(x,y)\xi_n(x) - \xi_n(y)\|$ converges pointwise to zero, up to restricting to a full measure subset of X.

Remark 7.1. Equivalently, one could take a weaker definition for almost invariant unit sections by asking that the sequence of functions $(x,y) \mapsto \|\pi(x,y)\xi_n(x) - \xi_n(y))\|$ converges to zero *in measure*. Since every sequence converging in measure to zero has a subsequence which converges pointwise to zero, this does not affect the definition of property (T) (see [Ana05, Lem. 4.1]).

Definition 7.2. A p.m.p. equivalence relation $\mathcal{R}$ has **property (T)** if whenever a unitary representation π of $\mathcal{R}$ has almost invariant unit sections, then it has an invariant unit section, i.e. we can find a section ξ such that for almost all $(x,y) \in \mathcal{R}$, we have $\pi(x,y)\xi(x) = \xi(y)$.

Remark 7.3. An ergodic p.m.p. equivalence relation $\mathcal{R}$ has property (T) iff every unitary representation with almost invariant unit sections has a non-zero invariant section. Indeed such a section must have an invariant norm which is thus constant by ergodicity. This constant must be non zero, so by rescaling we obtain an invariant unit section.

We now connect property (T) to the fundamental notion of strong ergodicity. Given a p.m.p. equivalence relation $\mathcal{R}$ on a standard probability space (X,μ), a sequence (A_n) of subsets of X is called **asymptotically invariant** if for all $T \in [\mathcal{R}]$, $\mu(T(A_n) \bigtriangleup A_n) \to 0$, and **asymptotically trivial** if $\mu(A_n)(1-\mu(A_n)) \to 0$.

Definition 7.4. A p.m.p. equivalence relation $\mathcal{R}$ on a standard probability space (X,μ) is **strongly ergodic** if every asymptotically invariant sequence of subsets of X has to be asymptotically trivial.

By considering, for an $\mathcal{R}$-invariant set A, the constant sequence equal to A, we see that strong ergodicity implies ergodicity. We now note the well-known fact (implicit in [Pic07]) that for an ergodic p.m.p. equivalence relation, property (T) implies strong ergodicity.

Proposition 7.5 (Folklore)**.** *Let $\mathcal{R}$ be an ergodic p.m.p. equivalence relation with property (T). Then $\mathcal{R}$ is strongly ergodic.*

Proof. Suppose by contradiction that $\mathcal{R}$ has (T) but is not strongly ergodic. By [JS87], there is a measure-preserving map $\pi : (X,\mu) \to (\{0,1\}^{\mathbb{N}}, \mathcal{B}(1/2)^{\odot\mathbb{N}})$ such that $(\pi\times\pi)(\mathcal{R}) = \mathcal{R}_0$, where $\mathcal{R}_0$ is as in Definition 3.20.

For every $k \in \mathbb{N}$, let $\mathcal{S}_k$ be the equivalence relation on $\{0,1\}^{\mathbb{N}}$ defined by $(x_n)\mathcal{S}_k(y_n)$ if and only if $x_n = y_n$ for all $n \geqslant k$. We already observed that $\mathcal{R}_0 = \bigcup_{k\in\mathbb{N}} \mathcal{S}_k$, so if $\mathcal{R}'_k := (\pi\times\pi)^{-1}(\mathcal{S}_k)$ then $\mathcal{R} = \bigcup_{k\in\mathbb{N}} \mathcal{R}'_k$.

But since $\mathcal{R}$ has property (T) it is not *approximable* : there is $k \in \mathbb{N}$ and a non-negligible Borel set $A \subseteq X$ such that $\mathcal{R} \cap (A\times A) = \mathcal{R}'_k \cap (A\times A)$ (see [Pic07, Prop. 16]; see also [GT16] for more on approximability). Now observe that each $\mathcal{S}_k$ has diffuse ergodic decomposition because it has only finite classes, hence so does each $\mathcal{R}'_k$. However $\mathcal{R}$ is ergodic, so we cannot have $\mathcal{R} \cap (A\times A) = \mathcal{R}'_k \cap (A\times A)$, which is the desired contradiction. □

Our aim is now to give a notion of Kazhdan pair (and triple) which provides us more quantitative versions of property (T) as in [BdlHV08, Prop. 1.1.9]. Such a characterization of property (T) will be given purely in terms of full unitary representations of the full group, using implicitely Theorem 6.12. This will rely crucially on the following spectral gap characterization of strong ergodicity due to Houdayer, Marrakchi and Verraedt [HMV19].

Theorem 7.6 (Houdayer, Marrakchi, Verraedt)**.** *Let $\mathcal{R}$ be a strongly ergodic p.m.p. equivalence relation. Then there is a finite set $F \subseteq [\mathcal{R}]$ and $\kappa > 0$ such that for all $\eta \in \mathrm{L}^2(X,\mu)$,*

$$\left\|\eta - \int_X \eta(x)d\mu(x)\right\|^2 \leqslant \kappa \sum_{T\in F} \left\|\eta - \eta\circ T^{-1}\right\|^2.$$

When we have a finite set $F \subseteq [\mathcal{R}]$ and $\kappa > 0$ as above, we say that (F,κ) is a **spectral gap pair** for $\mathcal{R}$.

7.2 Kazhdan pairs and proximity of invariant vectors

As explained before, one can associate to every unitary representation π of a p.m.p. equivalence relation $\mathcal{R}$ a full unitary representation $[\pi]$ of its full group on the space of square integrable sections: for all $T \in [\mathcal{R}]$ and all square-integrable section ξ, we let

$$([\pi](T)\xi)(T(x)) = \pi(x, T(x))\xi(x).$$

Observe that a square-integrable section is $[\pi]$-invariant if and only if it is a π-invariant section. Given a unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$ of a group G and a finite subset F of G, we say that a vector $\xi \in \mathcal{H}$ is (F, ϵ)**-invariant** if for all $g \in F$,

$$\|\pi(g)\xi - \xi\| < \epsilon \|\xi\|.$$

Definition 7.7. Let $\mathcal{R}$ be a p.m.p. equivalence relation, let F be a finite subset of $[\mathcal{R}]$ and let $\epsilon > 0$. Then (F, ϵ) is a **Kazhdan pair** for the equivalence relation $\mathcal{R}$ if whenever $[\pi]$ is a full unitary representation of $\mathcal{R}$ admitting an (F, ϵ)-invariant vector, then $[\pi]$ has a non-zero invariant vector.

Proposition 7.8. *A p.m.p. ergodic equivalence relation $\mathcal{R}$ has property (T) if an only if it admits a Kazhdan pair.*

Proof. Suppose (F, ϵ) is a Kazhdan pair for $\mathcal{R}$. By the dominated convergence theorem, if (ξ_n) is a sequence of almost invariant unit sections for a unitary representation π of $\mathcal{R}$, then for all $T \in F$ we have $\|\pi(T)\xi_n - \xi_n\| \to 0$. In particular for n large enough, ξ_n is (F, ϵ)-invariant and thus we have a non-zero invariant section, which by ergodicity and rescaling yields an invariant unit section.

Conversely, assume that $\mathcal{R}$ has (T) but no Kazhdan pair. Then $\mathcal{R}$ is strongly ergodic, so by Theorem 7.6 it has a spectral gap pair (F, κ). By the Feldman-Moore theorem, there is an increasing sequence (F_n) of finite subsets of $[\mathcal{R}]$ so that $\mathcal{R}$ is covered by the reunion over $n \in \mathbb{N}$ of the graphs of the elements of F_n. For every $n \in \mathbb{N}$, $(F \cup F_n, \frac{1}{n})$ is not a Kazhdan pair so we find a unitary representation π_n of $\mathcal{R}$ on $\mathcal{H}_n$ so that $[\pi_n]$ has an $(F \cup F_n, \frac{1}{n})$-invariant vector ξ_n but no non-zero invariant vector.

For each $n \in \mathbb{N}$, define $\eta_n(x) = \|\xi_n(x)\|$. By the reversed triangle inequality, we have for all $T \in [\mathcal{R}]$:

$$\begin{aligned}
\|[\pi](T)\xi_n - \xi_n\|^2 &= \int_X \|\pi(T)\xi_n(x) - \xi_n(x)\|^2 \, d\mu(x) \\
&\geqslant \int_X (\|\pi(T)\xi_n(x)\| - \|\xi_n(x)\|)^2 \, d\mu(x) \\
&= \int_X \left|\eta_n(T^{-1}(x)) - \eta_n(x)\right|^2 d\mu(x).
\end{aligned}$$

Since (F, κ) is a spectral gap pair, we then have

$$\left\|\eta_n - \int_X \eta_n(x) d\mu(x)\right\|^2 \leqslant \kappa \sum_{T \in F} \left\|\eta_n - \eta_n \circ T^{-1}\right\|^2 \leqslant \kappa \sum_{T \in F} \|\pi(T)\xi_n - \xi_n\|^2 \leqslant \frac{\kappa\,|F|}{n^2}.$$

Moreover since $\|\eta_n\| = 1$ we deduce from the reversed triangle inequality

$$\left|1 - \int_X \eta_n(x) d\mu(x)\right|^2 \leqslant \left\|\eta_n - \int_X \eta_n(x) d\mu(x)\right\|^2 \leqslant \frac{\kappa\,|F|}{n^2}.$$

so by the triangle inequality

$$\|\eta_n - 1\| \leqslant \frac{2\sqrt{\kappa\,|F|}}{n}.$$

Let $v_n \in \mathcal{H}_n$ be a fixed unit vector. Define $\xi_n' : X \to \mathcal{H}_n$ by

$$\xi_n'(x) = \begin{cases} \frac{1}{\eta_n(x)}\xi_n(x) & \text{if } \eta_n(x) \neq 0 \\ v_n & \text{otherwise.} \end{cases}$$

Each ξ_n' is by construction a unit section for π_n. We have

$$\begin{aligned} \|\xi_n - \xi_n'\|^2 &= \int_X \left(1 - \frac{1}{|\eta_n(x)|^2}\right) |\eta_n(x)|^2 \, d\mu(x) \\ &= \int_X |1 - \eta_n(x)|^2 \, d\mu(x) \\ &= \|\eta_n - 1\|^2 \\ &\leqslant \frac{4\kappa\,|F|}{n^2}. \end{aligned}$$

Since ξ_n is $(F_n, \frac{1}{n})$-invariant, it follows from the triangle inequality that each ξ_n' is an $\left(F_n, \frac{1}{n} + \frac{4\sqrt{\kappa|F|}}{n}\right)$-invariant unit section.

Consider the infinite direct sum $\mathcal{H} \coloneqq \bigoplus_n \mathcal{H}_n$, we have a unitary representation $\bigoplus_n \pi_n$ of $\mathcal{R}$ on $\mathcal{H}$. Each ξ_n' defines a unit section of $\bigoplus_n \pi_n$ and since $\mathcal{R}$ is the union of the graphs of the elements of F_n, by [Ana05, Lem. 4.1] after taking a subsequence the sequence (ξ_n') is almost invariant. But since no π_n had an invariant section, neither does their direct sum, contradicting that $\mathcal{R}$ had (T) as wanted. □

We can now give an even more quantitative version of property (T) by controlling how far our invariant vector will be from the (F, ϵ)-invariant vector, obtaining the desired p.m.p. equivalence relation version of [BdlHV08, Prop. 1.1.9] (see also [Pic07, Thm. 20] for a related sequential and pointwise version of what follows).

Proposition 7.9. *Let $\mathcal{R}$ be an ergodic p.m.p. equivalence relation with a Kazhdan pair (F_1, ϵ) and a spectral gap pair (F_2, κ). Suppose that $\pi : \mathcal{R} \to \mathcal{U}(\mathcal{H})$ is a unitary representation and that we have a section $\xi \in \mathrm{L}^2(X, \mu, \mathcal{H})$ which is $(F_1 \cup F_2, \delta\epsilon)$-invariant for some $\delta > 0$. Then $[\pi]$ admits an invariant vector which is at distance at most $\delta(1+\epsilon\sqrt{\kappa\,|F_2|})\,\|\xi\|$ from ξ.*

Proof. Let $\mathcal{K}_1$ be the subspace consisting of $[\pi]$-invariant vectors. Denote by p_1 the orthogonal projection onto $\mathcal{K}_1$. Let $\mathcal{K}$ be be the $\mathrm{L}^\infty(X)$-module spanned by $\mathcal{K}_1$. Observe that the restriction of $[\pi]$ to $\mathcal{K}$ is a multiple of $[\theta]$ where θ is the trivial unitary representation of $\mathcal{R}$ (on $\mathbb{C}$). Write $\xi = \xi' + \xi''$ with $\xi' \in \mathcal{K}$ and $\xi'' \in \mathcal{K}^\perp$, then by construction $p_1(\xi')$ is the invariant vector which is the closest to ξ.

Since $\mathcal{K}^\perp$ is a $[\pi]$-invariant $\mathrm{L}^\infty(X)$-module without non-zero invariant vectors and (F_1, ϵ) is a Kazhdan pair, there is some $T \in F_1$ such that $\|\xi' - [\pi](T)\xi'\| \geqslant \epsilon\,\|\xi'\|$. On the other hand, we have $\|\xi' - [\pi](T)\xi'\| \leqslant \|\xi - [\pi](T)\xi\| < \epsilon\delta\,\|\xi\|$, so $\|\xi'\| < \delta\,\|\xi\|$.

Finally, since the restriction of $[\pi]$ to $\mathcal{K}$ is a multiple of $[\theta]$, we have

$$\|\xi'' - p_1(\xi'')\|^2 \leqslant \kappa \sum_{T' \in F_2} \|\xi'' - [\pi](T')\xi''\|^2 \leqslant \kappa\delta^2\epsilon^2\,|F_2|\,\|\xi''\|^2 \leqslant \kappa\delta^2\epsilon^2\,|F_2|\,\|\xi\|^2.$$

Since $p_1(\xi) = p_1(\xi'')$, we finally have

$$\|\xi - p_1(\xi)\| \leqslant \|\xi'\| + \|\xi'' - p_1(\xi'')\| \leqslant \delta(1 + \epsilon\sqrt{\kappa\,|F_2|})\,\|\xi\|.$$ □

7.3 Some permanence properties

We begin this section with a result which is a direct consequence of Anantharaman-Delaroche's Theorem 5.15 from [Ana05]. We provide a version of her proof in terms of full unitary representations and Kazhdan pairs for the convenience of the reader.

Proposition 7.10. *Let $\mathcal{R}$ be an ergodic p.m.p. equivalence relation on (X, μ) with property (T), suppose $\mathcal{S}$ is a p.m.p. equivalence relation on (Y, ν) which is a class-bijective extension of $\mathcal{R}$. Then $\mathcal{S}$ has property (T).*

Proof. Let $\pi : Y \to X$ denote the class-bijective factor map from $\mathcal{S}$ to $\mathcal{R}$, and let $\rho_\pi : [\mathcal{R}] \to [\mathcal{S}]$ be the corresponding boolean action, which is a support-preserving extension of the inclusion $\iota : [\mathcal{R}] \to \mathrm{Aut}(X, \mu)$ (see 3.26). Let (F, ϵ) be a Kazhdan pair for $\mathcal{R}$, we will actually show that $(\rho_\pi(F), \epsilon)$ is a Kazhdan pair for $\mathcal{S}$.

Let $[\pi] : [\mathcal{S}] \to \mathcal{U}(\mathcal{H})$ be a full unitary representation of $\mathcal{S}$ with a $(\tilde{\alpha}(K), \epsilon)$-invariant vector, we view its space of square-integrable sections as an $\mathrm{L}^\infty(X \times Y)$-module. In particular, it is an $\mathrm{L}^\infty(X)$-module and it follows from the fact that $\tilde{\alpha}$ is a support-preserving extension of ι that $[\pi] \circ \tilde{\alpha}$ is a full unitary representation of $[\mathcal{R}]$. Since $[\pi]$ has a $(\tilde{\alpha}(F), \epsilon)$-invariant vector and (F, ϵ) is a Kazhdan pair for $\mathcal{R}$, there is a non-zero $\pi(\tilde{\alpha}([\mathcal{R}]))$-invariant vector $\xi \in \mathrm{L}^2(X \times Y, \mathcal{H})$. By fullness and the fact that the full group generated by $\tilde{\alpha}([\mathcal{R}])$ is equal to $[\mathcal{S}]$, this vector is actually $\pi([\mathcal{S}])$-invariant, which concludes the proof. □

We will now show that if $\mathcal{R}$ is a p.m.p. ergodic equivalence relation with property (T), then $\mathcal{R}^{\odot n}$ has property (T) as well. The following proposition already yields that $\mathcal{R}^n$ has property (T).

Lemma 7.11. *Let $\mathcal{R}$ and $\mathcal{S}$ be two p.m.p. ergodic equivalence relations on (X, μ) and (Y, ν) respectively. Suppose that both $\mathcal{R}$ and $\mathcal{S}$ have property (T). Then the equivalence relation $\mathcal{R} \times \mathcal{S}$ on $X \times Y$ is ergodic and has property (T).*

Proof. The ergodicity of $\mathcal{R} \times \mathcal{S}$ is well-known, and follows from the fact that ergodic full groups act transitively on sets of the same measure.

Let $(F_\mathcal{S}, \epsilon_\mathcal{S})$ be a Kazhdan pair for $\mathcal{S}$. By Proposition 7.9, we find a Kazhdan pair $(F_\mathcal{R}, \epsilon_\mathcal{R})$ such that given any full unitary representation of $[\mathcal{R}]$, if there is an $(F_\mathcal{R}, \epsilon_\mathcal{R})$-invariant unit vector, then there is an invariant vector at distance at most $\min(\frac{1}{2}, \frac{\epsilon_\mathcal{S}}{3})$ from it. Consider the two commuting inclusions $\iota_\mathcal{R} : [\mathcal{R}] \to [\mathcal{R} \times \mathcal{S}]$ and $\iota_\mathcal{S} : [\mathcal{S}] \to [\mathcal{R} \times \mathcal{S}]$, which are support-preserving for the projection on the first and second coordinate respectively. Let $F := \iota_\mathcal{R}(F_\mathcal{R}) \cup \iota_\mathcal{S}(F_\mathcal{S})$ and $\epsilon < \min(\frac{\epsilon_\mathcal{S}}{3}, \epsilon_\mathcal{R})$. We will show that (F, ϵ) is a Kazhdan pair for $\mathcal{R} \times \mathcal{S}$.

Let $[\pi] : [\mathcal{R} \times \mathcal{S}] \to \mathcal{U}(\mathcal{H})$ be a full unitary representation with an (F, ϵ)-invariant unit vector ξ, then we can view $\mathcal{H}$ both as an $\mathrm{L}^\infty(X)$ and as an $\mathrm{L}^\infty(Y)$-module. Then $[\pi] \circ \iota_\mathcal{R}$ is a full unitary representation of $[\mathcal{R}]$ and ξ is $(F_\mathcal{R}, \epsilon_\mathcal{R})$-invariant, so there is a $[\pi] \circ \iota_\mathcal{R}$-invariant vector η at distance at most $\min(\frac{1}{2}, \frac{\epsilon_\mathcal{S}}{3})$ from ξ, in particular η is not zero. By the triangle inequality, η is $(F, \epsilon + \frac{2\epsilon_\mathcal{S}}{3})$ invariant. In particular, η is $(\iota_\mathcal{S}(F_\mathcal{S}), \epsilon_\mathcal{S})$-invariant.

Let $\mathcal{K}$ denote the space of $[\pi] \circ \iota_\mathcal{R}$-invariant vectors, note that $\mathcal{K}$ is an $\mathrm{L}^\infty(Y)$-module which is $[\pi] \circ \iota_\mathcal{S}$-invariant. This module is non-zero by the previous paragraph and contains an $(F_\mathcal{S}), \epsilon_\mathcal{S})$-invariant vector, so it contains a non-zero invariant vector for $[\pi] \circ \iota_\mathcal{S}$. We thus have found a non-zero vector which is both $[\pi] \circ \iota_\mathcal{R}$ and $[\pi] \circ \iota_\mathcal{S}$ invariant. By fullness and the fact that $[\mathcal{R} \times \mathcal{S}] = [\iota_\mathcal{R}([\mathcal{R}]) \cup \iota_\mathcal{S}([\mathcal{S}])]$, this vector is $[\pi]$-invariant as wanted. □

Since $\mathcal{R}^n$ is a class-bijective extension of $\mathcal{R}^{\odot n}$, the following proposition is a consequence of the second item in [Ana05, Thm. 5.15] along with the previous lemma, but we provide a direct proof.

Proposition 7.12. *Let $\mathcal{R}$ be an ergodic p.m.p. equivalence relation on (X,μ) and $n \in \mathbb{N}$. Then $\mathcal{R}^{\odot n}$ has property (T).*

Proof. Let $\pi : \mathcal{R}^{\odot n} \to \mathcal{U}(\mathcal{H})$ be a unitary representation with a sequence (ξ_k) of almost invariant unit sections. Each ξ_k is thus an element of $\mathrm{L}^2(X^{\odot n}, \mathcal{H})$. We denote by $\hat{\pi} : \mathcal{R}^n \to \mathcal{U}(\mathcal{H})$ the unitary representation defined by $\hat{\pi}((x_1,\dots,x_n),(y_1,\dots,y_n)) = \pi([x_1,\dots,x_n],[y_1,\dots,y_n])$. Its space of square integrable sections is equal to $\mathrm{L}^2(X^n,\mathcal{H})$, and the subspace of $\mathfrak{S}_n$-invariant sections naturally identifies to $\mathrm{L}^2(X^{\odot n},\mathcal{H})$: every $\mathfrak{S}_n$-invariant section $X^n \to \mathcal{H}$ quotients down to a section $X^{\odot n} \to \mathcal{H}$ and conversely every section $\xi : X^{\odot n} \to \mathcal{H}$ yields a $\mathfrak{S}_n$-invariant section $\hat{\xi} : X^n \to \mathcal{H}$ given by $\hat{\xi}(x_1,\dots,x_n) = \xi([x_1,\dots,x_n])$.

With these identifications in mind, the orthogonal projection onto $\mathrm{L}^2(X^{\odot n},\mathcal{H})$ is the map which takes $\xi \in \mathrm{L}^2(X^n,\mathcal{H})$ to the average $\frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\sigma\xi$, where $\sigma\xi(x_1,\dots,x_n) = (x_{\sigma^{-1}(1)},\dots,x_{\sigma^{-1}(n)})$. Observe that $\mathcal{R}^n$ has property (T) as a consequence of the previous lemma. The sequence (ξ_k) is a sequence of almost invariant unit sections for $\hat{\pi}$ and they belong to the subspace $\mathrm{L}^2(X^{\odot n},\mathcal{H})$, hence by Proposition 7.9 there is $\eta \in \mathrm{L}^2(X^n,\mathcal{H})$ which is $\hat{\pi}$-invariant, has norm 1 and is at distance < 1 from $\mathrm{L}^2(X^{\odot n},\mathcal{H})$. Its orthogonal projection $\tilde{\eta} = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\sigma\eta$ onto $\mathrm{L}^2(X^{\odot n},\mathcal{H})$ is then non zero. Furthermore, observe that for every $\sigma \in \mathfrak{S}_n$,, the section $\sigma\eta$ is still $\hat{\pi}$-invariant. It follows that for every element $([x_1,\dots,x_n],[y_1,\dots,y_n]) \in \mathcal{R}^{\odot n}$, we have

$$\begin{aligned}\pi([x_1,\dots,x_n],[y_1,\dots,y_n])\tilde{\eta}(x_1,\dots,x_n) &= \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\hat{\pi}((x_1,\dots,x_n),(y_1,\dots,y_n))\sigma\eta(x_1,\dots,x_n)\\ &= \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\sigma\eta(y_1,\dots,y_n)\\ &= \tilde{\eta}(y_1,\dots,y_n).\end{aligned}$$

So $\tilde{\eta}$ is a non-zero invariant section for π, which by ergodicity and rescaling yields an invariant unit section. We conclude that $\mathcal{R}^{\odot n}$ has property (T) as wanted. □

7.4 Proof of the direct implication in Theorem C

In this section, we put together the previous results so as to obtain the direct implication from Theorem C. We need a natural definition.

Definition 7.13. Let G be a group. Given a boolean action $\rho : G \to \mathrm{Aut}(Y,\nu)$, a sequence (A_n) of subsets of X is asymptotically ρ-invariant when for all $g \in G$, $\nu(\rho(g)A_n \triangle A_n) = 0$, and asymptotically trivial when $\mu(A_n)(1-\mu(A_n)) \to 0$. Finally ρ is **strongly ergodic** if every asymptotically ρ-invariant sequence is asymptotically trivial.

Theorem 7.14. *Let $\mathcal{R}$ be an ergodic p.m.p. equivalence relation, suppose $\mathcal{R}$ has property (T). Then every ergodic non-free boolean action of $[\mathcal{R}]$ is strongly ergodic.*

Proof. Let $\rho : [\mathcal{R}] \to \mathrm{Aut}(Y,\nu)$ be p.m.p. boolean ergodic non-free action of $[\mathcal{R}]$. By Theorem 5.1 and ergodicity, there is some $n \geqslant 1$ such that ρ comes from a p.m.p. action of $\mathcal{R}^{\odot n}$. Since ρ is ergodic, the action of $\mathcal{R}^{\odot n}$ has to be ergodic as well, and since $\mathcal{R}^{\odot n}$ has

(T) the equivalence relation generated by ρ must have (T) as a consequence of Proposition 7.10. It is thus strongly ergodic by Proposition 7.5 and we conclude that the action α itself is strongly ergodic. $\square$

8 From strongly ergodic actions to property (T)

In this section, we prove a version of the Connes-Weiss characterization of property (T) for groups, but for p.m.p. equivalence relations, allowing us to complete our proof of Theorem C.

Theorem 8.1. *Let $\mathcal{R}$ be a p.m.p. ergodic equivalence relation. Then $\mathcal{R}$ has property (T) if and only if every ergodic p.m.p. action of $\mathcal{R}$ is strongly ergodic.*

The direct implication was already used in the previous section, and it is a direct consequence of Proposition 7.10 and Proposition 7.5. This section will thus be devoted to the converse, which is proved by contraposition. Towards this, we need as in the group case a weaker version of property (T) in terms of finite dimensional subrepresentations instead of invariant vectors [BV93]. This is where affine actions come in, as in the group case.

8.1 The Bekka-Valette characterization of property (T)

Definition 8.2. Let $\mathcal{R}$ be a p.m.p. equivalence relation and π be a unitary representation of $\mathcal{R}$ on a Hilbert space $\mathcal{H}$. **A finite dimensional subrepresentation of** π is a nontrivial $[\pi]$-invariant subspace $\mathcal{K}$ of $\mathrm{L}^2(X, \mu, \mathcal{H})$ such that there exists a finite set $F \subseteq \mathcal{K}$ with $\mathrm{L}^\infty(X, \mu)F$ dense in $\mathcal{K}$.

This section is devoted to the proof of the following theorem, which is the p.m.p. equivalence relation version of the Bekka-Valette theorem [BV93, Thm. 1].

Theorem 8.3. *Let $\mathcal{R}$ be an ergodic p.m.p. equivalence relation. Then $\mathcal{R}$ has property (T) if and only if every unitary representation of $\mathcal{R}$ which has almost invariant unit sections contains a finite dimensional subrepresentation.*

Note that the latter condition is a weaker version of property (T). Thus only right-to-left direction remains to be proven.

Proposition 8.4 ([DLHV89, Chap. 4])**.** *Let $\mathcal{H}$ be a real affine Hilbert space. Then for any $t > 0$, there exists a unique complex Hilbert space $\mathcal{H}_t$ and a continuous mapping $\xi \mapsto \xi_t$ from $\mathcal{H}$ to the unit sphere of $\mathcal{H}_t$ such that for any $\xi, \eta \in \mathcal{H}$, $\langle \xi_t, \eta_t \rangle = \exp(-t\|\xi - \eta\|^2)$ and the image of $\{\xi_t \colon \xi \in \mathcal{H}\}$ is total in $\mathcal{H}_t$.*

Let $\mathcal{R}$ be a p.m.p. equivalence relation. An **affine isometric action** of $\mathcal{R}$ is a homomorphism $\rho : \mathcal{R} \to \mathrm{Iso}(\mathcal{H})$, where $\mathrm{Iso}(\mathcal{H})$ denotes the group of affine isometries of a separable real Hilbert space $\mathcal{H}$.

For $t > 0$, we define a homorphism $\phi \mapsto \phi_t$ from the group of affine isometries $\mathrm{Iso}(\mathcal{H})$ of $\mathcal{H}$ to the unitary group $\mathcal{U}(\mathcal{H}_t)$ of $\mathcal{H}_t$, where ϕ_t is the only element of $\mathcal{U}(\mathcal{H}_t)$ such that $\forall \xi \in \mathcal{H}, \phi_t(\xi_t) = (\phi(\xi))_t$. This homomorphism is continuous. Indeed, let (ϕ_n) be a sequence in $\mathrm{Iso}(\mathcal{H})$ converging to ϕ. Then for $\xi, \eta \in \mathcal{H}$, $\langle (\phi_n)_t(\xi_t), \eta_t \rangle = \langle (\phi_n(\xi))_t, \eta_t \rangle = \exp(-t\|\phi_n(\xi) - \eta\|^2) \underset{n\to\infty}{\longrightarrow} \exp(-t\|\phi(\xi) - \eta\|^2)$.

Therefore, any affine isometric action ρ of an equivalence relation $\mathcal{R}$ on $\mathcal{H}$ gives rise to a unitary representation ρ_t on $\mathcal{H}_t$. We now give a cocycle-free proof of some results from [Ana05].

Lemma 8.5 ([Ana05, Lem. 3.20]). *Let $\mathcal{R}$ be an ergodic p.m.p. equivalence relation. Let ρ be an affine isometric action of $\mathcal{R}$ on a real Hilbert space $\mathcal{H}$, suppose that there is $A \subseteq X$ of positive measure and a Borel section $\xi : X \to \mathcal{H}$ such that for all $x \in A$, we have*

$$\sup_{y \in \mathcal{R}_x \cap A} \|\rho(x,y)\xi(x) - \xi(y)\| < +\infty.$$

Then there exists a section $\xi' : X \to \mathcal{H}$ such that for almost all $x \in X$, we have

$$\sup_{y \in \mathcal{R}_x} \|\rho(x,y)\xi'(x) - \xi'(y)\| < +\infty.$$

Proof. Fix a sequence (T_n) in $[\mathcal{R}]$ such that the graphs of the T_n cover $\mathcal{R}$. We define a Borel map $\theta : X \to X$ by letting $\theta : x \mapsto T_n x$ where n is the least integer such that $T_n x \in A$. Such an integer exists by ergodicity of $\mathcal{R}$.

Now let $\xi' : x \mapsto \rho(\theta x, x)\xi(\theta x)$. For $(x,y) \in \mathcal{R}$, we compute

$$\begin{aligned}
\|\rho(x,y)\xi'(x) - \xi'(y)\| &= \|\rho(x,y)\rho(\theta x, x)\xi(\theta x) - \rho(\theta y, y)\xi(\theta y)\| \\
&= \|\rho(\theta x, y)\xi(\theta x) - \rho(\theta y, y)\xi(\theta y)\| \\
&= \|\rho(y, \theta y)\rho(\theta x, y)\xi(\theta x) - \xi(\theta y)\| \\
&= \|\rho(\theta x, \theta y)\xi(\theta x) - \xi(\theta y)\|
\end{aligned}$$

which is bounded when y goes through $\mathcal{R}_x$. $\square$

Lemma 8.6 ([Ana05, Thm. 3.19]). *Let $\mathcal{R}$ be an ergodic p.m.p. equivalence relation. Let ρ be an affine isometric action of $\mathcal{R}$ on a real Hilbert space $\mathcal{H}$ and let $\xi : X \to \mathcal{H}$ be a Borel section, suppose that we have for all $x \in X$, we have $\sup_{y \in \mathcal{R}_x} \|\rho(x,y)\xi(x) - \xi(y)\| < +\infty$. Then ρ admits an invariant section.*

Proof. For all $(x,y) \in \mathcal{R}$, since $\rho(y,x)$ is an isometry which is the inverse of $\rho(x,y)$, we have $\sup_{y \in \mathcal{R}_x} \|\rho(y,x)\xi(y) - \xi(x)\| < +\infty$. Using a countable group Γ such that $\mathcal{R} = \mathcal{R}_\Gamma$, we see that the section η which takes $x \in X$ to the circumcenter of the closed convex hull of $\{\rho(y,x)\xi(y) : y \in \mathcal{R}_x\}$ is Borel. Such a section is easily checked to be fixed by ρ since $\rho(x',x)$ takes $\rho(y,x')\xi(y)$ to $\rho(y,x)\xi(y)$. $\square$

Proposition 8.7. *Let $\mathcal{R}$ be an ergodic p.m.p. equivalence relation. Let ρ be an affine isometric action of $\mathcal{R}$ on a real Hilbert space $\mathcal{H}$ and $t > 0$. Then ρ admits a fixed section if and only if the unitary representation ρ_t has an invariant unit section.*

Proof. If ξ is a fixed section for ρ then it is straightforward that $\xi_t : x \mapsto \xi(x)_t$ is an invariant unit section for ρ_t.

Conversely, if there exists an invariant unit section ζ for ρ_t, suppose by contradiction that ρ does not admit any fixed section. As in the proof of Theorem 4.12 in [Ana05], we fix a dense sequence $(h_n)_{n \geqslant 1} \in \mathcal{H}$ and for $m, n \in \mathbb{N}^*$, let $E_{m,n} = \{x \in X : |\langle (h_m)_t, \zeta(x)\rangle| \geqslant \frac{1}{n}\}$. Since the span of the $(h_m)_t$ for $m \geqslant 1$ is dense in $\mathcal{H}_t$ and ζ is a unit section, we have $\bigcup_{m,n \geqslant 1} E_{m,n} = X$ so there exist $m, n \in \mathbb{N}^*$ such that $\mu(E_{m,n}) > 0$.

Using Lemmas 8.6 to the constant section $x \mapsto h_m$ and 8.5, we get that for almost any $x \in E_{m,n}$,

$$\sup_{y \in \mathcal{R}_x \cap E_{m,n}} \|\rho(x,y)h_m - h_m\| = +\infty.$$

Therefore, for almost any $x \in E_{m,n}$, we can fix a sequence (y_k) in $\mathcal{R}_x \cap E_{m,n}$ such that $(\|\rho(x,y_k)h_m - h_m\|)_k$ tends to $+\infty$. In particular the sequence of vectors $(\rho(x,y_k)h_m)_k$ tends to ∞ in $\mathcal{H}$ as $k \to +\infty$.

Now for every $\xi \in \mathcal{H}$ we have

$$|\langle (\rho(x,y_k)h_m)_t, \xi_t\rangle| = \exp(-t\,\|\rho(x,y_k)h_m - \xi\|^2) \underset{k\to\infty}{\longrightarrow} 0.$$

Since the image of $\mathcal{H}$ is total in $\mathcal{H}_t$, we conclude that the sequence $((\rho(x,y_k)h_m)_t)_{k\in\mathbb{N}}$ weakly converges to 0 in $\mathcal{H}_t$. But on the other hand, for every $k \in \mathbb{N}$ we have

$$|\langle (\rho(x,y_k)h_m)_t, \zeta(x)\rangle| = |\langle (h_m)_t, \rho_t(y_k,x)\zeta(x)\rangle| = |\langle (h_m)_t, \zeta(y_k)\rangle| \geqslant \frac{1}{n}.$$

This contradicts the fact that $((\rho(x,y_k)h_m)_t)_{k\in\mathbb{N}}$ weakly converges to 0, which finishes the proof. $\square$

Lemma 8.8. *Let ρ be an affine isometric actions of a p.m.p. equivalence relation $\mathcal{R}$ on a real Hilbert space $\mathcal{H}$. Consider the family $(\mathcal{H}_t,\rho_t)_t$ associated to ρ as in Proposition 8.4. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence of positive reals converging to 0. Then $\bigoplus_{n\in\mathbb{N}} \rho_{t_n}$ has almost invariant unit sections.*

Proof. For $n \in \mathbb{N}$, ξ_n be the constant section with value 0_{t_n}. Then ξ_n is a unit section and moreover for every $(x,y) \in \mathcal{R}$,

$$\begin{aligned}\|\rho_{t_n}(x,y)\xi_n(x) - \xi_n(y)\|^2 &= \|(\rho(x,y)(0))_{t_n} - 0_{t_n}\|^2\\ &= \|(\rho(x,y)(0))_{t_n}\|^2 + \|0_{t_n}\|^2 - 2\langle(\rho(x,y)(0))_{t_n}, 0_{t_n}\rangle\\ &= 2\left(1-\exp(-t_n\|\rho(x,y)(0)\|^2)\right)\end{aligned}$$

It easily follows that (ξ_n), seen as a sequence of sections in $\bigoplus_{n\in\mathbb{N}}\mathcal{H}_{t_n}$ is a sequence of almost invariant unit sections for $\bigoplus_{n\in\mathbb{N}}\rho_{t_n}$. $\square$

Lemma 8.9. *Let ρ be an affine isometric action of a p.m.p. equivalence relation $\mathcal{R}$ on a real Hilbert space $\mathcal{H}$. Let us denote $\rho\oplus\rho$ the diagonal action on $\mathcal{H}\oplus\mathcal{H}$. Then for $t>0$, $(\rho\oplus\rho)_t = \rho_t\otimes\rho_t$.*

Proof. Fix $t>0$. We define a map $\Psi_t\colon \mathcal{H}\oplus\mathcal{H}\to\mathcal{H}_t\otimes\mathcal{H}_t$ by the formula $\Psi_t(\xi,\eta) \coloneqq \xi_t\otimes\eta_t$.

First, for $\xi,\xi',\eta,\eta' \in \mathcal{H}$ we have

$$\begin{aligned}\langle \Psi_t(\xi\oplus\eta), \Psi_t(\xi'\oplus\eta')\rangle &= \exp(-t\|\xi-\xi'\|^2)\exp(-t\|\eta-\eta'\|^2)\\ &= \exp(-t\|(\xi\oplus\eta)-(\xi'\oplus\eta')\|^2)\end{aligned}$$

and it is clear that the image of Ψ_t is total in $\mathcal{H}_t\otimes\mathcal{H}_t$. Moreover, for

$$\begin{aligned}\Psi_t(\rho\oplus\rho(x,y)(\xi\oplus\eta)) &= \Psi_t(\rho(x,y)\xi\oplus\rho(x,y)\eta)\\ &= (\rho(x,y)\xi)_t\otimes(\rho(x,y)\eta)_t\\ &= \rho_t(x,y)\xi_t\otimes\rho_t(x,y)\eta_t\\ &= (\rho_t\otimes\rho_t)(x,y)(\xi_t\otimes\eta_t)\\ &= (\rho_t\otimes\rho_t)(x,y)\Psi_t(\xi\oplus\eta).\end{aligned}$$

By uniqueness of the construction in Proposition 8.4, it follows that $\Psi_t(\xi\oplus\eta) = (\xi\oplus\eta)_t$ and $(\rho^2)_t = \rho_t\otimes\rho_t$. $\square$

Recall that given a complex Hilbert space $\mathcal{H}$, we get a conjugate Hilbert space $\overline{\mathcal{H}}$ which is as a set equal to $\mathcal{H}$, but endowed with a new structure where scalar multiplication by a complex number $c \in \mathbb{C}$ becomes $\xi \mapsto \bar{c}\xi$ and the scalar product becomes $\langle \xi, \eta \rangle_{\overline{\mathcal{H}}} = \langle \eta, \xi \rangle = \overline{\langle \xi, \eta \rangle}$. Given a unitary representation π on $\mathcal{H}$, we thus get a unitary representation $\bar{\pi}$ on $\overline{\mathcal{H}}$. Note that its invariant vectors are the same as those of π.

Lemma 8.10 ([GL17], Lemma 3.16.(2))**.** *Let $\mathcal{R}$ be a p.m.p equivalence relation and let π be a unitary representation of $\mathcal{R}$. The following assertions are equivalent:*

1. *$\pi \otimes \overline{\pi}$ has a non-zero invariant section,*
2. *there is a unitary representation σ of $\mathcal{R}$ such that $\pi \otimes \sigma$ has a non-zero invariant section,*
3. *π contains a finite dimensional subrepresentation.*

Finally, let us introduce the affine version of property (T), following Anantharaman-Delaroche:

Definition 8.11 (Property (FH), [Ana05], Section 4.3)**.** A p.m.p. equivalence relation $\mathcal{R}$ is said to have property (FH) if every affine isometric action of $\mathcal{R}$ on a separable real Hilbert space admits a fixed section.

Theorem 8.12 ([Ana05, Theorems 4.8 and 4.12])**.** *Let $\mathcal{R}$ be an ergodic p.m.p. equivalence relation. Then $\mathcal{R}$ has property (T) if and only if $\mathcal{R}$ has property (FH).*

We are now ready to prove Theorem 8.3.

Proof. Recall that left-to-right direction is trivial. For the other direction, suppose that every unitary representation of $\mathcal{R}$ which has almost invariant unit sections contains a finite dimensional subrepresentation and let us show that $\mathcal{R}$ has property (FH). We will use repeatedly Lemma 8.10.

Let ρ be an affine isometric action of $\mathcal{R}$ on a separable real Hilbert space $\mathcal{H}$. Let us consider the representation $\pi := \bigoplus_{n \in \mathbb{N}^*} \rho_{1/n}$. Then π admits almost invariant unit sections by Lemma 8.8, and therefore π contains a finite dimensional subrepresentation, or in other words, $\pi \otimes \overline{\pi}$ has an non-zero invariant section.

But $\pi \otimes \overline{\pi} = \bigoplus_{n,m \in \mathbb{N}^*} \rho_{1/n} \otimes \overline{\rho_{1/m}}$ so there must be $n_0, m_0 \in \mathbb{N}^*$ such that we have a non-zero invariant section for $\rho_{1/n_0} \otimes \overline{\rho_{1/m_0}}$. Then ρ_{1/n_0} contains a finite dimensional subrepresentation and finally $\rho_{1/n_0} \otimes \overline{\rho_{1/n_0}}$ has a non-zero invariant section, which by the definition of $\overline{\rho_{1/n_0}}$ is equivalent to $\rho_{1/n_0} \otimes \rho_{1/n_0}$ admitting a non-zero invariant section.

Applying Lemma 8.9 and Proposition 8.7 we get that $\rho \oplus \rho$ has a fixed section. Therefore ρ has a fixed section and $\mathcal{R}$ has property (FH). □

8.2 A reminder on Gaussian actions

We now briefly present the classical construction of Gaussian actions associated to unitary representations, which given our definitions of unitary representations and of boolean actions of p.m.p. equivalence relations (Definition 6.9) work exactly as in the group case. This can be phrased as the construction a homomorphism from the orthogonal group of a separable real Hilbert space to $\mathrm{Aut}(Z, \eta)$, but the details of this construction are important for the proofs. Here we simply state the facts which are relevant to us; the unacquainted reader should first read [KL16, Appendix E].

First we define the symmetric Fock space of a real Hilbert space $\mathcal{H}$. Let $n \in \mathbb{N}$ and $\sigma \in \mathfrak{S}_n$. Then σ induces an orthogonal operator O_σ on $\mathcal{H}^{\otimes n}$ defined on simple tensors by $O_\sigma(h_1 \otimes \cdots \otimes h_n) = h_{\sigma^{-1}(1)} \otimes \cdots \otimes h_{\sigma^{-1}(n)}$. We let the **n-th symmetric space of $\mathcal{H}$** be the subspace of $\mathcal{H}^{\otimes n}$ of elements that are fixed by all O_σ for $\sigma \in \mathfrak{S}_n$, and we denote it by $\mathcal{H}^{\odot n}$. For $n = 0$, we take the convention that $\mathcal{H}^{\odot 0} = \mathbb{R}$. The **symmetric Fock space of $\mathcal{H}$** is the direct sum $S(\mathcal{H}) \coloneqq \bigoplus_{n\in\mathbb{N}} \mathcal{H}^{\odot n}$.

Any orthogonal representation π on $\mathcal{H}$ then induces orthogonal representations $\pi^{\odot n}$ on $\mathcal{H}^{\odot n}$ and thus an orthogonal representation $S(\pi) \coloneqq \bigoplus_{n\in\mathbb{N}} \pi^{\odot n}$ on $S(\mathcal{H})$. These constructions also make sense in the complex case for unitary representations, and we will use the same notation.

Now let $\mathcal{R}$ be an ergodic p.m.p. equivalence relation on (X, μ) and let π be a unitary representation of $\mathcal{R}$ on a Hilbert space $\mathcal{H}$. Then one can associate to π a p.m.p. boolean action α_π on a standard atomless probability space (Z, η) called the **Gaussian action associated to** π so that its Koopman representation on the orthogonal of constant functions $\kappa^0_{\alpha_\pi}$ contains π as a direct summand as follows. One starts by considering π as an orthogonal representation $\pi_\mathbb{R}$ by viewing $\mathcal{H}$ as a real Hilbert space.

One constructs a *Gaussian Hilbert space* $\widetilde{\mathcal{H}} \subset \mathrm{L}^2(Z, \eta, \mathbb{R})$ of dimension dim $\mathcal{H}$ that does not contain the constant functions, such that the symmetric Fock space $S(\widetilde{\mathcal{H}})$ of $\widetilde{\mathcal{H}}$ is isometrically isomorphic to $\mathrm{L}^2(Y, \nu, \mathbb{R})$ via an isomorphism sending $\widetilde{\mathcal{H}}^{\odot 0}$ on $\mathbb{R}1_Y$ (see [KL16, Sec. E.3]). By the means of fixed isometric isomorphisms, let us identify $\widetilde{\mathcal{H}}$ and $\mathcal{H}$ as well as $S(\widetilde{\mathcal{H}})$ and $\mathrm{L}^2(Y, \nu, \mathbb{R})$. We can then consider $S(\pi_\mathbb{R})$ as an orthogonal representation on $\mathrm{L}^2(Y, \nu)$.

The construction of $S(\pi_\mathbb{R})$ then ensures that $S(\pi_\mathbb{R})$ preserves the subset $\mathrm{MAlg}(Y, \nu) \subseteq \mathrm{L}^2(Z, \nu, \mathbb{R})$ and acts by automorphism on it. Call α_π the co-restriction of $S(\pi_\mathbb{R})$ to $\mathrm{Aut}(Z, \eta)$. Then α_π is a Borel homomorphism and so it is a boolean measure-preserving $\mathcal{R}$-action on (Y, η).

Furthermore, by definition, the orthogonal Koopman representation of α_π on $\mathrm{L}^2(Y, \nu, \mathbb{R})$ is $S(\pi_\mathbb{R})$. Finaly the unitary Koopman representation of α_π on $\mathrm{L}^2(Y, \nu, \mathbb{C})$ is the complexification of the orthogonal Koopman representation. Since the complexification of $\pi_\mathbb{R}$ is $\pi \oplus \bar{\pi}$ (see [KL16, Prop. E.1]), we get that

$$\kappa_{\alpha_\pi} = S(\pi \oplus \bar{\pi}).$$

Moreover, the Koopman representation on the orthogonal of constant functions is $\kappa^0_{\alpha_\pi} = \bigoplus_{n\in\mathbb{N}^*} (\pi \oplus \bar{\pi})^{\odot n}$, in particular π is a direct summand of $\kappa^0_{\alpha_\pi}$.

8.3 The Connes-Weiss characterization of property (T)

Now we can prove the following theorem, which extends a theorem of Connes and Weiss from discrete countable groups to p.m.p. equivalence relation [CW80]. A version for p.m.p. groupoids can be found in the thesis of the second-named author [Gir22].

Theorem 8.13 (Connes-Weiss for p.m.p. equivalence relations)**.** *Let $\mathcal{R}$ be a p.m.p. equivalence relation. Then the following assertions are equivalent:*

(i) $\mathcal{R}$ has property (T).

(ii) Every ergodic measure-preserving $\mathcal{R}$-action is strongly ergodic.

Proof. The implication (i)⇒(ii) is a direct consequence of Proposition 7.5 and Proposition 7.10. We thus have to show (ii)⇒(ii), which we do via the contrapositive.

Let us thus suppose that $\mathcal{R}$ does not have property (T) and construct an ergodic action which is not strongly ergodic.

By Theorem 8.3, there is a unitary representation π of $\mathcal{R}$ which has almost invariant unit sections but does not admit any finite dimensional subrepresentation. We construct the Gaussian action α_π associated to π (see Section 8.2). Since π does not admit any finite dimensional subrepresentation, neither does $\pi \oplus \bar{\pi}$, and hence neither do $(\pi \oplus \bar{\pi})^{\otimes n}$ for $n \in \mathbb{N}^*$ by Lemma 8.10. In particular, for $n \in \mathbb{N}^*$, $(\pi \oplus \bar{\pi})^{\odot n}$ does not admit any finite dimensional subrepresentation. It follows again from Lemma 8.10 that for any $n, m \in \mathbb{N}^*$, $(\pi \oplus \bar{\pi})^{\odot n} \otimes \overline{(\pi \oplus \bar{\pi})^{\odot m}}$ does not have a non-zero invariant section.

So it follows from the above description that the Koopman representation on the orthogonal of constant functions $\kappa^0_{\alpha_\pi}$ of $\mathcal{R}$ satisfies $\kappa^0_{\alpha_\pi} \otimes \overline{\kappa^0_{\alpha_\pi}} = \bigoplus_{n,m \in \mathbb{N}^*} (\pi \oplus \bar{\pi})^{\odot n} \otimes \overline{(\pi \oplus \bar{\pi})^{\odot m}}$. Applying Lemma 8.10 once more, we see that $\kappa^0_{\alpha_\pi}$ has no finite dimensional subrepresentation, in particular it has no non-zero invariant section.

Now the Koopman representation of the boolean action $[\alpha_\pi]$ of $[\mathcal{R}]$ on $\mathrm{L}^2(X \times Z, \mu \otimes \eta)$ is equal to the direct sum of the standard representation κ_ι on $\mathrm{L}^2(X, \mu)$ (corresponding to fiberwise constant functions) and $[\kappa^0_{\alpha_\pi}]$ (corresponding to functions whose integral over each fiber is zero). By the previous paragraph, the latter has no non-zero invariant vectors. It then follows from the ergodicity of $\mathcal{R}$ that the only invariant vectors of our direct sum are given by constant functions in $\mathrm{L}^2(X \times Z, \mu \otimes \eta)$, so the $\mathcal{R}$-action α_π is ergodic[4] by Proposition 3.2.

We then show that the action is not strongly ergodic by constructing a sequence of asymptotically $[\alpha_\pi]$-invariant Borel sets $A_n \subset X \times Y$ which is not asymptotically trivial.

Let (ξ_n) be a sequence of almost invariant unit sections for π viewed as an orthogonal representation. Recall that through the construction of the Gaussian action α_π, we identified $\mathcal{H}$ to a Gaussian Hilbert space, so for all $x \in X$, $\xi_n(x, \cdot)$ is a centered gaussian random variable of variance 1. Write ξ_n^x for $\xi_n(x, \cdot)$ and for $x \in X$, let $A_n^x \subset Z$ be the set $\{z \in Z \colon \xi_n^x(z) \geqslant 0\}$. Since ξ_n^x is centered, we have $\nu(A_n^x) = \frac{1}{2}$.

Fix $(x, x') \in \mathcal{R}$. We write $\pi(x, x')\xi_n^x = \cos\theta_{n,x,x'}\xi_n^{x'} + \sin\theta_{n,x,x'}\eta_{n,x,x'}$ in $\mathcal{H}$, for some $\theta_{n,x,x'} \in [0, \pi]$ and $\eta_{n,x,x'} \in \left(\xi_n^{x'}\right)^\perp$. Then $\xi_n^{x'}$ and $\eta_{n,x,x'}$ are orthogonal gaussian random variables of variance 1 so they are independent and their joint distribution is a probability measure m on $\mathbb{R}^2$ which is the product of two gaussian measures (in particular, it is rotation invariant). Last, (ξ_n) is a sequence of almost invariant sections so for n large enough, $\theta_{n,x,x'} \in [0, \frac{\pi}{2})$. It follows that

$$\begin{aligned}
\eta(\alpha_\pi(x,x')A_n^x \,\triangle\, A_n^{x'}) &= \eta\big(\{z \in Z \colon \pi(x,x')\xi_n^x(z) \geqslant 0 \text{ and } \xi_n^{x'}(z) < 0\} \\
&\qquad \cup \{z \in Z \colon \pi(x,x')\xi_n^x(z) < 0 \text{ and } \xi_n^{x'}(z) \geqslant 0\}\big) \\
&= m\left(\{(a,b) \in \mathbb{R}^2 \colon \cos\theta_{n,x,x'}a + \sin\theta_{n,x,x'}b \geqslant 0 \text{ and } a < 0\}\right) \\
&\qquad + m\left(\{(a,b) \in \mathbb{R}^2 \colon \cos\theta_{n,x,x'}a + \sin\theta_{n,x,x'}b < 0 \text{ and } a \geqslant 0\}\right)
\end{aligned}$$

Using Figure 1 and the fact that these inequalities are about the signs of the scalar product of (a, b) with $(\cos\theta_{n,x,x'}, \sin\theta_{n,x,x'})$ and $(1, 0)$ respectively, we see that the first term is equal to the measure of the upper left gray part, while the second term is equal to the measure of the lower right gray part.

[4] This argument can actually be upgraded to show that the action is weakly mixing, see [GL17, Sec. 3.10] for the definition.

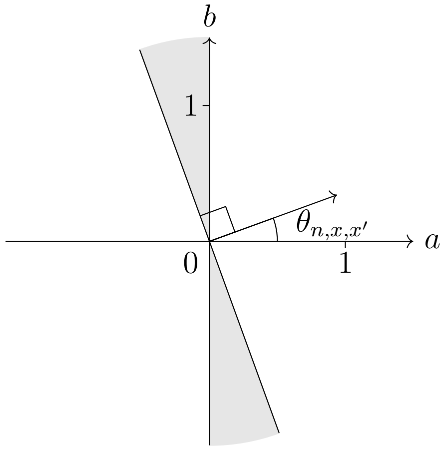

Figure 1: The m-measure of the gray part is equal to $\eta(\alpha_\pi(x,x')A_n^x \triangle A_n^{x'})$

It then follows from rotation invariance that

$$\nu(\alpha_\pi(x,x')A_n^x \triangle A_n^{x'}) = \frac{\theta_{n,x,x'}}{\pi}.$$

Let now $A_n = \{(x,z) \in X \times Z \colon z \in A_n^x\}$, note that A_n is measurable as a consequence of Lemma 6.4. Then $\mu \otimes \eta(A_n) = \frac{1}{2}$ so the sequence (A_n) is not asymptotically trivial. Let $T \in [\mathcal{R}]$, then

$$\begin{aligned}\mu \otimes \eta([\alpha_\pi](T)A_n \triangle A_n) &= \int_X \eta(\alpha_\pi(T^{-1}x,x)A_n^{T^{-1}x} \triangle A_n^x)\ d\mu(x) \\ &= \frac{1}{\pi}\int_X \theta_{n,T^{-1}x,x}\ d\mu(x)\end{aligned}$$

which converges to 0 by dominated convergence theorem. Therefore (A_n) is asymptotically invariant. We conclude that α_π is not strongly ergodic, although it is ergodic as shown before. Note that α_π was constructed as a boolean $\mathcal{R}$-action, but by Proposition 6.5 we can equally view it as a p.m.p. $\mathcal{R}$-action as defined in Section 3.4. This finishes the proof. □

8.4 Proof of the indirect implication in Theorem C

Theorem 8.14. *Let $\mathcal{R}$ be a p.m.p equivalence relation. Suppose that all non-free ergodic boolean actions of $[\mathcal{R}]$ are strongly ergodic. Then $\mathcal{R}$ has property (T).*

Proof. We use our characterization of property (T) obtained in Theorem 8.13. Let $\pi : (Y,\nu) \to (X,\mu)$ be an ergodic measure-preserving $\mathcal{R}$-action on (Y,ν), i.e. π is a class-bijective extension of $\mathcal{R}$ by a p.m.p. ergodic equivalence relation $\mathcal{S}$ on (Y,ν) . Then the corresponding boolean action ρ_π of $[\mathcal{R}]$ (see Definition 3.26) is ergodic and far from being free. Indeed, by Corollary 3.5 we can take $T \in [\mathcal{R}]$ such that $\mu(\operatorname{supp} T) \notin \{0,1\}$, and then $\operatorname{supp}[\alpha](T) = \pi^{-1}(\operatorname{supp} T)$ has measure $\mu(\operatorname{supp} T)$.

By our assumption the non-free ergodic boolean action ρ_π is strongly ergodic, which by definition means that the equivalence relation $\mathcal{S}$ is strongly ergodic. So every ergodic $\mathcal{R}$-action is strongly ergodic, and we conclude that $\mathcal{R}$ has property (T). □

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