Highly faithful actions and dense free subgroups in full groups

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Abstract

In this paper, we show that every measure-preserving ergodic equivalence relation of cost less than m comes from a "rich" faithful invariant random subgroup of the free group on m generators, strengthening a result of Bowen which had been obtained by a Baire category argument.

Our proof is completely explicit: we use our previous construction of topological generators for full groups and observe that these generators induce a totally non free action. We then twist this construction so that the action is moreover amenable onto almost every orbit and highly faithful.

In particular, we obtain that the full group of a measure-preserving ergodic equivalence of cost less than m contains a dense free subgroup on m generators.

Contents

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1 Introduction

A natural conjugacy-invariant for a measure-preserving action of a countable group Γ on a standard probability space (X, μ) is the associated *measure-preserving equivalence relation* $\mathcal{R}_\Gamma$ defined by $(x, y) \in \mathcal{R}_\Gamma$ if and only if $\Gamma x = \Gamma y$. Such equivalence relations are studied up to *orbit equivalence*, that is up to isomorphism and restrictions to full measure sets.

Measure-preserving actions of countable groups are often asked to be *free*: every non-trivial group element fixes almost no point. The study of free measure-preserving actions up to orbit equivalence is well developed and has fruitful connections to measured group theory and von Neumann algebras, see [Gab10] for a recent overview.

The theory of *cost*, introduced by Levitt [Lev95] and developed by Gaboriau [Gab00], has proven to be invaluable in this area. The cost of a measure-preserving equivalence relation $\mathcal{R}$ is the infimum of the measures of its generating sets[1], thus providing an analogue of the rank of a countable group for measure-preserving equivalence relations. A fundamental theorem due to Gaboriau is that every free action of the free group on m generators induces a measure-preserving equivalence relation of cost m (see [Gab00, Cor. 1]).

In this paper, we are interested in non-free actions of $\mathbb{F}_m$. Our starting point is the contrapositive of Gaboriau's aforementioned theorem: a measure preserving equivalence relation of cost less than m cannot come from a free action of $\mathbb{F}_m$. Moreover, an easy consequence of Gaboriau's results is that *every* ergodic measure-preserving equivalence relation of cost less than m comes from a non-free action of $\mathbb{F}_m$.

It is then natural to search for some strengthening of non-freeness for $\mathbb{F}_m$-actions so as to further classify measure-preserving equivalence relations of cost less than m. We thus ask:

Question 1. *Consider a measure-preserving ergodic equivalence relation of cost less than m. How non-free can the $\mathbb{F}_m$-actions that induce it be?*

We now list three ways a measure-preserving action of the free group on $m \geqslant 2$ generators can be thought of as "very" non-free.

1.1 Non freeness I: Amenability onto almost every orbit

Definition 1.1. An action of a countable group Γ on a set Y is called **amenable** if it admits a sequence of almost invariant sets, i.e. if there exists a sequence of finite subsets (F_n) of Y such that for all $\gamma \in \Gamma$,

$$\frac{|\gamma F_n \triangle F_n|}{|F_n|} \to 0 \, [n \to +\infty].$$

A countable group Γ is **amenable** if its left action onto itself by translation is amenable.

[1] See section 2 for a precise definition.

The group of integers $\mathbb{Z}$ is a key example of an amenable group (the sequence of intervals $[-n, n]$ is almost invariant). On the other hand, for any $n \geqslant 2$ the free group $\mathbb{F}_n$ is *not* amenable: for instance, one can build a *Ponzi scheme* on it (see [Gro99, Cor. 6.18]).

Definition 1.2. A measure-preserving action of a countable group Γ on a standard probability space (X, μ) is called **amenable onto almost every orbit** if for almost every $x \in X$, the Γ-action on $\Gamma \cdot x$ is amenable.

Example 1.3. Suppose that $\Gamma \curvearrowright (X, \mu)$ is a free measure-preserving action. Then for almost every $x \in X$, the Γ-equivariant map $\Gamma \to \Gamma \cdot x$ which takes $\gamma \in \Gamma$ to $\gamma \cdot x$ is a bijection. Hence almost all the actions on the orbits are conjugate to the left Γ-action onto itselft by translation. So a free Γ-action on (X, μ) is amenable onto almost every orbit if and only if Γ is amenable.

We deduce from the previous example that for all $n \geqslant 2$, a free measure-preserving action of the free group $\mathbb{F}_n$ is never amenable onto almost every orbit. In particular, measure-preserving actions of $\mathbb{F}_n$ which *are* amenable onto almost every orbit can be thought of as very non-free actions. Examples of non-amenable measure-preserving equivalence relations coming from $\mathbb{F}_m$-actions which are amenable onto almost every orbit were first constructed by Kaimanovich [Kai97].

1.2 Non freeness II: High transitivity onto almost every orbit

Definition 1.4. Let Γ be a countable group acting on a set Y. The action is **highly transitive** if for every $n \in \mathbb{N}$, the diagonal Γ-action on the set of n-tuples made of pairwise distinct elements of Y is transitive.

To be more precise , the action is highly transitive if for every $n \in \mathbb{N}$, every pairwise distinct $y_1, ..., y_n \in Y$ and every pairwise distinct $y'_1, ..., y'_n \in Y$, there exists $\gamma \in \Gamma$ such that for all $i \in \{1, ..., n\}$, we have $\gamma \cdot y_i = y'_i$.

As an example, the natural action of the group of finitely supported permutations of the integers is highly transitive. It has been an ongoing research theme to understand which countable groups admit faithful highly transitive actions; see [HO15] for a striking recent result in that area.

It is a well-known fact that a permutation group $\Gamma \leqslant \mathfrak{S}(Y)$ is highly transitive if and only if it is dense for the topology of pointwise convergence. Note that a nontrivial highly transitive action can never be free. The following definition was introduced by Eisenmann and Glasner [EG14] and can also be seen as a strengthening of non-freeness for measure-preserving actions.

Definition 1.5. Let $\mathcal{R}$ be a measure-preserving equivalence relation on a standard probability space (X, μ). A measure-preserving action of a countable group Γ on (X, μ) is **almost surely highly transitive on $\mathcal{R}$-classes** if for almost every $x \in X$, Γ preserves the equivalence class $[x]_{\mathcal{R}}$ and acts on it in a highly transitive manner.

There is a very nice sufficient condition for a group to act almost surely highly transitively on $\mathcal{R}$-classes and to state it we need to introduce full groups.

Definition 1.6. Let $\mathcal{R}$ be a measure-preserving equivalence relation. Its **full group**, denoted by $[\mathcal{R}]$, is the group of all measure-preserving Borel bijections T of (X, μ) such that for all $x \in X$, we have $T(x) \in [x]_{\mathcal{R}}$. Moreover, two such bijections are identified if they coincide up to a null set.

Whenever T and U are measure-preserving bijections of (X, μ), one can define the uniform distance between them by

$$d_u(T, U) := \mu(\{x \in X : T(x) \neq U(x)\}).$$

Whenever $\mathcal{R}$ is a measure-preserving equivalence relation, the uniform metric induces a complete separable metric on its full group which is thus a Polish group (see e.g. [Kec10, Prop. 3.2]).

We may now state Eisenmann and Glasner's result.

Theorem 1.7 ([EG14, Prop. 1.19])**.** *Let $\mathcal{R}$ be a measure-preserving equivalence relation. Let $\Gamma \leqslant [\mathcal{R}]$ be a countable dense subgroup of $[\mathcal{R}]$. Then Γ acts almost surely highly transitively on $\mathcal{R}$-classes.*

It is not true in general that any almost surely highly transitive action comes from a dense embedding into a full group[2]. However in the ergodic case the question of the converse was asked by Eisenmann and Glasner.

1.3 Non freeness III: Total non freeness

To state properly one last possible definition for an action to be very non free, we need to introduce invariant random subgroups, which are important invariants of non-free measure-preserving actions.

Let Γ be a countable group. We denote by $\mathrm{Sub}(\Gamma) \subseteq \{0,1\}^\Gamma$ the space of closed subgroups of Γ, which is a closed subspace of the compact metrizable space $\{0,1\}^\Gamma$ equipped with the product topology. With the induced topology, $\mathrm{Sub}(\Gamma)$ is thus a compact metrizable space naturally acted upon by Γ via conjugacy: for any $\Lambda \in \mathrm{Sub}(\Gamma)$ and any $\gamma \in \Gamma$, one lets $\gamma \cdot \Lambda := \gamma \Lambda \gamma^{-1}$.

Definition 1.8. An **invariant random subgroup** (or IRS) of a countable group Γ is a Γ-invariant Borel probability measure on $\mathrm{Sub}(\Gamma)$.

Let $\Gamma \curvearrowright (X, \mu)$ be a measure-preserving action. The map $\mathrm{Stab} : X \to \mathrm{Sub}(\Gamma)$ which maps $x \in X$ to $\mathrm{Stab}_\Gamma(x)$ is Γ-equivariant, so by pushing forward the measure μ we obtain an IRS $\mathrm{Stab}_*\mu$ of Γ. Abert, Glasner and Virag have shown that the converse is true: every IRS of Γ can be written as $\mathrm{Stab}_*\mu$ for some measure-preserving Γ-action on (X, μ) [AGV14, Prop. 13].

Definition 1.9 (Vershik)**.** Let $\Gamma \curvearrowright (X, \mu)$ be a measure-preserving action. It is called **totally non free** if the map $\mathrm{Stab} : (X, \mu) \to (\mathrm{Sub}(\Gamma), \mathrm{Stab}_*\mu)$ is a conjugacy.

[2]To see this, start with $\Gamma \curvearrowright (X, \mu)$ which is almost surely highly transitive on $\mathcal{R}$-classes. Then consider the Γ-action on two disjoint copies of (X, μ) and let $\mathcal{R}'$ be the associated equivalence relation. The new action is almost surely highly transitive on $\mathcal{R}'$-classes, but Γ is not dense in $[\mathcal{R}']$ since any element of the closed subgroup generated by Γ has to act the same on the two copies of (X, μ).

Note that since the map Stab is Γ-equivariant, the only thing one has to check in order to know that an action is totally non free is that Stab becomes injective when restricted to a suitable full measure subset of X. In the setting of full groups, our observation is the following.

Proposition 1.10 (see Proposition 2.4)**.** *Let $\mathcal{R}$ be a measure-preserving aperiodic*[3] *equivalence relation. If $\Gamma \leqslant [\mathcal{R}]$ is a dense countable subgroup, then the Γ-action is totally non free.*

Bowen obtained a satisfactory answer to Question 1 in the context of totally non-free actions: he showed by a Baire category argument that whenever $\mathcal{R}$ is an ergodic equivalence relation of cost less than n, there exists a totally non free action of the free group on n generators which induces the equivalence relation $\mathcal{R}$ [Bow15]. We remark that this result can also be obtained by combining the previous proposition with [LM14, Thm. 1].

1.4 Statement of the main result

Our main result is that the above conditions for non-freeness can be achieved all at once along with high faithfulness. The latter is a strengthening of the notion of faithfulness and is somehow dual to high transitivity (see Section 3 for more on this notion; our definition differs significantly from the one given by Fima, Moon and Stalder in [FMS15]).

Definition 1.11. A transitive action of a countable group Γ on a set Y is **highly faithful** if for all $n \in \mathbb{N}$ and all pairwise distinct $\gamma_1, ..., \gamma_n \in \Gamma$, there exists $y \in Y$ such that for all distinct $i, j \in \{1, ..., n\}$, we have $\gamma_i \cdot y \neq \gamma_j \cdot y$.

Note that the natural action of the group of finitely supported permutations of the integers is highly transitive faithful, but not highly faithful. It would be interesting to understand which countable groups admit highly faithful highly transitive actions.

A measure-preserving action of a countable group is called highly faithful if it is highly faithful onto almost every orbit. Here this notion was useful to us in order to obtain sequences of sets with nice disjointness properties (see item (5) in Theorem 3.5) and also to produce (highly) faithful actions for some free products via Theorem 4.5. We can now state our main result, which upgrades [LM14, Thm. 1].

Theorem 1.12. *Let $\mathcal{R}$ be an ergodic equivalence relation with finite cost. Then for all $m \in \mathbb{N}$ such that $m > \mathrm{Cost}(\mathcal{R})$ there is a dense free group on m generators in the full group of $\mathcal{R}$ whose action is moreover amenable onto almost every orbit and highly faithful.*

As a corollary, we can strengthen Bowen's Theorem and generalize a result that Eisenmann and Glasner had obtained for cost 1 ergodic measure-preserving equivalence relations by a Baire category argument [EG14, Cor. 21].

[3] A measure-preserving equivalence relation is aperiodic if almost all its classes are infinite.

Corollary 1.13. *Let $\mathcal{R}$ be an ergodic equivalence relation with finite cost. Then for all $m \in \mathbb{N}$ such that $m > \mathrm{Cost}(\mathcal{R})$ there is a totally non-free highly faithful action of $\mathbb{F}_m$ which induces the equivalence relation $\mathcal{R}$ and which is highly transitive and amenable onto almost every orbit.*

Proof. By Proposition 1.10 and Theorem 1.7, the $\mathbb{F}_m$-action obtained via Theorem 1.12 is totally non free and highly transitive onto almost every orbit. Since it is also highly faithful and amenable onto almost every orbit, we are done. □

Since total non-freeness implies that the stabiliser map is an isomorphism, the above result implies the following statement about invariant random subgroups (see [EG14] for the definitions of the terms used thereafter).

Corollary 1.14. *Let $\mathcal{R}$ be an ergodic equivalence relation with finite cost. Then for all $m \in \mathbb{N}$ such that $m > \mathrm{Cost}(\mathcal{R})$ there is an IRS of $\mathbb{F}_m$ which induces the equivalence relation $\mathcal{R}$ and which is core-free, co-highly transitive and co-amenable.*

All the above results admit non-ergodic counterparts where we require $\mathcal{R}$ to be aperiodic and its conditional cost to be almost surely less than m. However supposing that the equivalence relation is ergodic makes proofs much lighter and we hope this will help convey the ideas of this work. The interested reader will be able to "convert" the proofs presented here to their non-ergodic analogues by a careful reading of [LM15].

We now give an outline of this paper. The next section is devoted to notation and the proof of Proposition 1.10. In Section 3 we introduce and study high faithfulness. In Section 4 we build highly faithful actions of free products $\Gamma * \Lambda$ where Γ already acts highly faithfully and Λ is any residually finite group. Section 5 is devoted to a flexible construction of topological generators for the full group of the hyperfinite ergodic equivalence relation. Theorem 1.12 is finally proven in Section 6.

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2 Preliminaries

Let (X, μ) be a standard probability space. We will always work modulo sets of measure zero. Let us first briefly review some notation and definitions.

We denote by $\mathrm{Aut}(X, \mu)$ the group of all measure-preserving Borel bijections of (X, μ). Given $T \in \mathrm{Aut}(X, \mu)$, its **support** is the set $\mathrm{supp}\, T := \{x \in X : T(x) \neq x\}$.

Let A and B be Borel subsets of X, a **partial isomorphism** of (X, μ) of **domain** A and **range** B is a Borel bijection $f : A \to B$ which is measure-preserving for the measures induced by μ on A and B respectively. A **graphing** is a countable set $\Phi = \{\varphi_1, ..., \varphi_k, ...\}$ where the φ_k's are partial isomorphisms. It **generates** a **measure-preserving equivalence relation** $\mathcal{R}_\Phi$, defined to be the smallest equivalence relation containing the graphs of the partial isomorphisms belonging to Φ. The **cost** of a graphing Φ is the sum of the measures of the domains of the partial isomorphisms it contains. The **cost** of a measure-preserving equivalence relation $\mathcal{R}$ is the infimum of the costs of the graphings that generate it, we denote it by $\mathrm{Cost}(\mathcal{R})$.

The cost of $\mathcal{R}$ is **attained** if there exists a graphing Φ which generates $\mathcal{R}$ such that $\mathrm{Cost}(\Phi) = \mathrm{Cost}(\mathcal{R})$. We refer the reader to the lectures notes by Gaboriau[4] for an efficient overview of cost theory.

The **full group** of $\mathcal{R}$ is the group $[\mathcal{R}]$ of automorphisms of (X, μ) which preserve the $\mathcal{R}$-classes, that is

$$[\mathcal{R}] = \{\varphi \in \mathrm{Aut}(X, \mu) : \forall x \in X, \varphi(x)\, \mathcal{R}\, x\}.$$

It is a Polish group when equipped with the complete biinvariant metric d_u defined by

$$d_u(T, U) = \mu(\{x \in X : T(x) \neq U(x)\}.$$

One also defines the **pseudo full group** of $\mathcal{R}$, denoted by $[[\mathcal{R}]]$, which consists of all partial isomorphisms φ such that $\varphi(x)\, \mathcal{R}\, x$ for all $x \in \mathrm{dom}\ \varphi$.

Let $p \in \mathbb{N}$. A **pre-p-cycle** is a graphing $\Phi = \{\varphi_1, ..., \varphi_{p-1}\}$ such that the following two conditions are satisfied:

(i) $\forall i \in \{1, ..., p-2\}, \mathrm{rng}\ \varphi_i = \mathrm{dom}\ \varphi_{i+1}$.

(ii) The following sets are all disjoint:

$$\mathrm{dom}\ \varphi_1, \mathrm{dom}\ \varphi_2, ..., \mathrm{dom}\ \varphi_{p-1}, \mathrm{rng}\ \varphi_{p-1}.$$

A **p-cycle** is an element $C \in \mathrm{Aut}(X, \mu)$ whose orbits have cardinality 1 or p.

Given a pre-p-cycle $\Phi = \{\varphi_1, ..., \varphi_{p-1}\}$, we can extend it to a p-cycle $C_\Phi \in \mathrm{Aut}(X, \mu)$ as follows:

$$C_\Phi(x) = \begin{cases} \varphi_i(x) & \text{if } x \in \mathrm{dom}\ \varphi_i \text{ for some } i < p, \\ \varphi_1^{-1}\varphi_2^{-1} \cdots \varphi_{p-1}^{-1}(x) & \text{if } x \in \mathrm{rng}\ \varphi_{p-1}, \\ x & \text{otherwise.} \end{cases}$$

Say that a measure-preserving equivalence relation $\mathcal{R}$ is **ergodic** when every Borel $\mathcal{R}$-saturated set has measure 0 or 1. The following standard fact about ergodic measure-preserving equivalence relations is the main source of pre-p-cycles, and hence of p-cycles.

Proposition 2.1 (see e.g. [KM04], lemma 7.10.)**.** *Let $\mathcal{R}$ be an ergodic measure-preserving equivalence relation on (X, μ), let A and B be two Borel subsets of X such that $\mu(A) = \mu(B)$. Then there exists $\varphi \in [[\mathcal{R}]]$ of domain A and range B.*

The following theorem is fundamental for building dense subgroups of full groups.

Theorem 2.2 ([KT10], Thm. 4.7)**.** *Let $\mathcal{R}_1$, $\mathcal{R}_2$,... be measure-preserving equivalence relations on (X, μ), and let $\mathcal{R}$ be their join (i.e. the smallest equivalence relation containing all of them). Then $\left\langle \bigcup_{n\in\mathbb{N}} [\mathcal{R}_n] \right\rangle$ is dense in $[\mathcal{R}]$.*

An easy application is the following proposition.

[4]These are available online at `http://perso.ens-lyon.fr/gaboriau/Travaux-Publi/Copenhagen/Copenhagen-Lectures.html`.

Proposition 2.3 ([LM14, Prop. 10]). *If $\Phi = \{\varphi_1, ..., \varphi_{p-1}\}$ is a pre-p-cycle, then for all $i \in \{1, ..., p-1\}$, the full group of $\mathcal{R}_\Phi$ is topologically generated by $[\mathcal{R}_{\{\varphi_i\}}] \cup \{C_\Phi\}$.*

Let us finally turn to the relationship between dense subgroups of full groups and total non freeness that we mentioned in the introduction.

Proposition 2.4. *Let $\mathcal{R}$ be a measure-preserving aperiodic equivalence relation. If $\Gamma \leqslant [\mathcal{R}]$ is a dense countable subgroup, then the Γ-action is totally non free.*

Proof. We will work in the setting of measure algebras[5]: to see that $\mathrm{Stab} : (X, \mu) \to (\mathrm{Sub}(\Gamma), \mathrm{Stab}_*\mu)$ is a bijection between full measure sets, it suffices to show that the injective map $\mathrm{Stab}^{-1} : \mathrm{MAlg}(\mathrm{Sub}(\Gamma), \mathrm{Stab}_*\mu) \to \mathrm{MAlg}(X, \mu)$ is also surjective. Moreover since its image is closed, it suffices to show that its image is dense.

For all $\gamma \in \Gamma$, let $A_\gamma := \{\Lambda \in \mathrm{Sub}(\Gamma) : \gamma \notin \Lambda\}$. Then for all $\gamma \in \Gamma$, we have $\mathrm{Stab}^{-1}(A_\gamma) = \mathrm{supp}(\gamma)$. So it suffices to show that the family $(\mathrm{supp}(\gamma))_{\gamma \in \Gamma}$ is dense in $\mathrm{MAlg}(X, \mu)$. But this follows from the density of Γ in $[\mathcal{R}]$ and the well-known fact that for all $A \in \mathrm{MAlg}(X, \mu)$, there exists $T \in [\mathcal{R}]$ such that $\mathrm{supp}(T) = A$. □

3 Highly faithful actions

Let us now study in details the notion of a highly faithful action.

Definition 3.1. An action of a countable group Γ on a countable set Y is called n-faithful if for any $\gamma_1, ..., \gamma_n \in \Gamma \setminus \{1\}$, there exists $y \in Y$ such that $\gamma_i y \neq y$ for all $i = 1, ..., n$. The action is **highly faithful** if it is n-faithful for every $n \in \mathbb{N}$; in other words if for any finite subset $F \subseteq \Gamma \setminus \{1\}$, there exists $y \in Y$ such that $fy \neq y$ for all $f \in F$.

Note that every free action is highly faithful, and that an action is faithful iff it is 1-faithful. A simple example of a faithful action of an infinite group which is not highly faithful is given by the group $\mathfrak{S}_{(\infty)}$ of finitely supported permutations of $\mathbb{N}$ acting on $\mathbb{N}$. Note that this action is however highly transitive. I don't know if $\mathfrak{S}_{(\infty)}$ can have a highly transitive highly faithful action.

Lemma 3.2. *Let Γ be a countable group acting on a set Y. Then the action is highly faithful iff for all $n \in \mathbb{N}$ and all pairwise distinct $\gamma_1, ..., \gamma_n \in \Gamma$, there exists $y \in Y$ such that for all distinct $i, j \in \{1, ..., n\}$,*

$$\gamma_i y \neq \gamma_j y.$$

Proof. Apply the definition of high faithfulness to the finite set $F := \{\gamma_i \gamma_j^{-1} : i \neq j \in \{1, ..., n\}\}$. □

The previous lemma has the following nice geometric interpretation when Γ is a finitely generated group: a transitive action is highly faithful if and only if the associated Schreier graph contains arbitrarily large balls of the Cayley graph of (Γ, S) for some (or any) finite generating set S.

In this article our focus will be on the measured version of high faithfulness.

[5] See [Gla03, Chap. 2] for some background on measure algebras.

Definition 3.3. A measure-preserving action of a countable group Γ on a probability space (X, μ) is called **highly faithful** if for almost every $x \in X$, the Γ-action on $\Gamma \cdot x$ is highly faithful.

We will now give a useful characterization of highly faithful actions. The proof uses the following well-known lemma.

Lemma 3.4 (see e.g. [EG14, Lem. 5.1])**.** *Let $T \in \mathrm{Aut}(X, \mu)$, let $A \subseteq X$ such that $\mu(\{x \in A : T(x) \neq x\}) > 0$. Then there exists a positive measure set $A' \subseteq A$ such that A' and $T(A')$ are disjoint.*

Theorem 3.5. *Let Γ be a countable group and fix a measure-preserving ergodic Γ-action on (X, μ). Then the following are equivalent:*

(1) the Γ-action is highly-faithful;

(2) for all finite $F \subseteq \Gamma \setminus \{1\}$, the set $\{x \in X : \forall f \in F, fx \neq x\}$ has positive measure;

(3) for all finite $F \subseteq \Gamma$, there exists a positive measure set $A \subseteq X$ such that $(fA)_{f \in F}$ is disjoint;

(4) there exists an increasing exhaustive family (F_n) of finite subsets of Γ and a sequence of positive measure subsets (A_n) of X such that $(fA_n)_{f \in F_n, n \in \mathbb{N}}$ is disjoint;

(5) whenever (F_n) is an increasing exhaustive family of finite sets (F_n) of Γ, there exists a sequence of positive measure subsets (A_n) of X such that $(fA_n)_{f \in F_n, n \in \mathbb{N}}$ is disjoint.

Proof. The chain of implications (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) is straightforward. Note that by ergodicity given a countable family of Borel sets, all its members are of positive measure if and only if almost every Γ-orbit intersects each of its members. In particular condition (2) is satisfied if and only if the Γ-action onto almost every orbit is highly faithful, so the equivalence (1) $\Leftrightarrow$ (2) holds. Also (5) follows from (4) since given any two exhaustive increasing sequences (F_n), (F'_n) of subsets of Γ, there exists an increasing map $\varphi : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$ we have $F'_n \subseteq F_{\varphi(n)}$.

Let us show that (2) implies (3). Let F be a finite subset of Γ, consider the set $F' := \{f_2^{-1} f_1 : f_1 \in F, f_2 \in F, f_1 \neq f_2\}$. By (2) and an inductive application of Lemma 3.4, we find $A \subseteq X$ of positive measure such that for all $f \in F'$, $fA \cap A = \emptyset$. But then for all $f_1 \neq f_2 \in F$, we have $f_2^{-1} f_1 A \cap A = \emptyset$, so $f_1 A \cap f_2 A = \emptyset$, which establishes (3).

We now only have to prove that (3) implies (4), so let us assume (3). We fix an increasing exhausting sequence (F_n) of finite subsets of Γ such that $1 \in F_0$. Using (3) repeatedly, we obtain a sequence (B_n) of positive measure subsets of X such that for all $n \in \mathbb{N}$, the family $(fB_n)_{f \in F_n}$ is disjoint. By inductively taking smaller subsets, we may assume that for all $n \in \mathbb{N}$,

$$|F_n| \, |F_{n+1}| \, \mu(B_{n+1}) < \frac{1}{4} \mu(B_n).$$

This implies that for all $n \geqslant 0$ and $m \geqslant 1$, we have the inequality

$$|F_n|\,|F_{n+m}|\,\mu(B_{n+m}) < \frac{1}{4^m}\mu(B_n).$$

For all $n \in \mathbb{N}$, let $A_n := B_n \setminus \bigcup_{m\geqslant 1} F_n^{-1}F_{n+m}B_{n+m}$. Since $\sum_{m\geqslant 1}\frac{1}{4^m} < 1$, the previous inequality implies that each A_n has positive measure.

Let $n \geqslant 0$, $m \geqslant 1$, $f_1 \in F_n$ and $f_2 \in F_{n+m}$. By construction, the set A_n is disjoint from $f_1^{-1}f_2B_{n+m}$. Since B_{n+m} contains A_{n+m}, we deduce that A_n is disjoint from $f_1^{-1}f_2A_{n+m}$ so that f_1A_n is disjoint from f_2A_{n+m}. Since A_n is a subset of B_n whose F_n-translates are disjoint, this means that the sequence $(fA_n)_{f\in F_n, n\in\mathbb{N}}$ is made of pairwise disjoint sets as required. □

Remark. The non-ergodic version of the previous theorem is obtained by asking in (2), (3), (4) and (5) that the sets which are considered intersect almost every orbit.

4 Residually finite groups and high faithfulness

Let us first recast one definition of residual finiteness in terms of sequences of actions on finite sets.

Definition 4.1. Let Γ be a countable group, and let (X_n, α_n, o_n) be a sequence of pointed Γ-actions on finite sets. The sequence is **asymptotically free** if for all $\gamma \in \Gamma \setminus \{1\}$, there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$, one has $\gamma \cdot o_n \neq o_n$.

The following lemma is proven exactly as Lemma 3.2.

Lemma 4.2. *Let Γ be a countable group, and let (X_n, α_n, o_n) be an asymptotically free sequence of pointed Γ-actions on finite sets. Then for all finite $F \subseteq \Gamma$, there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$ and all distinct $\gamma, \gamma' \in F$, one has $\gamma o_n \neq \gamma' o_n$.*

Definition 4.3. A countable group Γ is **residually finite** if it admits an asymptotically free sequence of pointed actions on finite sets.

The following lemma is well-known and can be used to show that every residually finite group embeds into the full group of any ergodic measure-preserving equivalence relation (see. [Kec10, 4.(E)] for more on this). We include a proof for completeness.

Lemma 4.4. *Let $\mathcal{R}$ be a countable measure-preserving ergodic equivalence relation on (X, μ). Suppose that K is a finite set acted upon by a countable group Λ, and let $(C_k)_{k\in K}$ be a family of disjoint subsets of X, all of the same measure.*

Then there is a homomorphism $\iota : \Lambda \to [\mathcal{R}_{\restriction \bigcup_{k\in K} C_k}]$ such that for all $x \in \bigcup_{k\in K} C_k$, the Λ-action on the Λ-orbit of x is conjugate to the Λ-action on K, and moreover for all $\lambda \in \Lambda$ and all $k \in K$, one has $\iota(\lambda)(C_k) = C_{\lambda(k)}$.

Proof. Let n be the cardinality of the set K, then we can suppose that $K = \mathbb{Z}/n\mathbb{Z}$. Since $\mathcal{R}$ is ergodic and all the C_k's have the same measure, by Proposition 2.1 there is a pre-n-cycle $\Phi = \{\varphi_1, ..., \varphi_{n-1}\}$ such that for all $i \in \{1, ..., n-1\}$, we have

$\varphi_i(C_{i-1}) = C_i$. Let $T_\Phi \in [\mathcal{R}]$ be the associated n-cycle. Given $\lambda \in \Lambda$, we then define $\iota(\lambda)$ by:

$$\iota(\lambda)(x) = T_\Phi^{\lambda \cdot k - k}(x) \text{ where } k \in K \text{ is such that } x \in C_k.$$

Note that $\iota(\lambda)$ is well defined because $T^n = \mathrm{id}_X$. It is then straightforward to check that ι is a homomorphism satisfying the required assumptions. □

Theorem 4.5. *Let Γ be a countable group. Consider a measure-preserving highly faithful ergodic Γ-action on (X, μ) and let Λ be a residually finite countable group. Let (F_n) be an increasing exhaustive family of finite subsets (F_n) of Γ such that $1 \in F_0$, let (A_n) be a sequence of positive measure subsets (A_n) of X such that $(fA_n)_{f \in F_n, n \in \mathbb{N}}$ is disjoint*[6]*. Fix an asymptotically free sequence $(X_m, \alpha_m, o_m)_{m \in \mathbb{N}}$ of Λ-actions on finite pointed sets.*

Then there exists a measure-preserving Λ-action on (X, μ) which preserves the Γ-orbits such that the following assertions are true.

(1) The induced $\Gamma * \Lambda$-action is highly faithful.

(2) The Λ-action is supported on $\bigsqcup_{f \in F_n, n \in \mathbb{N}} fA_n$ and has only finite orbits.

(3) For all $x \in X$, either x is fixed by Λ or there exists $n \in \mathbb{N}$ such that the Λ-action on the Λ-orbit of x is conjugate to α_n.

(4) Any Λ-action which coincides with this action when restricted to $\bigsqcup_{f \in F_n, n \in \mathbb{N}} fA_n$ will induce a highly faithful $\Gamma * \Lambda$-action.

Proof. Let $(G_n)_{n \in \mathbb{N}}$ be an increasing exhaustive sequence of finite subsets of Λ such that $1 \in G_0$. For all $n \in \mathbb{N}$, let $G'_n := G_n \setminus \{1\}$ and $F'_n = F_n \setminus \{1\}$. For all $n \in \mathbb{N}$ and $k \in \{0, ..., n\}$, define the following finite subsets of $\Gamma * \Lambda$:

$$\begin{cases} I_{k,n} & := \underbrace{(G'_n F'_n)(G'_n F'_n) \cdots (G'_n F'_n)}_{k \text{ times}} G_n \text{ and} \\ J_{k,n} & := F'_n I_{k,n}, \end{cases}$$

where by convention $I_{0,n} = G_n$.

Then let $H_n = \bigcup_{k=0}^n (I_{k,n} \cup J_{k,n})$. The sequence (H_n) is clearly an increasing exhaustive sequence of finite subsets of $\Gamma * \Lambda$.

We will define the Λ-action piece by piece, so that for every $n \in \mathbb{N}$, the set $\bigcup_{f \in F_n} fA_n$ is Λ-invariant and A_n witnesses the fact that the $\Gamma * \Lambda$-action is highly faithful for the finite set H_n in the following sense: there is a smaller $A'_n \subseteq A_n$ such that the collection $(hA'_n)_{h \in H_n}$ is made of disjoint sets.

So let us fix $n \in \mathbb{N}$. Since the sequence of pointed Λ-actions $(X_m, \alpha_m, o_m)_{m \in \mathbb{N}}$ is asymptotically free, by Lemma 4.2 we find $m \in \mathbb{N}$ such that for all distinct $\lambda, \lambda' \in I_{0,n}$ we have $\alpha_m(\lambda)(o_m) \neq \alpha_m(\lambda')(o_m)$. Let k_m be the cardinality of the set X_m; we may as well assume that $X_m = \{0, ..., k_m - 1\}$ and $o_m = 0$. We also fix a subset A'_n of A_n of measure ϵ_n, where ϵ_n is a fixed positive real such that $\epsilon_n < \frac{\mu(A_n)}{k_m |H_n|}$.

Since $\epsilon_n < \frac{\mu(A_n)}{k_m + 1}$ we can find disjoint subsets $C_1, ..., C_{k_m - 1} \subseteq A_n$ of measure ϵ_n which are all disjoint from A'_n, and we let $C_0 := A'_n$.

[6]Such a sequence exists by Theorem 3.5.

We define the Λ-action on $\bigcup_{k=0}^{k_m-1} C_k$ via a chosen homomorphism $\iota : \Lambda \to [\mathcal{R}_{\restriction \bigcup_{k\in X_m} C_k}]$ provided by Lemma 4.4 applied to the Λ-action α_m on X_m, where $\mathcal{R}$ is the measure-preserving equivalence relation induced by the Γ-action on (X,μ). Recall that $A'_n = C_0$. Since $\lambda(C_0) = C_{\alpha_m(\lambda)(0)}$, we see that by construction the family $(\lambda(A'_n))_{\lambda\in I_{0,n}}$ is made of disjoint subsets of A_n. Moreover, since the family $(f(A_n))_{f\in F_n}$ is made of disjoint sets, we see that the equality $J_{0,n} = F'_n I_{0,n}$ yields that the family $(h(A'_n))_{h\in I_{0,n}\cup J_{0,n}}$ is made of disjoint sets.

The above setup initializes the following construction for $l = 0$: inductively on $l \in \{0,...,n\}$ we will now define the Λ-action on bigger and bigger sets. So suppose that for some $l \in \{0,...,n-1\}$, we have constructed a Λ-action on

$$\bigsqcup_{h\in\bigcup_{k=0}^{l} I_{k,n}} h(A'_n) \sqcup \bigsqcup_{h\in\bigcup_{k=0}^{l-1} J_{k,n}} h(A'_n)$$

satisfying the following assumptions

(a) the Λ-action is conjugate to α_m when restricted to any orbit,

(b) the Λ-action preserves the Γ-orbits and

(c_l) for any $k \leqslant l$ and any $h \in I_{k,n}$, the set $h(A'_n)$ is a subset of A_n.

Since the family $(f(A_n))_{f\in F_n}$ is made of disjoint sets, condition (c) and the fact that $(h(A'_n))_{h\in\bigcup_{k=0}^{l} I_{k,n}}$ is disjoint implies that the family

$$(h(A'_n))_{h\in\bigcup_{k=0}^{l} J_{k,n}}$$

is actually made of disjoint sets which are all disjoint from A_n. Let us fix a family

$$(C_{h,k})_{k\in\{1,...,k_m-1\},h\in J_{l,n}}$$

of disjoint subsets of A_n of measure ϵ_n such that they are also disjoint from the set $\bigsqcup_{h\in\bigcup_{k=0}^{l} I_{k,n}} h(A'_n)$ (here we fully use the condition $\epsilon_n < \frac{\mu(A_n)}{k_m|H_n|}$). For $h \in J_{l,n}$, let $C_{h,0} := h(A'_n)$. Using again Lemma 4.4, we define for every $h \in J_{l,n}$ the Λ-action on $\bigsqcup_{i=0}^{k_m-1} C_{h,i}$ so that it preserves the $\mathcal{R}$-classes, that it is conjugate to α_m when restricted to an orbit and that for all $i \in \{0,...,k_m\}$, we have $\lambda(C_{h,i}) = C_{h,\alpha_m(\lambda)(i)}$.

Now the Λ-action is also defined on $\bigsqcup_{h\in I_{l,n}} \bigsqcup_{k=0}^{k_m-1} C_{h,k}$. And every $h \in I_{l+1,n}$ is of the form $h = \lambda\bar{h}$ for some $\lambda \in G'_n$ and $\bar{h} \in J_{l,n}$ so that $h(A'_n) = \lambda\bar{h}(A'_n) = \lambda(C_{\bar{h},0}) = C_{\bar{h},\alpha_m(\lambda)(0)}$. Since $\alpha_m(\lambda)(0) \neq 0$, we see that the family $(h(A'_n))_{h\in\bigcup_{k=0}^{l+1} I_{k,n}}$ is made of disjoint subsets of A_n. This implies that the family $(h(A'_n))_{h\in\bigcup_{k=0}^{l+1} I_{k,n}\cup J_{k,n}}$ is disjoint, so we have now constructed a Λ-action on

$$\bigsqcup_{h\in\bigcup_{k=0}^{l+1} I_{k,n}} h(A'_n) \sqcup \bigsqcup_{h\in\bigcup_{k=0}^{l} J_{k,n}} h(A'_n)$$

which satisfies conditions (a), (b) and (c_{l+1}).

Now for every $n \in \mathbb{N}$ we have defined the Λ-action on a subset of $\bigcup_{f\in F_n} fA_n$ and we declare it to be trivial anywhere else. By construction, for every $n \in \mathbb{N}$ the family $(h(A'_n))_{h\in H_n}$ is made of disjoint sets so that the induced $\Gamma * \Lambda$-action is highly faithful: condition (1) is thus satisfied. Conditions (2), (3) and (4) also follow from the construction. □

5 Topological generators in the hyperfinite case

In this section, we get more flexibility in the construction from [LM15] of topological generators for the full group of the hyperfinite ergodic measure-preserving equivalence relation $\mathcal{R}_0$.

5.1 The equivalence relation $\mathcal{R}_0$

Recall that $\mathcal{R}_0$ is defined on the space of infinite binary sequences $\{0,1\}^{\mathbb{N}}$ equipped with the product Bernoulli probability measure $\bigotimes_{n\in\mathbb{N}} \frac{1}{2}(\delta_0+\delta_1)$. By definition, two sequences $(x_i)_{i\in\mathbb{N}}$ and $(y_i)_{i\in\mathbb{N}}$ are $\mathcal{R}_0$-equivalent if they are the same up to a finite number of indices, that is, if there is $N\in\mathbb{N}$ such that for all $i\geqslant N$, we have $x_i=y_i$.

Let us now introduce a bit of notation. Any finite binary sequence $s\in\{0,1\}^n$ defines a subset N_s of the product space $\{0,1\}^{\mathbb{N}}$ consisting of all the sequences starting by s, i.e.

$$N_s := \{x\in\{0,1\}^{\mathbb{N}} : x_i = s_i \text{ for } i\in\{0,...,n-1\}\}.$$

We can see elements $a\in\{0,1\}^n$ and $b\in\{0,1\}^{\mathbb{N}}\cup\bigcup_{n\in\mathbb{N}}\{0,1\}^n$ as words in $\{0,1\}$, and denote their concatenation by $a\frown b$. For $\epsilon\in\{0,1\}$ and $n\in\mathbb{N}$, ϵ^n is the word $(x_i)_{i=1}^n\in\{0,1\}^n$ defined by $x_i=\epsilon$.

Let $n\in\mathbb{N}$. The group $\mathfrak{S}_{\{0,1\}^n}$ is the group of permutations of the set $\{0,1\}^n$. There is a natural inclusion $\alpha_n:\mathfrak{S}_{\{0,1\}^n}\hookrightarrow\mathfrak{S}_{\{0,1\}^{n+1}}$ given by

$$\alpha_n(\sigma)(x_0,...,x_n) = (\sigma(x_0,...,x_{n-1}),x_n)$$

for $\sigma\in\mathfrak{S}_{\{0,1\}^n}$ and $(x_0,...,x_n)\in\{0,1\}^{n+1}$. Let $\mathfrak{S}_{\{0,1\}^{<\infty}}$ be the inductive limit of these groups, called the group of **dyadic permutations**.

The key feature of $\mathfrak{S}_{\{0,1\}^{<\infty}}$ is that it acts in a measure-preserving way on $(\{0,1\}^{\mathbb{N}},\lambda)$ as follows: for $\sigma\in\mathfrak{S}_{\{0,1\}^n}$, $s\in\{0,1\}^n$ and $x\in\{0,1\}^{\mathbb{N}}$,

$$\sigma(s\frown x) = \sigma(s)\frown x.$$

It is straightforward to check that the orbit equivalence relation induced by this action is $\mathcal{R}_0$. To avoid confusion, when we see $\mathfrak{S}_{\{0,1\}^n}$ as a subgroup of $[\mathcal{R}_0]$ we denote it by $\widetilde{\mathfrak{S}}_{\{0,1\}^n}$.

The following proposition belongs to the folklore, for a proof see [Kec10, Prop. 3.8].

Proposition 5.1. *The group of dyadic permutations is dense in the full group of $\mathcal{R}_0$.*

The **odometer** is the map $T_0\in\mathrm{Aut}(\{0,1\}^{\mathbb{N}},\lambda)$ defined by

$$(x_i)_{i\in\mathbb{N}}\in\{0,1\}^{\mathbb{N}}\mapsto 0^{n-1}1\frown(x_i)_{i>n},$$

where n is the first integer such that $x_n=0$ (note that this is well defined on a set of full measure). This can be understood as adding $(1,0,0,...)$ to $(x_i)_{i\in\mathbb{N}}$ with right carry. One can check that T_0 generates $\mathcal{R}_0$.

Let $n \in \mathbb{N}$, then we define a finite odometer $\sigma_n \in \mathfrak{S}_{\{0,1\}^n}$ by

$$\sigma_n((s_i)_{i=0}^{n-1}) = \begin{cases} 0^n & \text{if } (s_i) = 1^n \\ 0^{k-1}1 \frown (s_i)_{i>k} & \text{else, where } k \text{ is the first integer such that } s_k = 0. \end{cases}$$

We denote by T_n the corresponding element in $\widetilde{\mathfrak{S}}_{\{0,1\}^n}$. Note that by definition, T_n and T_0 coincide on $\{0,1\}^{\mathbb{N}} \setminus N_{1^n}$.

5.2 Modified topological generators in the hyperfinite case

Let $n \geqslant 2$, and define $\tau_n \in \mathfrak{S}_{\{0,1\}^n}$ to be the transposition which exchanges $0^{n-1}1$ and $1^{n-1}0$. Let U_n be the corresponding element of $\widetilde{\mathfrak{S}}_{\{0,1\}^n}$, that is, the element of $[\mathcal{R}_0]$ implementing the action of τ_n on $2^{\mathbb{N}}$. Note that the support of U_n is $N_{0^{n-1}1} \sqcup N_{1^{n-1}0}$, so that the supports of the U_n's are all disjoint.

The next lemma boils down to the well-known fact that the symmetric group over 2^n elements is generated by any 2^n-cycle σ along with a transposition τ exchanging two σ-consecutive elements. For a detailed proof see [LM15, Lem. 4.3].

Lemma 5.2. *The group $\widetilde{\mathfrak{S}}_{\{0,1\}^n}$ is contained in the group generated by T_0 and U_n.*

We see that if we could produce $U \in [\mathcal{R}_0]$ such that the closed subgroup generated by U contains infinitely many U_n's, the fact that $\widetilde{\mathfrak{S}}_{\{0,1\}^{<\infty}}$ is dense $[\mathcal{R}_0]$ coupled with the previous lemma would yield that T_0 and U generate a dense subgroup of $[\mathcal{R}_0]$. Although this cannot be done, the main idea of [LM15] is to find $U \in [\mathcal{R}_0]$ such that the closed subgroup generated by U contains infinitely many U_n's up to an error which tends very fast to zero, so that $\{T_0, U\}$ generates a dense subgroup of $[\mathcal{R}_0]$.

To this end, we now fix for every $n \in \mathbb{N}$ a constant $\kappa(n)$ such that any element of $\widetilde{\mathfrak{S}}_{\{0,1\}^n}$ can be written as a word in U_n and T_0 of length less than $\kappa(n)$. For all $p, q \in \mathbb{N}$, we will use the function

$$\sqrt[2^p]{\cdot} : \widetilde{\mathfrak{S}}(\{0,1\}^q) \to \widetilde{\mathfrak{S}}(\{0,1\}^{p+q})$$

defined in [LM15], which satisfies that for all $U \in \widetilde{\mathfrak{S}}(\{0,1\}^{<\infty})$, one has $\left(\sqrt[2^p]{U}\right)^{2^p} = U$ and $\sqrt[2^p]{U}$ has the same support as U.

If $T \in \mathrm{Aut}(X, \mu)$ and A is a Borel subset of X which is T-invariant, we define the induced transformation T_A with respect to A by: for all $x \in X$,

$$T_A(x) = \begin{cases} T(x) & \text{if } x \in A \\ x & \text{else.} \end{cases}$$

We can now state and prove a version of [LM15, Thm. 1.4] where we allow for some error. The argument is very close to the original one, but we give a full proof for the convenience of the reader.

Theorem 5.3. *Given any $\epsilon > 0$, there exists an increasing sequence of integers $(n_k)_{k\in\mathbb{N}}$ and a sequence of positive reals $(\delta_k)_{k\in\mathbb{N}}$ such that whenever we have for all*

$k \in \mathbb{N}$ a $\sqrt[2^{k-1}]{U_{n_k}}$-invariant set $B_k \subseteq \operatorname{supp} U_{n_k}$ with $\mu(B_k) > \mu(\operatorname{supp} U_{n_k}) - \delta_k$, if we let

$$U := \prod_{k=0}^{+\infty} \sqrt[2^k]{U_{n_k}}_{B_k},$$

then the set $\{T_0, U\}$ generates a dense subgroup of $[\mathcal{R}_0]$ and we have $\mu(\operatorname{supp} U) < \epsilon$.

Proof. Fix a sequence (ϵ_k) of positive real numbers such that $\epsilon_k \to 0$.

Claim. It suffices find sequences (n_k) and (δ_k) with $\sum_{k\in\mathbb{N}} 2^{-n_k} < \epsilon$ such that whenever we have for all $k \in \mathbb{N}$ a $\sqrt[2^{k-1}]{U_{n_k}}$-invariant set $B_k \subseteq \operatorname{supp} U_{n_k}$ with $\mu(B_k) > \mu(\operatorname{supp} U_{n_k}) - \delta_k$, if we let

$$U := \prod_{k=0}^{+\infty} \sqrt[2^{k-1}]{U_{n_k}}_{B_k},$$

then for all $k \in \mathbb{N}$, there exists $U' \in \langle U \rangle$ such that $d_u(U_{n_k}, U') < \epsilon_k/\kappa(n_k)$.

Proof of the claim. Assuming that the above conditions are satisfied, fix $k \in \mathbb{N}$ and $U' \in \langle U \rangle$ such that $d_u(U_{n_k}, U') < \epsilon_k/\kappa(n_k)$. Since every element of $\widetilde{\mathfrak{S}}_{\{0,1\}^{n_k}}$ can be written as a word in T_0 and U_{n_k} of length less than $\kappa(n_k)$ (see Lem. 5.2 and the definition of $\kappa(n_k)$), we deduce that every element of $\widetilde{\mathfrak{S}}_{\{0,1\}^{n_k}}$ belongs to $\langle T_0, U\rangle$ up to an error less than ϵ_k. Now $\epsilon_k \to 0$ so the closed group generated by $\{T_0, U\}$ contains $\widetilde{\mathfrak{S}}_{\{0,1\}^{<\infty}}$, hence $\overline{\langle T_0, U\rangle} = [\mathcal{R}_0]$ by Proposition 5.1. And since for all $k \in \mathbb{N}$ we have $\mu(\operatorname{supp} \sqrt[2^k]{U_{n_k}}_{B_k}) \leqslant \mu(\sqrt[2^k]{U_{n_k}}) = 2^{-n_k}$, we deduce that $\mu(\operatorname{supp} U) \leqslant \sum_{k\in\mathbb{N}} 2^{-n_k} < \epsilon$ as desired. □

We now build by induction an increasing sequence (n_k) such that $\sum_{k\in\mathbb{N}} 2^{-n_k} < \epsilon$ and for all $k \in \mathbb{N}$, we have

$$2^{-n_{k+1}-2} < \frac{\epsilon_k}{2\kappa(n_k)}. \tag{1}$$

Then we choose for every $k \in \mathbb{N}$ a positive δ_k such that

$$\delta_k < \frac{\epsilon_k}{2\kappa(n_k)} \tag{2}$$

Let us show that such sequences (n_k) and (δ_k) satisfy the hypotheses of the claim, which will end the proof. So suppose that for every $k \in \mathbb{N}$ we have a $\sqrt[2^{k-1}]{U_{n_k}}$-invariant set $B_k \subseteq \operatorname{supp} U_{n_k}$ with $\mu(B_k) > \mu(\operatorname{supp} U_{n_k}) - \delta_k$. First note that all the $\sqrt[2^k]{U_{n_k}}_{B_k}$ have disjoint supports, so they commute. We fix $k \in \mathbb{N}$ and compute

$$\begin{aligned} U^{2^k} &= \prod_{l=0}^{+\infty} \left(\sqrt[2^l]{U_{n_l}}_{B_l} \right)^{2^k} \\ &= \prod_{l=0}^{k-1} \left(U_{n_l B_l} \right)^{2^{k-l}} \cdot U_{n_k B_k} \cdot \prod_{l=k+1}^{+\infty} \left(\sqrt[2^l]{U_{n_l}}_{B_l} \right)^{2^k} \end{aligned}$$

Because the U_{n_l}'s are involution, the first product is equal to the identity, so that

$$U^{2^k} = U_{n_k B_k} \cdot \prod_{l=k+1}^{+\infty} \left(\sqrt[2^l]{U_{n_l B_l}} \right)^{2^k}. \tag{3}$$

We now check that the error term $W_k := \prod_{l=k+1}^{+\infty} \left(\sqrt[2^l]{U_{n_l B_l}} \right)^{2^k}$ is small. Because for every $l \in \mathbb{N}$, $\left(\sqrt[2^l]{U_{n_l B_l}} \right)^{2^k}$ has same support as $U_{n_l B_l}$, the support of W_k has measure smaller than

$$\sum_{l=k+1}^{+\infty} \lambda(\operatorname{supp} U_{n_l}) \leqslant \sum_{l=k+1}^{+\infty} \frac{1}{2^{n_l - 1}} \tag{4}$$

Since $(n_l)_{l \in \mathbb{N}}$ is increasing, we have for all $l \geqslant k+1$,

$$\frac{1}{2^{n_l - 1}} \leqslant \frac{1}{2^{n_{k+1} + (l-k-2)}}$$

We can now bound the right-hand term in (4) and get the inequality

$$\begin{aligned} \mu(\operatorname{supp} W_k) &\leqslant \frac{1}{2^{n_{k+1}}} \cdot \sum_{l=k+1}^{+\infty} \frac{1}{2^{l-k-2}} \\ &\leqslant \frac{4}{2^{n_{k+1}}} \\ &\leqslant \frac{\epsilon_k}{2\kappa(n_k)}, \end{aligned}$$

the latter inequality being a direct consequence of (1). From this and equation (3) we deduce

$$\begin{aligned} d_u(U^{2^k}, U_{n_k B_k}) &\leqslant \mu(\operatorname{supp}(W_k)) \\ &\leqslant \frac{\epsilon_k}{2\kappa(n_k)}. \end{aligned}$$

Since B_k is a U_{n_k}-invariant subset of the support of U_{n_k} such that $\mu(B_k) > \mu(\operatorname{supp} U_{n_k}) - \delta_k$, we have $d_u(U_{n_k B_k}, U_{n_k}) < \delta_k < \frac{\epsilon_k}{2\kappa(n_k)}$ by (2). We deduce that

$$d_u(U^{2^k}, U_{n_k}) < \frac{\epsilon_k}{\kappa(n_k)}$$

so that the theorem now follows from the claim. □

6 Proof of the main theorem

6.1 A lemma on commuting elements

Lemma 6.1. *Let $T, U \in \operatorname{Aut}(X, \mu)$ have disjoint supports, and suppose that there are two relatively prime numbers $p, q \geqslant 2$ such that*

- *every T-orbit is finite and its cardinality divides a power of p and*
- *every U-orbit is finite and its cardinality divides a power of q.*

Then both T and U belong to the closure of the group generated by TU for the uniform topology.

Proof. Since $U = T^{-1}(TU)$, it suffices to show that T belongs to the closure of $\langle TU\rangle$. Let $\epsilon > 0$, find $N \in \mathbb{N}$ large enough so that there is a Borel set A such that $\mu(A) > 1-\epsilon$ and for all $x \in A$;

$$|\mathrm{Orb}_T(x)| \leqslant p^N \text{ and } |\mathrm{Orb}_U(x)| \leqslant q^N$$

Since p^N and q^N are relatively prime, there is $l \in \mathbb{N}$ such that $lq^N \equiv 1 \bmod p^N$.

Note that T and U commute since they have disjoint support and fix $x \in A$. If x belongs to the support of T then $(TU)^{lq^N}(x) = T^{lq^N}(x) = T(x)$ because the cardinality of the T-orbit of x divides a power of p no greater than p^N and $lq^N \equiv 1 \bmod p^N$. If x belongs to the support of U then $(TU)^{lq^N}(x) = U^{lq^N}(x) = x$ because the cardinality of the U-orbit of x divides a power of q no greater than q^N, so $(TU)^{lq^N}(x) = T(x)$. And if x neither belongs to the support of T nor to the support of U, then $(TU)^{lq^N}(x) = x = T(x)$. So for all $x \in A$, we have $(TU)^{lq^N}(x) = T(x)$. As $\mu(A) > 1-\epsilon$, we deduce that $d_u((TU)^{lq^N}, T) < \epsilon$ and we conclude that T belongs to the closure of $\langle TU\rangle$. □

A proof by induction yields the following useful corollary.

Corollary 6.2. *Let $T_1, .., T_n \in \mathrm{Aut}(X, \mu)$ have disjoint supports, and suppose that there are n pairwise relatively prime numbers $p_1, ..., p_n \geqslant 2$ such that for every $k \in \{1, ..., n\}$, every T_k-orbit is finite and its cardinality divides a power of p_k.*

Then for all $k \in \{1, ..., n\}$, T_k belongs to the closure of the group generated by the product $T_1T_2 \cdots T_n$ for the uniform topology.

6.2 Proof of Theorem 1.12

Let us start with an ergodic equivalence relation $\mathcal{R}$ such that $\mathrm{Cost}(\mathcal{R}) < m+1$ for some $m \in \mathbb{N}$. Our goal is to find $m+1$ topological generators for the full group of $\mathcal{R}$ so that the induced $\mathbb{F}_{m+1}$-action is highly faithful and amenable onto almost every orbit.

By [Dye59, Thm. 4][7], we may and do assume that $X = \{0,1\}^{\mathbb{N}}$ equipped with the product Bernoulli probability measure $\mu = \bigotimes_{n\in\mathbb{N}} \frac{1}{2}(\delta_0 + \delta_1)$, and that the odometer T_0 belongs to the full group of $\mathcal{R}$.

Lemma III.5 in [Gab00] provides a graphing Φ such that $\mathrm{Cost}(\Phi) < m$ and $\{T_0\} \cup \Phi$ generates $\mathcal{R}$. Let

$$c = \frac{\mathrm{Cost}(\Phi)}{m} < 1,$$

and fix some odd $p \in \mathbb{N}$ such that $(\frac{p+2}{p})c < 1$. Splitting the domains of the partial automorphisms in Φ, we find Φ_1,...,Φ_m of cost c such that $\Phi = \Phi_1 \cup \cdots \cup \Phi_m$.

[7] See also [Zim84, 9.3.2] for a statement and a proof with a less operator-algebraic flavour.

The map T_0 induces a free (in particular highly faithful) ergodic $\mathbb{Z}$-action, so we apply item (5) of Theorem 3.5 and find a sequence $(A_n)_{n\in\mathbb{N}}$ of non-null Borel subsets of X such that the family $(T_0^i(A_n))_{|i|\leqslant n, n\in\mathbb{N}}$ is disjoint. Up to taking smaller non-null subsets $A'_n \subseteq A_n$ for all $n \in \mathbb{N}$, we can assume that $\mu(\bigsqcup_{n\in\mathbb{N}} \bigsqcup_{i=-n}^{n} T_0^i(A_n)) < \epsilon$.

Let $\epsilon := \frac{1-(\frac{p+2}{p})c}{2}$, using Theorem 5.3 we fix an increasing sequence of integers $(n_k)_{k\in\mathbb{N}}$ and a sequence of positive reals $(\delta_k)_{k\in\mathbb{N}}$ such that whenever we have for all $k \in \mathbb{N}$ a $\sqrt[2^{k-1}]{U_{n_k}}$-invariant set $B_k \subseteq \operatorname{supp} U_{n_k}$ with $\mu(B_k) > \mu(\operatorname{supp} U_{n_k}) - \delta_k$, if we let

$$U := \prod_{k=0}^{+\infty} \sqrt[2^k]{U_{n_k}}_{B_k},$$

then the set $\{T_0, U\}$ generates a dense subgroup of $[\mathcal{R}_0]$ and we have $\mu(\operatorname{supp} U) < \epsilon$, where $\mathcal{R}_0$ is the measure-preserving equivalence generated by T_0.

Claim. We can also assume that for all $k \in \mathbb{N}$,

$$\mu\left(\left(\bigsqcup_{n\in\mathbb{N}} \bigsqcup_{i=-n}^{n} T_0^i(A_n)\right) \cap \operatorname{supp} U_{n_k}\right) < \frac{\delta_k}{2^{k+1}}.$$

Proof. Let $n \in \mathbb{N}$. Find $l \in \mathbb{N}$ such that the set $\bigcup_{k\geqslant l} \bigcup_{i=-n}^{n} T_0^i(\operatorname{supp} U_{n_k})$ has measure less than $\mu(A_n)/2$, and set $A'_n := A_n \setminus \bigcup_{k\geqslant l} \bigcup_{i=-n}^{n} T_0^i(\operatorname{supp} U_{n_k})$ which is non-null. Then by construction for all $k \geqslant l$, the set $\bigsqcup_{i=-n}^{n} T_0^i(A'_n)$ is disjoint from $\operatorname{supp} U_{n_k}$.

We now take $A''_n \subseteq A'_n$ non-null such that for all $k < l$,

$$\mu\left(\bigsqcup_{i=-n}^{n} T_0^i(A''_n)\right) < \frac{\delta_k}{2^{k+n+2}}$$

Then for all $k, n \in \mathbb{N}$, we have

$$\mu\left(\bigsqcup_{i=-n}^{n} T_0^i(A''_n) \cap \operatorname{supp} U_{n_k}\right) < \frac{\delta_k}{2^{k+n+2}}$$

But now a straightforward calculation yields that for all $k \in \mathbb{N}$

$$\mu\left(\left(\bigsqcup_{n\in\mathbb{N}} \bigsqcup_{i=-n}^{n} T_0^i(A''_n)\right) \cap \operatorname{supp} U_{n_k}\right) < \frac{\delta_k}{2^k}$$

so the sequence (A''_n) is as desired. □

For all $k \in \mathbb{N}$, let

$$\overline{B}_k := \bigcup_{j=0}^{2^{k+1}-1} \sqrt[2^k]{U_{n_k}}^j \left(\operatorname{supp} U_{n_k} \cap \left(\bigsqcup_{n\in\mathbb{N}} \bigsqcup_{i=-n}^{n} T_0^i(A_n)\right)\right)$$

Since $\sqrt[2^k]{U_{n_k}}$ has order 2^{k+1}, the set $\overline{B}_k$ is $\sqrt[2^k]{U_{n_k}}$-invariant. Moreover by the previous claim $\mu(\overline{B}_k) < \delta_k$. We let $B_k := \operatorname{supp} U_{n_k} \setminus \overline{B}_k$, and then we define

$$U := \prod_{k=0}^{+\infty} \sqrt[2^k]{U_{n_k}}_{B_k},$$

By construction, the set $\{T_0, U\}$ generates a dense subgroup of $[\mathcal{R}_0]$ and we have $\mu(\operatorname{supp} U) < \epsilon$.

We now let $B := \left(\bigsqcup_{n\in\mathbb{N}} \bigsqcup_{i=-n}^{n} T_0^i(A_n)\right) \cup \operatorname{supp} U$. Note that by construction $\mu(B) < 1 - \frac{p+2}{p}c$. Let $D_1, ..., D_{p+2}$ be pairwise disjoint subsets of $X \setminus B$, of measure $\frac{c}{p}$ each. For all $i \in \{1, ..., m\}$ we use Proposition 2.1 to pre- and post-compose the partial isomorphisms of Φ_i by elements in $[[\mathcal{R}_0]]$ so that each Φ_i becomes a pre-$(p+1)$-cycle $\Phi_i = \{\varphi_1^i, \varphi_2^i, ..., \varphi_p^i\}$ where $\varphi_j^i : D_j \to D_{j+1}$ for all $j \in \{1, ..., p\}$. Note that this operation preserves the fact that $\mathcal{R}$ is generated by $\{T_0\} \cup \Phi_1 \cup \cdots \cup \Phi_m$.

Now choose $\psi \in [[\mathcal{R}_0]]$ with domain D_{p+1} and range D_{p+2}, and add it to every Φ_i. We get m pre-$(p+2)$-cycles $\tilde{\Phi}_i = \Phi_i \cup \{\psi\}$, and $\{T_0\} \cup \tilde{\Phi}_1 \cup \cdots \cup \tilde{\Phi}_n$ still generates $\mathcal{R}$. Consider the associated $(p+2)$-cycles $C_{\tilde{\Phi}_i}$.

Claim. The $m+2$ elements $T_0, U, C_{\tilde{\Phi}_1}, ..., C_{\tilde{\Phi}_m}$ generate a dense subgroup of the full group of $\mathcal{R}$.

Proof. Let G be the closed group generated by $\{T_0, U, C_{\tilde{\Phi}_1}, ..., C_{\tilde{\Phi}_m}\}$. Recall that T_0 and U have been chosen so that they generate together a dense subgroup of $[\mathcal{R}_0]$, so G contains $[\mathcal{R}_0]$.

Because ψ is a partial isomorphism of $\mathcal{R}_0$, we have $[\mathcal{R}_{\{\psi\}}] \subseteq [\mathcal{R}_0] \subseteq G$. Since for all $i \in \{1, ..., m\}$ we have $\psi \in \tilde{\Phi}_i$ and $C_{\tilde{\Phi}_i} \in G$, Proposition 2.3 implies that G contains $[\mathcal{R}_{\tilde{\Phi}_i}]$. But $\mathcal{R}$ is the join of $\mathcal{R}_0$, $\mathcal{R}_{\tilde{\Phi}_1}, ..., \mathcal{R}_{\tilde{\Phi}_m}$, so by Theorem 2.2 we are done. □

For each $n \in \mathbb{N}$, let A'_n and A''_n be two non-null disjoint subsets of A_n such that $A_n = A'_n \sqcup A''_n$. Let $q \in \mathbb{N}$ be an odd prime number which does not divide $p+2$. By [Hal50], the group $\mathbb{F}_m$ is a residually q-finite group, so we can find an asymptotically free sequence of pointed $\mathbb{F}_m$-actions (X_n, α_n, o_n) such that for all $n \in \mathbb{N}$ and all $\lambda \in \mathbb{F}_m$, the permutation $\alpha_n(\lambda)$ has order q^k for some $k \in \mathbb{N}$.

Let $A = \bigsqcup_{n\in\mathbb{N}} \bigsqcup_{i=-n}^{n} T_0^i(A'_n)$. We now apply Theorem 4.5 to $\Gamma = \mathbb{Z}$ through the action induced by T_0, $\Lambda = \mathbb{F}_m$, the sequence of actions (X_n, α_n, o_n) and the sequence of sets (A'_n) such that the sequence $(T_0^i(A'_n))_{i=-n}^{n}$ is made of disjoint sets. We thus obtain a $\mathbb{F}_m$-action supported on A which preserves the $\mathcal{R}$-classes and satisfies the following conditions.

(1) The induced $\mathbb{Z} * \mathbb{F}_m$-action is highly faithful.

(2) The $\mathbb{F}_m$-action is supported on A and has only finite orbits.

(3) For all $x \in X$, there exists $n \in \mathbb{N}$ such that the $\mathbb{F}_m$-action on the $\mathbb{F}_m$-orbit of x is conjugate to α_n.

(4) Any $\mathbb{F}_m$-action whose restriction to A coincides with this action will induce a highly faithful $\mathbb{Z} * \mathbb{F}_m$-action.

The $\mathbb{F}_m$-action we just obtained is determined by the elements of the full group induced by its standard generators which we denote by $V_1, ..., V_m \in [\mathcal{R}]$.

By our hypothesis on the sequence of actions on finite sets (α_n), for all $i \in \{1, ..., m\}$, every V_i-orbit has cardinality q^k for some $k \in \mathbb{N}$. Moreover, we have that

U, V_1 and $C_{\tilde{\Phi}_1}$ have disjoint supports and that for all $i \in \{2, ..., m\}$, V_i and $C_{\tilde{\Phi}_i}$ have disjoint supports.

By corollary 6.2, the elements U and $C_{\tilde{\Phi}_1}$ belong to the closure of the group generated by $UV_1C_{\tilde{\Phi}_1}$, and for all $i \in \{2, ..., m\}$, $C_{\tilde{\Phi}_i}$ belongs to the closure of the group generated by $V_iC_{\tilde{\Phi}_i}$. So by the previous claim the group generated by the $m + 1$ elements

$$T_0, UV_1C_{\tilde{\Phi}_1}, V_2C_{\tilde{\Phi}_2}, ..., V_mC_{\tilde{\Phi}_m}$$

is a dense subgroup of the full group of $\mathcal{R}$. Let us show that the associated $\mathbb{F}_{m+1}$-action has all the desired properties.

First, the fact that the m last generators $UV_1C_{\tilde{\Phi}_1}, V_2C_{\tilde{\Phi}_2}, ..., V_mC_{\tilde{\Phi}_m}$ act trivially on $\bigsqcup_{n\in\mathbb{N}} \bigsqcup_{i=-n}^{n} T_0^i(A_n'')$ implies that for almost all $x \in X$, the restriction of the Schreier graph of the $\mathbb{F}_{m+1}$-action on the orbit of x contains arbitrarily long intervals, so the $\mathbb{F}_{m+1}$-action is amenable onto almost every orbit.

Then recall that V_1,...,V_m induce an $\mathbb{F}_m$-action which satisfies conditions (1)-(4) above. Moreover, V_1 and $UV_1C_{\tilde{\Phi}_1}$ have the same restriction to A and for all $i \in \{2, ..., m\}$, V_i and $V_iC_{\tilde{\Phi}_i}$ have the same restriction to A. By (4), this implies that that the $\mathbb{Z} * \mathbb{F}_m = \mathbb{F}_{m+1}$-action that we have built is highly faithful, which ends the proof of Theorem 1.12.

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