Polish groups and non-free actions in the discrete or measurable context

François Le Maître

Université Paris Cité

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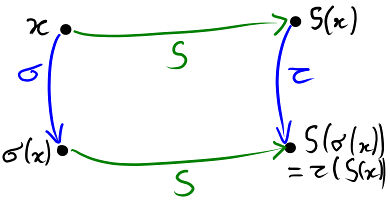
Conjugacy of permutations

Two permutations $\sigma \in \mathrm{Sym}(X)$ and $\tau \in \mathrm{Sym}(Y)$ are **conjugate** if $\exists S : X \to Y$ bijection such that

$$\forall x \in X, \quad S(\sigma(x)) = \tau(S(x)).$$

In other words $S \circ \sigma \circ S^{-1} = \tau$.
Throughout the talk, we identify $\sigma \in \mathrm{Sym}(X)$ to its *graph* $\mathcal{G}_\sigma$, whose vertex set is X and oriented edges are of the form $(x, \sigma(x))$.



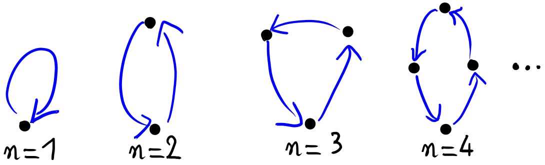
Lemma

A bijection S conjugates σ to τ iff it is a graph isomorphism $\mathcal{G}_\sigma \to \mathcal{G}_\tau$.

Cycle decomposition

Given $\sigma \in \mathrm{Sym}(X)$, the connected components of $\mathcal{G}_\sigma$ (a.k.a. the σ-orbits) are either:

- A finite cycle of length $n \geq 1$



- A biinfinite cycle, of length $n = \infty$

Proposition

Two permutations on countable sets are conjugate iff for all $n \in \mathbb{N} \cup \{\infty\}$ they have the same (finite or infinite) number of cycles of length n.

The permutation group of the integers

Definition

Let $\mathfrak{S}_\infty = \mathrm{Sym}(\mathbb{N})$ be the permutation group of the integers.

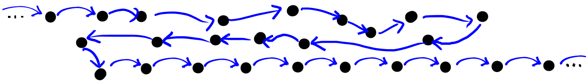
This is a topological group for the topology of pointwise convergence: $\sigma_n \to \sigma$ iff for all $x \in \mathbb{N}$, for $n \in \mathbb{N}$ large enough we have

$$\sigma_n(x) = \sigma(x).$$

This topology can be understood at the level of graphs as follows: the basic open neighborhood of $\sigma \in \mathfrak{S}_\infty$ associated to a finite set $F \Subset \mathbb{N}$ is

$$\mathcal{V}_F(\sigma) = \{\tau \in \mathfrak{S}_\infty : \mathcal{G}_\tau \textit{ induces the same graph on } F \text{ as } \mathcal{G}_\sigma\}$$

(**inducing** means only keeping edges whose source is in F).



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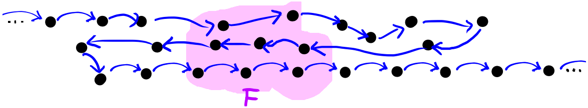

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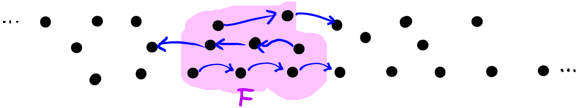


Theorem (Folklore)

The set $\mathcal{A}_1$ of all permutations $\sigma \in \mathfrak{S}_\infty$ such that

1. *all the σ-orbits are finite and*
2. *for all $k \geq 1$ finite, σ has infinitely many orbits of cardinality k*

is dense and has all its elements pairwise conjugate.

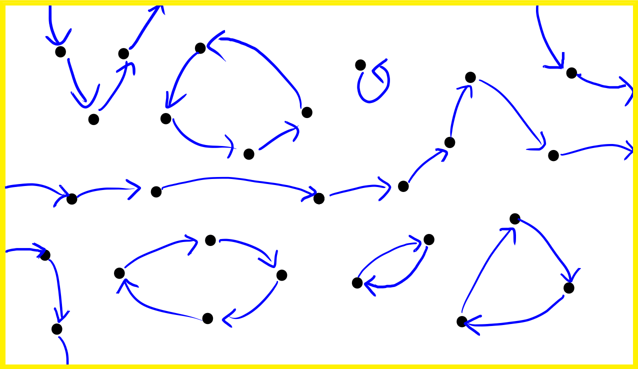
Proof.

Recall that two permutations are conjugate iff they have the same number of cycles of the same length, for every (possibly infinite) length.
So $\mathcal{A}_1$ consists of pairwise conjugate elements.
Only the density of $\mathcal{A}_1$ remains to be proven, cf. next slide.

Proof of the density of $\mathcal{A}_1$

Proof.

Let $\sigma \in \mathfrak{S}_\infty$, let $\mathcal{V}_F(\sigma)$ be an open neighborhood of σ.



The induced graph on F has only cycles or segments. Close the segments. Put infinitely many cycles of each possible finite length outside. The corresponding new permutation belongs to $\mathcal{A}_1 \cap \mathcal{V}_F(\sigma)$. $\square$

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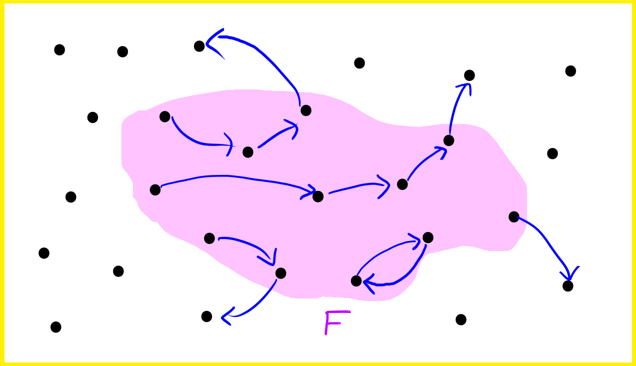

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Proof of the density of $\mathcal{A}_1$

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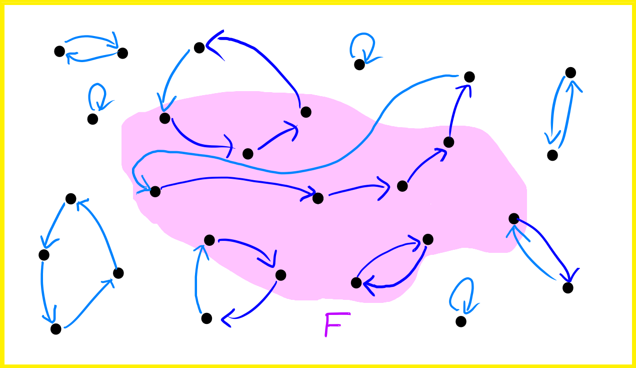

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$\mathfrak{S}_\infty$ is a Polish group

Definition

A **Polish space** is a topological space which is separable ($\exists$ countable dense subset) and whose topology comes from *some* complete metric.

Examples

- $\mathbb{N}$, $\mathbb{R}$, $]0, +\infty[$ are Polish spaces.
- Any countable product of Polish spaces is Polish.

Theorem (Alexandrov)

A subset of a Polish space is Polish for the induced topology iff it is G_δ.

A subset of a topological space is called G_δ when it can be written as a countable intersection of open subsets.

Example

The group $\mathfrak{S}_\infty$ is G_δ in the Polish space $\mathbb{N}^\mathbb{N}$, hence it is a Polish group.

Theorem (Baire category theorem)

Let X be a Polish space, let (U_n) be a countable family of dense open subsets, then $\bigcap_n U_n$ is also dense in X.

Call **dense** G_δ the subsets of X which can be written as $\bigcap_n U_n$, with each U_n dense open.
The Baire category theorem ensures us this terminology is sound.
Moreover, it implies that *every countable intersection of dense G_δ subsets is dense G_δ*. We thus have the following analogue of full measure subsets.

Definition

A subset of a Polish space is called **comeager** when it contains a dense G_δ subset.

Comeager conjugacy class

Theorem (Folklore)

The set $\mathcal{A}_1$ of all permutations σ all whose orbits are finite, and such that for all $k \geq 1$, σ has infinitely many orbits of cardinality k, is dense G_δ and consists of pairwise conjugate permutations.

Corollary

$\mathfrak{S}_\infty$ has a comeager conjugacy class: it contains a comeager set all whose elements are pairwise conjugate.

The notion of conjugacy makes sense in any group G: two elements $h_1, h_2 \in G$ are **conjugate** if there is $g \in G$ such that $gh_1g^{-1} = h_2$.
The above corollary can then be contrasted with:

Theorem (Wesolek 2016)

The only locally compact Polish group with a comeager conjugacy class is the trivial group.

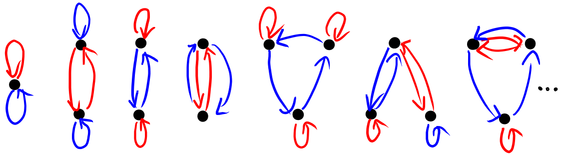
What about pairs (more generally r-tuples) of permutations?

Definition

$(\sigma_1, \sigma_2) \in \mathrm{Sym}(X)^2$ is **conjugate** to $(\tau_1, \tau_2) \in \mathrm{Sym}(Y)^2$ if there is $S : X \to Y$ such that for all $x \in X$, all $i \in \{1, 2\}$, $S(\sigma_i(x)) = \tau_i(S(x))$.

This can again be understood purely in terms of (Schreier) graphs $\mathcal{G}_{\sigma_1,\sigma_2}$, this time with colored edges of the form $(x, \sigma_1(x))$ and $(x, \sigma_2(x))$.Their connected components fall into two categories:

- Finite: still only countably many possibilities up to isomorphism!



- Infinite: huge mess (we will come back to that)

$\mathfrak{S}_\infty$ has a dense 2-diagonal conjugacy class

Let $(G_k)_{k\in\mathbb{N}}$ enumerate all possible *finite* connected components of all $\mathcal{G}_{\sigma_1,\sigma_2}$ up to colored graph isomorphism.

Theorem (Folklore)

The set $\mathcal{A}_2$ of all pairs $(\sigma_1, \sigma_2) \in \mathfrak{S}_\infty$ such that

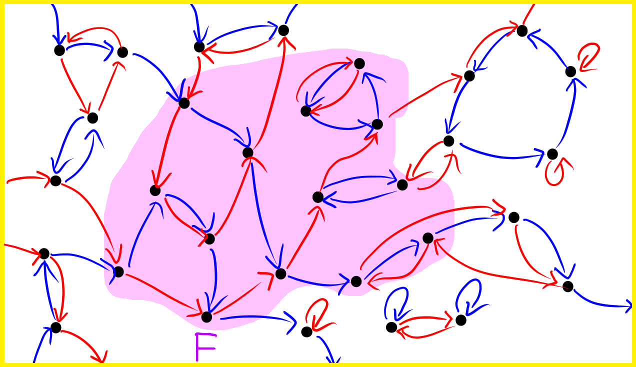
1. *$\mathcal{G}_{\sigma_1,\sigma_2}$ has only finite connected components,*
2. *and for all $k \in \mathbb{N}$, $\mathcal{G}_{\sigma_1,\sigma_2}$ contains infinitely many copies of G_k,*

is dense G_δ and consists of pairwise conjugate pairs of permutations.

Again the pairwise conjugacy part of the statement is straightforward to check, and $\mathcal{A}_2$ is G_δ for essentially the same reasons as $\mathcal{A}_1$. Let us prove the density.

Proof of density of $\mathcal{A}_2$.

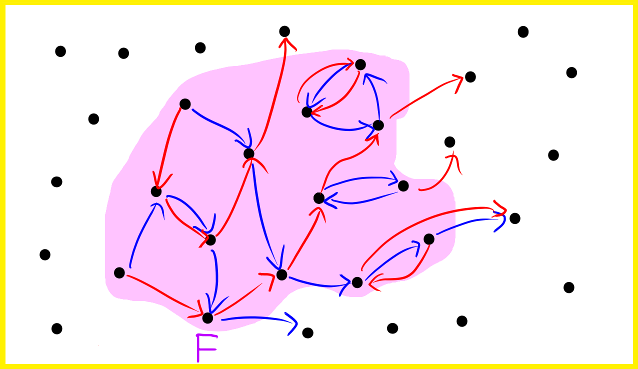
Let $(\sigma_1, \sigma_2) \in \mathfrak{S}_\infty^2$ and $\mathcal{V}_F(\sigma_1, \sigma_2)$ be a basic neighborhood of (σ_1, σ_2).



The induced graph on F has only cycles or segments. Close the segments. Put fixed points on incomplete degree > 0 vertices. Put infinitely many copies of each G_k outside. This belongs to $\mathcal{A}_2 \cap \mathcal{V}_F(\sigma_1, \sigma_2)$. □

Proof of density of $\mathcal{A}_2$.

Let $({\color{blue}\sigma_1}, {\color{red}\sigma_2}) \in \mathfrak{S}_\infty^2$ and $\mathcal{V}_F(\sigma_1, \sigma_2)$ be a basic neighborhood of (σ_1, σ_2).



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Proof of density of $\mathcal{A}_2$.

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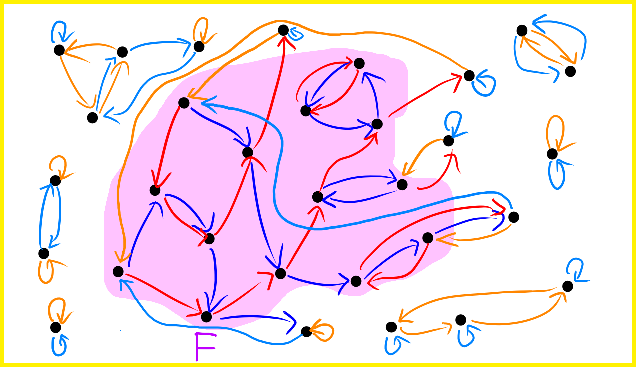


The induced graph on F has only cycles or segments. Close the segments. Put fixed points on incomplete degree > 0 vertices. Put infinitely many copies of each G_k outside. This belongs to $\mathcal{A}_2 \cap \mathcal{V}_F(\sigma_1, \sigma_2)$. $\square$

For any $r \geq 1$, say that a Polish group G has an r-**diagonal comeager conjugacy class** if its action by diagonal conjugacy on G^r, given by

$$g \cdot (g_1, \cdots, g_r) = (gg_1g^{-1}, \cdots, gg_rg^{-1}),$$

admits a comeager orbit. Say that G has **ample generics** if it has an r-diagonal comeager conjugacy class for every $r \geq 1$.
The fact that every $\mathcal{A}_r \subseteq \mathfrak{S}_\infty^r$ is a dense G_δ conjugacy class yields:

Theorem (Folklore)

$\mathfrak{S}_\infty$ *has ample generics.*

Other Polish groups with ample generics

$\mathfrak{S}_\infty$ is an example of a **non-archimedean** Polish group: its topology has a neighborhood basis consisting of open subgroups (pointwise stabilizers of finite sets). Many other such groups have ample generics, e.g.

Theorem (Kwiatkowska 2012)

The homeomorphism group of the Cantor space has ample generics.

Question (Kechris-Rosendal 2007)

Does every Polish group with ample generics have to be non-archimedean?

Theorem (Kaïchouh-LM, Malicki, 2015)

No! Malicki: totally disconnected examples (Tsankov groups).
Kaïchouh-LM: contractile (in particular connected) examples.

In our work with Kaïchouh, we show that ample generics is preserved by the L^0 construction. We also show that *every ergodic hyperfinite type III equivalence relation has a full group with ample generics*.
We will now discuss the remaining type II_∞ case, which was left open.

The full group generated by $\mathbb{Q}$

Full groups, defined by Dye in 1959, are groups of Borel bijections identified up to a conull set, stable under *countable cutting and pasting*. Denote by λ the Lebesgue measure on $\mathbb{R}$. Let us define the *full group* $[\mathbb{Q}]_\lambda$ *generated by* $\mathbb{Q}$ on $(\mathbb{R}, \lambda)$.

Definition

$[\mathbb{Q}]_\lambda = \{T\colon \mathbb{R} \to \mathbb{R}$ Borel bijection s.t. $\forall x \in \mathbb{R}, T(x) \in \mathbb{Q} + x\}/\simeq_\lambda$, where $T_1 \simeq_\lambda T_2$ iff $\lambda(\{x \in \mathbb{R}\colon T_1(x) \neq T_2(x)\}) = 0$.

For every $T \in [\mathbb{Q}]_\lambda$, we have a Borel partition $(A_q)_{q\in\mathbb{Q}}$ of $\mathbb{R}$ defined by

$$A_q = \{x \in \mathbb{R}\colon T(x) = q + x\}.$$

Since T is bijective, $(T(A_q))_{q\in\mathbb{Q}}$ is also a partition and since translations by $q \in \mathbb{Q}$ preserves the measure, it follows that *every element of* $[\mathbb{Q}]_\lambda$ *preserves* λ($\forall A \subseteq \mathbb{R}$ Borel, $\lambda(T^{-1}(A)) = \lambda(A)$).

Topology on the full group

Definition

$[\mathbb{Q}]_\lambda = \{T : \mathbb{R} \to \mathbb{R}$ Borel bijection s.t. $\forall x \in \mathbb{R}, T(x) \in \mathbb{Q} + x\}/\simeq_\lambda$, where $T_1 \simeq_\lambda T_2$ iff $\lambda(\{x \in \mathbb{R}\colon T_1(x) \neq T_2(x)\}) = 0$.

Basis of neighborhoods: given $F \subseteq \mathbb{R}$ Borel of finite measure and $\epsilon > 0$,a basic neighborhood of $T \in [\mathbb{Q}]_\lambda$ is given by

$$\mathcal{V}_{F,\epsilon}(T) = \{U \in [\mathbb{Q}]_\lambda : \lambda(\{x \in F\colon T(x) \neq U(x)\}) < \epsilon\}.$$

This makes $[\mathbb{Q}]_\lambda$ a Polish group. Replacing both $\mathbb{Q}$ and $\mathbb{R}$ by $\mathbb{Z}$, and λ by the counting measure, we get $\mathfrak{S}_\infty$ with its Polish group topology.

Theorem (LM)

$[\mathbb{Q}]_\lambda$ *has ample generics.*

A familiar statement

Let $r \geq 1$. As in the $\mathfrak{S}_\infty$ case, we have very concrete description of the r-diagonal comeager conjugacy class. Enumerate all possible finite connected graphs of the form $\mathcal{G}_{\sigma_1,\dots,\sigma_r}$ up to isomorphism as $(G^r_k)_{k\in\mathbb{N}}$.

Theorem

Let $\mathcal{A}_r$ denote the set of all $(T_1, \dots, T_r) \in [\mathbb{Q}]^r_\lambda$ such that

1. *All c.c. (connected components) of $\mathcal{G}_{T_1,\dots,T_r}$ are finite;*
2. $\forall k \in \mathbb{N}$, $\lambda(\{x \in \mathbb{R}\colon$ *the c.c. of x is isomorphic to* $G^r_k\})) = \infty$.

Then $\mathcal{A}_r$ is dense G_δ and consists of a single r-diagonal conjugacy class.

Replacing $\mathbb{R}$ and $\mathbb{Q}$ by $\mathbb{Z}$, and λ by the counting measure, we recover

Theorem (Folklore)

Let $\mathcal{A}_r$ denote the set of all $(\sigma_1, \dots, \sigma_r) \in \mathfrak{S}^r_\infty$ such that

1. *All connected components of $\mathcal{G}_{\sigma_1,\dots,\sigma_r}$ are finite;*
2. $\forall k \in \mathbb{N}$, $\mathcal{G}_{\sigma_1,\dots,\sigma_r}$ *contains infinitely many copies of* G^r_k.

Then $\mathcal{A}_r$ is dense G_δ and consists of a single r-diagonal conjugacy class.

About the proof

Crucial tool: approximate finiteness (Dye 1959).

Proposition (approximate finiteness of $[\mathbb{Q}]_\lambda$)

For all $T_1, ..., T_r \in [\mathbb{Q}]_\lambda$, $\forall F \subseteq \mathbb{R}$ Borel of finite measure, $\forall \epsilon > 0$, $\exists F_\epsilon \subseteq X$ Borel, $\lambda(F \setminus F_\epsilon) < \epsilon$ and the graph induced by $\mathcal{G}_{T_1,...,T_r}$ on F_ϵ has only finite connected components.

We also use the well-known fact that for every Borel $A, B \subseteq \mathbb{R}$ of the same measure, there is $\varphi : A \to B$ bijective (up to measure zero) such that for all $x \in A$, $\varphi(x) \in \mathbb{Q} + x$.

Remark (case $r = 1$, Rohlin's lemma)

Any ergodic type II_∞ full group has a comeager conjugacy class.

Conjecture

Let $\mathbb{G}$ be any type II_∞ ergodic full group. Suppose $\mathbb{G}$ is not approximately finite. Then $\mathbb{G}$ does not have ample generics.

Quasi non-archimedean groups

In this slide U and V are always neighborhoods of the identity.

A topological group is non-archimedean iff $\forall U, \exists V$ such that $\forall r \in \mathbb{N}$ and $\forall g_1, ..., g_r \in V$, the group generated by $\{g_1, ..., g_r\}$ is contained in U.

Definition (Gelander-LM)

A topological group is **quasi non-archimedean** if $\forall U\ \forall r \in \mathbb{N}$, $\exists V$ such that $\forall g_1, ..., g_r \in V$, the group generated by $\{g_1, ..., g_r\}$ is contained in U.

Examples

- Non-archimedean groups are quasi non-archimedean.
- $[\mathbb{Q}]_\lambda$ is quasi non-archimedean (let $U = \mathcal{V}_{F,\epsilon}(\mathrm{id}_\mathbb{R})$, let $r \in \mathbb{N}$, then $V = \mathcal{V}_{F,\epsilon/r}(\mathrm{id}_\mathbb{R})$ works) but contractile.

Question (Gelander-LM)

Is there a Polish group with ample generics which is not quasi non-archimedean?

For the rest of the talk $r = 2$. Let $\mathbb{F}_2 = \langle a, b \rangle$ be the free group on 2 generators.
From now on, we freely view pairs of permutations (σ_1, σ_2) as $\mathbb{F}_2$-actions $(a \mapsto \sigma_1,\ b \mapsto \sigma_2)$ and vice-versa.

Definition

$\mathcal{T}_2$ is the space of *transitive* $\mathbb{F}_2$-actions on $\mathbb{N}$, or equivalently the space of $(\sigma_1, \sigma_2) \in \mathfrak{S}_\infty^2$ such that $\mathcal{G}_{\sigma_1, \sigma_2}$ is connected.

$\mathcal{T}_2$ is G_δ in $\mathfrak{S}_\infty^2$, hence Polish. With Carderi and Gaboriau, we obtained *dense totipotent* $\mathbb{F}_2$ *in type* II_1 *ergodic full groups of cost* < 2, yielding

Corollary (Carderi-Gaboriau-LM)

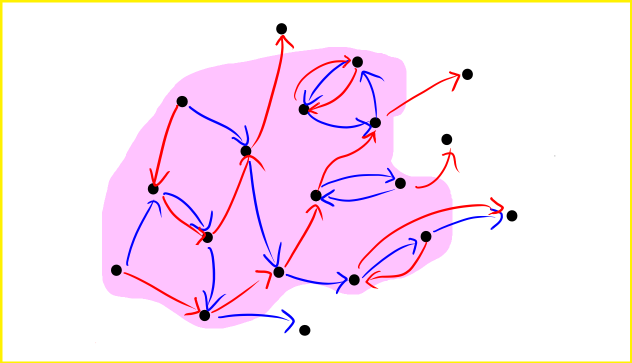
$\mathcal{T}_2$ contains a dense 2-diagonal conjugacy class and all 2-diagonal conjugacy classes are meager in $\mathcal{T}_2$. Moreover, the set of highly transitive $\mathbb{F}_2$-actions is comeager in $\mathcal{T}_2$.

We will focus on density, and then high transitivity.

A totipotent $\mathbb{F}_2$-action

Proof that there is a dense 2-diagonal conjugacy class in $\mathcal{T}_2$.

Let us enumerate all possible connected finite graphs induced by elements of $\mathcal{T}_2$ as $(H_k)_{k\in\mathbb{Z}}$. Add edges to obtain $\tilde{H}_k$ complete except one vertex lacking its two blue half edges.



Attach the $\tilde{H}_k$'s over a biinfinite blue line. This induces all H_k's! □

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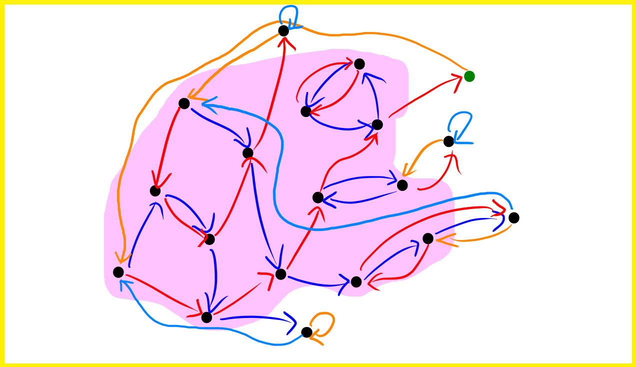

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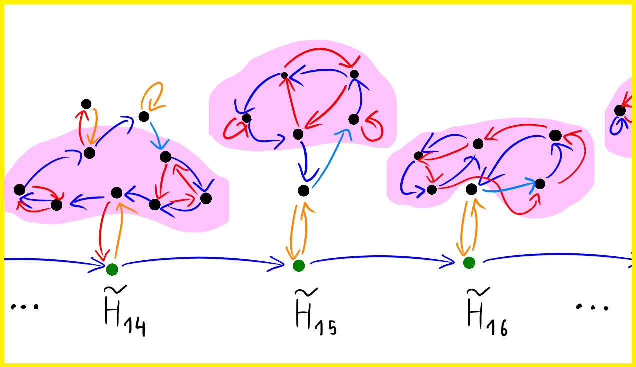

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High transitivity

Definition

An action α of a countable group Γ on an infinite set X is called **highly transitive** if α is k-**transitive** for all $k \in \mathbb{N}$:
given any $x_1, \ldots, x_k \in X$ and $y_1, \ldots, y_k \in X$ pairwise distinct, there is $\gamma \in \Gamma$ such that

$$\alpha(\gamma)x_i = y_i \text{ for all } i \in \{1, \ldots, k\}.$$

Γ is **highly transitive** if it admits a *faithful* highly transitive action.

Γ is highly transitive iff it is isomorphic to a dense subgroup of $\mathfrak{S}_\infty$.

Example

The group of finitely supported permutations of $\mathbb{N}$ is highly transitive.

Theorem (McDonough 1976)

$\mathbb{F}_2$ *is highly transitive.*

Generic high transitivity

Theorem (LM)

Let Γ_1 and Γ_2 be countable groups, with $|\Gamma_1| \geq 2$ and $|\Gamma_2| \geq 3$. Let $\Gamma = \Gamma_1 * \Gamma_2$. Then the set of highly transitive Γ-actions is comeager in the space of transitive Γ-actions on $\mathbb{N}$.

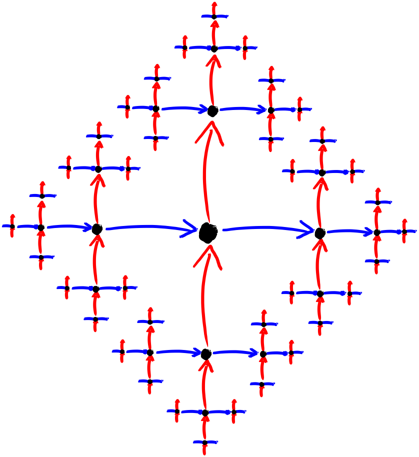
Theorem (Azuelos-Gaboriau 2023)

Take Γ as above. The space of transitive Γ-actions has a dense orbit.

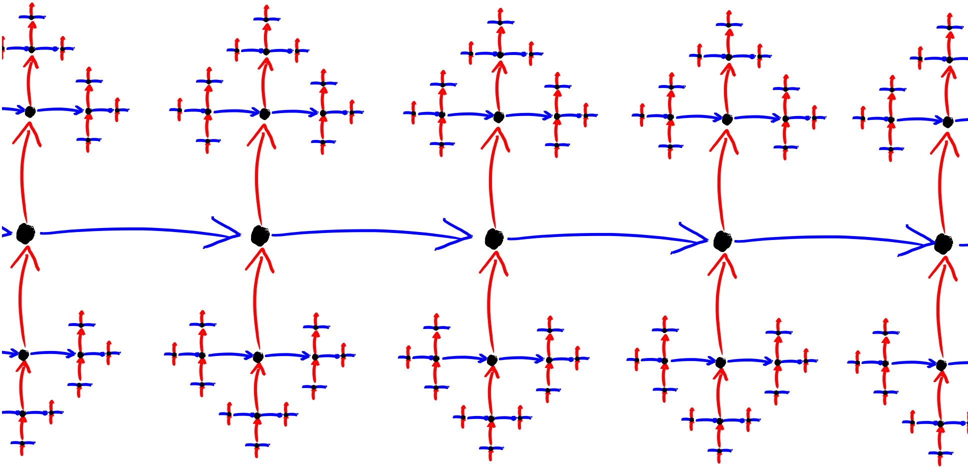
Corollary (Gunhouse 1990, Hickin 1992, Moon-Stalder 2013)

Γ as above is highly transitive.

We will now sketch the proof of the first theorem when $\Gamma_1 = \Gamma_2 = \mathbb{Z}$ so that $\Gamma = \mathbb{F}_2$.

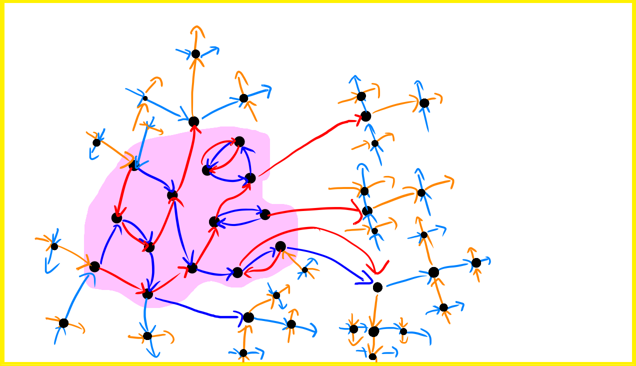


Interlude: the free transitive action of $\mathbb{F}_2$



Proof.

By Baire category, given $x_1, \ldots, x_k, y_1, \ldots, y_k$ pairwise $\neq$, enough to show that actions β s.t. $\exists\gamma$, $\beta(\gamma)x_i = y_i$ are dense in $\mathcal{T}_2$. Take $\alpha \in \mathcal{T}_2$, F finite defining a neighborhood of $\mathcal{V}_F(\alpha)$, WLOG F contains the x_i's and y_i's.

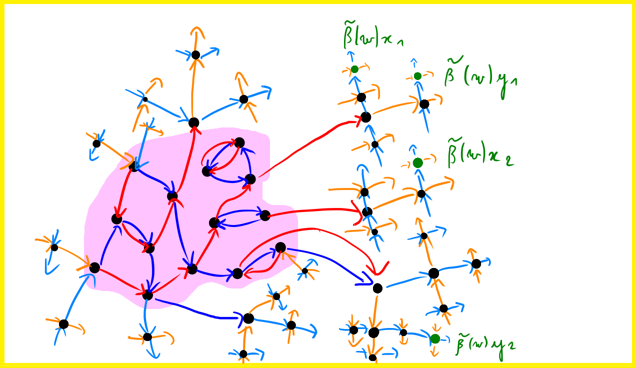


Freeify the induced graph on F, obtaining an action $\tilde{\beta}$.

Proof of generic high transitivity for $\mathbb{F}_2$

Proof.

By Baire category, given $x_1, \dots, x_k, y_1, \dots, y_k$ pairwise $\neq$, enough to show that actions β s.t. $\exists \gamma$, $\beta(\gamma)x_i = y_i$ are dense in $\mathcal{T}_2$. Take $\alpha \in \mathcal{T}_2$, F finite defining a neighborhood of $\mathcal{V}_F(\alpha)$, WLOG F contains the x_i's and y_i's.



Fix $w \in \mathbb{F}_2$ such that $\tilde{\beta}(w)x_1$ is in the tree.

Proof.

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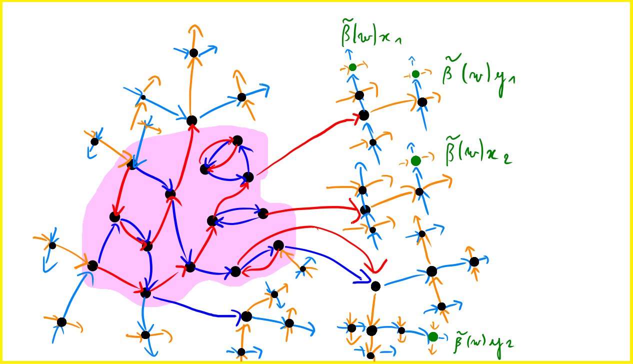

By prefixing many letters w starts with a and takes *all* x_i, y_i in the trees.

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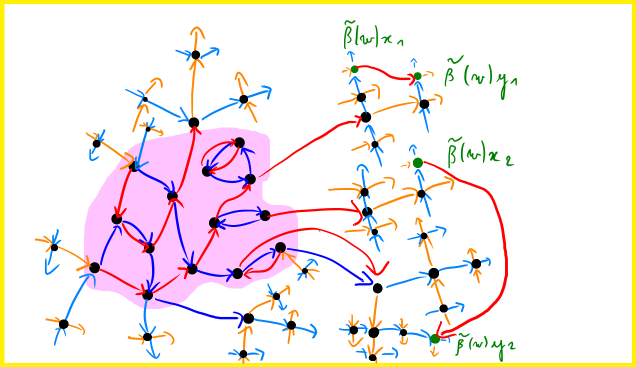


Modify $\tilde{\beta}$ so that b takes $\tilde{\beta}(w)x_i$ to $\tilde{\beta}(w)y_i$. Now $\tilde{\beta}(w^{-1}bw)x_i = y_i$! □

High transitivity for groups acting on trees

We used the tree structure coming from the free action of $\mathbb{F}_2$ so as to push things away. Using work of Le Boudec-Matte Bon and Bass-Serre theory, similar ideas yield:

Theorem (Fima-LM-Moon-Stalder 2022)

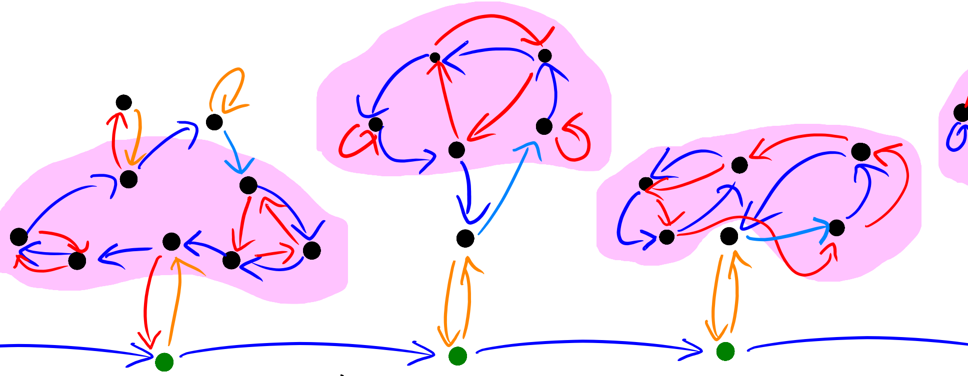
Let $\mathcal{T}$ be a tree. Let $\Gamma \curvearrowright \mathcal{T}$. Suppose that the action is faithful, minimal, of general type. The following are equivalent:

1. *Γ is highly transitive;*
2. *Γ admits a faithful 4-transitive action;*
3. *the induced Γ-action on the boundary of $\mathcal{T}$ is topologically free.*

Corollary

The Baumslag-Solitar group $\mathrm{BS}(2,3) = \langle t, a \colon ta^2t^{-1} = a^3 \rangle$ is highly transitive.

Carderi-Gaboriau-LM-Stalder: $\mathrm{BS}(2,3)$ is not generically highly transitive.



Merci !