ON STRONGLY JUST INFINITE PROFINITE BRANCH GROUPS

FRANÇOIS LE MAÎTRE AND PHILLIP WESOLEK

Abstract. For profinite branch groups, we first demonstrate the equivalence of the Bergman property, uncountable cofinality, Cayley boundedness, the countable index property, and the condition that every non-trivial normal subgroup is open; compact groups enjoying the last condition are called strongly just infinite. For strongly just infinite profinite branch groups with mild additional assumptions, we verify the invariant automatic continuity property and the locally compact automatic continuity property. Examples are then presented, including the profinite completion of the first Grigorchuk group. As an application, we show that many Burger–Mozes universal simple groups enjoy several automatic continuity properties.

1. Introduction

Given a Polish group[2], it is natural to study the extent to which topological properties are determined by the group's algebraic structure. A common approach is to study automatic continuity properties: For a Polish group G and some interesting class of topological groups $\mathcal{H}$, one asks if *every* homomorphism $\psi : G \to H$ with $H \in \mathcal{H}$ is continuous. Such questions explore the connection between algebraic and topological structure since homomorphisms from a Polish group must respect algebraic structure, but they do not necessarily respect the topology.

In the setting of non-locally compact Polish groups, there are now many groups known to enjoy the **automatic continuity property**, namely that every homomorphism into any Polish group is continuous; we refer the reader to the survey by C. Rosendal [25] for further discussion. For non-discrete locally compact Polish groups, however, much less is known. Indeed, the following fundamental question remains open:

Question 1.1 (Rosendal, [24]). *Is there a non-discrete locally compact Polish group which has the automatic continuity property?*

In the work at hand, we study automatic continuity properties for profinite branch groups. Our results fall just short of answering Question 1.1 positively; specifically, we obtain the weak Steinhaus property. Nonetheless, we do elucidate an interesting characterization of those groups which enjoy various weaker automatic continuity properties. Moreover, we connect these properties to combinatorial boundedness conditions; the strongest of these being the Bergman property.

[2]A **Polish group** is a separable topological group whose topology admits a compatible complete metric.

1.1. **Statement of results.**

Definition 1.2 (See Section 2.4)**.** A **profinite branch group** is a closed spherically transitive subgroup of the automorphism group of a rooted locally finite tree such that every rigid level stabilizer is an open subgroup.

Profinite branch groups form a rich, interesting class of compact Polish groups; an introduction to these groups and their structure can be found in [13].

A profinite group is **strongly just infinite** if every non-trivial normal subgroup is open. A straightforward adaption of an argument due to R. I. Grigorchuk, cf. [13, Theorem 4], gives a characterization of strongly just infinite profinite branch groups.

Proposition 1.3 (See Theorem 3.2)**.** *Suppose that $G \leqslant \mathrm{Aut}(T_\alpha)$ is a profinite branch group. Then the following are equivalent:*

(1) G is strongly just infinite.
(2) For every vertex $v \in T_\alpha$, the derived subgroup of $\mathrm{rist}_G(v)$ is open in $\mathrm{rist}_G(v)$.
(3) For every level $n \geqslant 1$, the derived subgroup of $\mathrm{rist}_G(n)$ is open.

The celebrated work of N. Nikolov and D. Segal [21] along with the previous proposition imply that all just infinite topologically finitely generated profinite branch groups are strongly just infinite.

Our main result establishes for profinite branch groups an equivalence between being strongly just infinite, automatic continuity properties, and combinatorial boundedness conditions.

Theorem 1.4 (See Theorem 4.8)**.** *Let G be a profinite branch group. Then the following are equivalent:*

(1) G is strongly just infinite.
(2) Every commensurated subgroup of G is either finite or open.
(3) G has the normal countable index property.
(4) G has the countable index property.
(5) G has the weak Steinhaus property.
(6) G has uncountable cofinality.
(7) G is Cayley-Bounded.
(8) G has property (FA).
(9) G has the Bergman property.

Remark 1.5. Infinite products of non-abelian finite simple groups are known to satisfy a similar characterization; see [27, 28].

Examples of groups satisfying the equivalent conditions of Theorem 1.4 include many iterated wreath products as well as the profinite completion of the Grigorchuk group. (See Section 7.)

Under slightly stronger hypotheses, we obtain additional automatic continuity properties.

Theorem 1.6 (See Theorem 5.10)**.** *If G is a strongly just infinite profinite branch group which locally has derangements and has uniform commutator widths, then G enjoys the equivalent properties of Theorem 1.4, the invariant automatic automatic continuity property, and the locally compact automatic continuity property.*

We go on to observe that profinite branch groups have a unique Polish group topology. Additionally, strongly just infinite branch groups admit exactly two locally compact group topologies: their profinite topology and the discrete topology. (See Section 6.)

Our study of profinite branch groups concludes by considering applications of our results. We give the first examples of non-discrete compactly generated locally compact Polish groups that are simple and enjoy the automatic continuity properties discussed herein.

Theorem 1.7 (See Theorem 8.4)**.** *Suppose that $F \leqslant \mathfrak{S}_d$ is non-trivial, perfect, and two transitive. Suppose further the point stabilizers of F are also perfect. The Burger–Mozes universal simple group $U(F)^+$ then enjoys the countable index property, the invariant automatic continuity property, and the locally compact automatic continuity property.*

The commensurated subgroups of these Burger–Mozes groups are additionally classified. Classifying commensurated subgroups gives information on the possible homomorphisms into totally disconnected locally compact groups, see [22]; a compelling example of such a classification is the work of Y. Shalom and G. Willis on commensurated subgroups of arithmetic groups [26].

Theorem 1.8 (see Theorem 8.6)**.** *Suppose that $F \leqslant \mathfrak{S}_d$ is non-trivial, perfect, and two transitive. Suppose further the point stabilizers of F are also perfect. Then every commensurated subgroup of $U(F)^+$ is either finite, compact and open, or equal to $U(F)^+$.*

The alternating group A_d for any $d \geqslant 6$ is an example of a finite group F that satisfies the hypotheses of the above theorems.

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2. Preliminaries

For a topological group G, the commutator of $g, h \in G$ is $[g,h] := ghg^{-1}h^{-1}$. The *set* of commutators of G is $[G,G] := \{[g,h] \mid g,h \in G\}$. We put

$$[G,G]^{*n} := \{[g_1,h_1]\dots[g_n,h_n] \mid g_i,h_i \in G\}.$$

The **derived subgroup** of G is $D(G) := \langle [G,G] \rangle$; in general this subgroup is not closed, hence we occasionally add the modifier "abstract" to call attention to this point.

The symmetric group on a set X is denoted $\mathfrak{S}(X)$. For all $d \in \mathbb{N}$, we let $[d]$ denote the set $\{0, ..., d-1\}$ and set $\mathfrak{S}_d := \mathfrak{S}([d])$.

2.1. **Generalities on groups.** We shall require a lemma likely well-known to mathematicians more familiar with the theory of uncountable abelian groups; we include a proof for completeness.

Lemma 2.1. *If A is an uncountable abelian group, then A has an infinitely generated countable quotient.*

Proof. Let $\mathrm{Tor}(A)$ be the torsion subgroup of A and form $\tilde{A} := A/\mathrm{Tor}(A)$. Suppose first $\tilde{A}$ is uncountable, so $\tilde{A}$ is an uncountable torsion free abelian group. The extension of scalars $\tilde{A}\otimes_{\mathbb{Z}}\mathbb{Q}$ is an uncountable $\mathbb{Q}$-vector space, and there is a canonical injection $\tilde{A}\hookrightarrow\tilde{A}\otimes_{\mathbb{Z}}\mathbb{Q}$, since $\tilde{A}$ is torsion free. We may find $\{a_i\otimes 1\}_{i\in\mathbb{N}}$ with $a_i\in\tilde{A}$ linearly independent vectors in $\tilde{A}\otimes_{\mathbb{Z}}\mathbb{Q}$. We then have a projection

$$\tilde{A}\otimes_{\mathbb{Z}}\mathbb{Q}\to\mathrm{span}(\{a_i\otimes 1\mid i\in\mathbb{N}\}) =: V.$$

The composition $A\to\tilde{A}\to\tilde{A}\otimes_{\mathbb{Z}}\mathbb{Q}\to V$ has a countably infinite image that is infinitely generated, verifying the lemma in this case.

If $A/\mathrm{Tor}(A)$ is countable, then it suffices to find a countable quotient of $\mathrm{Tor}(A)$ that is infinitely generated; we thus assume $A=\mathrm{Tor}(A)$. By [12, Theorem 8.4], we have a decomposition $A=\bigoplus_{p\text{ prime}}A_p$ where A_p are abelian p-groups, and since A is uncountable, there is a prime p so that A_p is uncountable. We may thus also assume A is an uncountable p-group.

Appealing to [12, Theorem 32.3], there is $B\leqslant A$ so that B is a direct sum of cyclic p-groups and A/B is divisible. Suppose first A/B is non-trivial. Divisible abelian groups are direct sums of copies of $\mathbb{Q}$ and Prüfer p-groups via [12, Theorem 23.1], and both of these are countably generated. Projecting onto one of these summands, we obtain a countable quotient of A which is infinity generated. If A/B is trivial, then A is an uncountable direct sum of cyclic p-groups. Projecting onto a countable direct sum gives the desired countable quotient which is infinitely generated. The lemma is thus verified. □

For a group G, define the function $l : D(G)\to\mathbb{Z}$ by

$$l(g) := \min\{n \mid g=\prod_{i=1}^{n}[h_i,k_i]\}.$$

The **commutator width** of G is then $\mathrm{cw}(G) := l(D(G))$.

For profinite groups, we note a useful sufficient condition for finite commutator width.

Lemma 2.2 (Hartley, [14, Lemma 1])**.** *If G is a profinite group so that $D(G)$ is open in G, then $\mathrm{cw}(G)<\infty$.*

Lastly, a topological group is called **just infinite** if all its non-trivial closed normal subgroups are of finite index. Discarding the topology, we arrive at the central notion of the current work.

Definition 2.3. A topological group G is called **strongly just infinite** if every nontrivial normal subgroup is open with finite index.

2.2. **Combinatorial boundedness conditions.** Our work primarily considers three boundedness conditions.

Definition 2.4. Let G be a group.

(i) G has the **Bergman property** if every G-action by isometries on a metric space has bounded orbits.
(ii) G has **uncountable cofinality** if there is no increasing chain $(G_n)_{n\in\mathbb{N}}$ of proper subgroups of G such that $\bigcup_{n\in\mathbb{N}} G_n = G$. Otherwise, G has **countable cofinality**.
(iii) G is **Cayley bounded** if for any symmetric generating set U containing 1, there is $n \geqslant 1$ so that $U^n = G$. Equivalently, every Cayley graph for G has finite diameter.

The Bergman property admits a useful algebraic reformulation.

Definition 2.5. A sequence $(A_n)_{n\in\mathbb{N}}$ of subsets of a group G is called a **Bergman sequence** if it is increasing, each of its elements is symmetric, $1 \in A_0$, and $\bigcup_{n\in\mathbb{N}} A_n = G$.

Theorem 2.6 (Cornulier, cf. [9, Proposition 2.7])**.** *Let G be a group. The following assertions are equivalent.*
(1) G has the Bergman property.
(2) If $(A_n)_{n\in\mathbb{N}}$ is a Bergman sequence such that $A_nA_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$, then there exists $k \in \mathbb{N}$ such that $A_k = G$.
(3) If $(A_n)_{n\in\mathbb{N}}$ is a Bergman sequence, then there exists $k, n \in \mathbb{N}$ such that $A_n^k = G$.

The Bergman property is related to uncountable cofinality and Cayley boundedness as follows:

Proposition 2.7 (Cornulier, [9, Proposition 2.4])**.** *A group G has the Bergman property if and only if it has uncountable cofinality and is Cayley bounded*

We also require a sufficient condition to be non-Cayley bounded, due to A. Khelif. Here we reproduce his unpublished proofs with his kind permission. In his work [17], his terminology differs from ours: what he calls “Bergman’s property” is for us “Cayley boundedness.”

Theorem 2.8 (Khelif, [17, Théorème 10])**.** *If Γ is a countable infinite subgroup of a compact group G, then Γ is not Cayley bounded.*

Proof. (Khelif) First note that we may assume G is metrizable: By the Peter-Weyl Theorem, for each $\gamma \in \Gamma$ there is a finite dimensional unitary representation $\pi_\gamma : G \to \mathcal{U}(\mathbb{C}^{n_\gamma})$ such that $\pi_\gamma(\gamma) \neq 1$, so Γ embeds into the compact metrizable group $\prod_{\gamma\in\Gamma} \mathcal{U}(\mathbb{C}^{n_\gamma})$. Furthermore, by taking the closure of Γ in G, we may assume that Γ is dense in G. Fix a compatible right-invariant metric d on G.

Enumerate $\Gamma = \{\gamma_n \mid n \in \mathbb{N}\}$ and for all $n \in \mathbb{N}$ let $\Gamma_n := \langle \gamma_0, \dots, \gamma_n\rangle$. Consider the sequence of continuous functions $f_n : G \to [0, +\infty[$ given by

$$f_n(g) = d(\Gamma_n, g) := \inf_{\gamma\in\Gamma_n} d(\gamma, g).$$

This sequence of functions decreases pointwise to zero, so by Dini’s theorem, they converge uniformly to zero. We can thus find a sequence $(u_n)_{n\in\mathbb{N}}$ such that $u_n \in \Gamma_n$ and $d(u_n, \gamma_{n+1})$ tends to zero.

Set $S := \{1, \gamma_0\} \cup \{\gamma_{n+1}u_n^{-1} \mid n \in \mathbb{N}\}$ and let $U := S \cup S^{-1}$. The set U is a symmetric generating set for Γ which contains 1, and since $\gamma_{n+1}u_n^{-1} \to 1$, the set U is also a compact subset of G. If Γ is Cayley bounded, we can find $k \in \mathbb{N}$ such that $\Gamma = U^k$. In particular, Γ is then compact, but by the Baire category theorem, there is no countable infinite compact group. We thus deduce that Γ is not Cayley bounded. $\square$

Corollary 2.9 ([17, Corollaire 11])**.** *If G is an infinite solvable-by-finite group, then G is not Cayley bounded.*

Proof. (Khelif) Let H be a finite index normal subgroup of G such that H is solvable and consider the derived series $H^{(n+1)} := D(H^{(n)})$ where $H^{(0)} := H$. Let n be the smallest integer such that $H/H^{(n)}$ is infinite. Since $H^{(n-1)}/H^{(n)}$ is abelian, the quotient group $G/H^{(n)}$ is infinite and abelian-by-finite. It thus suffices to show that no infinite abelian-by-finite group is Cayley bounded, so we assume G has a finite index normal subgroup N which is abelian.

By Lemma 2.1, the group N has a subgroup $B \leqslant N$ with countable index. Letting $x_1, \dots, x_k$ be coset representatives for N in G, the subgroup $L := \bigcap_{i=1}^k x_i B x_i^{-1}$ is a normal subgroup of G so that G/L is a countably infinite abelian-by-finite group. Passing to G/L, we may also assume G is countable.

Since N is abelian, Pontryagin duality gives a morphism $\psi : N \to K$ with countable infinite image where K is a compact group. Taking coset representatives $x_1, \dots, x_k$ for N in G, define $\rho : N \to K^k$ by

$$h \mapsto (\psi(x_1 h x_1^{-1}), \dots, \psi(x_k h x_k^{-1})).$$

The action of G on N by conjugation induces an action of G on $\rho(N)$ by defining $g.\rho(h) := \rho(ghg^{-1})$.

For each $g \in G$ and $h \in N$, it follows there is $\sigma \in \mathfrak{S}(k)$ so that

$$g.(\psi(x_1 h x_1^{-1}), \dots, \psi(x_k h x_k^{-1})) = (\psi(x_{\sigma(1)} h x_{\sigma(1)}^{-1}), \dots, \psi(x_{\sigma(k)} h x_{\sigma(k)}^{-1}))$$

Each element $g \in G$ thus acts on $\rho(N)$ by permuting coordinates. Such an action extends to an action on $\overline{\rho(N)}$ by continuous automorphisms.

Under this action, the group $\overline{\rho(N)} \rtimes G$ is a locally compact group. A straightforward calculation shows the subset $\{(\rho(h), h^{-1}) \mid h \in N\}$ is a discrete normal subgroup of $\overline{\rho(N)} \rtimes G$. The quotient is furthermore a compact group into which G embeds. The desired conclusion now follows via Theorem 2.8. $\square$

We lastly note that any quotient of a Cayley bounded group is again Cayley bounded.

2.3. **Automatic continuity properties.**

Definition 2.10. A topological group is **Polish** if the underlying topology is separable and admits a compatible, complete metric. A Polish group is called **non-archimedean** if the topology admits a basis at 1 of open subgroups.

The automatic continuity properties of primary interest have a useful algebraic description.

Definition 2.11. Let G be a Polish group.

(i) The group G has the **normal countable index property** if every countable index normal subgroup of G is open.
(ii) The group G has the **countable index property** if every countable index subgroup of G is open.

Lemma 2.12. *Let G be a Polish group.*

(1) The group G has the normal countable index property if and only if every homomorphism $\psi : G \to H$ with H a countable discrete group is continuous.
(2) The group G has the countable index property if and only if every homomorphism $\psi : G \to H$ with H a non-archimedean Polish group is continuous.

Proof. The first claim is immediate, so we prove the second. Suppose first G has the countable index property and $\psi : G \to H$ is a homomorphism with H a non-archimedean Polish group. Taking $O \leqslant H$ an open subgroup of H, the index of O is countable, since H is second countable. Hence, $\psi^{-1}(O)$ is a countable index subgroup of G. That G has the countable index property now ensures $\psi^{-1}(O)$ is open, and we deduce that ψ is continuous.

Conversely, suppose G is so that every homomorphism $\psi : G \to H$ is continuous where H is a non-archimedean Polish group. Take $O \leqslant G$ a countable index subgroup. The action of G on the left cosets G/O induces a homomorphism $\sigma : G \to \mathfrak{S}(G/O)$. The group $\mathfrak{S}(G/O)$ with the topology of pointwise convergence is a non-archimedean Polish group, hence σ is continuous. Furthermore, the collection of permutations in $\mathfrak{S}(G/O)$ which fix the coset O, denoted Σ, form an open subgroup of $\mathfrak{S}(G/O)$. We thus see that $\sigma^{-1}(\Sigma) = O$ is open in G, whereby G has the countable index property. $\square$

The countable index property is a weakening of the small index property which states that every subgroup of index less than the continuum is open. The latter has been studied in the context of automorphism groups of countable structures; see [10].

Our proofs will require analyzing σ-syndetic sets: For a Polish group G, a subset $A \subseteq G$ is called **σ-syndetic** if there is a sequence $(g_n)_{n\in\mathbb{N}}$ of elements of G so that $G = \bigcup_{n\in\mathbb{N}} g_n A$.

Definition 2.13. A Polish group G has the **weak Steinhaus property** if for any symmetric σ-syndetic set A, there exists $n \in \mathbb{N}$ such that the set A^n contains a neighborhood of 1.

Let us collect two technical results concerning σ-syndetic sets.

Lemma 2.14 ([29, Lemma 4])**.** *Suppose that G is a group and H is a subgroup of G. If A is a symmetric σ-syndetic set containing 1 for G, then $H \cap A^2$ is a symmetric σ-syndetic set containing 1 for H.*

Lemma 2.15. *Suppose G is a group and $g_0, \dots, g_n \in G$. If A is a symmetric σ-syndetic set containing 1, then $\bigcap_{i=0}^n g_i A^{2^n} g_i^{-1}$ is symmetric σ-syndetic set containing 1.*

Proof. The fact that $\bigcap_{i=0}^n g_i A^{2^n} g_i^{-1}$ is again symmetric and contains 1 is straightforward. To see this set is also σ-syndetic, one argues via the obvious induction. This induction comes down to the following claim, which we prove here: If $A, B \subseteq G$ are two σ-syndetic symmetric sets containing 1, then the set $B^2 \cap A^2$ is also σ-syndetic.

Let A, B be two σ-syndetic symmetric sets containing 1, fix a sequence $(h_k)_{k\in\mathbb{N}}$ so that $G = \bigcup_{k\in\mathbb{N}} h_k B$ and set

$$K := \{k \in \mathbb{N} \mid h_k B \cap A \neq \emptyset\}.$$

For each $k \in K$, fix $a_k \in h_k B \cap A$ and say $a_k = h_k b_k$ with $b_k \in B$.

For $a \in A$, there is $k \in K$ and $b \in B$ so that $a = h_k b$, hence $a = a_k b_k^{-1} b$. We also have $a = a_k a_k^{-1} a$, so $a \in a_k(B^2 \cap A^2)$. It now follows $A \subseteq \bigcup_{k \in K} a_k(B^2 \cap A^2)$. Since the set A is σ-syndetic, we conclude $B^2 \cap A^2$ is σ-syndetic, completing the proof. □

2.4. **Rooted trees and profinite branch groups.** A **rooted tree** T is a locally finite tree with a distinguished vertex r called the **root**. Letting d be the usual graph metric, the **levels** of T are the sets $V_n := \{v \in T \mid d(v, r) = n\}$. The **children** of a vertex $v \in V_n$ is collection of $w \in V_{n+1}$ so that there is an edge from v to w.

We think of the levels of the tree as linearly ordered so that the orders cohere. That is to say, if $v_0 < v_1$ in V_n with w_0 a child of v_0 and w_1 a child of v_1, then $w_0 < w_1$ in V_{n+1}. This ordering allows us to take the right-most branch: the **right-most branch** of T is the unique infinite path from the root $(v_i)_{i \in \mathbb{N}}$ so that v_i is the maximal element of V_i for all $i \in \mathbb{N}$.

When vertices k and w lie on the same path to the root and $d(k, r) \leqslant d(w, r)$, we write $k \leqslant w$. Given a vertex $s \in T$, the **tree below** s, denoted T^s, is the collection of t so that $s \leqslant t$ along with the induced graph structure.

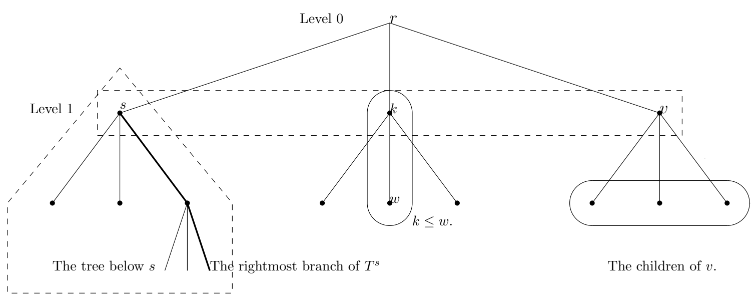

FIGURE 1. Levels, children, and trees below vertices

We call a rooted tree **spherically homogeneous** if all $v, w \in V_n$ the number of children of v is the same as the number of children of w. A spherically homogeneous tree is completely determined by specifying the number of children of the vertices at each level. These data are given by an infinite sequence $\alpha \in \mathbb{N}^{\mathbb{N}}$ so that $\alpha(i) \geqslant 2$ for all $i \in \mathbb{N}$; the condition $\alpha(i) \geqslant 2$ ensures non-triviality. We denote a spherically homogeneous tree by T_α for $\alpha \in \mathbb{N}^{\mathbb{N}}_{\geqslant 2}$. When $\alpha \equiv d$, we write T_d. The group of rooted tree automorphisms, denoted $\mathrm{Aut}(T_\alpha)$, is naturally a profinite group.

Profinite branch groups are certain closed subgroups of $\mathrm{Aut}(T_\alpha)$; our approach to branch groups follows closely Grigorchuk's presentation in [13]. For $G \leqslant \mathrm{Aut}(T_\alpha)$ a closed subgroup and for a vertex $v \in T_\alpha$, the **rigid stabilizer of** v in G is defined to be

$$\mathrm{rist}_G(v) := \{g \in G \mid g.w = w \text{ for all } w \in T_\alpha \setminus T_\alpha^v\}.$$

The rigid stabilizer acts non-trivially only on the subtree T_α^v.

The n**-th rigid level stabilizer** in G is defined to be

$$\mathrm{rist}_G(n) := \langle \mathrm{rist}_G(v) \mid v \in V_n \rangle.$$

It is easy to see that $\mathrm{rist}_G(n) \simeq \prod_{v\in V_n} \mathrm{rist}_G(v)$, and as a consequence, $\mathrm{rist}_G(n)$ is a closed subgroup of G.

For a level n, we denote the pointwise stabilizer in G of V_n by $\mathrm{st}_G(n)$. The subgroup $\mathrm{st}_G(n)$ is called the **n-th level stabilizer** of G. Observe that it can be the case that $\mathrm{rist}_G(n) < \mathrm{st}_G(n)$, even for profinite branch groups.

Definition 2.16. A profinite group G is said to be a **profinite branch group** if there is a tree T_α for some $\alpha \in \mathbb{N}^{\mathbb{N}}_{\geqslant 2}$ so that the following hold:

(i) G is isomorphic to a closed subgroup of $\mathrm{Aut}(T_\alpha)$.
(ii) G acts transitively on each level of T_α.
(iii) For each level n, the index $|G : \mathrm{rist}_G(n)|$ is finite.

We shall always identify a profinite branch group G with the isomorphic closed subgroup of $\mathrm{Aut}(T_\alpha)$.

The rigid level stabilizers form a basis at 1 for the topology on a profinite branch group G. The transitivity of the action on the levels ensures that $\mathrm{rist}_G(v) \simeq \mathrm{rist}_G(w)$ for all $v, w \in V_n$. The transitivity further insures that profinite branch groups are always infinite.

Lemma 2.17. *Suppose that $G \leqslant \mathrm{Aut}(T_\alpha)$ is a profinite branch group and $v \in T_\alpha$. Then each $g \in C_G(\mathrm{rist}_G(v))$ fixes pointwise T^v_α. In particular, the center of $\mathrm{rist}_G(v)$ is trivial for all $v \in T_\alpha$.*

Proof. Fix $v \in T_\alpha$ and suppose for contradiction there is $w \geqslant v$ so that $g.w \neq w$. The subgroup $\mathrm{rist}_G(w)$ is non-trivial since G is infinite, so we may find $y \in \mathrm{rist}_G(w)$ and $u \geqslant w$ with $y.u \neq u$. The element gy then sends u to $g.(y.u)$, but the element yg sends u to $g.u$. Hence, $gy \neq yg$ contradicting that g centralizes $\mathrm{rist}_G(v)$. □

As an immediate corollary, we obtain a description of $\mathrm{rist}_G(v)$.

Lemma 2.18. *Suppose $G \leqslant \mathrm{Aut}(T_\alpha)$ is a profinite branch group. If $v \in T_\alpha$ is at level n, then*

$$\mathrm{rist}_G(v) = \bigcap_{w\in V_n\setminus\{v\}} C_G(\mathrm{rist}_G(w)).$$

3. Commutators and diagonalization in branch groups

3.1. **The commutator trick.** The following lemma is well-known and dates back to the fifties, where it was used by G. Higman [15] to show the simplicity of various permutation groups. It has since become a cornerstone to proofs that various groups are simple.

The **support** of a permutation $\sigma \in \mathfrak{S}(X)$ is the set

$$\mathrm{supp}(\sigma) := \{x \in X : \sigma(x) \neq x\}.$$

Lemma 3.1. *Let X be a set and let $G \leqslant \mathfrak{S}(X)$ be a permutation group with $\tau \in G$. If $\sigma_1, \sigma_2 \in G$ are so that $\tau(\mathrm{supp}(\sigma_1))$ is disjoint from $\mathrm{supp}(\sigma_1) \cup \mathrm{supp}(\sigma_2)$, then the commutator $[\sigma_1, \sigma_2]$ is the product of four conjugates of $\tau^{\pm 1}$ by elements of G.*

Proof. Whenever two permutations have disjoint support, they commute. Moreover,

$$\tau(\mathrm{supp}\,\sigma_1) = \mathrm{supp}(\tau\sigma_1\tau^{-1}),$$

so by our hypothesis, $\tau\sigma_1\tau^{-1}$ commutes with both σ_1 and σ_2. It follows $\tau\sigma_1^{-1}\tau^{-1}$ also commutes with both σ_1 and σ_2.

Setting $\bar{\sigma_1} := [\sigma_1, \tau] = \sigma_1(\tau\sigma_1^{-1}\tau^{-1})$, the fact that $\tau\sigma_1^{-1}\tau^{-1}$ commutes with both σ_1 and σ_2 yields that $[\sigma_1, \sigma_2] = [\bar{\sigma}_1, \sigma_2]$. The permutation $\bar{\sigma}_1 = (\sigma_1\tau\sigma_1^{-1})\tau^{-1}$ is the product of two conjugates of $\tau^{\pm 1}$ by elements of the group G. Hence, $[\bar{\sigma}_1, \sigma_2]$ is the product of four conjugates of $\tau^{\pm 1}$ by elements of G, verifying the lemma. □

We now adapt Grigorchuk's argument [13, Theorem 4] to characterize strongly just infinite profinite branch groups.

Theorem 3.2. *Suppose that $G \leqslant \mathrm{Aut}(T_\alpha)$ is a profinite branch group. Then the following are equivalent:*
(1) G is strongly just infinite;
(2) For all $v \in T_\alpha$, the abstract derived subgroup $D(\mathrm{rist}_G(v))$ is open in $\mathrm{rist}_G(v)$.
(3) For all $n \geqslant 1$, the abstract derived subgroup $D(\mathrm{rist}_G(n))$ is open in G.

Proof. For (1) ⇒ (2), we prove the contrapositive. Suppose for some $v \in T_\alpha$ the abstract commutator $D(\mathrm{rist}_G(v))$ is not open in $\mathrm{rist}_G(v)$; Lemma 2.17 ensures $D(\mathrm{rist}_G(v))$ is also non-trivial. Letting n be the level of v, we have $D(\mathrm{rist}_G(n)) = \prod_{w\in V_n} D(\mathrm{rist}_G(w))$ is a characteristic subgroup of $\mathrm{rist}_G(n)$ which is non-trivial and not open. Since $\mathrm{rist}_G(n)$ is a normal subgroup of G, we deduce that G has a non-trivial normal subgroup which is not open, hence G is not strongly just infinite.

The implication (2) ⇒ (3) is immediate.

For (3) ⇒ (1), let H be a non-trivial normal subgroup of G and let $\tau \in H \setminus \{1\}$. There exists a vertex v such that $\tau(v) \neq v$; let n be its level. Taking two elements σ_1, σ_2 in $\mathrm{rist}_G(v)$, their support is a subset of T_α^v, and since $\tau(T_\alpha^v) = T_\alpha^{\tau.v}$ is disjoint from T_α^v, we apply the commutator trick and deduce that the commutator $[\sigma_1, \sigma_2]$ is the product of four conjugates of $\tau^{\pm 1}$.

The commutator group $D(\mathrm{rist}_G(v))$ is thus a subgroup of H. Since H is normal and G acts spherically transitively on T_α, it follows the open subgroup

$$D(\mathrm{rist}_G(n)) = \prod_{w\in V_n} D(\mathrm{rist}_G(w))$$

is a subgroup of H, hence H is open. □

The next lemma establishes a version of the commutator trick for certain large sets.

Definition 3.3. For $G \leqslant \mathrm{Aut}(T_\alpha)$, we call a subset A of G **full above** the vertex $v \in T_\alpha$ if every element of $\mathrm{rist}_G(v)$ coincides with an element of A restricted to T_α^v.

For a group G with $\mathrm{rist}_G(v) = \{1\}$, any set containing 1 is full above v. In profinite branch groups, however, rigid stabilizers are *necessarily* infinite, so this trivial case never occurs.

The relevance of this definition stems from the following observation.

Lemma 3.4. *Suppose that $G \leqslant \mathrm{Aut}(T_\alpha)$ is a strongly just infinite profinite branch group and that $A \subseteq G$ is full above v. If $A \cap \mathrm{rist}_G(v) \nsubseteq \{1\}$, then there is $w \geqslant v$ so that $D(\mathrm{rist}_G(w)) \leqslant A^{10k}$ with $k := \mathrm{cw}(\mathrm{rist}_G(w))$.*

Proof. Take $x \in A \cap \mathrm{rist}_G(v) \setminus \{1\}$ and find $w \geqslant v$ so that $x.w \neq w$. We now consider $g, h \in \mathrm{rist}_G(w)$. Since A is full above v, there are $\tilde{g}, \tilde{h} \in A$ with the same action on T_α^v as g and h, respectively. The element x is supported on $\mathrm{rist}_G(v)$, so $\tilde{h}x^{-1}\tilde{h}^{-1} = hx^{-1}h^{-1}$. In particular, $[\tilde{h}, x] = [h, x]$. The element $[h, x]$ is again supported on T_α^v, so we have that $[\tilde{g}, [h, x]] = [g, [h, x]]$.

Now $xh^{-1}x^{-1}$ commutes with both h and g since $\mathrm{supp}(xh^{-1}x^{-1}) \subseteq T_\alpha^{x.w}$, so we further have that $[g, h] = [g, [h, x]]$. Therefore,

$$[g, h] = [g, [h, x]] = [\tilde{g}, [h, x]] = [\tilde{g}, [\tilde{h}, x]],$$

and we deduce that $[g, h] \in A^{10}$.

The set A^{10} thus contains every commutator of $\mathrm{rist}_G(w)$. In view of Lemma 2.2, Theorem 3.2 implies that $k := \mathrm{cw}(\mathrm{rist}_G(w))$ is finite, so $D(\mathrm{rist}_G(w)) \leqslant A^{10k}$. □

3.2. **The diagonalization trick.** We now show we can find full sets under certain mild conditions. This lemma was already present in the work of J. Dixon, P. Neumann, and S. Thomas on the small index property for permutation groups [10].

Lemma 3.5. *Let $G \leqslant \mathrm{Aut}(T_\alpha)$ be a closed subgroup and let $(A_n)_{n\in\mathbb{N}}$ be a countable family of subsets of G such that $G = \bigcup_{n\in\mathbb{N}} A_n$. Then for any vertex $w \in T_\alpha$, there exists a vertex $v \geqslant w$ and $n \in \mathbb{N}$ so that A_n is full above v.*

Proof. Let $(w_n)_{n\in\mathbb{N}}$ enumerate the vertices of the rightmost branch of T_α^w. For each n, let v_n be a child of w_n different from w_{n+1}. Let us prove by contradiction there is some $n \in \mathbb{N}$ such that A_n is full above v_n; this implies the lemma.

If not, for each $n \in \mathbb{N}$ there is a tree automorphism $g_n \in G$ supported on $T_\alpha^{v_n}$ such that its restriction to $T_\alpha^{v_n}$ does not extend to an element of A_n. The products $\prod_{i=0}^n g_i$ converge to a limit g, and g extends g_i for all i. Since G is closed in $\mathrm{Aut}(T_\alpha)$, we have that $g \in G$, but for all $n \in \mathbb{N}$, the restriction of g to $T_\alpha^{v_n}$ does not extend to an element of A_n. We conclude $g \notin A_n$ for any n, contradicting the assertion $\bigcup_{n\in\mathbb{N}} A_n = G$. □

Sets which are σ-syndetic are natural sources of sets full above a vertex.

Lemma 3.6. *Let $G \leqslant \mathrm{Aut}(T_\alpha)$ be a closed subgroup and let A be a symmetric σ-syndetic subset of G. Then for any $w \in T_\alpha$, there exists $v \geqslant w$ such that A^2 is full above v.*

Proof. We many find a sequence $(g_n)_{n\in\mathbb{N}}$ of elements of G such that $\bigcup_{n\in\mathbb{N}} g_n A = G$. Lemma 3.5 provides $n \in \mathbb{N}$ and a vertex $v \geqslant w$ such that $g_n A$ is full above v. Since $1 \in \mathrm{rist}_G(v)$, there is $a \in g_n A$ so that $a \restriction_{T_\alpha^v}$ is the identity. It now follows that $Ag_n^{-1}g_n A = A^2$ is full above v. □

4. Strongly just infinite profinite branch groups

4.1. **Combinatorial boundedness conditions.** Our characterization of profinite branch groups with boundedness conditions requires a general observation.

Lemma 4.1. *If G is a profinite group with uncountable cofinality, then the derived group $D(O)$ is open for every open normal $O \trianglelefteq G$.*

Proof. We prove the contrapositive. Suppose that $O \trianglelefteq G$ is open but that $D(O)$ is not. The subgroup $D(O)$ is then meagre in O, hence the quotient group $O/D(O)$ is uncountable. Appealing to Lemma 2.1, we may find $D(O) \leqslant A \trianglelefteq O$ so that O/A is countable and infinitely generated.

Taking $g_0, \dots, g_n$ left coset representatives for O in G, the subgroup $\tilde{A} := \bigcap_{i=0}^n g_i A g_i^{-1}$ is normal in G, and $G/\tilde{A}$ is countable. The group $G/\tilde{A}$ must be infinitely generated since $O/\tilde{A}$ is a finite index subgroup. Since infinitely generated countable groups plainly have countable cofinality, we conclude that G has countable cofinality. $\square$

We note one further boundedness condition, which we will obtain for free from results in the literature: A group G satisfies **property (FA)** if whenever G acts on a tree without edge inversions, then it fixes a vertex.

Theorem 4.2. *Let $G \leqslant \mathrm{Aut}(T_\alpha)$ be a profinite branch group. Then the following are equivalent:*

(1) G is strongly just infinite.
(2) G has the Bergman property.
(3) G has uncountable cofinality.
(4) G has property (FA).
(5) G is Cayley bounded.

Proof. For (1) $\Rightarrow$ (2), let $(A_n)_{n\geqslant 0}$ be a Bergman sequence. Via Lemma 3.5, there is $n_0 \geqslant 0$ and $v \in T_\alpha$ so that A_{n_0} is full above v. Appealing to Lemma 3.4, we may find $w \geqslant v$ on some level l so that $D(\mathrm{rist}_G(w)) \subseteq A_{n_0}^{10k}$ for $k := \mathrm{cw}(\mathrm{rist}_G(w))$.

Since G acts transitively on the levels, there are $g_1, \dots, g_n \in G$ so that

$$\{g_1.w, \dots, g_n.w\} = V_l,$$

and that $(A_n)_{n\geqslant 0}$ is a Bergman sequence ensures there is $n_1 \geqslant n_0$ for which $g_1, \dots, g_n \in A_{n_1}$. It now follows

$$\prod_{i=1}^{n} g_i A_{n_0}^{10k} g_i^{-1} \subseteq A_{n_1}^{10nk+2n}.$$

We infer $D(\mathrm{rist}_G(l)) \leqslant A_{n_1}^{10nk+2n}$.

The set $A_{n_1}^{10nk+2n}$ contains the open subgroup $D(\mathrm{rist}_G(l))$ of G, which has finite index. Letting $h_1, \dots, h_m$ be left coset representatives for this subgroup, there is $n_2 \geqslant n_1$ so that $h_1, \dots, h_m \in A_{n_2}$. We deduce that $A_{n_2}^{10nk+2n+1} = G$, hence G has the Bergman property.

The implications (2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (5) are given by Proposition 2.7. H. Bass' work [3] establishes the equivalence (3) $\Leftrightarrow$ (4). It thus remains to show (3) $\Rightarrow$ (1) and (5) $\Rightarrow$ (1). The former is an easy exercise: The contrapositive follows from Lemma 4.1 and Theorem 3.2.

To show (5) $\Rightarrow$ (1), we prove by contrapositive. Suppose G is not strongly just infinite. In view of Theorem 3.2, there is a level k so that $D(\mathrm{rist}_G(k))$ is not open in G. The quotient $G/D(\mathrm{rist}_G(k))$ is then an infinite abelian-by-finite group. Applying Corollary 2.9, $G/D(\mathrm{rist}_G(k))$ is not Cayley bounded, whereby G is not Cayley bounded. $\square$

4.2. **Automatic continuity properties.** We now consider automatic continuity properties.

Theorem 4.3. *Suppose G is a profinite branch group. Then the following are equivalent:*

(1) G is strongly just infinite.
(2) G has the weak Steinhaus property.
(3) G has the countable index property.

(4) G has the normal countable index property.

Proof. The implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are immediate. The contrapositive of (4) $\Rightarrow$ (1) follows from Lemma 2.1 and Theorem 3.2.

For (1) $\Rightarrow$ (2), suppose G is strongly just infinite and let $A \subseteq G$ be a σ-syndetic symmetric set. The set $\overline{A}$ is then also σ-syndetic. Applying the Baire category theorem, some left translate of $\overline{A}$ has non-empty interior, so A is dense in some open set V. Since $V^{-1}V$ is a neighborhood of the identity and A is symmetric, we deduce that A^2 is dense in a neighborhood of the identity. There is thus a level $n \in \mathbb{N}$ such that A^2 is dense in the pointwise stabilizer of V_n, denoted $\mathrm{st}_G(n)$. Fixing a symmetric set of right coset representatives $g_0, \dots, g_l$ for $\mathrm{st}_G(n)$ in G, put $B := \bigcap_{i=0}^{l} g_i^{-1} A^{2^l} g_i$. Via Lemma 2.15, B is again a symmetric σ-syndetic set.

For $v \in V_n$, we apply Lemma 3.6 to find $w \geqslant v$ so that B^2 is full above w. By Lemma 2.14, the set $B^4 \cap \mathrm{rist}_G(w)$ is σ-syndetic in $\mathrm{rist}_G(w)$, and since $\mathrm{rist}_G(w)$ is uncountable, we have $B^4 \cap \mathrm{rist}_G(w) \not\subseteq \{1\}$. Lemma 3.4 now implies we may find $s \geqslant w$ so that $D(\mathrm{rist}_G(s)) \subseteq (B^4)^{10k} = B^{40k}$.

Let m be the level of s. The group G acts transitively on V_m, so for all $t \in V_m$, there is $z \in G$ so that $z.s = t$. We may write $z = xg_i$ for some $x \in \mathrm{st}_G(n)$ and g_i one of the previously fixed right coset representatives. Since A^2 is dense in $\mathrm{st}_G(n)$, there is $h \in A^2$ so that $hg_i.s = xg_i.s = t$. We now have that $hg_iD(\mathrm{rist}_G(s))g_i^{-1}h^{-1} = D(\mathrm{rist}_G(t))$. Moreover,

$$hg_iD(\mathrm{rist}_G(s))g_i^{-1}h^{-1} \subseteq hg_iB^{40k}g_i^{-1}h^{-1} \subseteq A^{40k2^l+4}.$$

We conclude that A^{40k2^l+4} contains $D(\mathrm{rist}_G(t))$ for all $t \in V_m$. The open subgroup $D(\mathrm{rist}_G(m))$ is thus contained in $A^{(40k2^l+4)|V_m|}$, whereby G enjoys the weak Steinhaus property. $\square$

We pause for a moment to observe that the branch assumption in Theorem 4.3 is necessary.

Proposition 4.4. *For $n \geqslant 2$, the profinite group $PSL_n(\mathbb{Z}_p)$ fails the countable index property but is strongly just infinite.*

Proof. The profinite group $PSL_n(\mathbb{Z}_p)$ is strongly just infinite via the main theorem of the appendix of [23].

On the other hand, via [16, Theorem 1], there is an injective homomorphism $\xi : GL_n(\mathbb{C}) \to \mathfrak{S}_\infty$. Considered as abstract fields, the algebraic closure of $\mathbb{Q}_p$ is isomorphic to $\mathbb{C}$, hence we may see $SL_n(\mathbb{Q}_p) \leqslant GL_n(\mathbb{C})$. For each $\alpha \in \mathrm{Aut}(\mathbb{C})$, the map ϕ_α given by applying α to the entries of a matrix is an automorphism of $GL_n(\mathbb{C})$. We therefore obtain maps $\xi \circ \phi_\alpha : SL_n(\mathbb{Q}_p) \to \mathfrak{S}_\infty$ for each $\alpha \in \mathrm{Aut}(\mathbb{C})$.

For α and β in $\mathrm{Aut}(\mathbb{C})$, the maps $\xi \circ \phi_\alpha$ and $\xi \circ \phi_\beta$ are equal if and only if

$$\phi_\alpha \restriction_{SL_n(\mathbb{Q}_p)} = \phi_\beta \restriction_{SL_n(\mathbb{Q}_p)} .$$

The maps ϕ_α and ϕ_β agree on $SL_n(\mathbb{Q}_p)$ if and only if $\phi_{\alpha^{-1}\circ\beta}$ is the identity on $SL_n(\mathbb{Q}_p)$. The group $SL_n(\mathbb{Q}_p)$ contains elementary matrices $E_{i,j}(a)$ for $i \neq j$ and $a \in \mathbb{Q}_p$ where $E_{i,j}(a)$ has ones on the diagonal, a in the (i,j)-entry, and zeros elsewhere. We conclude $\alpha^{-1} \circ \beta(a) = a$, so $\alpha^{-1} \circ \beta$ fixes $\mathbb{Q}_p$ pointwise. Therefore, $\xi \circ \phi_\alpha$ and $\xi \circ \phi_\beta$ are equal if and only if $\alpha^{-1} \circ \beta \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}_p)$.

It is well-known $|\mathrm{Aut}(\mathbb{C})| = 2^{\mathfrak{c}}$. On the other hand, $\mathrm{Aut}(\mathbb{C}/\mathbb{Q}_p)$ is a second countable profinite group and thus has size $\mathfrak{c}$. We conclude there are $2^{\mathfrak{c}}$ many distinct left cosets of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q}_p)$ in $\mathrm{Aut}(\mathbb{C})$. In view of the previous paragraph, there must be $2^{\mathfrak{c}}$ many distinct homomorphisms $\xi \circ \phi_\alpha : SL_n(\mathbb{Q}_p) \to \mathfrak{S}_\infty$. Since there can be at most continuum many continuous homomorphisms, we conclude that $SL_n(\mathbb{Q}_p)$ fails the countable index property, and as $SL_n(\mathbb{Z}_p)$ is an open subgroup of $SL_n(\mathbb{Q}_p)$, the group $SL_n(\mathbb{Z}_p)$ also fails the countable index property. It now follows $PSL_n(\mathbb{Z}_p)$ fails the countable index property. □

4.3. **Commensurated subgroups.** We finally give a characterization of strongly just infinite profinite branch groups in terms of commensurated subgroups.

For a group G, subgroups H and K are **commensurate**, denoted $H \sim_c K$, if $|H : H \cap K|$ and $|K : H \cap K|$ are both finite. We say $H \leqslant G$ is **commensurated** if $H \sim_c gHg^{-1}$ for all $g \in G$.

We shall need an important feature of the commensuration relation.

Theorem 4.5 (Bergman–Lenstra, [4, Theorem 6])**.** *Let G be a group with subgroups H and K. Then the following are equivalent:*

(1) $\sup_{k \in K} |H : H \cap kHk^{-1}| < \infty$.

(2) There is N normalized by K so that $N \sim_c H$.

Via Theorem 4.5, groups with the Bergman property have strong restrictions on commensurated subgroups:

Proposition 4.6. *If a Polish group G has the Bergman property, then every commensurated subgroup is commensurate to a normal subgroup.*

Proof. Suppose $C \leqslant G$ is commensurated. For each $n \geqslant 1$, set

$$\Omega_n := \{g \in G \mid |C : C \cap gCg^{-1}| \leqslant n \text{ and } |gCg^{-1} : C \cap gCg^{-1}| \leqslant n\}$$

The sets Ω_n are symmetric, and since C is commensurated, $G = \bigcup_{n \geqslant 1} \Omega_n$. For all $n, m \geqslant 1$, an easy computation further verifies $\Omega_n \Omega_m \subseteq \Omega_{nm}$. Since G has the Bergman property, there is n, k so that $\Omega_n^k = G$, whereby $\Omega_{n^k} = G$. Appealing to Theorem 4.5, there is $L \trianglelefteq G$ which is commensurate with C, verifying the proposition. □

Theorem 4.7. *Suppose $G \leqslant \mathrm{Aut}(T_\alpha)$ is a profinite branch group. Then G is strongly just infinite if and only if every commensurated subgroup of G is either finite or open.*

Proof. Suppose that G is strongly just infinite and that $C \leqslant G$ is a commensurated subgroup. Since G is strongly just infinite, G has the Bergman property via Theorem 4.2. Proposition 4.6 then supplies $D \trianglelefteq G$ so that $C \sim_c D$, and since G is strongly just infinite, D is either open or trivial. If D is trivial, then C is finite. If D is open, then C has finite index in G. That G is strongly just infinite implies the normal core of C in G is open, whereby C is open.

Conversely, suppose every commensurated subgroup of G is either finite or open. Fix a level n of T_α. The subgroup $D(\mathrm{rist}_G(n))$ is normal in G, so a fortiori, it is commensurated. Suppose toward a contradiction that $D(\mathrm{rist}_G(n))$ is finite. The subgroups $\mathrm{st}_G(k)$ form a basis at 1 for G, so we may find $m \geqslant n$ for which $\mathrm{st}_G(m) \cap D(\mathrm{rist}_G(n)) = \{1\}$ and $\mathrm{st}_G(m) \leqslant \mathrm{rist}_G(n)$.

The group $\mathrm{st}_G(m)$ thus injects into $\mathrm{rist}_G(n)/D(\mathrm{rist}_G(n))$ and therefore is abelian. For any $v \in V_m$, the rigid stabilizer $\mathrm{rist}_G(v)$ then has a non-trivial center, but this is absurd in view of Lemma 2.17. We conclude $D(\mathrm{rist}_G(n))$ is open for all levels n. Theorem 3.2 now implies G is strongly just infinite. □

Bringing together Theorems 4.2, 4.3, and 4.7, we have established the claimed equivalences.

Theorem 4.8. *Let $G \leqslant \mathrm{Aut}(T_\alpha)$ be a profinite branch group. Then the following are equivalent:*
(1) G is strongly just infinite.
(2) Every commensurated subgroup of G is either finite or open.
(3) G has the normal countable index property.
(4) G has the countable index property.
(5) G has the weak Steinhaus property.
(6) G has uncountable cofinality.
(7) G is Cayley bounded.
(8) G has property (FA).
(9) G has the Bergman property.

5. Invariant and locally compact automatic continuity properties

We now consider two further automatic continuity properties in the setting of profinite branch groups.

5.1. **Preliminaries.** A Polish group is called a **small invariant neighborhood** group, abbreviated SIN group, if it admits a basis of conjugation invariant neighborhoods at 1. By integrating a compatible left-invariant metric on a compact metrizable group against the Haar measure, we obtain a two-sided invariant metric, so every compact metrizable group is a SIN group.

Definition 5.1. Let G be a Polish group.
(i) The group G has the **invariant automatic continuity property** if every homomorphism $\psi : G \to H$ with H a SIN Polish group is continuous.
(ii) The group G has the **locally compact automatic continuity property** if every homomorphism $\psi : G \to H$ with H a locally compact Polish group is continuous.

The invariant automatic continuity property has an associated Steinhaus property. A subset $A \subseteq G$ is called **invariant** if $gAg^{-1} = A$ for all $g \in G$.

Definition 5.2. A Polish group G has the **invariant Steinhaus property** if there is $N > 0$ such that for any symmetric invariant σ-syndetic set A, the set A^N contains a neighborhood of 1.

Proposition 5.3 (Dowerk–Thom, [11])**.** *If a Polish group G has the invariant Steinhaus property, then G has the invariant automatic continuity property.*

Remark 5.4. In [11], Ph. Dowerk and A. Thom show finite-dimensional unitary groups satisfy the invariant automatic continuity property, but these groups fail the automatic continuity property by a result of Kallman [16].

For a finite permutation group (F, Ω), a **derangement** of Ω is a permutation $f \in F$ so that f fixes no point in Ω. It is an easy, amusing exercise to see that every finite transitive permutation group (F, Ω) with $|\Omega| > 1$ contains a derangement; this observation is originally due to C. Jordan. Given a derangement $f \in F$, we may write f as a product of disjoint cycles $f = c_1 \dots c_n$, and each c_i has length at least 2.

Definition 5.5. We say a profinite branch group $G \leqslant \mathrm{Aut}(T_\alpha)$ **locally has derangements** if for each $n \geqslant 0$ there is $N \geqslant n$ for which $\mathrm{st}_G(n)$ contains a derangement of V_N.

Groups built by iterated wreath products of transitive permutation groups are easy examples of branch groups which locally have derangements. Indeed, suppose $G \leqslant \mathrm{Aut}(T_\alpha)$ is such an iterated wreath product. For each $v \in V_n$, the rigid stabilizer $\mathrm{rist}_G(v)$ acts transitively on the children of v in V_{n+1}, so there is $x_v \in \mathrm{rist}_G(v)$ a derangement of the children of v in V_{n+1}. The element $\prod_{v\in V_n} x_v$ is then an element of $\mathrm{rist}_G(n)$ which is a derangement of V_{n+1}. The reader is encouraged to look ahead to Section 7 to see examples of such constructions.

We remark that we do not know of a profinite branch group which fails to locally have derangements. As this seems an independently interesting question, we set it out explicitly:

Question 5.6. *Does every (profinite) branch group locally have derangements?*

Lastly, let us isolate a class of profinite branch groups with well-behaved commutator widths of rigid stabilizers.

Definition 5.7. A profinite branch group $G \in \mathrm{Aut}(T_\alpha)$ is said to have **uniform commutator widths** if $\sup\{\mathrm{cw}(\mathrm{rist}_G(v)) \mid v \in T_\alpha\} = c < \infty$. The value c is called a **uniform bound** for the commutator widths.

Examples of profinite branch groups with uniform commutator widths are also presented in Section 7.

5.2. **Automatic continuity results.** Let us begin with a general, elementary lemma.

Lemma 5.8. *Let G be a Polish group. If G has the invariant automatic continuity property and the Bergman property, then G has the locally compact automatic continuity property.*

Proof. Let H be a locally compact Polish group and $\varphi : G \to H$ a homomorphism. Since H is locally compact and Polish, it is σ-compact, so we may write $H = \bigcup_{n\in\mathbb{N}} K_n$ where $(K_n)_{n\in\mathbb{N}}$ is an increasing sequence of compact subsets. We may assume that $1 \in K_0$ and up to replacing K_n by $K_n \cup K_n^{-1}$ we may also assume that each K_n is symmetric.

The sequence $(\varphi^{-1}(K_n))_{n\in\mathbb{N}}$ is a Bergman sequence in G, so there exists $k, n \in \mathbb{N}$ such that $\varphi^{-1}(K_n)^k = G$. We deduce that $\varphi(G) \subseteq K_n^k$. Thus, $\varphi(G)$ has compact closure, and since compact groups are SIN groups, the conclusion follows from the invariant automatic continuity property. □

Proposition 5.9. *Suppose G is a profinite branch group which locally has derangements and has uniform commutator widths with uniform bound c. If $A \subseteq G$ is an invariant symmetric σ-syndetic set, then $D(\mathrm{rist}_G(k)) \leqslant A^{24c}$ for some level k.*

Proof. Since A is σ-syndetic, the Baire category theorem implies that A is dense in some open set V. The set A^2 is then dense in the neighborhood of the identity $V^{-1}V$. Let n be so that $\mathrm{st}_G(n)$ is contained in $\overline{A^2}$.

As G locally has derangements, there is $k \geqslant n$ and $y \in \mathrm{st}_G(n)$ so that y is a derangement of V_k. The set A^2 is dense in $\mathrm{st}_G(n)$, whereby we may find $z \in A^2$ so that $z.w = y.w$ for all $w \in V_k$. Hence, z acts as a derangement on V_k.

Let the action of z on V_k be given by the product of disjoint cycles $c_1 \dots c_m$. This action is a derangement, so each cycle has length at least 2. A cycle c may be written as a tuple $(w_{i_0}, \dots, w_{i_{l-1}})$ of vertices from V_k so that $c : w_{i_j} \to w_{i_{j+1 \mod l}}$. We may thus choose every other vertex appearing in c; that is to say, we take $w_{i_1}, w_{i_3}, \dots$. Let Z list every other vertex from each of the $c_1, \dots, c_m$.

Consider the subgroup $H := \prod_{v \in Z} \mathrm{rist}_G(v)$. For each $h \in H$, we see that $(z.\,\mathrm{supp}(h)) \cap \mathrm{supp}(h) = \emptyset$. Lemma 3.1 therefore implies that every commutator $[g,t]$ with $g,t \in H$ is a product of four conjugates of $z^{\pm 1}$. Recalling A is conjugation invariant, we conclude that $[H,H] \subseteq A^8$. Since the derived subgroup of H is $\prod_{v \in Z} D(\mathrm{rist}_G(v))$, the group H has commutator width at most c, hence $D(H) \leqslant A^{8c}$.

Since z is a derangement, it follows that $z^{-1}Zz \cup Z \cup zZz^{-1} = V_k$, and as A is conjugation invariant, we infer that

$$\prod_{w \in zZz^{-1}} D(\mathrm{rist}_G(w)) \cup \prod_{w \in z^{-1}Zz} D(\mathrm{rist}_G(w)) \subseteq A^{8c}.$$

Hence, $D(\mathrm{rist}_G(k)) \leqslant A^{24c}$, verifying the proposition. □

Theorem 5.10. *If G is a strongly just infinite profinite branch group which locally has derangements and has uniform commutator widths, then G has the invariant automatic continuity property.*

Proof. In view of Proposition 5.3, it suffices to show that G satisfies the invariant Steinhaus property.

Let $c > 0$ be a uniform bound on the commutator widths and suppose $A \subseteq G$ is an invariant σ-syndetic subset of G. Proposition 5.3 ensures A^{24c} contains $D(\mathrm{rist}_G(n))$ for some level n. The group G is strongly just infinite, so $D(\mathrm{rist}_G(n))$ is open via Theorem 3.2. The subset A^{24c} therefore contains a neighborhood of 1, whereby G has the invariant Steinhaus property with constant $24c$. □

Corollary 5.11. *If G is a strongly just infinite profinite branch group which locally has derangements and has uniform commutator widths, then G has the locally compact automatic continuity property.*

Proof. By Lemma 5.8 we need only to check that G has the Bergman property and the invariant automatic continuity property, and these are given by Theorems 4.2 and 5.10. □

Neither the condition that G is strongly just infinite nor the condition that G has uniform commutator widths are implied by the other hypotheses. In a private communication, Nikolov explained to us an example of a profinite branch group which is strongly just infinite and locally has derangements, but it fails to have uniform commutator widths.

On the other hand, letting $(p_i)_{i \in \mathbb{N}}$ be a sequence of distinct primes, the iterated wreath product $W(C_{p_i}, [p_i])$, as defined in Section 7, is topologically two generated via [5, Corollary 3.2] and the discussion thereafter. All rigid stabilizers are also topologically two generated, so work of Nikolov and Segal [21, Theorem 1.2] implies there is a uniform bound on the commutator widths. Additionally, $W(C_{p_i}, [p_i])$ locally has derangements as it is built via iterated wreath products. However, it surjects onto an infinite profinite abelian group, so it is not strongly just infinite. Uniform commutator widths and local derangements therefore do not imply strongly just infinite.

6. Rigidity of the group topology

We now consider the group topologies a profinite branch group admits.

Theorem 6.1. *The profinite topology of a profinite branch group is its unique Polish group topology as well as its unique compact Hausdorff group topology.*

Proof. Suppose $G \leqslant \mathrm{Aut}(T_\alpha)$ is a profinite branch group and suppose $\psi : G \to H$ is a bijective homomorphism with H a topological group. For each $v \in T_\alpha$ with $v \in V_n$, Corollary 2.18 gives that

$$\psi(\mathrm{rist}_G(v)) = \bigcap_{w \in V_n \setminus \{v\}} \psi(C_G(\mathrm{rist}_G(w))) = \bigcap_{w \in V_n \setminus \{v\}} C_H(\psi(\mathrm{rist}_G(w))).$$

As centralizers are always closed, we conclude that $\psi(\mathrm{rist}_G(v))$ is closed in H.

If H is a Polish group, we deduce that the subgroup $\psi(\mathrm{rist}_G(n))$ is analytic, as it is a finite product of closed sets, so it is Baire measurable for all levels n. The subgroup $\psi(\mathrm{rist}_G(n))$ is also finite index in H, so via the Baire category theorem, $\psi(\mathrm{rist}_G(n))$ is indeed open. We deduce that the map $\psi^{-1} : H \to G$ is continuous, whereby ψ is continuous since both G and H are Polish. It now follows that G has a unique Polish group topology.

If H is a compact group, for every $n \in \mathbb{N}$ the subgroup $\psi(\mathrm{rist}_G(n))$ is compact as the product of finitely many compact sets, so it is closed. Since $\psi(\mathrm{rist}_G(n))$ has finite index, it must be open. We conclude that the map $\psi^{-1} : H \to G$ is continuous, so by compactness, φ is a homeomorphism. Hence, G has a unique compact group topology. $\square$

Under the additional assumption of being strongly just infinite, we can upgrade our rigidity results.

Theorem 6.2. *A strongly just infinite profinite branch group admits exactly two locally compact Hausdorff group topologies: the discrete topology and the profinite topology of a profinite branch group.*

Proof. Let $G \leqslant \mathrm{Aut}(T_\alpha)$ be a strongly just infinite profinite branch group and suppose $\psi : G \to H$ is a bijective homomorphism with H a locally compact group. Consider first the connected component $H^\circ \trianglelefteq H$. Since G is strongly just infinite $\psi^{-1}(H^\circ)$ is either trivial or open with finite index.

Let us eliminate the latter case first. In this case, H is almost connected, and since connected locally compact groups are compactly generated, H is compactly generated. Theorem 4.2 ensures H Cayley bounded, hence compact generation implies H is compact. This is absurd since G has a unique compact group topology by Theorem 6.1.

It is therefore the case that H° is trivial, so H is a totally disconnected locally compact group. Assume that H is non discrete. By van Dantzig's theorem, H admits a basis at 1 of infinite compact open subgroups. Let U be such a subgroup. Since compact open subgroups are necessarily commensurated, $\psi^{-1}(U)$ is a commensurated subgroup of G, whereby Theorem 4.7 implies $\psi^{-1}(U)$ is open in G. It now follows the map ψ is continuous, hence $G \simeq H$ as topological groups. $\square$

7. Examples

We write (G, X) for a permutation group, where G is a group acting on the set X. Let (A, X) and (B, Y) be finite permutation groups. We may form the group $B \wr (A, X) := B^X \rtimes A$ where $A \curvearrowright B^X$ by permuting the domain. The group $B^X \rtimes A$ is a permutation group via the following canonical action on $X \times Y$:

$$(f, a).(x, y) := (a.x, f(a.x).y).$$

The **wreath product** of (B, Y) with (A, X), denoted $(B, Y) \wr (A, X)$, is the permutation group $(B \wr (A, X), X \times Y)$. Wreath products defined in this way are associative.

For an infinite sequence $((A_i, X_i))_{i\in\mathbb{N}}$ of finite permutation groups, the set of finite wreath products $(A_n, X_i) \wr \cdots \wr (A_0, X_0)$ forms an inverse system via the obvious quotient maps

$$(A_{n+1}, X_{n+1}) \wr \cdots \wr (A_0, X_0) \to (A_n, X_n) \wr \cdots \wr (A_0, X_0).$$

We define

$$W((A_i, X_i)_{i\in\mathbb{N}}) := \varprojlim_{n\in\mathbb{N}} \left((A_n, X_n) \wr \cdots \wr (A_0, X_0)\right).$$

The action of the finite wreath products $(A_n, X_n) \wr \cdots \wr (A_0, X_0)$ on the product $X_0 \times \cdots \times X_n$ induces an action of the group $W((A_i, X_i)_{i\in\mathbb{N}})$ on the tree T_α where $\alpha(i) := |X_i|$. When the permutation groups (A_n, X_n) are transitive and non-trivial for all n, the action of the group $W((A_i, X_i)_{i\in\mathbb{N}})$ on T_α witnesses that $W((A_i, X_i)_{i\in\mathbb{N}})$ is a profinite branch group. The rigid stabilizers are also easy to understand: If $v \in T_\alpha$ lies on level n, then

$$\mathrm{rist}_G(v) \simeq W((A_i, X_i)_{i>n}).$$

7.1. Iterated wreath products.

Many of the groups $W((A_i, X_i)_{i\in\mathbb{N}})$ are strongly just infinite, locally have derangements, and have uniform commutator widths. The latter requires a theorem due to Nikolov.

Theorem 7.1 (Nikolov, [20, Corollary 1.4])**.** *Suppose $((A_i, X_i))_{i\in\mathbb{N}}$ is a sequence of finite perfect permutation groups. If $\sup\{\mathrm{cw}(A_i) \mid i \in \mathbb{N}\} = N < \infty$, then $W((A_i, X_i)_{i\in\mathbb{N}})$ is perfect as an abstract group, and $\mathrm{cw}(W((A_i, X_i)_{i\in\mathbb{N}})) \leqslant N$.*

Via Theorem 7.1, we isolate a rich family of profinite branch groups to which our results apply.

Proposition 7.2. *Suppose $((A_i, X_i))_{i\in\mathbb{N}}$ is a sequence of finite non-trivial perfect transitive permutation groups. If $\sup\{\mathrm{cw}(A_i) \mid i \in \mathbb{N}\} < \infty$, then $W((A_i, X_i)_{i\in\mathbb{N}})$ is strongly just infinite, locally has derangements, and has uniform commutator widths.*

Proof. Suppose $\sup\{\mathrm{cw}(A_i) \mid i \in \mathbb{N}\} = N$, set $G := W((A_i, X_i)_{i\in\mathbb{N}})$, and let T_α be the rooted tree on which G acts, as discussed above. For each $v \in T_\alpha$ with $v \in V_n$, we have that $\mathrm{rist}_G(v) \simeq W((A_i, X_i)_{i>n})$, so Theorem 7.1 implies $\mathrm{rist}_G(v)$ is abstractly perfect and $\mathrm{cw}(\mathrm{rist}_G(v)) \leqslant N$. The group G thus has uniform commutator widths, and Theorem 3.2 ensures G is strongly just infinite. That G locally has derangements follows since $\mathrm{rist}_G(v) \simeq W((A_i, X_i)_{i>n})$. □

Remark 7.3. We note a weak converse: if infinitely many of the A_i are not perfect, then $W((A_i, X_i)_{i\in\mathbb{N}})$ fails to be just infinite. Indeed, if $((A_i, X_i))_{i\in\mathbb{N}}$ is any sequence of finite transitive permutation groups, the group $W((A_i, X_i)_{i\in\mathbb{N}})$ surjects continuously onto the abelian group $\prod_{i\in\mathbb{N}} A_i/D(A_i)$.

An easy example of this situation is provided by the full automorphism group of T_α. The group $\mathrm{Aut}(T_\alpha)$ may be written as $W((\mathfrak{S}(\alpha_i), [\alpha_i])_{i\in\mathbb{N}})$ and thus surjects onto $\prod_{i\in\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$.

Corollary 7.4. *Suppose $((A_i, X_i))_{i\in\mathbb{N}}$ is any sequence of non-abelian finite simple transitive permutation groups. Then, $W((A_i, X_i)_{i\in\mathbb{N}})$ is strongly just infinite, locally has derangements, and has uniform commutator widths.*

Proof. Non-abelian finite simple groups have commutator width one by the celebrated solution to the Ore conjecture [18]. The desired result then follows from Proposition 7.2. □

7.2. **The profinite completion of the first Grigorchuk group.** Our discussion of the first Grigorchuk group follows V. Nekrashevych's work [19]. The first Grigorchuk group, denoted G_1, is a four generated group that acts on the two regular rooted tree T_2. Identifying the vertices of T_2 with finite binary sequences in the obvious way, the generators a, b, c, d of G_1 are defined recursively as follows:

$$\begin{array}{cc} a.(0^\frown\alpha) := 1^\frown\alpha & a.(1^\frown\alpha) := 0^\frown\alpha \\ b.(0^\frown\alpha) := 0^\frown a.\alpha & b.(1^\frown\alpha) := 1^\frown c.\alpha \\ c.(0^\frown\alpha) := 0^\frown a.\alpha & c.(1^\frown\alpha) := 1^\frown d.\alpha \\ d.(0^\frown\alpha) := 0^\frown\alpha & d.(1^\frown\alpha) := 1^\frown b.\alpha \end{array}$$

Letting σ be the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$, these generators can be given by so-called wreath-recursion as follows: $a := (1, 1)\sigma$, $b := (a, c)$, $c := (a, d)$, and $d := (1, b)$. These are forms of the generators upon realizing $\mathrm{Aut}(T_2)$ as $\mathrm{Aut}(T_2)^2 \rtimes \mathbb{Z}/2\mathbb{Z}$.

Fact 7.5 (Grigorchuk)**.** *The first Grigorchuk group G_1 enjoys the following properties:*

(1) It is an infinite two group. (See [19, Theorem 1.6.1].)
(2) It is a branch group. (See [13, Corollary of Proposition 8].)
(3) It is just infinite. (See [13, Corollary of Proposition 9].)
(4) For all $v \in T_2$, $\mathrm{rist}_{G_1}(v)$ is at most four generated. (See [2, Theorem 4.3])

Lemma 7.6. *For every $n \geqslant 0$, the rigid level stabilizer $\mathrm{rist}_{G_1}(n)$ contains a derangement of V_{n+2}.*

Proof. Via [13, Proposition 8], the subgroup $K := \langle x, (x, 1), (1, x)\rangle \leqslant G_1$ with $x := (ca, ac)$ is self-replicating. I.e. for each level n, the group K contains K^{V_n} where v-th coordinate acts on the tree below the vertex v. The element x obviously acts on as a derangement on the level two vertices of T_2.

For an arbitrary level $n \geqslant 0$, that K is self-replicating ensures $K \hookrightarrow \mathrm{rist}_K(v) \leqslant \mathrm{rist}_{G_1}(v)$ for all $v \in V_n$. We may thus find $x_v \in \mathrm{rist}_K(v)$ so that x_v is a derangement of level 2 of the tree T_2^v. The element $z := \prod_{v\in V_m} x_v \in \mathrm{rist}_{G_1}(n)$ is then a derangement of V_{n+2}, verifying the lemma. □

We are now ready to prove the desired result.

Theorem 7.7. *The profinite completion of the first Grigorchuk group is a profinite branch group which is strongly just infinite, locally has derangements, and has uniform commutator widths.*

Proof. Via [13, Theorem 9, Propostion 10], the profinite completion $\widehat{G}_1$ is isomorphic to the topological closure $\overline{G_1}$ in $\mathrm{Aut}(T_2)$. For each level n, the rigid level stabilizer $\mathrm{rist}_{G_1}(n)$ additionally contains $\mathrm{st}_{G_1}(m)$ for some $m \geqslant n$ via [13, Proposition 10]. The subgroup $\overline{\mathrm{rist}_{G_1}(n)}$ is then open in $\overline{G_1}$, so $\overline{G_1}$ is a profinite branch group. It follows further that $\mathrm{rist}_{\overline{G_1}}(v) = \overline{\mathrm{rist}_{G_1}(v)}$.

Lemma 7.6 now implies that $\overline{G_1}$ locally has derangements. Fact 7.5 ensures the rigid stabilizer $\mathrm{rist}_{\overline{G_1}}(v)$ is topologically 4 generated for any $v \in T_2$, and since $\overline{G_1}$ is a pro-2 group, $\mathrm{rist}_{\overline{G_1}}(v)$ has commutator width at most 4 by classical results - e.g. [8, Proposition 4.7]. The group $\overline{G_1}$ therefore also has uniform commutator widths.

Finally, that $\mathrm{rist}_{\overline{G_1}}(v)$ has finite commutator widths implies $D(\mathrm{rist}_{\overline{G_1}}(v))$ is closed. On the other hand, $\mathrm{rist}_{G_1}(v)$ is dense in $\mathrm{rist}_{\overline{G_1}}(v)$, and $D(\mathrm{rist}_{G_1}(v))$ has finite index in $\mathrm{rist}_{G_1}(v)$, since G_1 is just infinite. It follows $D(\mathrm{rist}_{\overline{G_1}}(v))$ has finite index in $\mathrm{rist}_{\overline{G}_1}(v)$. Theorem 3.2 now implies $\overline{G_1}$ is strongly just infinite. □

8. Application: Burger-Mozes universal groups

Let $\mathcal{T}_d$ be the d-regular *unrooted* tree for $d \geqslant 3$. Let $V\mathcal{T}_d$ and $E\mathcal{T}_d$ denote the collection of vertices and edges. For a vertex $v \in V\mathcal{T}_d$, the set $E(v)$ denotes the collection of edges with origin v. Recall $[d] := \{0, 1, \dots, d-1\}$ and fix a map $c : E\mathcal{T}_d \to [d]$ so that $c \restriction_{E(v)} =: c_v$ is a bijection for each $v \in V$; the map c is called a **coloring**. For each $v \in V\mathcal{T}_d$ and $g \in \mathrm{Aut}(\mathcal{T}_d)$, there is a bijection $g_v : E(v) \to E(g.v)$ arising from the action of g.

Definition 8.1. Let $d \geqslant 3$, $F \leqslant \mathfrak{S}_d$, and $c : E\mathcal{T}_d \to [d]$ be a coloring. The **Burger–Mozes universal group** is defined to be

$$U(F) := \{g \in \mathrm{Aut}(\mathcal{T}_d) \mid c_{g.v} \circ g_v \circ c_v^{-1} \in F \text{ for all } v \in V\mathcal{T}_d\}.$$

The subgroup generated by all pointwise edge stabilizers in $U(F)$ is denoted $U(F)^+$.

The group $U(F)$ is closed in $\mathrm{Aut}(\mathcal{T}_d)$ and so is a totally disconnected locally compact Polish group. When F is transitive, the isomorphism type of the group $U(F)$ is independent of the coloring c; indeed, any two groups built using different colorings are conjugate in $\mathrm{Aut}(\mathcal{T}_d)$. We therefore will suppress the coloring.

Proposition 8.2 (Burger–Mozes, [6, Proposition 3.2.1])**.** *Let $d \geqslant 3$ and let $F \leqslant \mathfrak{S}_d$ be transitive and generated by its point stabilizers. Then, $U(F)^+$ is a non-discrete compactly generated totally disconnected locally compact Polish group which is simple and has index two in $U(F)$.*

The group $U(F)^+$ is often called the **Burger–Mozes universal simple group**. The structure of a compact open subgroup of $U(F)^+$ is well-understood.

Proposition 8.3 (Burger–Mozes, [6, Section 3.2] (cf. [7, Proposition 4.3]))**.** *Let $d \geqslant 3$, let $F \leqslant \mathfrak{S}_d$ be transitive and generated by its point stabilizers, and set $H := \mathrm{Stab}_F(v)$ for some $v \in [d]$. The group $U(F)^+$ has a compact open subgroup isomorphic to $W((A_i, X_i)_{i\in\mathbb{N}})$ where*

$$(A_i, X) := \begin{cases} (F, [d]) & \text{if } i = 0 \\ (H, [d] \setminus \{v\}) & \text{else.} \end{cases}$$

We are now ready to apply our results on branch groups. Since $\mathfrak{S}_n$ is solvable for $n \leqslant 4$, we shall have to consider $\mathcal{T}_d$ with $d \geqslant 6$.

Theorem 8.4. *Suppose that $d \geqslant 6$ and that $F \leqslant \mathfrak{S}_d$ is perfect and two transitive. Suppose further the point stabilizers of F are also perfect. The group $U(F)^+$ then enjoys the countable index property, the invariant automatic continuity property, and the locally compact automatic continuity property.*

Proof. Let $H := \mathrm{Stab}_F(v)$ for some $v \in [d]$. Since F is generated by its point stabilizers, Proposition 8.3 ensures that $U(F)^+$ has a compact open subgroup U isomorphic to $W((A_i, X_i))_{i\in\mathbb{N}}$ where

$$(A_i, X) := \begin{cases} (F, [d]) & \text{if } i = 0 \\ (H, [d] \setminus \{v\}) & \text{else.} \end{cases}$$

The groups $(F, [d])$ and $(H, [d] \setminus \{v\})$ are perfect transitive permutation groups, so Proposition 7.2 implies the compact open subgroup U is strongly just infinite, locally has derangements, and has uniform commutator widths. Theorems 4.8 and 5.10 and Corollary 5.11 imply U has the countable index property, the invariant automatic continuity property, and the locally compact automatic continuity property.

Let K be either a non-archimedean Polish group, a SIN Polish group, or a locally compact Polish group and suppose $\psi : U(F)^+ \to K$ is a homomorphism. The restriction $\psi \restriction_U : U \to K$ must be continuous, so taking $O \subseteq K$ open, $(\psi \restriction_U)^{-1}(O)$ contains an open subset of U. Therefore, $\psi^{-1}(O)$ contains an open subset of $U(F)^+$, and it follows that ψ is continuous. $\square$

Corollary 8.5. *For $A_n \leqslant \mathfrak{S}_n$ the alternating group with $n \geqslant 6$, the Burger-Mozes universal simple group $U(A_n)^+$ has the countable index property, the invariant automatic continuity property, and the locally compact automatic continuity property.*

Proof. The group A_n is perfect, two transitive on $[n]$, and generated by its point stabilizers. The point stabilizers are A_{n-1}, so they are also perfect. Theorem 8.4 now implies the corollary. $\square$

Of course, that the Burger–Mozes universal simple groups have these automatic continuity properties follow from our results on profinite groups. As it is known that similar results hold for infinite products of non-abelian finite simple groups, [27, 28], one naturally asks if such examples can be found using infinite products instead. It turns out this is not possible: *There is no compactly generated locally compact group which is topologically simple and contains an infinite product of non-trivial finite groups as a compact open subgroup.* This follows by considering the quasi-center of such a group and applying [1, Theorem 4.8].

We conclude by classifying the commensurated subgroups of certain Burger–Mozes universal simple groups.

Theorem 8.6. *Suppose that $d \geqslant 6$ and that $F \leqslant \mathfrak{S}_d$ is perfect and two transitive. Suppose further the point stabilizers of F are also perfect. Then every commensurated subgroup of $U(F)^+$ is either finite, compact and open, or equal to $U(F)^+$.*

Proof. Let U be the compact open subgroup given by Proposition 8.3 and let $O \leqslant G := U(F)^+$ be a commensurated subgroup of G. The group $O \cap U$ is then a commensurated subgroup of U, and since U is strongly just infinite, Theorem 4.8 implies $O \cap U$ is either open or finite. If $O \cap U$ is open, then O is open. Via [7, Proposition 4.1], O is either compact and open or equal to G; in either case we are done.

We thus suppose that $O \cap U$ is finite. If O is finite, we are done, so we suppose for contradiction that O is infinite. The sets

$$\Omega_n := \{u \in U \mid |O : O \cap uOu^{-1}| \leqslant n \text{ and } |uOu^{-1} : O \cap uOu^{-1}| \leqslant n\}$$

form an increasing exhaustion of U by symmetric sets since O is commensurated; recall for all $n, m \geqslant 1$, we have $\Omega_n \Omega_m \subseteq \Omega_{nm}$. The group U has the Bergman property via Theorem 4.8, whereby $\Omega_m = U$ for some sufficiently large m. Appealing to Theorem 4.5, we conclude there is $O' \sim_c O$ so that U normalizes O'.

The intersection $O' \cap U$ is again finite, hence O' is a non-trivial discrete subgroup of G. The group U normalizers O', whereby each $o \in O'$ has an open centralizer. The collection of elements with open centralizer in G, denoted $QZ(G)$, forms a normal subgroup. Since G is abstractly simple, we conclude that $QZ(G) = G$. This is absurd in view of [1, Theorem 4.8]. □

Corollary 8.7. *For $A_n \leqslant \mathfrak{S}_n$ the alternating group with $n \geqslant 6$, every commensurated subgroup of $U(A_n)^+$ is either finite, compact and open, or $U(A_n)^+$.*

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Institut de Mathématiques de Jussieu-PRG, Université Paris Diderot, Sorbonne Paris Cité, 75205 Paris cedex 13, France
Email address: `francois.le-maitre@imj-prg.fr`

Université catholique de Louvain, Institut de Recherche en Mathématiques et Physique (IRMP), Chemin du Cyclotron 2, box L7.01.02, 1348 Louvain-la-Neuve, Belgique
Email address: `phillip.wesolek@uclouvain.be`