On the space of subgroups of Baumslag-Solitar groups II: High transitivity

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Abstract

We continue our study of the perfect kernel of the space of transitive actions of Baumslag-Solitar groups by investigating high transitivity. We show that actions of finite phenotype are never highly transitive, except when the phenotype is 1, in which case high transitivity is actually generic. In infinite phenotype, high transitivity is generic, except when $|m| = |n|$ where it is empty. We also reinforce the dynamical properties of the action by conjugation on the perfect kernel that we had established in our first paper, replacing topological transitivity by high topological transitivity.

Keywords: Baumslag-Solitar groups; space of subgroups; perfect kernel; high transitivity; topologically transitive actions; Bass-Serre theory.

MSC-classification: 20B22; 37B; 20E06; 20E08; 20F65.

Contents

1 Introduction

In order to study the possible dynamics of a given countable group Γ, a natural first step is to study its transitive actions. One particularly striking property a transitive action can have is **high transitivity** (see Definition 2.1). A basic example is the natural action of the group $\mathrm{Sym}_f(\mathbb{N})$ of finitely supported permutations of $\mathbb{N}$. The first example of a finitely generated group with a highly transitive faithful action was exhibited by B. H. Neumann who pointed out that the natural action of $\mathrm{Sym}_f(\mathbb{Z}) \rtimes \mathbb{Z}$ on $\mathbb{Z}$ is also highly transitive. Since the latter group is two-generated, it follows that the free group $\mathbb{F}_2$ admits a (non-faithful!) highly transitive action. As for faithful highly transitive actions of the free group, they were initially constructed by McDonough [McD77]. This strong property was eventually proved to be generic among all transitive $\mathbb{F}_2$-actions [Dix90].

Subsequently, many groups were shown to admit faithful highly transitive actions, see e.g. [Cha12, Kit12, MS13, FMS15, HO16, FLMMS22] and references therein. The proofs always rely in a more or less explicit way on the Baire category theorem in a tailored space of actions (see Remark 2.11 for details on how to recast Chaynikov's and Hull-Osin's results as Baire category arguments).

Since transitive actions are completely encoded by the conjugacy class of the associated stabilizer subgroup, one is naturally led to the study of the Γ-action by conjugation on its space of subgroups $\mathrm{Sub}(\Gamma)$. This space carries a natural compact Polish topology. High transitivity only makes sense for actions on infinite sets, so we focus on the Polish subspace $\mathrm{Sub}_{[\infty]}(\Gamma)$ of infinite index subgroups. Denote by $\mathcal{HT}(\Gamma)$ the set of subgroups $\Lambda \in \mathrm{Sub}_{[\infty]}(\Gamma)$ such that $\Lambda\backslash\Gamma \curvearrowleft \Gamma$ is highly transitive. This set is G_δ (see Lemma 2.8).

Question 1. When is it true that $\mathcal{HT}(\Gamma)$ is dense in $\mathrm{Sub}_{[\infty]}(\Gamma)$?

We are thus asking what is the class of countable groups whose *generic* transitive actions on an infinite set are actually highly transitive.

It sometimes happens that the Γ-action on $\mathrm{Sub}_{[\infty]}(\Gamma)$ is topologically transitive [AG23]. In this case, the topological zero-one law [Kec95, Theorem 8.46] ensures that either the G_δ set $\mathcal{HT}(\Gamma)$ is dense in $\mathrm{Sub}_{[\infty]}(\Gamma)$, or its complement contains a dense G_δ set. For instance, when Γ is the group $\mathrm{Sym}_f(\mathbb{N})$, even though Γ acts topologically transitively on $\mathrm{Sub}_{[\infty]}(\Gamma)$ (see for instance [LM24, Ex. 9.43 and Prop. 9.44]), $\mathcal{HT}(\Gamma)$ consists of a single (meager) conjugacy class by [LBMB22, Prop. 2.4].

A first example with dense $\mathcal{HT}(\Gamma)$ is provided by free groups as a consequence of Dixon's aforementioned result. More generally, this holds for free products $\Gamma_1 * \Gamma_2$, with $|\Gamma_1| \geq 2$ and $|\Gamma_2| \geq 3$ by [LM24, Thm. 10.30]. A natural next step is to consider groups acting on trees. In the aforementionned work [FLMMS22], it was shown that when Γ admits a faithful minimal action of general type on a tree, then Γ admits a faithful highly transitive action as soon as the action on the boundary of $\mathcal{T}$ is topologically free. A key example is given by Baumslag-Solitar groups. For these groups, we previously showed that the Γ-action on $\mathrm{Sub}_{[\infty]}(\Gamma)$ is not topologically transitive.

To be more precise, let us fix some parameters $m, n \in \mathbb{Z}$ with $|m| \geq 2$ and $|n| \geq 2$ and consider the Baumslag-Solitar group

$$\Gamma = \mathrm{BS}(m,n) := \left\langle b, t \mid tb^m t^{-1} = b^n \right\rangle.$$

In [CGLMS22] we unveiled a natural partition of $\mathrm{Sub}(\Gamma)$ into Γ-invariant subsets provided by a conjugation invariant map $\mathbf{Ph}_{n,m} : \mathrm{Sub}(\Gamma) \to \mathbb{Z}_{\geq 1} \cup \{\infty\}$ that we call the *(m,n)-phenotype map*.

This phenotype $\mathbf{Ph}_{m,n}(\Lambda)$ is obtained from the index $[\langle b\rangle : \langle b\rangle \cap \Lambda]$ as follows: if $[\langle b\rangle : \langle b\rangle \cap \Lambda] = \infty$, then $\mathbf{Ph}_{m,n}(\Lambda) = \infty$. Otherwise, remove from the prime factors decomposition of $[\langle b\rangle : \langle b\rangle \cap \Lambda]$ all the prime numbers p that appear in m or n, except those such that $|m|_p = |n|_p < |[\langle b\rangle : \langle b\rangle \cap \Lambda]|_p$, where $|k|_p$ denotes the p-adic valuation of k (see Section 3.7 for details). Thus $\mathbf{Ph}_{m,n}(\langle b\rangle) = 1$ and $\mathbf{Ph}_{m,n}(\{\mathrm{id}\}) = \infty$ and $\mathbf{Ph}_{m,n}(\langle b^q\rangle) = q$ for every $q \in \mathbb{Z}_{\geq 1}$ prime with both m and n. In particular, the set of possible phenotypes $\mathcal{Q}_{m,n} \coloneqq \mathbf{Ph}_{n,m}(\mathrm{Sub}(\Gamma))$ is infinite and always contains 1 and ∞, independently of (m,n). Observe that $\mathbf{Ph}_{m,n}^{-1}(\infty)$ contains only infinite index subgroups.

Next, we proved that the perfect kernel $\mathcal{K}(\Gamma)$ of the space $\mathrm{Sub}(\mathrm{BS}(m,n))$ is the set of subgroups $\Lambda \leq \Gamma$ such that the double quotient $\Lambda\backslash\Gamma/\langle b\rangle$ is infinite. Letting $\mathcal{K}_q \coloneqq \mathbf{Ph}_{m,n}^{-1}(q) \cap \mathcal{K}(\Gamma)$ for all phenotype $q \in \mathcal{Q}_{m,n}$, we obtain a Γ-invariant decomposition

$$\mathcal{K}(\Gamma) = \bigsqcup_{q\in\mathcal{Q}_{n,m}} \mathcal{K}_q.$$

All the pieces $\mathcal{K}_q$ for $q < \infty$ are open in $\mathcal{K}(\Gamma)$, while $\mathcal{K}_\infty$ is closed[1]. A key result from [CGLMS22] is the fact that the $\mathrm{BS}(m,n)$-action on any each of the pieces $\mathcal{K}_q$ of the above partition is topologically transitive.

Furthermore, the perfect kernel is almost equal to the space of infinite index subgroups: if we define the remaining piece as

$$\mathcal{C}_\infty \coloneqq \mathrm{Sub}_{[\infty]}(\Gamma) \smallsetminus \mathcal{K}(\Gamma),$$

then $\mathcal{C}_\infty$ is a countable open subset of $\mathbf{Ph}_{m,n}^{-1}(\infty)$ which is empty exactly when $|m| \neq |n|$. The complete picture for the space of infinite index subgroups is thus as follows:

$$\mathrm{Sub}_{[\infty]}(\mathrm{BS}(m,n)) = \underbrace{\mathcal{C}_\infty \sqcup \overbrace{\mathcal{K}_\infty}}_{\mathrm{Sub}_{[\infty]}\cap\mathbf{Ph}^{-1}(\infty)} \overbrace{\sqcup \bigsqcup_{q\in\mathcal{Q}_{n,m}\smallsetminus\{\infty\}} \mathcal{K}_q}^{\mathcal{K}(\mathrm{BS}(m,n))}$$

The main goal of this article is to highlight a striking phenomenon: the subgroups giving rise to highly transitive actions are located in certain very specific pieces $\mathcal{K}_q$ of the perfect kernel (depending on the parameters m and n). On the contrary, the subgroups of the other pieces never even correspond to primitive actions. The precise statement is as follows:

[1]Further, when $|m| = |n|$, all the pieces $\mathcal{K}_q$ for $q < \infty$ are clopen.

Theorem A. *Let $m, n \in \mathbb{Z}$ such that $|m| \geq 2$ and $|n| \geq 2$, let $\Gamma = \mathrm{BS}(m,n)$. Let q be an (m,n)-phenotype and let*

$$\mathcal{HT}_q \coloneqq \{\Lambda \in \mathcal{K}_q \colon \Lambda\backslash\Gamma \curvearrowleft \Gamma \text{ is highly transitive}\}.$$

Then $\mathcal{HT}_q$ is dense G_δ in $\mathcal{K}_q$ exactly when either

- *$q = 1$, or*
- *$q = \infty$ and $|m| \neq |n|$.*

Otherwise, $\mathcal{HT}_q$ is empty, and infinite index subgroups of phenotype q actually never even correspond to primitive actions.

Since it is more enjoyable, in connection with high transitivity, to consider faithful actions, we use the results of [CGLMS22] to localize the subgroups that give rise to faithful actions.

Proposition B (Proposition 3.7)**.** *The set of faithful actions forms a dense G_δ set inside $\mathcal{K}_q$ for every phenotype q when $|m| \neq |n|$, or for $q = \infty$ and $|m| = |n|$; otherwise, $\mathcal{K}_q$ contains no faithful action.*

Moreover, if Λ belongs to $\mathcal{C}_\infty$, then the action $\Lambda\backslash\mathrm{BS}(m,n) \curvearrowleft \mathrm{BS}(m,n)$ is not faithful.

Theorem A is proved in Section 6. We summarize the results of Theorem A and Proposition B about high transitivity and faithfulness in $\mathcal{K}_q$ in the following tables.

Case $|m| \neq |n|$

Phenotypes	*High transitivity*	*Faithfulness*
$q = 1$	dense G_δ	dense G_δ
$q = \infty$	dense G_δ	dense G_δ
other phenotypes	$\emptyset$	dense G_δ

Case $|m| = |n|$

Phenotypes	*High transitivity*	*Faithfulness*
$q = 1$	dense G_δ	$\emptyset$
$q = \infty$	$\emptyset$	dense G_δ
other phenotypes	$\emptyset$	$\emptyset$

These tables can be completed by mentioning that if one concentrates on $\mathrm{Sub}_{[\infty]}(\mathrm{BS}(m,n))$ instead of $\mathcal{K}(\mathrm{BS}(m,n))$, we should add in the case $|m| = |n|$ a line for $\mathcal{C}_\infty$, whose elements are never highly transitive nor faithful.

High transitivity admits a topological version which is an enrichment of topological transitivity. Namely, an action $X \curvearrowleft \Gamma$ on an infinite perfect Polish space X, is called **highly topologically transitive** when the diagonal action $X^d \curvearrowleft \Gamma$ is topologically transitive for every $d \geq 1$.

We have established in [CGLMS22] the topological transitivity of the action of $\mathrm{BS}(m,n)$ on each $\mathcal{K}_q$. We now upgrade this to high topological transitivity.

Theorem C. *Let m, n be integers such that $|m|, |n| \geq 2$. Then for every phenotype $q \in \mathcal{Q}_{m,n}$ the action by conjugation of $\mathrm{BS}(m,n)$ on the invariant subspace $\mathcal{K}_q(\mathrm{BS}(m,n))$ is highly topologically transitive.*

Theorem A was implicitly proved in [FLMMS22, Theorem 4.4] in the case $\mathcal{K}_\infty(\mathrm{BS}(m,n))$ for $|m| \neq |n|$. We adapt these techniques and develop a unified framework (see in particular Lemma 4.12) towards establishing high transitivity or high topological transitivity. This framework is based on pre-actions and their maximal forest saturations, as in our first paper [CGLMS22]. A notable difference is that we often have to stay at the level of pre-actions instead of working with the more flexible (m,n)-graphs. We clarify the definition of maximal forest saturations by showing that each pre-action admits a *unique* maximal forest saturation (this result is a special case of Theorem 4.6). We also identify precisely the corresponding stabilizer subgroup (see Corollary 4.11).

We would also like to highlight the work of Sasha Bontemps [Bon24], who extended the main results of [CGLMS22] to Generalized Baumslag-Solitar groups in a work which should appear on the arXiv very soon. She discovered the right notion of phenotype in this wider context and obtained a description analogous to ours of the perfect kernel. Her phenotype yields a decomposition of the perfect kernel into invariant pieces where she shows the action is highly topologically transitive, thus obtaining a natural generalization of Theorem C.

Let us finally mention a result of independent interest that relates high transitivity to its topological counterpart:

Theorem D (HT + TT $\implies$ HTT, Theorem 2.12)**.** *For any countable group Γ, if $\mathcal{P} \subseteq \mathrm{Sub}(\Gamma)$ is a Γ-invariant G_δ subset such that*
(1) $\mathcal{HT}(\Gamma) \cap \mathcal{P}$ is dense in $\mathcal{P}$, and
(2) the Γ-action on $\mathcal{P}$ is topologically transitive,
then the Γ-action on $\mathcal{P}$ is highly topologically transitive.

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2 General results on high transitivity and its topological counterpart

This section collects some results on high transitivity which are not specific to Baumslag-Solitar groups. Most of them are well-known, but Theorem 2.12 appears to be new. We also propose a slight modification of the usual definition of high topological transitivity so that it encompasses high transitivity (see Lemma 2.5 for the comparison with the definition used in [AG23]).

2.1 Preliminaries on high (topological) transitivity

We begin by recalling the definition of high transitivity. To this end, recall that given a right action α of a group Γ on a set X with at least $d \in \mathbb{N}$ elements, we have a natural diagonal action $\alpha^{(d)}$ on the set $X^{(d)} \subseteq X^d$ made of d-tuples consisting of pairwise distinct elements of X. We say then say that α is d-**transitive** when $\alpha^{(d)}$ is transitive.

Definition 2.1. A right action α of a group Γ on an infinite set X is called **highly transitive (HT)** when it is d-transitive for every $d \in \mathbb{N}$, namely: for every $x_1, \ldots, x_d \in X$ pairwise distinct and $y_1, \ldots, y_d \in X$ pairwise distinct, there is $\gamma \in \Gamma$ such that for all $i \in \{1, \ldots, d\}$, we have $x_i\alpha(\gamma) = y_i$.

We will make use of the following two well-known lemmas. The first says that we may as well assume $x_i \neq y_j$ for all $i, j \in \{1, \ldots, d\}$ in the above definition.

Lemma 2.2 ([FLMMS22, Lem. 2.2])**.** *Let α be a right Γ-action on an infinite set X, then the action is highly transitive iff for all $d \in \mathbb{N}$ and all $x_1, \ldots, x_{2d} \in$*

X pairwise distinct, there is $\gamma \in \Gamma$ such that

$$\forall i \in \{1, \ldots, d\}, \quad x_i \alpha(\gamma) = x_{i+d}.$$

Lemma 2.3 ([MS13, Lem. 1.4(2)])**.** *Let α be a highly transitive Γ-action on an infinite set X. If N is a normal subgroup of Γ, then the restriction of α to N is either trivial or highly transitive.*

We now introduce the topological versions of these notions. We choose a definition which slightly differs from the classical one (they are equivalent when the underlying space is infinite Hausdorff perfect, see Lemma 2.5). The advantage of our choice is that an action on an infinite set X is highly transitive if and only if it is highly topologically transitive when endowing X with the discrete topology. In general, $X^{(d)}$ is endowed with the topology induced form the product topology on X^d.

Definition 2.4. Let α be a right action of a group Γ on a Hausdorff topological space X by homeomorphisms.

1. The action is called **topologically transitive (TT)** if whenever U, V are nonempty open subsets of X, there is $\gamma \in \Gamma$ such that

$$U\alpha(\gamma) \cap V \neq \emptyset.$$

2. Given $d \in \mathbb{N}$, if X contains at least d elements, the action is furthermore called **d-topologically transitive (d-TT)** when the diagonal action $\alpha^{(d)}$ on $X^{(d)}$ is topologically transitive.
3. If X is infinite, α is called **highly topologically transitive (HTT)** when it is d-topologically transitive for every $d \in \mathbb{N}$.

Note that in the definition of topological transitivity, we may always restrict to U and V belonging to a basis of the topology of X. The next lemma connects this definition with the more classical definition, where $X^{(d)}$ is replaced by X^d. Recall that a topological space is called **perfect** when it has no isolated points.

Lemma 2.5. *Let α be a Γ-action by homeomorphisms on a Hausdorff infinite topological space X. Then α is highly topologically transitive in the sense of Definition 2.4 iff one of the following holds:*

- *X is perfect and for every $d \in \mathbb{N}$, the diagonal action α^d on X^d is topologically transitive;*

- *The set of isolated points of X is dense and the restriction of α to this set is highly transitive (in particular, this set consists of a single orbit).*

Proof. First observe that since X is Hausdorff, the sets $U_1 \times \cdots \times U_d$, where $U_i \subseteq X$ is open and the U_i's are pairwise disjoint, form a basis for the topology of $X^{(d)}$. It follows that α is highly topologically transitive iff for all $d \in \mathbb{N}$, for all $U_1, \ldots, U_d \subseteq X$ pairwise disjoint, open, and nonempty, and for all $V_1, \ldots, V_d \subseteq X$ pairwise disjoint, open, and nonempty, there is $\gamma \in \Gamma$ such that for all $i \in \{1, \ldots, d\}$, $U_i\alpha(\gamma) \cap V_i \neq \emptyset$. Let us distinguish two cases.

Case 1: X is perfect. Assuming first that α is highly topologically transitive, we now show that the action α^d on X^d is topologically transitive. To this end, by the above paragraph it suffices to show that given any nonempty open sets $U_1, \ldots, U_d \subseteq X$, there are *pairwise disjoint* nonempty open sets $U'_1 \subseteq U_1, \ldots, U'_d \subseteq U_d$. Since X is perfect, every U_i is infinite, so we can select pairwise distinct $x_i \in U_i$, and then since X is Hausdorff each x_i admits a neighborhood $U'_i \subseteq U_i$ such that $(U'_i)_{i=1}^d$ consists of pairwise disjoint open sets, as required.

Conversely, it is obvious that, for every d, topological transitivity of α^d implies that $\alpha^{(d)}$ is also topologically transitive. Conversely, it is obvious that, for every d, topological transitivity of α^d implies that $\alpha^{(d)}$ is also topologically transitive.

Case 2: X is not perfect. Let us first assume that α is highly topologically transitive. By definition, X contains at least one isolated point x_0, and topological transitivity ensures us that for every non empty open set U, there is some $\gamma \in \Gamma$ such that $\{x_0\}\alpha(\gamma) \cap U \neq \emptyset$, i.e. the isolated point $x_0\alpha(\gamma)$ belongs to U. So the set of isolated points is dense. Using singletons as open sets, it is then straightforward to see that the restriction of α to the set of isolated points is highly transitive as desired.

Conversely, assume that the set of isolated points is dense and that the restriction of α to this set is highly transitive. Let $U_1, \ldots, U_d$ and $V_1, \ldots, V_d$ be pairwise disjoint nonempty open subsets of X, for every $i \in \{1, \ldots, d\}$ we pick $x_i \in U_i$ isolated and $y_i \in V_i$ isolated, then the x_i's and the y_i's are pairwise distinct by disjointness, so by assumption we find $\gamma \in \Gamma$ taking each x_i to y_i, in particular $U_i\alpha(\gamma) \cap V_i \neq \emptyset$ as desired. □

Remark 2.6. Similar considerations yield that Lemma 2.2 can be generalized as follows: a Γ-action α by homeomorphisms on an infinite Hausdorff topological space X is highly topologically transitive if and only if for every $d \in \mathbb{N}$, and every **pairwise disjoint** nonempty open sets $U_1, \ldots, U_{2d}$, there is $\gamma \in \Gamma$ such that for all $i \in \{1, \ldots, d\}$, we have $U_i\alpha(\gamma) \cap U_{i+d} \neq \emptyset$.

On the contrary, the natural generalization of Lemma 2.3 fails badly: high topological transitivity does not pass to normal subgroups acting faithfully. Here is a counterexample: let $\Gamma = \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and consider the natural Γ-action on $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ where $\mathbb{Z}$ acts by shift and $\mathbb{Z}/2\mathbb{Z}$ acts diagonally by translation. The $\mathbb{Z}$-action is highly topologically transitive, in particular the Γ-action is highly topologically transitive. However, the normal subgroup $\mathbb{Z}/2\mathbb{Z}$ acts faithfully, but its action is not even topologically transitive.

Remark 2.7. Letting $X^{(\infty)} = \{(x_n) \in X^{\mathbb{N}} \colon \forall i \neq j, x_i \neq x_j\}$, one can conveniently reformulate high topological transitivity as the fact that the diagonal action $\alpha^{(\infty)}$ on $X^{(\infty)}$ is topologically transitive. If X is discrete, then α is highly transitive if and only if $\alpha^{(\infty)}$ is topologically transitive.

2.2 Space of subgroups and high transitivity

Recall that $\mathrm{Sub}(\Gamma)$ is the set of subgroups of the countable group Γ. It is a Polish space when equipped with the topology defined by a basis of open (in fact clopen) subsets given by $\mathcal{V}(\mathcal{I}, \mathcal{O}) = \{\Lambda \leq \Gamma \mid \mathcal{I} \subseteq \Lambda \text{ and } \mathcal{O} \cap \Lambda = \emptyset\}$ where $\mathcal{I}, \mathcal{O}$ are finite subsets of Γ.

As in [CGLMS22, Section 2.2], we freely identify the compact Polish space $\mathrm{Sub}(\Gamma)$ with the space of isomorphism classes of transitive pointed right Γ-actions. Such isomorphism classes are usually denoted $[\alpha, x_0]$. Recall that Γ acts on $\mathrm{Sub}(\Gamma)$ by conjugacy, and that in terms of isomorphism classes of transitive pointed right actions, this action is given by moving the base point:

$$[\alpha, x_0] \cdot \gamma \coloneqq [\alpha, x_0\alpha(\gamma)].$$

We denote by $\mathrm{Sub}_{[\infty]}(\Gamma) \subseteq \mathrm{Sub}(\Gamma)$ the set of infinite index subgroups of Γ and by $\mathcal{HT}(\Gamma) \subseteq \mathrm{Sub}_{[\infty]}(\Gamma)$ the set of infinite index subgroups Λ such that $\Lambda\backslash\Gamma \curvearrowleft \Gamma$ is highly transitive.

Recall that a subset of a topological space is G_δ if it can be written as a countable intersection of open subsets. Every closed subset of a Polish space is G_δ. Furthermore, a subset of a Polish space is Polish for the induced topology if and only if it is G_δ [Kec95, Theorem 3.11].

Lemma 2.8. *The set* $\mathrm{Sub}_{[\infty]}(\Gamma)$ *is a* G_δ *subset of* $\mathrm{Sub}(\Gamma)$ *which is closed when* Γ *is finitely generated. The set* $\mathcal{HT}(\Gamma)$ *is a* G_δ *subset of* $\mathrm{Sub}(\Gamma)$ *(and of* $\mathrm{Sub}_{[\infty]}(\Gamma)$*). In particular,* $\mathcal{HT}(\Gamma)$ *and* $\mathrm{Sub}_{[\infty]}(\Gamma)$ *are both Polish spaces for their induced toology.*

We will merge the proof of Lemma 2.8 together with the proof of the following application of the Baire Category Theorem.

Lemma 2.9. *Let* Γ *be a countable group. Let* $\mathcal{P} \subseteq \mathrm{Sub}_{[\infty]}(\Gamma)$ *be a* G_δ *(hence Polish) subspace. Then the set* $\mathcal{HT}_{\mathcal{P}} \coloneqq \mathcal{HT}(\Gamma) \cap \mathcal{P}$ *is dense in* $\mathcal{P}$ *if and only if the following holds:*

- *(*) For every* $[\alpha, x] \in \mathcal{P}$*, for every* $d \in \mathbb{N}$*, for every* $g_1, \ldots, g_{2d} \in \Gamma$ *such that* $(x\alpha(g_i))_{i=1}^{2d}$ *consists of pairwise distinct elements, and for every neighborhood* $\mathcal{V}$ *of* $[\alpha, x]$*, there is* $[\alpha', x'] \in \mathcal{P} \cap \mathcal{V}$ *and* $\gamma \in \Gamma$ *such that for all* $i \in \{1, \ldots, d\}$*, we have*

$$x'\alpha'(g_i\gamma) = x'\alpha'(g_{i+d}).$$

Proof of Lemmas 2.8 and 2.9. For every n, the set of subgroups of index at least n is the open set of those Λ for which there exist pairwise distinct $g_1, \ldots g_n \in \Gamma$ such that $g_i g_j^{-1} \notin \Lambda$ for all $i \neq j$. Thus $\mathrm{Sub}_{[\infty]}(\Gamma) = \bigcap_{n \in \mathbb{N}} \bigcup_{(g_i)_i \in \Gamma^{(n)}} \mathcal{V}(\emptyset, g_i g_j^{-1})$ is G_δ. When Γ is finitely generated, its finite index subgroups are isolated, thus $\mathrm{Sub}_{[\infty]}(\Gamma)$ is closed.

We now show that the space $\mathcal{HT}(\Gamma)$ is G_δ in $\mathrm{Sub}_{[\infty]}(\Gamma)$, thus also showing that $\mathcal{HT}_{\mathcal{P}} = \mathcal{HT}(\Gamma) \cap \mathcal{P}$ is G_δ in $\mathcal{P}$. First observe that, by Lemma 2.2, a pointed transitive action $[\alpha, x]$ on an infinite set is highly transitive iff for all $d \in \mathbb{N}$, and for all $g_1, \ldots, g_{2d} \in \Gamma$ such that $x\alpha(g_i) \neq x\alpha(g_j)$ for all distinct $i, j \in \{1, \ldots, 2d\}$, there is $\gamma \in \Gamma$ such that $x\alpha(g_i\gamma) = x\alpha(g_{i+d})$ for all $i \in \{1, \ldots, d\}$. For every $g_1, \ldots, g_{2d} \in \Gamma$, let us thus denote by $\mathcal{U}_{g_1, \ldots, g_{2d}}$ the set of all infinite index subgroups $\Lambda \leq \Gamma$ such that at least one of the following holds:

- there is some $i \neq j \in \{1, \ldots, 2d\}$ such that $g_i g_j^{-1} \in \Lambda$;
- there is $\gamma \in \Gamma$ such that $g_i \gamma g_{i+d}^{-1} \in \Lambda$, for all $\forall i \in \{1, \ldots, d\}$.

It is clear from their definition that all $\mathcal{U}_{g_1, \ldots, g_{2d}}$ are open. Our first observation combined with the correspondence between pointed actions and subgroups yield that

$$\mathcal{HT}(\Gamma) = \bigcap_{d \geq 1} \bigcap_{g_1 \ldots g_{2d} \in \Gamma} \mathcal{U}_{g_1, \ldots, g_{2d}}.$$

It follows that $\mathcal{HT}(\Gamma)$ is G_δ as wanted.

Now if $\mathcal{HT}_{\mathcal{P}}$ is dense in $\mathcal{P}$, Condition $(*)$ of Lemma 2.9 clearly holds. Conversely, assume that Condition $(*)$ of Lemma 2.9 holds. By the Baire category theorem applied to the Polish space $\mathcal{P}$, it suffices to show that each intersection $\mathcal{P} \cap \mathcal{U}_{g_1,\dots,g_{2d}}$ is dense in $\mathcal{P}$.

Let us thus fix some subgroup $\Lambda \in \mathcal{P}$ and $g_1, \dots, g_{2d} \in \Gamma$. We need to show that Λ is a limit of elements of $\mathcal{P} \cap \mathcal{U}_{g_1,\dots,g_{2d}}$. If for all $i \neq j$, we have $g_i g_j^{-1} \notin \Lambda$, Condition $(*)$ of Lemma 2.9 precisely yields that conclusion. Otherwise, we have some $i \neq j$ such that $g_i g_j^{-1} \in \Lambda$, but then Λ is already in $\mathcal{U}_{g_1,\dots,g_{2d}}$, so we are done as well. □

Observe that if Condition $(*)$ of Lemma 2.9 holds for an arbitrary subset $\mathcal{P} \subseteq \mathrm{Sub}_{[\infty]}(\Gamma)$ then it also holds for its closure in $\mathrm{Sub}_{[\infty]}(\Gamma)$. This immediately yields the following result.

Corollary 2.10. *Let Γ be a countable group. Let $\mathcal{P} \subseteq \mathrm{Sub}_{[\infty]}(\Gamma)$ be a subset satisfying Condition $(*)$ of Lemma 2.9. Let $\overline{\mathcal{P}}$ denote the closure of $\mathcal{P}$ in* $\mathrm{Sub}_{[\infty]}(\Gamma)$. *Then the set $\mathcal{HT}_{\overline{\mathcal{P}}}$ of highly transitive pointed actions in $\overline{\mathcal{P}}$ is dense G_δ in $\overline{\mathcal{P}}$.* □

Remark 2.11. Corollary 2.10 can be used to show Hull and Osin's result on high transitivity a bit differently (compare with [HO16, Sec. 3.3]). Recall that their result is that every acylindrically hyperbolic group with trivial finite radical is highly transitive. For this, they crucially use the notion of *small subgroup* of such a group (see [HO16, Definition 2.10]), which always has infinite index. Denote by $\mathcal{P}$ the set of all small subgroups of a fixed acylindrically hyperbolic group Γ with trivial finite radical. The fact that Γ is highly transitive can then be deduced directly from the following two statements, denoting by $\overline{\mathcal{P}}$ the closure of $\mathcal{P}$ in $\mathrm{Sub}_{[\infty]}(\Gamma)$:

- By [HO16, Prop. 3.1], the set $\mathcal{P}$ of small subgroups satisfies Condition $(*)$ of Lemma 2.9. So we can apply Corollary 2.10 and obtain that $\mathcal{HT}_{\overline{\mathcal{P}}}$ is dense G_δ in $\overline{\mathcal{P}}$ (note that their result is slightly stronger than Condition $(*)$ of Lemma 2.9).
- By [HO16, Lem. 2.11], every element of $\mathcal{P}$ is faithful. Since the set of faithful actions is G_δ, we conclude that the set of faithful actions is dense G_δ in $\overline{\mathcal{P}}$.

By the Baire category theorem and the two above items, the set of highly transitive faithful actions is dense G_δ in $\overline{\mathcal{P}}$, in particular it is not empty and hence Γ is highly transitive. A similar approach works for Chaynikov's result on high transitivity for hyperbolic groups [Cha12], replacing small subgroups by quasi-convex subgroups.

We now observe a natural connection between high transitivity and high topological transitivity. This statement will not be used in the proof of Theorem C, so the hasty reader can safely skip it, but it initially led us to some cases which we state as Corollary 2.13.

Theorem 2.12 (HT + TT ⇒ HTT)**.** *Let Γ be a countable group and let $\mathcal{P} \subseteq \mathrm{Sub}(\Gamma)$ be a Γ-invariant nonempty Polish subset. Assume that the set of highly transitive actions $[\alpha, x_0] \in \mathcal{P}$ is dense in $\mathcal{P}$ and that the Γ-action on $\mathcal{P}$ is topologically transitive. Then the Γ-action on $\mathcal{P}$ is highly topologically transitive.*

Proof. Following Remark 2.6, we fix $U_1, \dots, U_{2d}$ non empty pairwise disjoint open subsets of $\mathcal{P}$ and need to find $\gamma \in \Gamma$ such that $U_j \cdot \gamma \cap U_{j+d} \neq \emptyset$ for all $j \in \{1, \dots, d\}$. By topological transitivity, letting $g_1 = \mathrm{id}$, we can inductively find $g_2, \dots, g_{2d} \in \Gamma$ so that the open set $U \coloneqq \bigcap_{i=1}^{2d} U_i \cdot g_i^{-1}$ is not empty.

By density, we may and do fix a highly transitive action $[\alpha, x_0] \in U$. Then by the definition of U, for all $i \in \{1, \dots, 2d\}$ we have $[\alpha, x_0\alpha(g_i)] \in U_i$. Since $U_1, \dots, U_{2d}$ are pairwise disjoint, the elements $x_0\alpha(g_1), \dots, x_0\alpha(g_{2d})$ are pairwise distinct.

By high transitivity, there is then $\gamma \in \Gamma$ such that for all $j \in \{1, \dots, d\}$, we have $x_0\alpha(g_j)\alpha(\gamma) = x_0\alpha(g_{j+d})$, hence

$$[\alpha, x_0\alpha(g_j)] \cdot \gamma = [\alpha, x_0\alpha(g_{j+d})] \quad \text{in } \mathrm{Sub}(\Gamma).$$

Using that $[\alpha, x_0\alpha(g_i)] \in U_i$ for all $i \in \{1, \dots, 2d\}$, we conclude that $U_j \cdot \gamma \cap U_{j+d} \neq \emptyset$ for all $j \in \{1, \dots, d\}$ as wanted. □

As an application to Baumslag-Solitar groups, we obtain the following particular case of Theorem C from pre-existing results.

Corollary 2.13. *Let $\Gamma = \mathrm{BS}(m, n)$ where $|m|, |n| \geq 2$, $|m| \neq |n|$, and let $\mathcal{K}_\infty(\Gamma)$ be the set of infinite phenotype subgroups in the perfect kernel. Then, the action $\mathcal{K}_\infty(\Gamma) \curvearrowleft \Gamma$ is highly topologically transitive.*

Proof. Let us fix $x_0 \in \mathbb{N}$. Let $\mathcal{A}$ denote the set of transitive Γ-actions α on $\mathbb{N}$ such that $[\alpha, x_0] \in \mathcal{K}_\infty(\Gamma)$. This is exactly the set of transitive Γ-actions α on $\mathbb{N}$ such that the permutation $\alpha(b)$ has infinitely many infinite orbits and no finite orbit (in particular, $\mathcal{A}$ does not depend on the choice of x_0).

We endow $\mathcal{A}$ with the pointwise convergence topology of $\mathrm{Hom}(\Gamma, \mathrm{Sym}(\mathbb{N}))$. For every $\sigma \in \mathrm{Sym}(\mathbb{N})$ with infinitely many infinite orbits and no finite orbit, the set of $\alpha \in \mathcal{A}$ such that $\alpha(b) = \sigma$ is a homeomorphic copy of the space **TA** of [FLMMS22, Definition 4.2]; and these copies form a partition of $\mathcal{A}$. Thus, Theorem 4.4 in the same article shows that highly transitive actions form a dense subset in $\mathcal{A}$ (in fact a dense G_δ subset in each copy of **TA**).

Since the map $\mathcal{A} \to \mathcal{K}_\infty(\Gamma); \alpha \mapsto [\alpha, x_0]$ is continuous and surjective, we obtain that highly transitive pointed actions form a dense subset in $\mathcal{K}_\infty(\Gamma)$. Finally, the action $\mathcal{K}_\infty(\Gamma) \curvearrowleft \Gamma$ is topologically transitive by [CGLMS22, Theorem B]. Hence, Theorem 2.12 applies. □

3 Background around Baumslag-Solitar groups

3.1 Graphs

We keep the same definitions and conventions about graphs as in [CGLMS22]. Let us nevertheless emphasize the following points concerning graphs.

- Paths are sequences of edges, and by convention paths of length 0 are vertices.
- Given a vertex v, the ball of radius $R \in \mathbb{N}$ is the graph induced by the set of vertices reachable from v by a path of length $\leq R$. In particular, it may contain edges which connect two vertices at distance R from the identity, but which do not not belong to any path of length $\leq R$ starting from v.
- Given an edge e, the half-graph of e is the connected component of $\mathtt{t}(e)$ obtained when removing the edge e from the ambient graph. When this connected component is a tree, we rather say the half-graph is a **half-tree**. This is typically the case when the ambient graph is a tree, but we will mostly use this terminology in the forest saturation, which will be reviewed in the next section.

3.2 Pre-actions

We continue our preliminaries by recalling what pre-actions are, motivated by the presentation of Baumslag-Solitar groups:

$$\mathrm{BS}(m,n) \coloneqq \left\langle b, t \mid tb^m t^{-1} = b^n \right\rangle.$$

Now, a (right) **preaction** of $\mathrm{BS}(m,n)$ on a set X is a couple (β,τ) where β is a bijection of X and τ is a *partial* bijection of X such that $\mathrm{dom}(\tau)$ is β^n-invariant, $\mathrm{rng}(\tau)$ is β^m-invariant and for all $x \in \mathrm{dom}\,\tau$ we have $x\tau\beta^m = x\beta^n\tau$. The set X is called the **domain** of the pre-action. We also sometimes write pre-actions as triples (X,β,τ) in order to make the domain explicit. We say that a pre-action is **saturated** when $\mathrm{dom}\,\tau = \mathrm{rng}\,\tau = X$. Note that saturated pre-actions are the exact same thing as $\mathrm{BS}(m,n)$-actions.

An **extension** of a pre-action (X,β,τ) is a pre-action (X',β',τ') such that β' extends β (in particular $X \subseteq X'$) and τ' extends τ. Conversely, (X,β,τ) is the **restriction** to $X \subseteq X'$ of a pre-action (X',β',τ') if it is itself a pre-action that is extended by (X',β',τ'). As we will recall in Section 4.1, every pre-action can be canonically extended to a saturated pre-action that we call the **maximal forest saturation**.

Finally, by a **pointed pre-action**, we simply mean a couple (α,x) where α is a pre-action on a set X and $x \in X$.

3.3 Schreier graphs

To each pre-action (X,β,τ) we associate a **Schreier graph** whose underlying vertex set is X, endowed with the following labeled edges:

- for every $x \in \mathrm{dom}\,\beta$, we put a positive b labeled edge from x to $\beta(x)$, and an opposite negative b^{-1}-labeled edge from $\beta(x)$ to x;
- for every $x \in X$, we put a positive t-labeled edge from x to $\tau(x)$, and an opposite t^{-1}-labeled edge from $\tau(x)$ to x.

3.4 Bass-Serre graphs and (m,n)-graphs

To every pre-action (X,β,τ) we associate another graph coarser than the Schreier graph, that we call its Bass-Serre graph, roughly obtained by shrinking b-orbits to vertices while keeping track of their cardinality. To be more precise, the **Bass-Serre graph** associated to a pre-action $\alpha = (X,\beta,\tau)$ of $\mathrm{BS}(m,n)$ is the oriented labeled graph $\mathbf{BS}(\alpha)$ defined by

$$V(\mathbf{BS}(\alpha)) \coloneqq X/\langle\beta\rangle\,, \quad \begin{cases} E^+(\mathbf{BS}(\alpha)) \coloneqq \mathrm{dom}(\tau)/\langle\beta^n\rangle\,, \\ E^-(\mathbf{BS}(\alpha)) \coloneqq \mathrm{rng}(\tau)/\langle\beta^m\rangle\,, \end{cases}$$

where for every $x \in \mathrm{dom}\,\tau$ we set

$$\mathtt{s}(x\langle\beta^n\rangle) \coloneqq x\langle\beta\rangle\,,\ \mathtt{t}(x\langle\beta^n\rangle) \coloneqq x\tau\langle\beta\rangle\,,\ \text{and}\ \overline{x\langle\beta^n\rangle} \coloneqq x\tau\langle\beta^m\rangle = x\langle\beta^n\rangle\tau.$$

It is endowed with the **label map** $L\colon V(\mathbf{BS}(\alpha))\sqcup E(\mathbf{BS}(\alpha))\to\mathbb{Z}_{\geq 1}\cup\{\infty\}$ given by

$$L(x\langle\beta\rangle)\coloneqq|x\langle\beta\rangle|\,,\quad L(x\langle\beta^n\rangle)\coloneqq|x\langle\beta^n\rangle|\,,\quad L(y\langle\beta^m\rangle)\coloneqq|y\langle\beta^m\rangle|\,.$$

The labeled oriented graphs obtained as Bass-Serre graphs can be axiomatized and are called (m,n)-**graphs** (see [CGLMS22, Definition 3.12]): every positive (t-labeled) edge e has to satisfy the **transfer equation**, obtained by writing down the combinatorial condition that the sizes of β-orbits have to satisfy in order for τ to be able to take one β^n-suborbit to a β^m-suborbit:

$$\frac{L(\mathtt{s}(e))}{\gcd(L(\mathtt{s}(e)),n)}=L(e)=\frac{L(\mathtt{t}(e))}{\gcd(L(\mathtt{t}(e)),m)};\tag{3.1}$$

Writing down what this means for p-adic valuations, we can reformulate this as: for every prime p,

$$\max(|L(\mathtt{s}(e))|_p-|n|_p\,,0)=|L(e)|_p=\max(|L(\mathtt{t}(e))|_p-|m|_p\,,0).\tag{3.2}$$

Note that if one of the two β-orbits is infinite, the other one has to be as well, and we convene that the transfer equation means that both labels are infinite.

Moreover, one can write down the number of β^m and β^n orbits that a single β-orbit splits into, yielding the following constraints on the outgoing and ingoing degree of every vertex v:

$$\deg_{\text{out}}(v)\leq\gcd(L(v),n)\quad\text{and}\quad\deg_{\text{in}}(v)\leq\gcd(L(v),m),\tag{3.3}$$

where $\gcd(k,\infty)=k$ for every nonzero $k\in\mathbb{Z}$. Note that a pre-action is saturated iff all the vertices v of its Bass-Serre graphs are saturated, i.e. they satisisfy

$$\deg_{\text{out}}(v)=\gcd(L(v),n)\quad\text{and}\quad\deg_{\text{in}}(v)=\gcd(L(v),m).\tag{3.4}$$

In our previous paper, we showed the expected fact that (m,n)-graphs do come from $\mathrm{BS}(m,n)$-actions, but more interestingly that any (m,n)-graph extension of the Bass-Serre graph of a pre-action comes from some extension of the pre-action (see [CGLMS22, Thm. 4.13]). However, the results of the present paper won't make use of these facts, as we rather directly work at the level of pre-actions.

3.5 Morphisms

A **morphism of pre-actions** from $\alpha_1 = (X_1, \beta_1, \tau_1)$ to $\alpha_2 = (X_2, \beta_2, \tau_2)$ is a map $\varphi : X_1 \to X_2$ such that one has $\varphi(x\beta_1) = \varphi(x)\beta_2$ for all $x \in X_1$ and $\varphi(x\tau_1) = \varphi(x)\tau_2$ for all $x \in \mathrm{dom}(\tau_1)$. It is an **isomorphism** if it is bijective and $\varphi^{-1} : X_2 \to X_1$ is also a morphism.

Notice that a morphism $\varphi : (X_1, \beta_1, \tau_1) \to (X_2, \beta_2, \tau_2)$ always maps $\mathrm{dom}(\tau_1)$ into $\mathrm{dom}(\tau_2)$. It is an isomorphism if and only if it is bijective and maps $\mathrm{dom}(\tau_1)$ *onto* $\mathrm{dom}(\tau_2)$. Also note that every pre-action (iso)morphism induces a labeled graph (iso)morphism at the level of Schreier graphs. The same is true at the level of Bass-Serre graphs, except that morphism may fail to respect the label map. They are nevertheless graph morphisms.

3.6 Connection to Bass-Serre theory

Finally, we recall the connection to Bass-Serre theory: to the free transitive $\Gamma = \mathrm{BS}(m, n)$-action by right-translation on itself one can associate a Bass-Serre graph $\mathcal{T} = \mathbf{BS}(\Gamma \curvearrowleft \Gamma)$ which is actually a tree. Indeed, by definition the latter is nothing but the Bass-Serre tree associated to the standard decomposition of $\mathrm{BS}(m, n)$ as an HNN extension, see [Ser80, Chapter I, § 5].

The fact that $\mathcal{T}$ is a tree is tightly connected to the uniqueness of normal forms in HNN extensions, which depend on the choice of coset representatives. Here, a natural choice is to take $\{b^k : 0 \leq k \leq m-1\}$ and $\{b^k : 0 \leq k \leq n-1\}$ as coset representatives, and then by Theorem 2.1 from [LS01, Chapter IV], every $\gamma \in \mathrm{BS}(m, n)$ is uniquely written as a **normal form**

$$\gamma = b^{k_1} t^{\epsilon_1} \cdots b^{k_d} t^{\epsilon_d} b^k$$

where $d \geq 0$, $\epsilon_i = \pm 1$ for $1 \leq i \leq d$, and $\varepsilon_i = +1$ implies $0 \leq k_i \leq n-1$, $\varepsilon_i = -1$ implies $0 \leq k_i \leq m-1$, $k \in \mathbb{Z}$, and there is no subword of the form $t^{\varepsilon} b^0 t^{-\varepsilon}$. Note that the case $d = 0$ corresponds to elements in $\langle b \rangle$.

Going back to the Bass-Serre tree $\mathcal{T}$, given a transitive right $\mathrm{BS}(m, n)$-action $\Lambda \backslash \Gamma \curvearrowleft \Gamma$, one can form the quotient graph $\Lambda \backslash \mathcal{T}$ whose vertices are thus double cosets $\Lambda \backslash \Gamma / \langle b \rangle$ and thus identify naturally to the vertex set of the Bass-Serre graph of $\Lambda \backslash \Gamma \curvearrowleft \Gamma$. It is not hard to check that a similar identification holds at the edge level: the Bass-Serre graphs that we are considering can simply be stated as quotients of the Bass-Serre tree, suitably decorated by the label map[2]. However, they also make sense for pre-actions, making them more flexible.

Finally, given a right action α on a set X and $x \in X$, we have a unique Γ-equivariant map $\pi_x : \Gamma \to X$ such that $\pi(\mathrm{id}) = x$, which when viewed as a pre-action morphism yields a graph morphism from $\mathbf{BS}(\Gamma \curvearrowleft \Gamma) = \mathcal{T}$ to $\mathbf{BS}(\alpha)$. As we will see, many of our arguments will be stated at the level of this graph morphism, which takes a particularly nice form on the forest part of α when α is a maximal forest saturation (as defined in Section 4.1).

3.7 Phenotype

Given a $\mathrm{BS}(m,n)$-action (or more generally pre-action), the transfer equation that the edges of its Bass-Serre graph has to satisfy induces a natural equivalence relation on integers, defined as the smallest equivalence relation $\sim$ such that for all $L, L' \in \mathbb{Z}_{\geq 1}$,

$$\frac{L}{\gcd(L,n)} \sim \frac{L'}{\gcd(L',m)}. \tag{3.5}$$

A basic result from our first paper is the description of this equivalence relation as *having equal phenotype* [CGLMS22, Proposition 4.6]. The phenotype of an integer $L \in \mathbb{Z}_{\geq 1}$ is defined as follows:

- We first let $\mathcal{P}_{m,n}(L) \coloneqq \left\{ p \in \mathcal{P} \colon \ |m|_p = |n|_p \text{ and } |L|_p > |n|_p \right\}$.
- Then the (m,n)-**phenotype** of L is the following positive integer:
$$\mathrm{Ph}_{m,n}(L) \coloneqq \prod_{p \in \mathcal{P}_{m,n}(L)} p^{|L|_p}.$$
If $L = \infty$, we set $\mathrm{Ph}_{m,n}(L) \coloneqq \infty$.

The phenotype $\mathbf{Ph}_{m,n}(\Lambda)$ of a subgroup $\Lambda \leq \mathrm{BS}(m,n)$ is obtained as the phenotype $\mathrm{Ph}_{m,n}([\langle b \rangle : \langle b \rangle \cap \Lambda])$ of the index of $\langle b \rangle \cap \Lambda$ in $\langle b \rangle$.

As we will see, the constructions that we present here towards proving high topological transitivity and high transitivity still rely on constructions similar to [CGLMS22, Theorem 4.17] which allow to connect vertices of equal phenotype, but we carry out such constructions directly at the level of pre-actions since we need to control exactly where points are taken. These constructions will be carried out in details so that no knowledge of the proofs from [CGLMS22] is required.

[2] These labels also arise naturally from the description of Λ as the fundamental group of a graph of group, see [CGLMS22, Proposition 2.9 and Proposition 3.34] for details.

Finally, recall from [CGLMS22, Theorem A] that the perfect kernel of $\mathrm{Sub}(\mathrm{BS}(m,n))$ is the space

$$\mathcal{K}(\mathrm{Sub}(\mathrm{BS}(m,n)) = \{\Lambda \leq \mathrm{BS}(m,n) \colon \Lambda\backslash\mathcal{T} \text{ is infinite}\},$$

where $\mathcal{T}$ is the standard Bass-Serre tree of $\mathrm{BS}(m,n)$, and that the phenotype map yields a partition $\mathcal{K}(\mathrm{BS}(m,n)) = \bigsqcup_{q\in\mathcal{Q}_{n,m}} \mathcal{K}_q$ each of whose piece $\mathcal{K}_q \coloneqq \mathcal{K}(\mathrm{BS}(m,n)) \cap \mathbf{Ph}^{-1}_{n,m}(q)$ is topologically transitive [CGLMS22, Theorem B]. If we denote the remaining piece in the space of infinite index subgroups by $\mathcal{C}_\infty \coloneqq \mathrm{Sub}_{[\infty]}(\mathrm{BS}(m,n)) \smallsetminus \mathcal{K}(\mathrm{BS}(m,n))$, then $\mathcal{C}_\infty$ is empty when $|m| \neq |n|$, and otherwise is it a countable open subset of $\mathbf{Ph}^{-1}(\infty)$.

3.8 High faithfulness and proof of Proposition B

Definition 3.6. Let Γ be a countable group. An action $X \curvearrowleft \Gamma$ is **highly faithful** if for any finite set $F \subseteq \Gamma \smallsetminus \{1\}$, there is $x \in X$ such that for all $\gamma \in F$, we have $x \cdot \gamma \neq x$.

Clearly every highly faithful action is faithful. Moreover, $\Lambda\backslash\Gamma \curvearrowleft \Gamma$ is highly faithful iff there is a sequence of conjugates of Λ converging to the trivial subgroup $\{\mathrm{id}\}$ (equivalently, Λ is not a confined subgroup, see [LBMB22]). We can now state and prove a slight strengthening of Proposition B.

Proposition 3.7. *Let $|m|, |n| \geq 2$, let $\Gamma = \mathrm{BS}(m,n)$. The set of highly faithful actions forms a dense G_δ set inside $\mathcal{K}_q$ for every phenotype q when $|m| \neq |n|$ and for $q = \infty$ when $|m| = |n|$; otherwise, $\mathcal{K}_q$ contains no faithful action. Moreover, if Λ belongs to $\mathcal{C}_\infty$, then the action $\Lambda\backslash\mathrm{BS}(m,n) \curvearrowleft \mathrm{BS}(m,n)$ is not faithful.*

Proof of Proposition 3.7. Observe that a subgroup Λ gives rise to a highly faithful transitive action if and only if its orbit (under the action by conjugation) accumulates on $\{\mathrm{id}\}$. The set of subgroups corresponding to highly faithful actions is thus the G_δ subset obtained from the basic open neighborhoods $\mathcal{V}(\emptyset, F)$ of $\{\mathrm{id}\}$ as follows:

$$\bigcap_{F \Subset \Gamma\smallsetminus\{\mathrm{id}\}} \bigcup_{\gamma\in\Gamma} \mathcal{V}(\emptyset, F)\cdot\gamma.$$

When the phenotype is infinite, by topological transitivity of the action on $\mathcal{K}_\infty$ ([CGLMS22, Theorem 5.14]), there is a dense G_δ subset of $\mathcal{K}_\infty$, consisting of dense orbits, thus of orbits accumulating on $\{\mathrm{id}\} \in \mathcal{K}_\infty$. For $|m| \neq |n|$, the trivial subgroup {id} belongs to the closure of each $\mathcal{K}_q$ (for any phenotype) (by [CGLMS22, Theorem D (2)]). Thus every subgroup with dense orbit in $\mathcal{K}_q$ has its orbit accumulating on {id}. By topological transitivity ([CGLMS22, Theorem 5.14]), these dense orbits form a dense G_δ subset of $\mathcal{K}_q$. On the contrary, when $|m| = |n|$, every subgroup of finite phenotype q contains the non-trivial normal subgroup $\langle\langle b^{s(q,m,n)} \rangle\rangle$ [CGLMS22, Theorem 5.20 (3)]. The corresponding transitive actions are not faithful.

It remains to consider the case $\Lambda \in \mathcal{C}_\infty$, i.e. the case of infinite index subgroups that do not belong to the perfect kernel (which occurs only when $|m| = |n|$). This means that $\Lambda\backslash\Gamma \curvearrowleft \Gamma$ has finitely many b-orbits, each of which being infinite since Λ has infinite index. In particular, there are finitely many b^m-orbits. Let $k \in \mathbb{N}$ be the number of such orbits. Consider the non-oriented Schreier graph of the b^m-action: it consists of k bi-infinite lines, and observe that since $tb^mt^{-1} = b^{\pm m}$ and b commutes with b^m, Γ acts by automorphism on this graph. Since the automorphism group of this non-oriented Schreier graph is the amenable group $(\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z})^k \rtimes \mathrm{Sym}(k)$ and Γ is not amenable, the kernel of the action $\Lambda\backslash\Gamma \curvearrowleft \Gamma$ is not trivial as wanted. □

4 Pre-actions and forest saturations

4.1 Forest saturations of graphs and of pre-actions

In this section, we carry out some constructions from [CGLMS22] in a wider context. The goal is to extend any pre-action to an action, or equivalently to a saturated pre-action.

In order to do this, it is natural to extend the Bass-Serre graph by connecting one vertex which is not yet saturated, i.e. which does not satisfy Equation (3.4), to a new vertex. This new vertex must have a label which respects the transfer equation (3.5), but this leaves out several possibilities for the label in general, which is why we introduce the following notion.

Definition 4.1. A **transfer rule** is a map

$$\mathtt{r} : (\mathbb{Z}_{\geq 1} \cup \{\infty\}) \times \{+,-\} \longrightarrow \mathbb{Z}_{\geq 1} \cup \{\infty\}$$

such that for all $L \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$ and $\epsilon \in \{\pm\}$, if we set $L' = \mathtt{r}(L, \epsilon)$ then the transfer equation is satisfied, namely

$$\frac{L}{\gcd(L,n)} = \frac{L'}{\gcd(L',m)} \text{ if } \varepsilon = + \quad ; \quad \frac{L}{\gcd(L,m)} = \frac{L'}{\gcd(L',n)} \text{ if } \varepsilon = -.$$

Important examples of transfer rules are the **maximum rule** and the **minimum rule**, where $\mathtt{r}(L,\varepsilon)$ is the maximum L', respectively minimum L', such that the above relation is satisfied. The maximum rule can explicitly be described by

$$\mathtt{r}(L,\varepsilon) = |m|\,\frac{L}{\gcd(L,n)} \text{ if } \varepsilon = + \text{ ; } \mathtt{r}(L,\varepsilon) = |n|\,\frac{L}{\gcd(L,m)} \text{ if } \varepsilon = -. \quad (4.2)$$

Let us now recall the notion of forest saturation introduced in [CGLMS22, Definition 4.21].

Definition 4.3. Let $\mathcal{G}_0$ be a (m,n)-graph. A **forest extension** of $\mathcal{G}_0$ is a (m,n)-graph $\mathcal{G}$ containing $\mathcal{G}_0$ and such that:

- the subgraph of $\mathcal{G}$ induced by the vertices of $\mathcal{G}_0$ is exactly $\mathcal{G}_0$;
- the subgraph of $\mathcal{G}$ induced by the other vertices is a forest $\mathcal{F}$;
- each connected component $\mathcal{A}$ of $\mathcal{F}$ is connected to $\mathcal{G}_0$ by a single pair of opposite edges in $\mathcal{G}$. The endpoint of these edges which is in $\mathcal{A}$ is called the **root** of $\mathcal{A}$.

If $\mathcal{G}$ is moreover saturated, we call it a **forest saturation** of $\mathcal{G}_0$.

In the context of Definition 4.3, the **parent** of a vertex v of $\mathcal{F}$ is the unique neighbour of v which is closer to $\mathcal{G}_0$ that v. This parent is in $\mathcal{G}_0$ if v is a root; otherwise, it is the neighbour of v closer to the root of the connected component of $\mathcal{F}$. We say also that v is a **child** of its parent.

We can finally formalize forest extensions where the label of any vertex v' in the forest only depends on the label of its parent v and on the orientation of the edge from the parent to v'.

Definition 4.4. Let $\mathcal{G}$ be a forest extension of $\mathcal{G}_0$ and let $\mathcal{F}$ denote the forest. We say that the extension **satisfies the transfer rule** $\mathtt{r}$ if for any vertex v' in $\mathcal{F}$, if we denote by v its parent then $L(v') = \mathtt{r}(L(v),\varepsilon)$, where ε is the orientation of the edge from v to v'.

It is called **maximal**, respectively **minimal**, if it satisfies the maximum rule, respectively the minimum rule.

Remark 4.5. It follows from Equation (3.4) and Equation (4.2) that in any maximal forest saturation, if v' is a child of v and e is the unique edge with source v and target v', then:

- if e is positive, then the incoming degree of v' is $|m|$;
- if e is negative, then the outgoing degree of v' is $|n|$.

These definitions extend naturally to pre-actions by considering their Bass-Serre graphs, for instance a **maximal forest saturation of a pre-action** α_0 is a pre-action α extending α_0 whose Bass-Serre graph is a maximal forest saturation of the Bass-Serre graph of α. We can now state the main result of this section.

Theorem 4.6. *Let α_0 be a pre-action and let $\mathtt{r}$ be a transfer rule. Then α_0 admits a forest saturation α satisfying $\mathtt{r}$, and any two such saturations are isomorphic. In particular, up to isomorphism, α_0 admits a unique maximal forest saturation and a unique minimal forest saturation.*

The proof of the existence relies on Zorn's lemma, the key point being to constructively extend non-saturated pre-actions by adding one orbit. We isolate this as the following construction.

Let $\alpha_1 = (Y, \beta_1, \tau_1)$ be a pre-action that is not saturated, let Y' be an infinite set disjoint from Y. Since α_1 is not saturated, one of the following cases must occur.

- $Y \setminus \mathrm{dom}(\tau_1) \neq \emptyset$. Let us fix $x \in Y \setminus \mathrm{dom}(\tau_1)$. Let L be the cardinality of the β_1-orbit of x. Let $O' \subseteq Y'$ have cardinality $L' = \mathtt{r}(L, +)$, let $\beta \in \mathrm{Sym}(O')$ be an L'-cycle and define $\beta_2 = \beta_1 \sqcup \beta$.
  Finally, fix some $y \in O'$ and define $\tau : x\langle \beta^n \rangle \to y\langle \beta^m \rangle$ as the unique (β^n, β^m)-equivariant map taking x to y, namely
  $$x\beta^{jn}\tau \coloneqq y(\beta')^{jm} \text{ for all } j \in \mathbb{Z}.$$
  This makes sense precisely because the transfer rule $\mathtt{r}$ satisfies the transfer equation. Letting $\tau_2 = \tau_1 \sqcup \tau$, we have that $\alpha_2 = (Y \sqcup O', \beta_2, \tau_2)$ is as desired. We then say that α_2 is a **positive one point free** $\mathtt{r}$-extension of (α_1, x)
- $Y \setminus \mathrm{rng}(\tau_1) \neq \emptyset$. We fix this time $x \in Y \setminus \mathrm{rng}(\tau_1)$ and proceed similarly as before: let L be the cardinality of the β_1-orbit of x. Let $O' \subseteq Y'$ have cardinality $L' = \mathtt{r}(L, -)$, let $\beta \in \mathrm{Sym}(O')$ be an L'-cycle and define $\beta_2 = \beta_1 \sqcup \beta$.
  Fix some $y \in O'$ and define $\tau : x\langle \beta^m \rangle \to y\langle \beta^n \rangle$ as the unique (β^m, β^n)-equivariant map taking x to y, which makes sense again thanks to the fact that $\mathtt{r}$ satisfies the transfer equation. Letting $\tau_2 = \tau_1 \sqcup \tau$, we have again that $\alpha_2 = (Y \sqcup O', \beta_2, \tau_2)$ is as desired. We then say that α_2 is a **negative one point free** $\mathtt{r}$-extension of (α_1, x).

Lemma 4.7. *Let $\varphi : X_1 \to X_2$ be an isomorphism between non-saturated pre-actions $\alpha_1 = (X_1, \beta_1, \tau_1)$ to $\alpha_2 = (X_2, \beta_2, \tau_2)$. Assume α_1' is a one orbit free* $\mathtt{r}$*-extension of (α_1, x_1) and α_2' is a one orbit free* $\mathtt{r}$*-extension of $(\alpha_2, \varphi(x_1))$ both positive or both negative. Then, φ extends to an isomorphism from α_1' to α_2'.*

Proof. Set $\alpha_i' = (X_i', \beta_i', \tau_i')$ for $i = 1, 2$ and $x_2 = \varphi(x_1)$. We treat the positive case. The negative case is similar and left to the reader.

Recall that $X_i' = X_i \sqcup y_i \langle \beta_i' \rangle$ for $i = 1, 2$. The orbits of y_1, y_2 both have cardinal $M = \mathtt{r}(L, +)$ where $L = |x \langle \beta_1 \rangle| = |\varphi(x) \langle \beta_2 \rangle|$. Thus, one can extend φ to a bijection $\varphi' : X_1' \to X_2'$ by setting

$$\varphi'(y_1 (\beta_1')^k) \coloneqq y_2 (\beta_2')^k \text{ for all } k \in \mathbb{Z}.$$

It remains to check that φ' is an isomorphism between α_1' and α_2'. The relation $\varphi'(x\beta_1') = \varphi'(x)\beta_2'$, for every $x \in X_1'$, is obvious. Moreover, one has

$$\mathrm{dom}(\tau_i') = \mathrm{dom}(\tau_i) \sqcup x_i \langle \beta_i^n \rangle \qquad \text{for } i = 1, 2,$$

so that φ' maps $\mathrm{dom}(\tau_1')$ onto $\mathrm{dom}(\tau_2')$. Finally, for every $k \in \mathbb{Z}$, one has

$$\begin{aligned} \varphi'(x_1 \beta_1^{kn} \tau_1') &= \varphi'(x_1 \tau_1' (\beta_1')^{km}) = \varphi'(y_1 (\beta_1')^{km}) = y_2 (\beta_2')^{km} \\ &= y_1 \tau_2' (\beta_2')^{km} = \varphi'(x_1) \beta_2^{kn} \tau_2' = \varphi'(x_1 \beta_1^{kn}) \tau_2'. \end{aligned}$$

Thus $\varphi'(x\tau_1') = \varphi'(x)\tau_2'$ for all $x \in \mathrm{dom}(\tau_1')$; the proof is complete. □

We now have all the tools in order to prove Theorem 4.6.

Proof of Theorem 4.6. Let α_0 be a non-saturated pre-action on a set X, we first need to show that α_0 admits a saturated forest extension satisfying the transfer rule $\mathtt{r}$.

First observe that any forest extension of α_0 must have an underlying set which is countable or finite if X was finite, and of the same cardinality as X if X was infinite. Let us then fix a set Z containing X whose cardinality is strictly greater than that of X.

Zorn's lemma provides us a forest extension $\alpha = (Y, \beta, \tau)$ of α_0 satisfying $\mathtt{r}$ whose underlying set is a subset of Z and which is maximal among such extensions. Assume by contradition α is not saturated, then by our previous observation $Z \smallsetminus Y$ is infinite and we can thus construct a positive or negative one point free $\mathtt{r}$-extension of α with underlying set contained in Z

as explained right after the statement of Theorem 4.6. This contradicts the maximality of α, which is thus saturated as desired.

We now prove uniqueness: assume α_1, α_2 are forest saturations of α_0 satisfying $\mathtt{r}$. For $i = 0, 1, 2$, write $\alpha_i = (X_i, \beta_i, \tau_i)$. Consider the set of partial isomorphisms, from a forest extension $\alpha_1' = (X_1', \beta_1', \tau_1')$ contained in α_1 to a forest extension $\alpha_2' = (X_2', \beta_2', \tau_2')$ contained in α_2, that fix X_0 pointwise. This is a non-empty inductive poset. Hence, it contains a maximal element $\varphi : X_1' \to X_2'$ by Zorn's Lemma.

Let us prove by contradiction that $\alpha_1' = \alpha_1$ and $\alpha_2' = \alpha_2$; this will complete the proof. If this is not the case, we may assume that $\alpha_1' \neq \alpha_1$ (the case $\alpha_2' \neq \alpha_2$ is similar). In this case, α_1' is non-saturated, hence one can find $x \in X_1'$ and a restriction α_1'' of α_1 which is a one-orbit free $\mathtt{r}$-extension of (α_1', x_1). As $\varphi : X_1' \to X_2'$ is an isomorphism the orbit $\varphi(x)\langle\beta_2\rangle$ lies in X_2' while $\varphi(x)\tau_2^{\varepsilon}\langle\beta_2\rangle$ does not. So, the restrictions of β_2, τ_2 to $X_2'' \coloneqq X_2' \sqcup \varphi(x)\tau_2^{\varepsilon}\langle\beta_2\rangle$ give a one-orbit free $\mathtt{r}$-extension α_2'' of α_2', contained in α_2. Notice that $x\langle\beta_1\rangle$ and $\varphi(x)\langle\beta_2\rangle$ have the same cardinal since φ is an isomorphism. Lemma 4.7 now implies that φ extends to an isomorphism from α_1'' to α_2'', contradicting the maximality of φ. □

Remark 4.8. When $\mathbf{BS}(\alpha_0)$ is countable, one can in fact obtain any forest saturation α by a sequence of one orbit free extensions. More precisely, there exists a sequence

$$\alpha_0 \subseteq \alpha_1 \subseteq \cdots \subseteq \alpha_n \subseteq \alpha_{n+1} \subseteq \cdots$$

of one orbit free extensions such that $\alpha = \bigcup_{i\geq 0} \alpha_i$. To see this, it suffices to enumerate the edges of the forest wisely, e.g. by doing a simultaneous breadth-first exploration of each connected component the forest.

4.2 Stabilizer of a pointed pre-action

Let $\Gamma \coloneqq \mathrm{BS}(m, n)$. Let $\alpha = (X, \beta, \tau)$ be a pre-action of Γ. Every path c in $\mathbf{Sch}(\alpha)$ between two vertices is labelled by a word on letters t, t^{-1}, b, b^{-1}; this word represents an element of Γ that we denote $\Psi(c)$. Notice that we have $\mathtt{t}(c) = \mathtt{s}(c) \cdot \alpha(\Psi(c))$ in x.

In particular, for every choice of a basepoint $x_0 \in X$, the map Ψ defines a group homomorphism $\Psi_{x_0} : \pi_1(\mathbf{Sch}(\alpha), x_0) \to \Gamma$ from the fundamental group of the connected component of x_0 to Γ.

Definition 4.9. The image of Ψ_{x_0} is called the α-**stabilizer** of x_0, and denoted $\mathrm{Stab}_\alpha(x_0)$.

If x_1 is another vertex in the same connected component of $\mathbf{Sch}(\alpha)$, the stabilizers of x_0 and x_1 are conjugate: one has

$$\mathrm{Stab}_\alpha(x_0) = \Psi(c)\,\mathrm{Stab}_\alpha(x_1)\,\Psi(c)^{-1},$$

where c is any path from x_0 to x_1. If the pre-action α is saturated, i.e. it is a genuine Γ-action, then $X \curvearrowleft^\alpha \Gamma$ is isomorphic to $\mathrm{Stab}_\alpha(x_0)\backslash\Gamma \curvearrowleft \Gamma$ and $\mathrm{Stab}_\alpha(x_0)$ is the usual stabilizer of x_0 with respect to the action α.

Proposition 4.10 (Stabilizer of a one orbit free extension)**.** *Let α_0 be a non-saturated pre-action and α be a one orbit free extension of α_0 from (x, L_x) to (y, L_y). Set $\varepsilon = 1$ if this extension is positive and $\varepsilon = -1$ if it is negative.*

1. *If α_0 has finite phenotype, then the α-stabilizer of the basepoint x_0 is generated by the α_0-stabilizer and the element $\Psi(c)\left(t^\varepsilon b^{L_y} t^{-\varepsilon}\right)\Psi(c)^{-1}$, where c is any path from x_0 to x in $\mathbf{Sch}(\alpha_0)$. In formula:*

$$\mathrm{Stab}_\alpha(x_0) = \left\langle \mathrm{Stab}_{\alpha_0}(x_0), \Psi(c)\left(t^\varepsilon b^{L_y} t^{-\varepsilon}\right)\Psi(c)^{-1}\right\rangle.$$

2. *If the extension α is maximal, or if the phenotype is infinite, then the stabilizers coincide:*

$$\mathrm{Stab}_\alpha(x_0) = \mathrm{Stab}_{\alpha_0}(x_0).$$

Proof. Up to conjugating the situation by $\Psi(c)$, it is equivalent to prove $\mathrm{Stab}_\alpha(x) = \left\langle \mathrm{Stab}_{\alpha_0}(x), t^\varepsilon b^{L_y} t^{-\varepsilon}\right\rangle$ in Item 1, respectively $\mathrm{Stab}_\alpha(x) = \mathrm{Stab}_{\alpha_0}(x)$ in Item 2. Clearly, the free group $\pi_1(\mathbf{Sch}(\alpha), x)$ is obtained from $\pi_1(\mathbf{Sch}(\alpha_0), x)$ by adding the following generators:

- if the extension is positive, the cycles c_j, for $j \in \mathbb{Z}$, based at x and labelled by $tb^{jm}t^{-1}b^{-jn}$;
- if the extension is negative, the cycles c'_j, for $j \in \mathbb{Z}$, based at x and labelled by $t^{-1}b^{jn}tb^{-jm}$;
- if the phenotype is finite, the cycle c based at x and labelled by $t^\varepsilon b^{L_y} t^{-\varepsilon}$.

(these cases are not mutually exclusive). Notice that all the c_j have trivial image under Ψ_{x_0}, hence $\mathrm{Stab}_\alpha(x) = \mathrm{Stab}_{\alpha_0}(x)$ if the phenotype is infinite, concluding the proof in this case.

If the phenotype is finite, we get $\mathrm{Stab}_\alpha(x) = \left\langle \mathrm{Stab}_{\alpha_0}(x), t^\varepsilon b^{L_y} t^{-\varepsilon}\right\rangle$, so Item 1 is established. Finally, if the extension is morover maximal, it suffices to show that $t^{\varepsilon}b^{L_y}t^{-\varepsilon}$ already belongs to $\mathrm{Stab}_{\alpha_0}(x_0)$. In the positive case, one has $L_y = m\frac{L_x}{\gcd(L_x,n)}$, so we have in Γ:

$$\left(tb^{L_y}t^{-1}\right) = (tb^m t^{-1})^{\frac{L_x}{\gcd(L_x,n)}} = (b^n)^{\frac{L_x}{\gcd(L_x,n)}} = (b^{L_x})^{\left(\frac{n}{\gcd(L_x,n)}\right)}.$$

As b^{L_x} belongs to $\mathrm{Stab}_{\alpha_0}(x)$, this case is over. The negative case is similar. □

Corollary 4.11. *Let (α_0, x_0) be a pointed transitive pre-action and α be the associated maximal forest saturation. Then $\mathrm{Stab}_{\alpha}(x_0) = \mathrm{Stab}_{\alpha_0}(x_0)$. In particular, the maximal forest saturation identifies with $\mathrm{Stab}_{\alpha_0}(x_0)\backslash\Gamma \curvearrowleft \Gamma$.*

Proof. As in Remark 4.8, the maximal forest saturation is obtained by a sequence

$$\alpha_0 \subseteq \alpha_1 \subseteq \cdots \subseteq \alpha_n \subseteq \alpha_{n+1} \subseteq \cdots$$

of **maximal** one orbit free extensions such that $\alpha = \bigcup_{i\geq 0} \alpha_i$. By Proposition 4.10 (item 2), the sequence of natural injections

$$\mathrm{Stab}_{\alpha_0}(x_0) \hookrightarrow \mathrm{Stab}_{\alpha_1}(x_0) \hookrightarrow \cdots \hookrightarrow \mathrm{Stab}_{\alpha_n}(x_0) \hookrightarrow \mathrm{Stab}_{\alpha_{n+1}}(x_0) \hookrightarrow \cdots$$

consists only in identity maps. Thus $\mathrm{Stab}_{\alpha}(x_0) = \mathrm{Stab}_{\alpha_0}(x_0)$. □

More generally, Proposition 4.10 allows to describe the stabilizer of any forest saturation α of α_0 satisfying a transfer rule: just write the saturation as a sequence of one-orbit free extensions as before; then Proposition 4.10 provides a list of generators to add to $\mathrm{Stab}_{\alpha_0}(x_0)$ so as to obtain $\mathrm{Stab}_{\alpha}(x_0)$.

4.3 Leaving finite pre-actions through their maximal forest saturation

The following technical lemma underlies both our results on high transitivity and on high topological transitivity. It allows us to get out of any finite transitive pre-action through its maximal forest saturation, with some additional control on the labels that we end up at.

Even better, we can leave *simulateneously* finitely many finite transitive pre-actions. The key observation, already apparent in [MS13, Claim 3.5], is that once a reduced path has arrived in the maximal forest saturation, it will stay there (see Claim 4.13). This allows us to inductively construct the path we seek, working out each pre-action one by one.

Lemma 4.12. *Let (α_1^0, x_1),$\ldots$, (α_k^0, x_k) be transitive non-saturated pointed pre-actions with finite Bass-Serre graphs. Let α_1,$\ldots$, α_k be the maximal forest saturations of these pre-actions, with underlying sets $X_1, \ldots, X_k$. Recall that $\mathcal{T}$ is the Bass-Serre tree of* $\mathrm{BS}(m,n)$. *For every $i \in \{1, \ldots, k\}$, consider the graph morphism $\pi_i : \mathcal{T} \to \mathbf{BS}(\alpha_i)$ associated to the Γ-equivariant map $\hat{\pi}_i : \Gamma \to X_i$ taking* id *to x_i.*

Then there is a reduced path c in $\mathcal{T}$ from the base vertex $\langle b \rangle$ to a vertex $g\langle b \rangle$, whose last edge f is positive and such that for all $i \in \{1, \ldots, k\}$:

(1) the edge $\pi_i(f)$ separates $\mathbf{BS}(\alpha_i^0)$ from $v_i \coloneqq \pi_i(g\langle b \rangle) = x_i \alpha_i(g\langle b \rangle)$;
(2) if the phenotype q_i of α_i^0 is finite, then the label of the terminal vertex v_i satisfies
   - *$|L(v_i)|_p = |m|_p$ for all prime p such that $|m|_p < |n|_p$, and*
   - *$|L(v_i)|_p = \max(|q_i|_p, |m|_p)$ for all prime p such that $|m|_p = |n|_p$.*

Moreover, any extension of c using only positive edges satisfies the same conclusions.

Proof. We first take care of condition (1). For every $i \in \{1, ..., k\}$, denote by $\mathcal{F}_i$ the forest that was added to the Bass-Serre graph of α_i^0 when building its maximal forest saturation α_i. We rely on the following crucial claim.

Claim 4.13. *Let $i \in \{1, \ldots, k\}$. If c is a reduced path in $\mathcal{T}$ starting at the base vertex $\langle b \rangle$ and whose projection $\pi_i(c)$ contains an edge $\pi_i(e)$ in $\mathcal{F}_i$, then the subpath of $\pi_i(c)$ starting with $\pi_i(e)$ has no backtracking, hence it follows a geodesic in $\mathcal{F}_i$.*

Proof of the claim. Without loss of generality, we may and will assume that e is the first edge in c such that $\pi_i(e)$ lies in $\mathcal{F}_i$. This implies that the source $\mathtt{s}(\pi_i(e))$ lies in $\mathbf{BS}(\alpha_i^0)$.

Recall from [CGLMS22] (see the proof of Lemma 4.21), that the maximal forest saturation is constructed by iteratively adding at each step as many new vertices v as possible so that

- v is connected to the graph constructed at the previous step by a unique edge a and
- the label of v satisfies $L(v) = mL(a)$ if a is positive and $L(v) = nL(a)$ if a is negative.

The second item implies $\deg_{\mathrm{in}} v = m$ if a is positive, and $\deg_{\mathrm{out}} v = n$ if a is negative (however, one may fail to have both $\deg_{\mathrm{in}} v = m$ and $\deg_{\mathrm{out}} v = n$). Since $\mathcal{T}$ is a regular tree with incoming degree m and outgoing degree n, this may be reformulated as: for any pull-back $\tilde{v} \in \mathcal{T}$ of v, the map π_i is injective on the set $E_{\tilde{v},a}$ of edges attached to $\tilde{v}$ which have the same orientation as a.

Assume by contradiction that $\pi_i(c)$ does backtrack at some point. Let e_1, e_2 be the first pair of consecutive edges of c after e whose π_i-images form a backtracking and let $\tilde{v} := \mathtt{t}(e_1) = \mathtt{s}(e_2)$ be their common vertex. Let $v := \pi_i(\tilde{v}) = \mathtt{t}(\pi_i(e_1)) = \mathtt{s}(\pi_i(e_2))$. Since $\pi_i(c)$ has no backtracking between $\pi_i(e)$ and $\pi_i(e_1)$, the source $\mathtt{s}(\pi_i(e_1))$ is in the same connected component as the image of the base-point in $\mathbf{BS}(\alpha_i) \smallsetminus \pi_i(e_1)$, i.e. the target $v = \mathtt{t}(\pi_i(e_1))$ appears after $\mathtt{s}(\pi_i(e_1))$ in the construction of the maximal forest saturation and $\pi_i(e_1)$ is the unique edge connecting v to the previously constructed stages.

Since $\pi_i(e_1) = \pi_i(\bar{e}_2)$, the edges $e_1, \bar{e}_2$ have the same orientation and they belong to $E_{\tilde{v},e_1}$, a set on which π_i is injective. This contradicts the fact that c is reduced. $\square_{\text{claim}}$

We can now construct the path c satisfying condition 1 as the last term of an inductive construction $(c_i)_{i=0}^k$ with c_i extending c_{i-1} so that for all $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, i\}$, the path $\pi_j(c_i)$ ends in $\mathcal{F}_j$. The first term c_0 is chosen to be any edge with origin $\langle b \rangle$ in $\mathcal{T}$. For the inductive step, assume c_i has been constructed for some $i < k$. Let e_i be the last edge of c_i. By [FLMMS22, Lemma 2.17] we can extend $\pi_{i+1}(e_i)$ to a reduced path $\pi_{i+1}(e_i)q_{i+1}$ in $\mathbf{BS}(\alpha_{i+1})$ whose terminal edge disconnects the graph $\mathbf{BS}(\alpha_{i+1})$ and points towards a tree. Since $\mathbf{BS}(\alpha_{i+1})$ has no vertex of degree ≤ 1, this tree is unbounded. As the complement of $\mathcal{F}_{i+1}$ is finite, we can extend q_{i+1} if necessary so that it ends in $\mathcal{F}_{i+1}$.

Let $e_i\tilde{q}_{i+1}$ be a lift in $\mathcal{T}$ of $\pi_{i+1}(e_i)q_{i+1}$ starting by e_i, then $c_{i+1} := c_i\tilde{q}_{i+1}$ has the following properties:

- c_{i+1} is a reduced path;
- $\pi_{i+1}(c_{i+1})$ ends in $\mathcal{F}_{i+1}$ by construction;
- for every $j \in \{1, \ldots, i\}$, the path $\pi_j(c_{i+1})$ ends in $\mathcal{F}_j$, by Claim 4.13.

This concludes the construction of $c = c_k$ if the last edge of c_k is positive. If not, we extend c_k to $c = c_k f$ by adding a last positive edge f without reduction: this is possible since the in-degree and the out-degree of every vertex in $\mathcal{T}$ is ≥ 2 when $|m|, |n| > 1$. Claim 4.13 ensures that $c_k f$ continues to satisfy condition 1, namely that $\pi_i(f) \in \mathcal{F}_i$ for all $i \in \{1, \ldots, k\}$.

If the phenotype is infinite, we are done since condition 2 is empty in this case. Let us thus assume that the phenotype is finite. Notice that by Claim 4.13 any reduced extention of c keeps satisfying condition 1. Thus it suffices to extend c to a longer reduced path such that it also satisfies condition 2.

Consider first a geodesic ray r consisting of positive edges and starting at the terminal vertex $\mathtt{t}(c)$ of c. The path cr and its projection $\pi_i(cr)$ are reduced, since the last edge in c was positive. For every i, the projection $\pi_i(r)$ lies in $\mathcal{F}_i$ and for each edge e of $\pi_i(r)$, the source of e was built before its target in the maximal forest saturation.

For each edge e of $\pi_i(r)$, and for any prime p such that $|m|_p \leq |n|_p$, we claim that:

$$|L(\mathtt{t}(e))|_p = \max\big(|m|_p, |L(\mathtt{s}(e))|_p - (|n|_p - |m|_p)\big). \tag{4.14}$$

Indeed, recall the label transfer equation (3.2):

$$\max(|L(\mathtt{s}(e))|_p - |n|_p, 0) = |L(e)|_p = \max(|L(\mathtt{t}(e))|_p - |m|_p, 0).$$

If $|L(\mathtt{s}(e))|_p \leq |n|_p$, then $|L(e)|_p = 0$, hence the maximality in the construction of the maximal forest saturation gives $|L(\mathtt{t}(e))|_p = |m|_p$. If $|L(\mathtt{s}(e))|_p > |n|_p$, then $|L(e)|_p = |L(\mathtt{s}(e))|_p - |n|_p > 0$, hence Equation (3.2) gives $|L(\mathtt{t}(e))|_p = |L(\mathtt{s}(e))|_p - |n|_p + |m|_p$. This proves Equation (4.14).

Now, for any edge e in $\pi_i(r)$ and all prime p such that $|m|_p = |n|_p$, we have

$$\begin{aligned} |L(\mathtt{t}(e))|_p &= \max\big(|m|_p, |L(\mathtt{s}(e))|_p\big), \text{ by Equation (4.14)} \\ &= \max\big(|m|_p, |q_i|_p\big), \text{ by definition of the phenotype.} \end{aligned}$$

Finally, Equation (4.14) yields that for any vertex $g\langle b\rangle$ far enough in the ray r, for all prime p such that $|m|_p < |n|_p$ and all $i \in \{1, \dots, k\}$ we have $|L(\pi_i(g\langle b\rangle))|_p = |m|_p$.

So replacing c by cr', where r' is any sufficiently long initial segment of r, we get that c is the desired path.

The last statement about extensions of c by positive edges follows from Equation (4.14), using the same argument as before. This finishes the proof of Lemma 4.12. □

5 High topological transitivity results

Recall that for any phenotype q (finite or not), $\mathcal{K}_q(\mathrm{BS}(m, n))$ is a perfect compact space [CGLMS22, Remark 5.12].

Theorem 5.1 (multiple topological transitivity)**.** *Let m, n be integers such that $|m|, |n| \geq 2$. Then for every phenotype $q \in \mathcal{Q}_{m,n}$ the action by conjugation of* $\mathrm{BS}(m,n)$ *on the invariant subspace* $\mathcal{K}_q(\mathrm{BS}(m,n))$ *is highly topologically transitive.*

Proof. Let d be a positive integer and let $U_1, \ldots, U_{2d}$ be nonempty open subsets of $\mathcal{K}_q(\mathrm{BS}(m,n))$. We want to prove that there exists $\gamma \in \mathrm{BS}(m,n)$ such that,

$$\forall i \in \{1, \ldots, d\}, \quad U_i \cdot \gamma \cap U_{i+d} \neq \emptyset$$

We split this proof into several steps.

Step 1. Preliminary reductions. Up to shrinking the U_i's, we may as well assume they are all of the form

$$U_i = \mathcal{N}([\alpha_i, x_i], R)$$

for some $R > 0$, i.e. all the (classes of) pointed transitive actions with the same R-ball of Schreier graph as some pointed transitive action (α_i, x_i) on an infinite set X_i. Recall that X_i contains infinitely many b-orbits (in other words, $\mathbf{BS}(\alpha_i)$ is infinite) since we are in the perfect kernel.

Then, for any $i \in \{1, \ldots, 2d\}$, let B_i be the union of the b-orbits that meet the ball $B(x_i, R)$ in the Schreier graph of α_i and take the restriction $\alpha_i^0 \coloneqq \alpha_{i \restriction B_i}$. Notice that the Bass-Serre graph $\mathbf{BS}(\alpha_i^0)$ is finite and, consequently, α_i^0 is not an action. We may moreover assume that α_i is the maximal forest-saturation, (as defined in Section 4) of α_i^0. Indeed, $B(x_i, R)$ is contained in B_i. Therefore, replacing α_i by the maximal forest saturation of α_i^0 doesn't change the ball $B(x_i, R)$ in the Schreier graph, hence doesn't change $\mathcal{N}([\alpha_i, x_i], R)$.

Step 2. Going out of balls "uniformly". We are now in the situation of Lemma 4.12. So, let us consider the graph morphism $\pi_i : \mathcal{T} \to \mathbf{BS}(\alpha_i)$ and pick a reduced path c in $\mathcal{T}$, from $\langle b \rangle$ to a vertex $g \langle b \rangle$, given by this lemma. Notice that, up to changing the representative g of the coset $g \langle b \rangle$, we may assume that its normal form w, as defined in Section 3.6 ends on the right by $t^{\pm 1}$.

Notice also that w defines a reduced path from id to g in $\mathbf{Cay}(\mathrm{BS}(m,n))$. This path projects onto c, since c is the unique reduced path from $\langle b \rangle$ to $g \langle b \rangle$ in $\mathcal{T}$ and w is a normal form. Observe that, since the last edge of c is positive, w ends by a positive power of t .

In $\mathbf{Sch}(\alpha_i)$, consider the path from x_i labeled by w, and its target $y_i \coloneqq x_i\alpha_i(g)$. Notice that its projection in $\mathbf{BS}(\alpha_i)$ is the path $\pi_i(c)$. Set moreover $v_i \coloneqq \pi_i(\mathtt{t}(c))$, which is also the projection of y_i.

Step 3. From actions to pre-actions. Let $\mathcal{G}_i$ be the subgraph of $\mathbf{BS}(\alpha_i)$ consisting of the union of $\mathbf{BS}(\alpha_i^0)$ with $\pi_i(c)$. By Condition (1) from Lemma 4.12, the vertex v_i is disconnected from the rest of $\mathcal{G}_i$ by the last edge of $\pi_i(c)$. Moreover, by the maximal transfer rule, as the last edge of c is positive, the label $L(v_i)$ is a multiple of $|m|$ and hence v_i has ingoing degree $|m|$.

By contrast, we have seen that v_i has degree 1 in $\mathcal{G}_i$.

Consider the subgraph of $\mathbf{Sch}(\alpha_i)$ consisting of the b-orbits that are shrunk to vertices of $\mathcal{G}_i$ and all the edges that are shrunk to edges of $\mathcal{G}_i$. It is the Schreier graph of a pre-action that we denote ξ_i. Observe that ξ_i extends α_i^0 and that its Bass-Serre graph is $\mathcal{G}_i$.

The former discussion about $\mathcal{G}_i$ shows that the $\xi_i(b)$-orbit of y_i does not intersect the domain of $\xi_i(t)$ and intersects the target of $\xi_i(t)$ only along the $\xi_i(b)^m$-orbit of y_i.

Step 4. Conclusion in case of infinite phenotype. In this case, the orbits $y_i\xi_i(\langle b\rangle)$, for $i \in \{1,\dots,2d\}$, are moreover infinite. For each $i \in \{1,\dots,d\}$, we shall weld ξ_i and ξ_{i+d} into a new transitive preaction η_i as follows. The domain of η_i is the disjoint union of those of ξ_i and of ξ_{i+d} and the bijection $\eta_i(b)$ is the disjoint union of $\xi_i(b)$ and $\xi_{i+d}(b)$. In order to define $\eta_i(t)$, let us start with $\eta_i(t) \coloneqq \xi_i(t) \sqcup \xi_{i+d}(t)$. Then, since $y_i\xi_i(b)^{kn} \notin \operatorname{dom}(\xi_i(t))$ and $y_{i+d}\xi_{i+d}(b)^{mk+1} \notin \operatorname{rng}(\xi_{i+d}(t))$, we can set

$$y_i\xi_i(b)^{kn}\eta_i(t) \coloneqq y_{i+d}\xi_{i+d}(b)^{mk+1}$$

for all $k \in \mathbb{Z}$. We get in particular $y_i\eta_i(t) = y_{i+d}\xi_{i+d}(b) = y_{i+d}\eta_i(b)$.

Afterwards, we extend η_i to a transitive $\mathrm{BS}(m,n)$-action $\tilde{\eta}_i$, e.g. using the maximal forest saturation. Since $\tilde{\eta}_i$ extends ξ_i, the pointed action $[\tilde{\eta}_i, x_i]$ belongs to $U_i = \mathcal{N}([\alpha_i, x_i], R)$, and since $\tilde{\eta}_i$ extends ξ_{i+d}, the pointed action $[\tilde{\eta}_i, x_{i+d}]$ belongs to $U_{i+d} = \mathcal{N}([\alpha_{i+d}, x_{i+d}], R)$. By construction,

$$x_i\tilde{\eta}_i(g) = y_i \text{ and } y_i\tilde{\eta}_i(t)\tilde{\eta}_i(b)^{-1} = y_{i+d} \text{ and } x_{i+d}\tilde{\eta}_i(g) = y_{i+d}.$$

It follows that by letting

$$\gamma \coloneqq gtb^{-1}g^{-1}$$

we have $x_i\tilde{\eta}_i(\gamma) = x_{i+d}$ for every $i \in \{1, \dots, d\}$. Since $[\tilde{\eta}_i, x_i] \in U_i$ and $[\tilde{\eta}_i, x_{i+d}] \in U_{i+d}$, this proves that for all $i \in \{1, \dots, d\}$, we have $U_i\gamma \cap U_{i+d} \neq \emptyset$ as wanted.

Step 4 bis. In case of finite phenotype: U-turn. We restart from the end of Step 3 and, from now on, we assume that the phenotype is finite. Let $y_i^0 \coloneqq y_i \cdot \xi_i(b) \neq y_i$. Define inductively a sequence of labels $(l_i^j)_j$ as follows: $l_i^0 \coloneqq L(v_i)$, which is also the cardinality of the $\xi_i(b)$-orbit of y_i^0, and then for all $j \geq 0$, l_i^{j+1} is the smallest positive integer satisfying the equation

$$\frac{l_i^{j+1}}{\gcd(l_i^{j+1}, n)} = \frac{l_i^j}{\gcd(l_i^j, m)}. \tag{5.2}$$

Claim 5.3. *From a certain rank r, we have $l_i^j = q$ for all i and for all $j \geq r$ (i.e., we eventually reach the phenotype for all i).*

Let us finish our argument before proving the claim: fix r as above. We extend ξ_i by first adding a collection of r extra $\xi_i(b)$-orbits $(O_i^j)_{1\leq j\leq r}$, with each O_i^j of cardinality l_i^j. Then pick a point y_i^j in each O_i^j. The transfer equation (5.2) guarantees that the $\xi_i(b^n)$-orbit of y_{i+1} and the $\xi_i(b^m)$-orbit of y_i have the same cardinality. Morover, as $|m| \geq 2$, observe that y_i and y_i^0 belong to disjoint $\xi_i(b)^m$-orbits, so that y_i^0 is not in the target of $\xi_i(t)$. Therefore, we can extend $\xi_i(t)$ by letting for every $0 \leq j \leq r-1$ and $k \in \mathbb{Z}$:

$$y_i^{j+1} \cdot \xi_i(t) \coloneqq y_i^j \quad \text{and} \quad y_i^{j+1}\xi_i(b)^{nk} \cdot \xi_i(t) \coloneqq y_i^j\xi_i(b)^{mk}.$$

We now have $2d$ transitive preactions ξ_i which extend the Schreier balls of radius R centered at x_i of our original actions α_i, and a large word wbt^{-r}, with $r \geq 0$, such that for all $i \in \{1, \dots, 2d\}$, by following the word wbt^{-r} and starting at x_i, we end up at the point y_i^r whose b-orbit's cardinality is equal to the phenotype q.

Proof of Claim 5.3. Our claim is equivalent to stating that, from a certain rank, $\left|l_i^j\right|_p = 0$ for all i and for every prime p that does not divide the phenotype. Let us thus fix such a prime p.

First, the Transfer Equation (3.2) implies:

$$\max(\left|l_i^{j+1}\right|_p - |n|_p, 0) = \max(\left|l_i^j\right|_p - |m|_p, 0). \tag{5.4}$$

We consider three cases, depending on the sign of $|m|_p - |n|_p$.

- If $|m|_p < |n|_p$, then Lemma 4.12 (Condition (1)) gives $|l_i^0|_p = |L(v_i)|_p = |m|_p$, and Equation (5.4) allows $\left|l_i^{j+1}\right|_p = 0$ starting from $j = 0$ (which happens since we choose l_i^{j+1} as small as possible).
- If $|m|_p > |n|_p$, then, as long as $\left|l_i^j\right|_p > 0$, we have:

$$\left|l_i^{j+1}\right|_p = \begin{cases} \left|l_i^j\right|_p - |m|_p + |n|_p & \text{if } \left|l_i^j\right|_p > |m|_p\,; \\ 0 & \text{if } \left|l_i^j\right|_p \leq |m|_p\,. \end{cases}$$

  This follows from equation (5.4). Hence, from a certain rank r_p, we get $\left|l_i^j\right|_p = 0$.
- If $|m|_p = |n|_p$, we have $|l_i^0|_p \leq |m|_p$ since p is assumed not to divide the phenotype. As before, Equation (5.4) allows $\left|l_i^j\right|_p = 0$ starting from $j = 1$.

Taking r larger than the finitely many r_p's arising when $|m|_p > |n|_p$, the claim is proven. $\square_{\text{claim}}$

Step 5. Final welding. For every $i \in \{1, ..., d\}$, we weld ξ_i and ξ_{i+d} into a new transitive preaction η_i as follows. The domain of η_i is the disjoint union of the domain of ξ_i, the domain of ξ_{i+d}, and a set O_i of cardinal equal to $\frac{nq}{\gcd(n,q)}$, i.e. the least common multiple n and q. The bijection $\eta_i(b)$ is the disjoint union of $\xi_i(b)$, and $\xi_{i+d}(b)$, and of a cycle of length $\frac{nq}{\gcd(n,q)}$ on O_i.

In order to define $\eta_i(t)$, let us start with $\eta_i(t) \coloneqq \xi_i(t) \sqcup \xi_{i+d}(t)$. Then, we fix some $z_i \in O_i$ and we want to extend $\eta_i(t)$ in such a way that z_i is sent onto y_i^r and $z_i\eta_i(b)$ is sent onto y_{i+d}^r. In order to enforce the equivariance condition in the definition of preactions, observe that the following orbits all have the same cardinal, namely $\frac{q}{\gcd(n,q)} = \frac{q}{\gcd(m,q)}$: the $\eta_i(b)^n$-orbits of z_i and of $z_i\eta_i(b)$, the $\eta_i(b)^m$-orbits of y_i^r and of y_{i+d}^r. Moreover, since the cardinality of the $\eta_i(b)$-orbit of z_i is a multiple of n, the $\eta_i(b)^n$-orbits of z_i and $z_i\eta_i(b)$ are disjoint. This allows us to extend $\eta_i(t)$ as wanted by setting

$$\left(z_i\eta_i(b)^{nk}\right) \cdot \eta_i(t) \coloneqq y_i^r\eta_i(b)^{mk} \quad \text{and} \quad \left(z_i\eta_i(b)^{nk+1}\right) \cdot \eta_i(t) \coloneqq y_{i+d}^r\eta_i(b)^{mk}$$

for all $k \in \mathbb{Z}$. Then, we extend η_i to a transitive $\mathrm{BS}(m,n)$-action $\tilde{\eta}_i$, e.g. using the maximal forest saturation as described in Section 4.

Since $\tilde{\eta}_i$ extends ξ_i, the pointed action $[\tilde{\eta}_i, x_i]$ belongs to U_i, and since $\tilde{\eta}_i$ extends ξ_{i+d}, the pointed action $[\tilde{\eta}_i, x_{i+d}]$ belongs to U_{i+d}. By construction,

$$x_i\tilde{\eta}_i(wbt^{-(r+1)}) = z_i \text{ and } x_{i+n}\tilde{\eta}_i(wbt^{-(r+1)}) = z_i\tilde{\eta}_i(b).$$

It follows that by letting

$$\gamma := \left(wbt^{-(r+1)}\right) b \left(wbt^{-(r+1)}\right)^{-1}$$

we have $x_i\tilde{\eta}_i(\gamma) = x_{i+d}$ for every $i \in \{1, \dots, d\}$. Since $[\tilde{\eta}_i, x_i] \in U_i$ and $[\tilde{\eta}_i, x_{i+d}] \in U_{i+d}$, this proves that for all $i \in \{1, \dots, d\}$, we have $U_i\gamma \cap U_{i+d} \neq \emptyset$ as wanted. $\square$

6 High transitivity results

In this section, we provide the proof for Theorem A. Section 6.1 describes the pieces of the phenotypical partition where there are no highly transitive actions (Theorem 6.1 and Proposition 6.8). Section 6.2 then shows the genericity of high transitivity in the remaining pieces (Theorem 6.9), thus completing the proof of Theorem A.

6.1 Phenotypical obstruction to high transitivity

Recall that an action is **primitive** when it preserves no non-trivial equivalence relation, and note that 2-transitive (in particular highly transitive) actions are always primitive. The purpose of this section is to prove the following result.

Theorem 6.1. *Let $m, n \in \mathbb{Z} \setminus \{0\}$, let q be an (m,n)-phenotype such that $q \neq 1$ and $q \neq \infty$. Then every transitive $\mathrm{BS}(m,n)$-action of phenotype q on an infinite set fails to be primitive. In particular, there are no highly transitive $\mathrm{BS}(m,n)$-actions of phenotype q.*

The proof goes by exhibiting a natural nontrivial equivalence relation preserved by actions of phenotype $q \notin \{1, \infty\}$ which we will shortly introduce. We begin with a preliminary definition.

Definition 6.2. Given a permutation $\sigma \in \mathrm{Sym}(X)$ and a finite σ-orbit O, a σ-**suborbit** of O is a subset $O' \subseteq O$ of which is a σ^ℓ-orbit for some $\ell \in \mathbb{N}$.

Given a any (finite) cycle $\sigma \in \mathrm{Sym}(X)$ of length k and any r dividing k, the permutation $\sigma^{\frac{k}{r}}$ generates the unique subgroup G_r of order r in $\langle\sigma\rangle$. The support O of σ admits a unique partition of O into σ-suborbits of cardinality

r, which is the partition into G_r-orbits. This is also the partition of O into $\sigma^{\frac{k}{r}}$-orbits, and σ acts by permuting the pieces of this partition.

We now relate these remarks to our setup as follows. Given a transitive action $X \curvearrowleft^{\alpha} \mathrm{BS}(m,n)$ such that $\mathbf{Ph}(\alpha) < \infty$, we call **reduced phenotype** the integer

$$\mathbf{Ph}_{\mathrm{red}}(\alpha) \coloneqq \frac{\mathbf{Ph}(\alpha)}{\gcd(\mathbf{Ph}(\alpha), m)} = \frac{\mathbf{Ph}(\alpha)}{\gcd(\mathbf{Ph}(\alpha), n)} \tag{6.3}$$

and notice that $\mathbf{Ph}_{\mathrm{red}}(\alpha) > 1$ whenever $\mathbf{Ph}(\alpha) > 1$ by definition of the phenotype. Every orbit O of $\alpha(b) \in \mathrm{Sym}(X)$ has cardinality divisible by $\mathbf{Ph}(\alpha)$, hence also by $\mathbf{Ph}_{\mathrm{red}}(\alpha)$.

Remark 6.4. Observe that when m and n are coprime, the phenotype is always coprime with both m and n, so in this case the reduced phenotype coincides with the phenotype. More generally, the reduced version alters the p-adic valuation of the phenotype only for those p's such that

$$0 < |m|_p = |n|_p < |\mathbf{Ph}(\alpha)|_p\,,$$

in which case it becomes $|\mathbf{Ph}_{\mathrm{red}}(\alpha)|_p = |\mathbf{Ph}(\alpha)|_p - |m|_p = |\mathbf{Ph}(\alpha)|_p - |n|_p$.

Definition 6.5. Given a transitive action α of finite phenotype on a set x, we call the **reduced b-orbit relation** of α the equivalence relation $\mathcal{R}^{\mathrm{red}}_{\alpha(b)}$ on x whose classes are the $\alpha(b)$-suborbits of cardinality $\mathbf{Ph}_{\mathrm{red}}(\alpha)$.

Proposition 6.6. *For every transitive action $X \curvearrowleft^{\alpha} \mathrm{BS}(m,n)$ such that $\mathbf{Ph}(\alpha) < \infty$, the reduced b-orbit relation $\mathcal{R}^{\mathrm{red}}_{\alpha(b)}$ is invariant under α.*

Proof. First, $\mathcal{R}^{\mathrm{red}}_{\alpha(b)}$ is invariant under $\alpha(b)$, since inside every $\alpha(b)$-orbit, the suborbits of cardinality $r \coloneqq \mathbf{Ph}_{\mathrm{red}}(\alpha)$ are permuted by $\alpha(b)$.

It remains to prove that $\alpha(t)$ also permutes the $\alpha(b)$-suborbits of cardinal r. Let us fix $x \in X$. Denote by O the $\alpha(b)$-orbit of x and by O' the $\alpha(b)$-suborbit of x of cardinality r. Its image $O' \cdot \alpha(t)$ clearly has the same cardinal r, so we only have to show $O' \cdot \alpha(t)$ is still an $\alpha(b)$-suborbit. Let σ denote the restriction of $\alpha(b)$ to O; this is a transitive permutation of O of order $k \coloneqq |O|$ and the orbit of x under $\sigma' \coloneqq \sigma^{\frac{k}{r}}$ is equal to O'.

Let $q \coloneqq \mathbf{Ph}(\alpha)$. By Equation (6.3), we have $\gcd(q,n) = \frac{q}{r}$. Bézout's identity provides some $u, v \in \mathbb{Z}$ such that $\gcd(q,n) = uq + vn$. So if $s \in \mathbb{N}$ satisfies $k = sq$, we obtain

$$\frac{k}{r} = s \cdot \frac{q}{r} = s \cdot \gcd(q,n) = s(uq + vn).$$

Then we get

$$\sigma' = \sigma^{\frac{k}{r}} = \sigma^{suq+svn} = \sigma^{uk+svn} = \sigma^{svn}.$$

Hence, one has $O' = x \cdot \langle \sigma' \rangle = x \cdot \langle \sigma^{svn} \rangle = x \cdot \langle \alpha(b)^{nsv} \rangle$. Thus, using the defining relation $tb^m = b^n t$ in $\mathrm{BS}(m,n)$, we get

$$O' \cdot \alpha(t) = \big(x \cdot \langle \alpha(b)^{nsv} \rangle\,\big) \cdot \alpha(t) = \big(x \cdot \alpha(t)\big) \cdot \langle \alpha(b)^{msv} \rangle\,.$$

The set $O' \cdot \alpha(t)$ is therefore an $\alpha(b)$-suborbit as desired. □

We can now easily prove the main result of this section.

Proof of Theorem 6.1. Let α be a $\mathrm{BS}(m,n)$-action of phenotype $q \notin \{1, \infty\}$ on an infinite set X. The phenotypical relation $\mathcal{R}^{\mathrm{red}}_{\alpha(b)}$ is non-trivial: all its equivalence classes have cardinal $\mathbf{Ph}_{\mathrm{red}}(\alpha) > 1$ as soon as $\mathbf{Ph}(\alpha) > 1$ hence they are neither singletons nor equal to X which is infinite. Finally, Proposition 6.6 ensures us that $\mathcal{R}^{\mathrm{red}}_{\alpha(b)}$ is α-invariant, thus witnessing the non-primitivity of α. □

Remark 6.7. Let α be a transitive $\mathrm{BS}(m,n)$-action of phenotype $q \notin \{1, \infty\}$ on a set X. Since the equivalence relation $\mathcal{R}^{\mathrm{red}}_{\alpha(b)}$ is α-invariant, we have a quotient $\mathrm{BS}(m,n)$-action $\tilde{\alpha}$ on the quotient set $X/\mathcal{R}^{\mathrm{red}}_{\alpha(b)}$. By construction, given an $\alpha(b)$-orbit of cardinal k, the corresponding $\tilde{\alpha}(b)$-orbit has cardinal l, where l satisfies that for all prime p that if $0 < |m|_p = |n|_p < |\mathbf{Ph}(\alpha)|_p$, then

$$\begin{aligned}
|l|_p &= |k|_p - |\mathbf{Ph}_{\mathrm{red}}(\alpha)|_p \\
&= |\mathbf{Ph}(\alpha)|_p - (|\mathbf{Ph}(\alpha)|_p - |m|_p) \\
&= |m|_p = |n|_p\,,
\end{aligned}$$

while $|l|_p = |k|_p$ otherwise. It follows that the quotient action $\tilde{\alpha}$ has phenotype 1.

We conclude this section by examining the case of infinite phenotype when $|m| = |n|$, where a similar and easier argument holds.

Proposition 6.8. *Let $|m| = |n| \geq 2$, then $\mathrm{BS}(m,n)$ has no primitive transitive action of infinite phenotype.*

Proof. Let α be a transitive $\mathrm{BS}(m,n)$-action of infinite phenotype. By definition b acts freely, so the partition into b^m-orbits is non trivial. Since $\langle b^m \rangle$ is a normal subgroup, this partition is invariant, and α is thus not primitive. □

6.2 Genericity of highly transitive actions

In this section, we prove the following:

Theorem 6.9. *Let $|m| \neq 1$ and $|n| \neq 1$ and let q be a (m,n)-phenotype. Let $\Gamma = \mathrm{BS}(m,n)$ and let $\mathcal{K}_q$ be the subset of the perfect kernel $\mathcal{K}(\Gamma)$ consisting of actions of phenotype q. The set of highly transitive actions is dense G_δ in $\mathcal{K}_q$ when*

- *either $q = 1$,*
- *or $q = \infty$ and $|m| \neq |n|$.*

Proof of Theorem 6.9. We decompose this proof into three parts: the first part does not depend on the phenotype, and then the proof splits between the case $q = 1$ and the case $q = \infty$.

Common part. First recall that $\mathcal{K}_q(\Gamma)$ is always Polish by [CGLMS22, Theorem B] and the fact that open and closed subspaces of Polish spaces are Polish for the induced topology. We thus use the characterization from Lemma 2.9 and show that Condition $(*)$ of Lemma 2.9 therefrom holds for $\mathcal{P} = \mathcal{K}_q$.

Let us thus fix some $[\alpha, x] \in \mathcal{K}_q$ and $g_1, \ldots, g_{2d} \in \Gamma$ such that the elements $x\alpha(g_1), \ldots, x\alpha(g_{2d})$ are pairwise distinct. Let $\mathcal{V}$ be a neighborhood of $[\alpha, x]$, we must exhibit a pointed transitive action $[\alpha', x'] \in \mathcal{V} \cap \mathcal{K}_q$ and $\gamma \in \Gamma$ such that for all $i \in \{1, \ldots, 2d,\}$, we have

$$x'\alpha'(g_i)\alpha'(\gamma) = x\alpha'(g_{i+d})$$

Shrinking $\mathcal{V}$ if necessary, we assume that $\mathcal{V} = \mathcal{N}([\alpha, x], R)$ for some $R > 0$. Since we are working up to pointed isomorphism, we equivalently need to find a transitive action α' whose Schreier R-ball around the basepoint $x' = x$ coincides with that of α. Furthermore, we may as well assume that R is larger than the length of each g_i with respect to the generating set $\{t, b\}$. For $i \in \{1, \ldots, 2d\}$, let $x_i = x\alpha(g_i)$. Our aim is now to find $[\alpha', x] \in \mathcal{K}_q$ and $\gamma \in \Gamma$ such that $B_{\leq}^{\alpha'}(x, R) = B_{\leq}^{\alpha}(x, R)$ and for all $i \in \{1, \ldots, d\}$,

$$x_i\alpha'(\gamma) = x_{i+d}.$$

Let b be the reunion of the b-orbits that intersect the ball $B_{\leq}(x, R+1)$. Since $[\alpha, x] \in \mathcal{K}_q(\Gamma)$, the set b is a proper subset of x.

Consider the transitive pre-action α^0 obtained by restricting α to b . We now replace α by the maximal forest saturation of α^0. This does not affect the definition of $\mathcal{V} = \mathcal{N}([\alpha, R])$ by construction, nor the definition of each $x_i = x\alpha(g_i)$ since R was taken larger than the length of each g_i.

For $i \in \{1, \dots, 2d\}$, let $\alpha_i^0 = \alpha_0$ be pointed at x_i, and let $\pi_i : \mathcal{T} \to \mathbf{BS}(\alpha)$ be the graph morphism associated to the unique Γ-equivariant map $\hat{\pi}_i : \Gamma \to X$ taking id to x_i.

Let c be a path provided by Lemma 4.12 for the finite family of pointed pre-actions $(\alpha^0, x_i)_{i=1}^{2d}$. Let r be a geodesic with the same length as c, such that $\mathtt{s}(r) = \mathtt{t}(c)$, and consisting only of positive edges only. The path cr still satisfies the conclusions of Lemma 4.12 (by its "Moreover" part).

It follows that for every i, j, if the target $\pi_j(\mathtt{t}(cr))$ belongs to $\pi_i(cr)$, then, in fact, $\pi_j(\mathtt{t}(cr))$ belongs to $\pi_i(r)$ (i.e. to the image of the second half of cr). Thus, the unique path from $\pi_j(\mathtt{t}(cr))$ to $\pi_i(\mathtt{t}(cr))$ is made of positive edges. We now split the proof in two cases.

Case $q = \infty$ and $|m| \neq |n|$. Since $\mathrm{BS}(m, n)$ and $\mathrm{BS}(n, m)$ are isomorphic, we may and do actually assume in this case that $|m| < |n|$. Let us then fix a prime p such that $|m|_p < |n|_p$.

Let $g \in \Gamma$ be such that $g\langle b\rangle = \mathtt{t}(cr)$. By construction for every $i \in \{1, \dots, 2d\}$ we have $x_i\alpha(g) \in \pi_i(\mathtt{t}(cr))$.

Claim 6.10. *For large enough $k \in \mathbb{N}$ the images $x_1\alpha(gt^k), \dots, x_{2d}\alpha(gt^k)$ all belong to different b-orbits.*

Proof of the claim. Let us fix $i \neq j \in \{1, \dots, 2d\}$. It suffices to show that for large enough k, the points $x_i\alpha(gt^k)$ and $x_j\alpha(gt^k)$ belong to distinct b-orbits. To see this, first note that by construction of the maximal forest saturation, if for some $k \geq 1$ we have that the two elements $x_i\alpha(gt^k)$, $x_j\alpha(gt^k)$ belong to the same b-orbit then they must actually belong to the same b^m-orbit (and hence their predecessors $x_i\alpha(gt^{k-1})$, $x_j\alpha(gt^{k-1})$ belong to the same b^n-orbit, in particular to the same b-orbit).

Now assume we have $K \in \mathbb{Z}$ such that $x_i\alpha(g)\alpha(b^K) = x_j\alpha(g)$ (witnessing that $x_i\alpha(g)$ and $x_j\alpha(g)$ are in the same b-orbit) and $L \in \mathbb{Z}$ such that $x_i\alpha(gt)\alpha(b^L) = x_j\alpha(gt)$ (witnessing that their t-images are also in the same b-orbit). Then by the observation we just made we must have n divides K. But then using the Baumslag-Solitar relation $tb^mt^{-1} = b^n$ and $|m|_p < |n|_p$, we obtain that $|L|_p < |K|_p$. Iterating this argument, we conclude that once

$k > |K|_p$, the elements $x_i\alpha(gt^k)$ and $x_j\alpha(gt^k)$ must belong to distinct b-orbits. This finishes the proof of the claim. $\square_{\text{claim}}$

We now fix k large enough so that the points $x_1\alpha(gt^k), \ldots, x_{2d}\alpha(gt^k)$ belong to pairwise disjoint b-orbits.

Let $\mathcal{G}$ be the connected subgraph of $\mathbf{BS}(\alpha)$ obtained as the union of $\mathbf{BS}(\alpha_0)$ with all the paths $\pi_i(cr)$ and the positive edges $x_i\alpha(gt^l)\alpha(\langle b^n\rangle)$ extending them for $l \in \{1, \ldots, k\}$. Note that for every $i \in \{1, \ldots, 2d\}$, the vertex corresponding to the b-orbit of $x_i\alpha(gt^k)$ is in the complement of $\mathbf{BS}(\alpha^0)$. Moreover, this vertex has ingoing degree m in $\mathbf{BS}(\alpha)$, while it has ingoing degree 1 in $\mathcal{G}$ by the observation we made right before splitting the proof in two cases.

Let G be the subgraph of the Schreier graph of α consisting of the b-orbits that are shrunk to vertices of $\mathcal{G}$ and all the edges that are shrunk to edges of $\mathcal{G}$. It is the Schreier graph of a pre-action that we denote ξ.

We now extend back ξ as follows. We first add d new infinite $\xi(b)$-orbits $O_1, \ldots, O_d$, and pick for each $j \in \{1, \ldots, d\}$ a point $z_j \in O_j$. Observe that z_j and $z_j\xi(b)$ belong to distinct b^m-orbits. We can then extend ξ further by letting, for every $l \in \mathbb{Z}$:

$$z_j\xi(b^{lm})\xi(t) = y_j\xi(b)\xi(b^{ln}) = \text{ and } z_j\xi(b^{lm+1})\xi(t) = y_{d+j}\xi(b).$$

Finally let α' be an arbitrary forest-saturation of ξ, we claim that α' is the transitive action we are after. Indeed, since we did not modify the path $\pi_i(cr)$ nor the positive edges $x_i\alpha(gt^l)\alpha(\langle b^n\rangle)$ following it for $l \in \{1, \ldots, k\}$, we have $x_i\alpha(gt^k) = x_i\alpha'(gt^k) = y_i$ for all $i \in \{1, \ldots, 2d\}$. By construction, we now have, letting $\gamma = (gt^kbt^{-1})b(gt^kbt^{-1})^{-1}$:

$$\begin{aligned} x_j\alpha'(\gamma) &= x_j\alpha'(gt^kbt^{-1}btb^{-1}t^{-k}g^{-1}) \\ &= y_j\alpha'(bt^{-1}btb^{-1}t^{-k}g^{-1}) \\ &= z_j\alpha'(btb^{-1}t^{-k}g^{-1}) \\ &= y_{j+d}\alpha'(b^{-1}t^{-k}g^{-1}) \\ &= x_{j+d}. \end{aligned}$$

Moreover, since α' extends α_0, it belongs to $\mathcal{V} = \mathcal{N}(\alpha, R)$ as wanted.

Case $q = 1$. Let $\mathcal{G}$ be the subgraph of $\mathbf{BS}(\alpha)$ consisting of the union of $\mathbf{BS}(\alpha^0)$ with all the $\pi_i(cr)$, for $i = 1, \cdots, 2d$. As in the previous case, each

$\pi_i(\mathtt{t}(cr))$ is in the complement of $\mathbf{BS}(\alpha^0)$. Since it comes from a positive edge, its label satisfies $\gcd(L(\pi_i(\mathtt{t}(cr))), m) = |m|$ by the transfer rule. So it has ingoing degree $|m|$ in $\mathbf{BS}(\alpha)$, while it has ingoing degree 1 in $\mathcal{G}$.

Let G be the subgraph of the Schreier graph of α consisting of the b-orbits that are shrunk to vertices of $\mathcal{G}$ and all the edges that are shrunk to edges of $\mathcal{G}$. As in the first case, G is the Schreier graph of a pre-action ξ which extends α_0 and whose Bass-Serre graph is $\mathcal{G}$.

We now replace α by the minimal forest-saturation of ξ, which we still denote α. Then α still extends ξ and thus it also extends α^0. In particular, this new modification of α does not modify the open set $\mathcal{N}([\alpha, R])$ nor the elements $x_i = x\alpha(g_i)$. Moreover, for every $i \in \{1, \dots, 2d\}$ the path $\pi_i(cr)$ is also left unchanged since α extends ξ. Its terminal vertex has the same label as before and hence has the same (maximal) ingoing degree as before, namely m.

We now extend the path cr in $\mathcal{T}$ to a reduced (infinite) path crs by adding a geodesic ray only made of negative edges. Since both $\mathtt{t}(cr)$ and $\pi_i(\mathtt{t}(cr))$ have m ingoing edges, the path $\pi_i(crs)$ is reduced at the vertex $\pi_i(\mathtt{t}(cr))$. Moreover, the orientation of the two edges incident to $\pi_i(\mathtt{t}(cr))$ in $\pi_i(crs)$ being opposite, while the ongoing degree of $\pi_i(\mathtt{t}(cr))$ in $\mathcal{G}$ is 1, it follows that the path $\pi_i(s)$ is a geodesic whose intersection with $\mathcal{G}$ is reduced to its source $\mathtt{s}(\pi_i(s))$.

Let us denote by $(v_{i,k})_{k\geq 0}$ the vertices of $\pi_i(s)$ in $\mathbf{BS}(\alpha)$. The following version of Claim 5.3 holds.

Claim 6.11. *For k large enough, the labels satisfy $L(v_{i,k}) = 1$ for $i = 1, \dots, 2d$. In other words, the b-orbits corresponding to $v_{i,k}$ are singletons.*

Proof of the claim. Let us fix some index i. It suffices to prove $L(v_{i,k}) = 1$ for k large enough. So let us abbreviate $v_{i,k}$ as v_k. For each k, let e_k be the positive edge from v_{k+1} to v_k. The Transfer Equation (3.2) implies:

$$\max(|L(v_{k+1})|_p - |n|_p, 0) = |L(e_k)|_p = \max(|L(v_k)|_p - |m|_p, 0).$$

Since we are in the minimal forest saturation, we then have

$$|L(v_{k+1})|_p = \begin{cases} 0 & \text{if } |L(v_k)|_p \leq |m|_p \\ |L(v_k)|_p - |m|_p + |n|_p & \text{if } |L(v_k)|_p > |m|_p . \end{cases} \tag{6.12}$$

Let us now fix a prime p. We have two cases to consider.

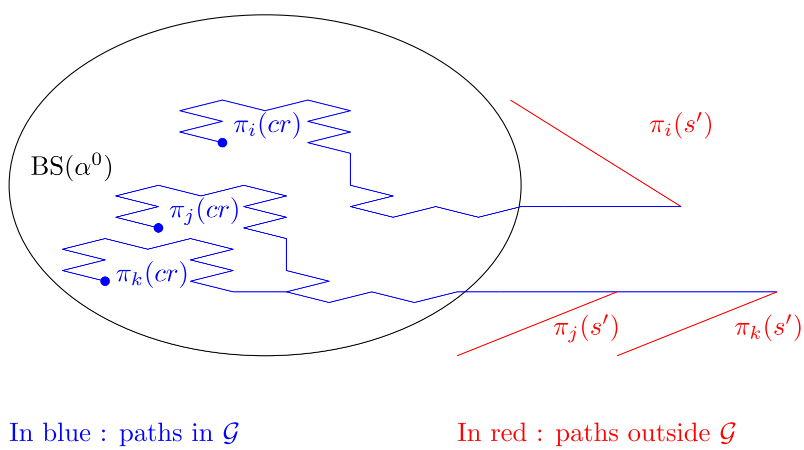
1. If $|m|_p \leq |n|_p$, Using that cr satisfies the conclusions of Lemma 4.12, that $v_0 = \mathtt{t}(\pi_i(cr))$ and that the phenotype is 1, we have: $|L(v_0)|_p = |m|_p$. The transfer equation (6.12) (first case) gives $|L(v_k)|_p = 0$ for every $k \geq 1$.
2. If $|m|_p > |n|_p$, we prove that $|L(v_k)|_p = 0$ for k large enough. Indeed, the transfer equation (6.12) (second case) forces $|L(v_{k+1}))|_p < |L(v_k)|_p$ as long as $|L(v_k)|_p > |m|_p$. Thus, we get $|L(v_{k_0})|_p \leq |m|_p$ for some index k_0 and then $|L(v_k)|_p = 0$ for all $k \geq k_0 + 1$ by Equation (6.12) (first case).

Since there are only finitely many primes such that $|m|_p > |n|_p$, we conclude $L(v_k) = 1$ for k large enough. $\square_{\text{claim}}$

By the above Claim 6.11, we can pick an initial segment s' of s such that $\pi_i(\mathtt{t}(s'))$ has label 1 for every $i \in \{1, \dots, 2d\}$. For $i \in \{1, \dots, 2d\}$, let y_i be the unique element of the label 1 vertex $\pi_i(\mathtt{t}(s'))$, viewed as a b-orbit.

Fix $g \in \mathrm{BS}(m,n)$ such that $g\langle b \rangle = \mathtt{t}(crs')$ in $\mathcal{T}$. For $i \in \{1, \dots, 2d\}$, we have by equivariance and the definition of π_i that $y_i = x_i \alpha(g)$. Since $\alpha(g)$ is a bijection, we conclude that the y_i's are pairwise distinct. In particular, their image in the Bass-Serre graph, i.e. the vertices $\pi_i(\mathtt{t}(s'))$, are pairwise distinct. This is the point where it is crucial that the label is 1, and thus to be in phenotype 1.

We now carry out one last modification of α: let $\mathcal{H}$ be the subgraph of $\mathbf{BS}(\alpha)$ consisting of the union of $\mathbf{BS}(\alpha^0)$ with all the paths $\pi_i(crs')$, for $i = 1, \dots, 2d$. In fact, $\mathcal{H}$ is the union of $\mathcal{G}$ and the geodesics $\pi_i(s')$, for $i = 1, \dots, 2d$.



Since $\pi_i(s)$ is a geodesic whose intersection with $\mathcal{G}$ is reduced to its source $\mathtt{s}(\pi_i(s))$, all $\pi_i(\mathtt{t}(crs'))$ are at the same distance from $\mathcal{G}$ in $\mathcal{H}$, namely the length of s', thus they all have degree 1 in $\mathcal{H}$. In particular, for every i, the (unique) positive edge arriving at $\pi_i(\mathtt{t}(s')) = \{y_i\}$ lies outside $\mathcal{H}$.

As before, the subgraph of the Schreier graph obtained by pulling back $\mathcal{H}$ is the Schreier graph of a pre-action that we denote η. Observe that η extends ξ, hence also extends α^0; its Bass-Serre graph is $\mathcal{H}$.

We now extend η as follows: we add d new $\eta(b)$-orbits O_j of cardinal m, for $j \in \{1, \ldots, d\}$, and we pick a point z_j in O_j. We connect O_j to both y_j and y_{j+d} by declaring $z_j\eta(t) = y_j$ and $z_j\eta(b)\eta(t) = y_{j+d}$. This is a genuine pre-action since O_j decomposes as m $\eta(b^m)$-orbits of cardinality 1. Let α' be a forest-saturation of η, let us show that α' is the action we seek. First, since we did not modify the path $\pi_i(crs')$, we have $x_i\alpha(g) = x_i\alpha'(g) = y_i$ for all $i \in \{1, \ldots, 2d\}$. By construction, we now have

$$\begin{aligned} x_j\alpha'(gt^{-1}btg^{-1}) &= y_j\alpha'(t^{-1}btg^{-1}) \\ &= z_j\alpha'(btg^{-1}) \\ &= y_{j+d}\alpha'(g^{-1}) \\ &= x_{j+d}. \end{aligned}$$

Moreover, since α' extends α_0, it belongs to $\mathcal{V} = \mathcal{N}(\alpha, R)$ as wanted. This finishes the proof of the second and last case, so Theorem 6.9 is proven. ☐

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