On dense totipotent free subgroups in full groups

A. Carderi, D. Gaboriau, F. Le Maître

Abstract

We study probability measure preserving (p.m.p.) non-free actions of free groups and the associated IRS's. The perfect kernel of a countable group Γ is the largest closed subspace of the space of subgroups of Γ without isolated points. We introduce the class of totipotent ergodic p.m.p. actions of Γ: those for which almost every point-stabilizer has dense conjugacy class in the perfect kernel. Equivalently, the support of the associated IRS is as large as possible, namely it is equal to the whole perfect kernel. We prove that every ergodic p.m.p. equivalence relation $\mathscr{R}$ of cost $< r$ can be realized by the orbits of an action of the free group $\mathbf{F}_r$ on r generators that is totipotent and such that the image in the full group $[\mathscr{R}]$ is dense. We explain why these actions have no minimal models. This also provides a continuum of pairwise orbit inequivalent invariant random subgroups of $\mathbf{F}_r$, all of whose supports are equal to the whole space of infinite index subgroups. We are led to introduce a property of topologically generating pairs for full groups (we call evanescence) and establish a genericity result about their existence. We show that their existence characterizes cost 1.

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Contents

1 Introduction

In this context, clarifying precisely what is meant by "totipotency" and how it is experimentally determined will both avoid unnecessary controversy and potentially reduce inappropriate barriers to research.
— M. Condic, [Con14]

Let Γ be a countable discrete group. Denote by $\mathrm{Sub}(\Gamma)$ the space of subgroups of Γ. It is equipped with the compact totally disconnected topology of pointwise convergence and with the continuous Γ-action by conjugation. Let β be a Borel Γ-action on the standard Borel space $X \simeq [0,1]$. Its **stabilizer map**

$$\begin{aligned} \mathrm{Stab}^{\beta} \quad : \quad & X \mapsto \mathrm{Sub}(\Gamma) \\ & x \mapsto \{\gamma \in \Gamma \colon \beta(\gamma)x = x\} \end{aligned}$$

is Γ-equivariant. If μ is a probability measure on X which is preserved by β, then the push-forward measure $\mathrm{Stab}^{\beta}_{*}\mu$ is invariant under conjugation. It is the prototype of an Invariant Random Subgroup (**IRS**). When μ is atomless and the stabilizer map is essentially injective (a.k.a. the action β is **totally non-free**), the support of the associated IRS $\mathrm{Stab}^{\beta}_{*}(\mu)$ has no isolated points: it is a perfect set. The largest closed subspace of $\mathrm{Sub}(\Gamma)$ with no isolated points is called the **perfect kernel** of $\mathrm{Sub}(\Gamma)$. We say that an ergodic probability measure-preserving (**p.m.p.**) action is **totipotent** when the support of its IRS is equal to the perfect kernel of $\mathrm{Sub}(\Gamma)$. By ergodicity, the following stronger property holds: *almost every element of the associated IRS has dense orbit in the perfect kernel* (see Proposition 2.3). We call such an IRS **totipotent**.

Given a p.m.p. action $\Gamma \curvearrowright^{\beta} (X,\mu)$, we consider the associated **p.m.p. equivalence relation**

$$\mathscr{R}^{\beta} \coloneqq \{(x,y) \in X \times X : \beta(\Gamma)x = \beta(\Gamma)y\}$$

and its **full group** $[\mathscr{R}^\beta]$ as the group of all measure-preserving transformations whose graph is contained in $\mathscr{R}^\beta$. The (bi-invariant) **uniform distance** between two measure-preserving transformations S and T is defined by $d_u(T,S) \coloneqq \mu(\{x \in X : S(x) \neq T(x)\})$. It endows the full group $[\mathscr{R}^\beta]$ with a Polish group structure. The **cost** is a numerical invariant attached to the equivalence relation $\mathscr{R}^\beta$. If β is a p.m.p. action of the free group $\mathbf{F}_r$ on r generators, then *the cost of $\mathscr{R}^\beta$ is exactly r when β is free and the cost of $\mathscr{R}^\beta$ is $< r$ when β is non-free* [Gab00].

The main result of [LM14a] is that for any ergodic p.m.p. equivalence relation $\mathscr{R}$, *if $\mathscr{R}$ has cost $< r$ for some integer $r \geq 2$, then there exists a homomorphism $\tau : \mathbf{F}_r \to [\mathscr{R}]$ with dense image.*

This result has been sharpened in order to ensure that *the homomorphism τ is injective*. Actually, the associated (almost everywhere defined) p.m.p. action α_τ can be made to satisfy the following two opposite conditions: *high faithfulness* and *amenability on μ-almost every orbit* [LM18].

These two conditions can be phrased in terms of the support of the IRS associated to the action: the first one means that the support contains the trivial subgroup, and one can show that the second one is equivalent to the support containing a co-amenable subgroup (which in the construction of [LM18] is the kernel of a certain surjective homomorphism $\mathbf{F}_r \to \mathbb{Z}$).

The purpose of the present paper is to show that the homomorphism can be chosen so that the support of the associated IRS is actually the largest perfect subspace of $\mathrm{Sub}(\mathbf{F}_r)$, which consists in all its infinite index subgroups (see Proposition 2.1).

Theorem. *Let $\mathscr{R}$ be an ergodic p.m.p. equivalence relation whose cost is $< r$ for some integer $r \geq 2$. Then there exists a homomorphism $\tau : \mathbf{F}_r \to [\mathscr{R}]$ whose image is dense and whose associated p.m.p. action α_τ is totipotent.*

The density in $[\mathscr{R}]$ of the image of τ implies that $\mathscr{R}^{\alpha_\tau} \simeq \mathscr{R}$ and that the stabilizer map $\mathrm{Stab}^{\alpha_\tau}$ is essentially injective [LM18, Prop. 2.4]. In particular, the IRS of α_τ determines $\mathscr{R}$. It follows that up to isomorphism, *every p.m.p. ergodic equivalence relation of cost $< r$ comes from a totipotent IRS of $\mathbf{F}_r$ (actually, from continuum many different totipotent IRS's of $\mathbf{F}_r$, see Remark 5.1).*

Such a statement is optimal since p.m.p. equivalence relations of cost $\geq r$ cannot come from a non-free $\mathbf{F}_r$ action. To our knowledge, it was not even clear until now whether $\mathbf{F}_r$ admits ergodic totipotent IRS's. Since there are continuum many pairwise non-isomorphic ergodic p.m.p. equivalence relations of cost $< r$, our approach provides continuum many pairwise distinct ergodic totipotent IRS's of the free group on r generators, whose associated equivalence relations are even non-isomorphic.

Another interesting fact about totipotent p.m.p. $\mathbf{F}_r$-actions is that they have no minimal model, i.e., they cannot be realized as minimal actions on a compact space. Indeed, it follows from a result of Glasner-Weiss [GW15, Cor. 4.3] that as soon as the support of the IRS of a given p.m.p. action contains two distinct minimal subsets (e.g. when it contains two distinct fixed points), the action does not admit a minimal model (see Theorem 2.5). In our case the perfect kernel of $\mathrm{Sub}(\mathbf{F}_r)$ contains a continuum of fixed points (namely, all infinite index normal subgroups), so that totipotent p.m.p. actions of $\mathbf{F}_r$ are actually very far from admitting a minimal model.

Let us now recall the context around our construction. The term IRS was coined by Abert-Glasner-Virag [AGV14] and has become an important subject on its own at the intersection of group theory, probability theory and dynamical systems. The notion of IRS is a natural generalization of a normal subgroup, especially in the direction of superrigidity type results. It has thus been present implicitly in the work of many authors, a famous landmark being the Stuck-Zimmer Theorem [SZ94], which gives examples of groups admitting very few IRS's. On the contrary, some groups admit a "zoo" of IRS's, starting with free groups [Bow15] (see [BGK15, BGK17, KQ19] for other examples).

In particular, Bowen proved that every p.m.p. ergodic equivalence relation of cost $< r$ comes from some IRS of $\mathbf{F}_r$. He obtained this result through a Baire category argument which required that the first generator acts freely. In particular, such IRS's can never be totipotent.

Eisenmann and Glasner then used homomorphisms $\mathbf{F}_r \to [\mathscr{R}]$ with dense image so as to obtain interesting IRS's of $\mathbf{F}_r$ [EG16]. They proved that given a homomorphism $\Gamma \to [\mathscr{R}]$ with dense image, the associated IRS is always co-highly transitive almost surely, which means that for almost every $\Lambda \leq \Gamma$, the Γ-action on Γ/Λ is n-transitive for every $n \in \mathbb{N}$. They also showed that the IRS's of $\mathbf{F}_r$ obtained by Bowen for cost 1 equivalence relations are faithful and moreover almost surely co-amenable.

The third-named author then used a modified version of his result on the topological rank of full groups to show that every p.m.p. ergodic equivalence relation of cost $< r$ comes from a co-amenable, co-highly transitive and faithful IRS of $\mathbf{F}_r$ [LM18]. Also in this construction, the first generator continues to act freely, thus preventing totipotency. Let us now briefly explain how our new construction (Section 5) allows us to circumvent this.

The main idea is to use a smaller set $Y \subsetneq X$ such that the restriction of $\mathscr{R}$ to Y still has cost $< r$, so that we can find some homomorphism $\mathbf{F}_r \to [\mathscr{R}_{\restriction Y}]$ with dense image. This provides us with some extra space in order to obtain totipotency via a well-chosen perturbation of the above homomorphism.

This perturbation is obtained by mimicking all Schreier balls on $X \setminus Y$ and then merging these amplifications with the action on Y so as to obtain both density in $[\mathscr{R}]$ and totipotency. The use of evanescent pairs of topological generators (see Definition 4.1) with Theorem 4.5 and Proposition 3.8 will grant us that this perturbation maintains the density. We establish in Theorem 4.6 that the existence of an evanescent pair of topological generators is equivalent with $\mathscr{R}$ having cost 1.

Finally, let us mention the case of the free group on infinitely many generators $\mathbf{F}_\infty$. Here, the space of subgroups is already perfect (see Proposition 2.1), and one can easily adapt our arguments to show that: *For every ergodic p.m.p. equivalence relation $\mathscr{R}$, there exists a homomorphism $\tau : \mathbf{F}_\infty \to [\mathscr{R}]$ whose image is dense and whose associated p.m.p. action α_τ is totipotent.*

This result could however also be obtained by a purely Baire-categorical argument: it is not hard to see that the space of such homomorphisms is dense G_δ in the Polish space of all homomorphisms $\tau : \mathbf{F}_\infty \to [\mathscr{R}]$.

Going back to the case of finite rank, it is not even true that a generic homomorphism $\tau : \mathbf{F}_r \to [\mathscr{R}]$ generates the equivalence relation $\mathscr{R}$. In order to hope for a similar genericity statement, one should first answer the following question.

Question. Consider a p.m.p. ergodic equivalence relation $\mathscr{R}$ of cost $< r$. Is it true that, in the space of homomorphisms $\tau : \mathbf{F}_r \to [\mathscr{R}]$ whose image generates $\mathscr{R}$, those with dense image are dense?

The fact that Bowen and then Eisenmann-Glasner had to work in the even smaller space where the first generator acts freely indicates that a Baire-categorical approach to our main result is out of reach at the moment, if not impossible.

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2 Perfect kernel for groups and minimal models

Let Γ be a countable discrete group. The topology on its space of subgroups $\mathrm{Sub}(\Gamma)$ admits as a basis of open sets the $V(\mathcal{I}, \mathcal{O}) \coloneqq \{\Lambda \in \mathrm{Sub}(\Gamma) \colon \mathcal{I} \subseteq \Lambda \text{ and } \mathcal{O} \cap \Lambda = \emptyset\}$ where $\mathcal{I}$ and $\mathcal{O}$ are finite subsets of Γ. By the Cantor-Bendixson theorem, $\mathrm{Sub}(\Gamma)$

decomposes in a unique way as the disjoint union of a perfect set, called the **perfect kernel** $\mathcal{K}(\Gamma)$ of $\mathrm{Sub}(\Gamma)$, and of a countable set. We indicate some isolation properties of subgroups:

1. If $\Lambda \in \mathrm{Sub}(\Gamma)$ is not finitely generated, then writing $\Lambda = (\lambda_j)_{j\in\mathbb{N}}$ we obtain Λ as the non-trivial limit of the infinite index (both in Λ and in Γ) of the finitely generated subgroups $\Lambda_n := \langle \lambda_0, \lambda_1, \cdots, \lambda_n \rangle$.

2. If Γ is finitely generated, then its finite index subgroups are isolated. Indeed, a finite index subgroup Λ is finitely generated as well and it is alone in the open subset defined by a finite family $\mathcal{I}$ of generators and a finite family $\mathcal{O}$ of representatives of its cosets Γ/Λ except $\{\Lambda\}$.

3. If Γ is not finitely generated, then its finite index subgroups are also not finitely generated and thus are not isolated by Property 1.

Let us denote by $\mathrm{Sub}_{\infty i}(\Gamma)$ the subspace of infinite index subgroups of Γ. The following is probably well-known but we were not able to locate a proof in the literature.

Proposition 2.1. *For the free group $\mathbf{F}_r$ on r generators, $2 \leq r \leq \infty$,*

(i) for finite $r \geq 2$, $\mathcal{K}(\mathbf{F}_r) = \mathrm{Sub}_{\infty i}(\Gamma)$;

(ii) for r infinite, $\mathcal{K}(\mathbf{F}_\infty) = \mathrm{Sub}(\mathbf{F}_\infty)$.

Proof. We first show that if $\Lambda \in \mathrm{Sub}_{\infty i}(\mathbf{F}_r)$, $2 \leq r \leq \infty$, then it is a non-trivial limit of finitely generated infinite index subgroups of $\mathbf{F}_r$. If Λ is not finitely generated, then Property 1 above applies. Thus assume Λ is finitely generated. If r is infinite, then Λ is already of infinite index in some finitely generated subgroup $\mathbf{F}_s \leq \mathbf{F}_\infty$. We can thus assume that the rank r is finite. Then by Hall theorem, Λ is a free factor of a finite index subgroup $\Lambda * \Delta$ of the free group $\mathbf{F}_r$. Since Λ has infinite index, Δ is not $\{1\}$. A sequence of (for instance finite index) subgroups $\Delta_n \leq \Delta$ tending to $\{1\}$ allows to express Λ as the non-trivial limit of $\Lambda * \Delta_n$. This (with Property 2, resp., Property 3 above) shows that $\mathcal{K}(\mathbf{F}_r) = \mathrm{Sub}_{\infty i}(\mathbf{F}_r)$ for $r < \infty$, resp., $\mathcal{K}(\mathbf{F}_\infty) = \mathrm{Sub}(\mathbf{F}_\infty)$. □

Remark 2.2. This also shows that the Cantor-Bendixson rank of $\mathrm{Sub}(\mathbf{F}_r)$ equals 1 when r is finite and equals 0 when $r = \infty$.

Computations of the perfect kernel for some other groups have been performed in [BGK15, SW20].

The following is a classical result: *Assume* Γ *acts by homeomorphisms on a Polish space* Z *and* ν *is an ergodic* Γ*-invariant probability measure on* Z*, then the orbit of almost every point* $z \in Z$ *is dense in the support of* ν*.* In particular:

Proposition 2.3. *If* $\Gamma \curvearrowright (X, \mu)$ *is a p.m.p. ergodic action on a standard probability space, then the stabilizer* $\mathrm{Stab}(x)$ *of almost every point* $x \in X$ *is dense in the support of the associated IRS* $\nu = \mathrm{Stab}_* \mu$ *of* $\mathrm{Sub}(\Gamma)$*.*

Thus, our main theorem produces IRS's on $\mathrm{Sub}(\mathbf{F}_r)$ for which almost every $\mathbf{F}_r$-orbit (under conjugation) is dense in $\mathcal{K}(\mathbf{F}_r) = \mathrm{Sub}_{\infty i}(\mathbf{F}_r)$. In other words, for almost every subgroup Λ the Schreier graph of the action $\mathbf{F}_r \curvearrowright \mathbf{F}_r/\Lambda$ contains arbitrarily large copies of Schreier ball of every infinite transitive $\mathbf{F}_r$-action.

Remark 2.4. In the introduction, we defined an IRS to be totipotent when almost every subgroup has dense orbit in the perfect kernel. But an IRS can also be considered as a p.m.p. dynamical system whose associated IRS can be different. The connections between the two notions of totipotency are unclear to us. Note however that since the actions that we construct are totally non-free, our IRS's are totipotent in both senses.

Moreover, this proposition can be combined with [GW15, Cor. 4.3] to give the following result.

Theorem 2.5. *Let* $\Gamma \curvearrowright (X, \mu)$ *be a p.m.p. ergodic action on a standard probability space. Suppose that the support of the associated IRS contains at least two distinct minimal subsets. Then the action has no minimal model.*

This is in wide contrast with free actions of countable groups: they always admit minimal models [Wei12].

Proof. By the previous proposition, the orbit closure of the stabilizer of μ-almost every point is equal to the support of the IRS, and hence contains two distinct minimal subsets. Admitting a minimal model would thus be incompatible with [GW15, Cor. 4.3]. □

3 Full groups and density

We fix once and for all a standard probability space (X, μ) and denote by $\mathrm{Aut}(X, \mu)$ the group of all its measure-preserving transformations, two such transformations being identified if they coincide on a full measure set. In order to ease notation, we will always neglect what happens on null sets. Given an element $T \in \mathrm{Aut}(X, \mu)$, its set of fixed points is denoted

$$\mathrm{Fix}(T) \coloneqq \{x \in X : T(x) = x\}.$$

A **partial isomorphism** of (X, μ) is a partially defined Borel bijection $\varphi : \operatorname{dom} \varphi \to \operatorname{rng} \varphi$, with $\operatorname{dom} \varphi, \operatorname{rng} \varphi$ Borel subsets of X, such that φ is measure-preserving for the measures induced by μ on its domain $\operatorname{dom} \varphi$ and its range $\operatorname{rng} \varphi$. In particular, we have $\mu(\operatorname{dom} \varphi) = \mu(\operatorname{rng} \varphi)$. The **support** of φ is the set

$$\operatorname{supp} \varphi \coloneqq \{x \in \operatorname{dom} \varphi : \varphi(x) \neq x\} \cup \{x \in \operatorname{rng} \varphi : \varphi^{-1}(x) \neq x\}.$$

Given two partial isomorphisms with φ, ψ disjoint domains and ranges, one can form their **union**, which is the partial isomorphism

$$\begin{aligned} \varphi \sqcup \psi : \operatorname{dom} \varphi \sqcup \operatorname{dom} \psi &\to \operatorname{rng} \varphi \sqcup \operatorname{rng} \psi \\ x &\mapsto \begin{cases} \varphi(x) & \text{if } x \in \operatorname{dom} \varphi, \\ \psi(x) & \text{if } x \in \operatorname{dom} \psi. \end{cases} \end{aligned}$$

A **graphing** is a countable set of partial isomorphisms Φ. Its **cost** $\mathcal{C}(\Phi)$ is the sum of the measures of the domains of its elements, which is also equal to the sum of the measures of their ranges since they preserve the measure.

Given a graphing Φ, the smallest equivalence relation which contains all the graphs of the elements of Φ is denoted by $\mathscr{R}_\Phi$ and called the **equivalence relation generated** by Φ. When $\Phi = \{\varphi\}$, we also write it as $\mathscr{R}_\varphi$ and call it the equivalence relation generated by φ.

The equivalence relations that can be generated by graphings are called **p.m.p. equivalence relations**, they are Borel as subsets of $X \times X$ and have countable classes. The **cost** $\mathcal{C}(\mathscr{R})$ of a p.m.p. equivalence relation $\mathscr{R}$ is the infimum of the costs of the graphings which generate it.

Whenever $\alpha : \Gamma \to \mathrm{Aut}(X, \mu)$ is a p.m.p. action, we denote by $\mathscr{R}^\alpha$ the equivalence relation generated by $\alpha(\Gamma)$.

Given a p.m.p. equivalence relation $\mathscr{R}$, the set of partial isomorphisms whose graph is contained in $\mathscr{R}$ is denoted by $[[\mathscr{R}]]$ and called the **pseudo full group** of $\mathscr{R}$. Here is a useful way of obtaining elements of the pseudo full group that we will use implicitly. Say that $\mathscr{R}$ is **ergodic** when every Borel $\mathscr{R}$-saturated set has measure 0 or 1. Under this assumption, given any two Borel subsets $A, B \subseteq X$ of equal measure, there is $\varphi \in [[\mathscr{R}]]$ such that $\operatorname{dom} \varphi = A$ and $\operatorname{rng} \varphi = B$ [KM04, Lem. 7.10].

The **full group** of $\mathscr{R}$ is the subgroup $[\mathscr{R}]$ of $\mathrm{Aut}(X,\mu)$ consisting of almost everywhere defined elements of the pseudo full group. Endowed with the **uniform metric** given by $d_u(S,T) = \mu(\{x \in X : S(x) \neq T(x)\})$, it becomes a Polish group. Observe that $d_u(T, \mathrm{id}_X) = \mu(\operatorname{supp} T)$.

For more material about this section, we refer to [KM04, Gab11] and the references therein.

3.1 Around a theorem of Kittrell-Tsankov

In this paper, we will be interested in p.m.p. actions $\tau : \mathbf{F}_r \to [\mathscr{R}]$ with dense image in $[\mathscr{R}]$. To that end, the following result of Kittrell and Tsankov is very useful. Given a family $(\mathscr{R}_i)$ of equivalence relations on the same set X, we define $\bigvee_{i\in I} \mathscr{R}_i$ as the smallest equivalence relations which contains each $\mathscr{R}_i$.

Theorem 3.1 ([KT10, Thm. 4.7]). *Let $\mathscr{R}$ be a p.m.p. equivalence relation on (X,μ), suppose that $(\mathscr{R}_i)_{i\in I}$ is a family of Borel subequivalence relations such that $\mathscr{R} = \bigvee_{i\in I} \mathscr{R}_i$. Then $[\mathscr{R}] = \overline{\left\langle \bigcup_{i\in I} [\mathscr{R}_i] \right\rangle}$.*

We will also use two easy corollaries of their result which require us to set up a bit of notation.

Definition 3.2. Given an equivalence relation $\mathscr{R}$ on a set X and $Y \subseteq X$, we define the equivalence relation $\mathscr{R}_{\restriction Y}$ **restricted** to Y and the equivalence relation $\mathscr{R}_{\updownarrow Y}$ **induced** on Y by

$$\mathscr{R}_{\restriction Y} \coloneqq \mathscr{R} \cap Y \times Y = \{(x,y) \in \mathscr{R} : x, y \in Y\} \subseteq Y \times Y;$$
$$\mathscr{R}_{\updownarrow Y} \coloneqq \mathscr{R}_{\restriction Y} \cup \{(x,x) : x \in X\} \subseteq X \times X.$$

Observe that given a p.m.p. equivalence relation $\mathscr{R}$, we have a natural way of identifying the full group of the restriction $\mathscr{R}_{\restriction Y}$ to the full group of the induced equivalence relation $\mathscr{R}_{\updownarrow Y}$ by making its elements act trivially outside of Y.

Corollary 3.3. *Let $\mathscr{R}$ be an ergodic p.m.p. equivalence relation on (X,μ). Let $T \in [\mathscr{R}]$ and $Y \subseteq X$ measurable such that $\mu(Y \cap TY) > 0$ and put $Y_T \coloneqq \cup_{n\in\mathbb{Z}} T^n Y$. Then $\overline{\langle T, [\mathscr{R}_{\updownarrow Y}] \rangle} \geq [\mathscr{R}_{\updownarrow Y_T}]$.*

Proof. Since $\mu(Y \cap TY) > 0$ and $\mathscr{R}$ is ergodic, we have that $\mathscr{R}_{\updownarrow Y \cup TY} = \mathscr{R}_{\updownarrow Y} \vee \mathscr{R}_{\updownarrow TY}$. Therefore Theorem 3.1 implies that

$$\overline{\left\langle [\mathscr{R}_{\updownarrow Y}], T[\mathscr{R}_{\updownarrow Y}]T^{-1} \right\rangle} = [\mathscr{R}_{Y\cup TY}].$$

Now observe that $(Y \cup TY) \cap T(Y \cup TY) \supseteq TY$ has positive measure. Therefore Theorem 3.1 implies that $\overline{\langle T, [\mathscr{R}_{\upharpoonright Y}] \rangle}$ contains $[\mathscr{R}_{\upharpoonright (Y \cup TY \cup T^2 Y)}]$ and the corollary follows by induction. $\square$

Corollary 3.4. *Consider an ergodic p.m.p. equivalence relation $\mathscr{R}$ on (X, μ) and let $Y \subseteq X$ be a positive measure subset. Let α be a p.m.p. action of Γ on (X, μ) such that $\alpha(\Gamma) \leq [\mathscr{R}]$, $\mu(\alpha(\Gamma)Y) = 1$ and $[\mathscr{R}_{\upharpoonright Y}] \leq \overline{\alpha(\Gamma)}$. Then either $\overline{\alpha(\Gamma)} = [\mathscr{R}]$, or Γ preserves a finite partition $\{Y_i\}_{i=1}^k$ of X, with $Y \subseteq Y_1$ and $[\mathscr{R}_{\upharpoonright Y_i}] \leq \overline{\alpha(\Gamma)}$ for each $i \leq k$.*

In particular if $\mu(Y) > 1/2$, then $k = 1$ and hence $\overline{\alpha(\Gamma)} = [\mathscr{R}]$.

Proof. Let $B \supset Y$ a subset of maximal measure such that $\overline{\alpha(\Gamma)} \geq [\mathscr{R}_{\upharpoonright B}]$. Then by the above corollary for every $\gamma \in \Gamma$ such that $\alpha(\gamma)B \neq B$, we must have that $\mu(B \cap \alpha(\gamma)B) = 0$, hence B is an atom of a finite partition preserved by the Γ-action α. $\square$

3.2 From graphings to density

The following is a slight variation of [LM14a, Def. 8].

Definition 3.5. Let $n \geq 2$. A **pre-cycle** of **length** n is a partial isomorphism φ such that if we set $B \coloneqq \operatorname{dom} \varphi \setminus \operatorname{rng} \varphi$ (the **basis** of the pre-cycle), then $\{\varphi^i(B)\}_{i=0,\dots,n-2}$ is a partition of $\operatorname{dom} \varphi$ and $\{\varphi^i(B)\}_{i=1,\dots,n-1}$ is a partition of $\operatorname{rng} \varphi$.

We say that $T \in \operatorname{Aut}(X, \mu)$ **extends** φ if $Tx = \varphi x$ for every $x \in \operatorname{dom}(\varphi)$.

Observe that a pre-cycle of length 2 is an element $\varphi \in [[\mathscr{R}]]$ such that $\operatorname{dom}(\varphi) \cap \operatorname{rng}(\varphi) = \emptyset$. If φ is a pre-cycle of length n, then $\mu(\operatorname{supp} \varphi) = n\mu(B)$ and $\mu(\operatorname{dom} \varphi) = (n-1)\mu(B)$.

A n**-cycle** is a measure-preserving transformation all whose orbits have cardinality either 1 or n. Given a pre-cycle φ of length n, we can extend it to an n-cycle $U_\varphi \in [\mathscr{R}_\varphi]$ as follows:

$$U_\varphi(x) \coloneqq \begin{cases} \varphi(x) & \text{if } x \in \operatorname{dom} \varphi; \\ \varphi^{-(n-1)}(x) & \text{if } x \in \operatorname{rng} \varphi \setminus \operatorname{dom} \varphi; \\ x & \text{otherwise.} \end{cases}$$

This n-cycle U_φ is called the **closing cycle** of φ and $\operatorname{supp} U_\varphi = \operatorname{supp} \varphi$.

Remark 3.6. Note that if $\{\varphi_1, ..., \varphi_{n-1}\}$ is a pre-n-cycle in the sense of [LM14a, Def. 8], then $\varphi_1 \sqcup \cdots \sqcup \varphi_{n-1}$ is a pre-cycle of length n in our sense, and that if φ is a pre-cycle of length n in our sense then $\{\varphi_{\restriction \varphi^i(B)}\colon i=0,...,n-2\}$ is a pre-n-cycle in the sense of [LM14a, Def. 8]. The reason for this change of terminology will become apparent in the statement of the next lemma, which was proved for $U=U_\varphi$ in [LM14a, Prop. 10].

Lemma 3.7. *Suppose φ is a pre-cycle of basis B, let $\psi \coloneqq \varphi_{\restriction B}$, and suppose $U \in \mathrm{Aut}(X,\mu)$ extends φ. Then $[\mathscr{R}_\varphi]$ is contained in the closure of the group generated by $[\mathscr{R}_\psi]\cup\{U\}$.*

Proof. Let n be the length of φ. For $i=0,...,n-2$ let $\psi_i=\varphi_{\restriction \varphi^i(B)}$, then we have $\mathscr{R}_\varphi=\bigvee_{i=0}^{n-2}\mathscr{R}_{\psi_i}$. Since U extends φ, we have $U\psi_iU^{-1}=\psi_{i+1}$ for all $i=0,...,n-3$, and hence $U[\mathscr{R}_{\psi_i}]U^{-1}=[\mathscr{R}_{\psi_{i+1}}]$. Since $\psi_0=\psi$, the group generated by $U\cup[\mathscr{R}_\psi]$ contains $[\mathscr{R}_{\psi_i}]$ for all $i=0,...,n-2$. Theorem 3.1 finishes the proof. □

The following proposition is obtained by a slight modification of the proof of the main theorem of [LM14a].

Proposition 3.8. *Let $\mathscr{R}$ be a p.m.p. ergodic equivalence relation on X and let $Y\subseteq X$ be a positive measure subset. Let $\mathscr{R}_0\leq \mathscr{R}_{\updownarrow Y}$ be a hyperfinite equivalence relation ergodic restricted to Y (and trivial on $X\setminus Y$). Suppose that $\mathcal{C}(\mathscr{R}_{\updownarrow Y})<r\mu(Y)$ for some integer $r\geq 2$. Then there are $r-1$ pre-cycles $\varphi_2,\varphi_3,\dots,\varphi_r\in[[\mathscr{R}_{\updownarrow Y}]]$ such that $\mu(\mathrm{supp}(\varphi_i))<\mu(Y)$ and such that whenever $U_2,U_3,\dots,U_r\in[\mathscr{R}]$ extend $\varphi_2,\varphi_3,\dots,\varphi_r$, we have that $\overline{\langle[\mathscr{R}_0],U_2,U_3,\dots,U_r\rangle}\geq[\mathscr{R}_{\updownarrow Y}]$.*

For instance, one can take $U_2,U_3,\dots,U_r$ to be the closing cycles of $\varphi_2,\varphi_3,\dots,\varphi_r$.

Proof. Let $T\in[\mathscr{R}_0]$ be such that its restriction to Y is ergodic. Our assumption $\mathcal{C}(\mathscr{R}_{\updownarrow Y})<r\mu(Y)$ means that the normalized cost of the restriction $\mathscr{R}_{\restriction Y}$ is less than r. Lemma III.5 from [Gab00] then provides a graphing Φ on Y of normalized cost $<(r-1)$ such that $\{T_{\restriction Y}\}\cup\Phi$ generates the restriction $\mathscr{R}_{\restriction Y}$. We now view Φ has a graphing on X, so that $\{T\}\cup\Phi$ generates $\mathscr{R}_{\updownarrow Y}$, and $\mathcal{C}(\Phi)<(r-1)\mu(Y)$. Let $c\coloneqq\mathcal{C}(\Phi)/(r-1)<\mu(Y)$. We take $p\in\mathbb{N}$ so large that $c(p+2)/p<\mu(Y)$.

Pick $\psi\in[[\mathscr{R}_0]]$ a pre-cycle of length 2 whose domain B has measure c/p. By cutting and pasting the elements of Φ and by conjugating them by elements of $[\mathscr{R}_0]$, we may as well assume that $\Phi=\{\varphi_2,...,\varphi_r\}$ where each φ_i is a pre-cycle of length $p+2$ extending ψ of basis B whose support is a strict subset of Y. Assume that $U_i\in[\mathscr{R}]$ extends φ_i for every $i=2,3,\dots,r$. Since $\psi\in[[\mathscr{R}_0]]$, then $[\mathscr{R}_\psi]\leq[\mathscr{R}_0]$. We can apply Lemma 3.7 and obtain that the closure of the group generated by $[\mathscr{R}_0]$ and U_i contains $[\mathscr{R}_{\varphi_i}]$. Since $\mathscr{R}_{\updownarrow Y}=\mathscr{R}_0\vee\mathscr{R}_{\varphi_2}\vee\cdots\vee\mathscr{R}_{\varphi_r}$, we can conclude the proof of the theorem using Theorem 3.1. □

Remark 3.9. Observe that we have a lot of freedom in the construction of the pre-cycles $\varphi_2, \varphi_3, \ldots, \varphi_r$ of Proposition 3.8. To start with, their length can be chosen to be any integer $n = p + 2$ large enough that $\frac{\mathcal{C}(\Phi)}{(r-1)} \frac{n}{n-2} < 1$. Actually, they could even have been chosen with any (possibly different) lengths $n_2, n_3, \cdots, n_r$, large enough integers so that $\frac{\mathcal{C}(\Phi)}{(r-1)} \frac{n_j}{n_j-2} < 1$: simply pick $r-1$ pre-cycles $\psi_j \in [[\mathscr{R}_0]]$ of length 2 whose domain B_j has measure $\frac{\mathcal{C}(\Phi)}{(r-1)} \frac{1}{n_j-2}$ and proceed as in the proof above.

In particular, the periodic closing cycles $U_2, U_3, \cdots, U_r$ can be assumed to have any large enough period $n_2, n_3, \cdots, n_r$ and domains contained in Y of measure $< \mu(Y)$. Up to conjugating by elements of $[\mathscr{R}_0]$, one can further assume the closing cycles have a non-null common subset of fixed points: $\mu(\mathrm{Fix}(U_2) \cap \mathrm{Fix}(U_3) \cap \cdots \cap \mathrm{Fix}(U_r) \cap Y) > 0$.

4 Evanescent pairs and topological generators

In this section our main goal is to obtain two topological generators of the full group of a hyperfinite ergodic equivalence relation with new flexibility properties relying on the following definition.

Definition 4.1. A pair (T, V) of elements of the full group $[\mathscr{R}]$ of the p.m.p. equivalence relation $\mathscr{R}$ is called an **evanescent pair of topological generators** of $\mathscr{R}$ if

1. V is periodic; and
2. for every $n \in \mathbb{N}$, the full group $[\mathscr{R}]$ is topologically generated by the conjugates of V^n by the powers of T, i.e., $\overline{\langle T^j V^n T^{-j} : j \in \mathbb{Z} \rangle} = [\mathscr{R}]$.

In particular, if (T, V) is an evanescent pair of topological generators, then the following hold:

- the pair (T, V) topologically generates $[\mathscr{R}]$,
- (T, V^n) is an evanescent pair of topological generators for any $n \in \mathbb{N}$,
- $d_u(V^{n!}, \mathrm{id}_X)$ tends to 0 when n tends to ∞.

We will show in Theorem 4.5 that the odometer T_0 can be completed to form an evanescent pair (T_0, V) of topological generators for $\mathscr{R}_{T_0}$, and that the set of possible V is actually a dense G_δ.

In this section, we set $X = \{0,1\}^{\mathbb{N}}$ endowed with the Bernoulli 1/2 measure $\mu = (\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1)^{\otimes\mathbb{N}}$. Given $s \in \{0,1\}^n$, we define the basic clopen set

$$N_s \coloneqq \{x \in \{0,1\}^{\mathbb{N}} \colon x_i = s_i \text{ for } 1 \leq i \leq n\}.$$

The **odometer** T_0 is the measure-preserving transformation of this space defined as adding the binary sequence $(1,0,0,\cdots)$ with carry to the right. More precisely, for each sequences $x \in \{0,1\}^{\mathbb{N}}$, if k is the (possibly infinite) first integer such that $x_k = 0$, then $y = T_0(x)$ is defined by

$$y_n \coloneqq \begin{cases} 0 & \text{if } n < k, \\ 1 & \text{if } n = k, \\ x_n & \text{if } n > k. \end{cases}$$

For each $n \in \mathbb{N}$, the permutation group $\mathrm{Sym}(\{0,1\}^n)$ has a natural action α_n on $\{0,1\}^{\mathbb{N}} \simeq \{0,1\}^n \times \{0,1\}^{\mathbb{N}}$ given for $x \in \{0,1\}^{\mathbb{N}}$ and $\sigma \in \mathrm{Sym}(\{0,1\}^n)$ by:

$$\alpha_n(\sigma)(x_1, ..., x_n, x_{n+1}, ...) \coloneqq (\sigma(x_1, ..., x_n), x_{n+1}, ...).$$

The sequence $(\alpha_n(\mathrm{Sym}(\{0,1\}^n)))_{n\in\mathbb{N}}$ is an increasing sequence of subgroups of the full group $[\mathscr{R}_{T_0}]$ whose reunion is dense in $[\mathscr{R}_{T_0}]$ (see [Kec10, Prop. 3.8]).

We now define a sequence of involutions $U_n \in [\mathscr{R}_{T_0}]$ with disjoint supports as in [LM16, Sect. 4.2]: $U_n \coloneqq \alpha_n(\upsilon_n)$ where $\upsilon_n \in \mathrm{Sym}(\{0,1\}^n)$ is the 2-points support transposition that exchanges $0^{n-1}1$ and $1^{n-1}0$. Observe that U_n is the involution with support $N_{1^{n-1}0} \sqcup N_{0^{n-1}1}$ (of measure 2^{-n+1}) which is equal to T_0 on $N_{1^{n-1}0}$ and T_0^{-1} on $N_{0^{n-1}1}$.

Recall that if $\tau_n \in \mathrm{Sym}(\{0,1\}^n)$ is 2^n-cycle and w_n is a transposition which exchanges two τ_n-consecutive elements, then the group $\mathrm{Sym}(\{0,1\}^n)$ is generated by the conjugates of w_n by powers of τ_n (actually $2^n - 1$ of them are enough). A straightforward modification gives the following (see [LM16, Lem. 4.3] for a detailed proof):

Lemma 4.2. *For every $n \in \mathbb{N}$, the group $\alpha_n(\mathrm{Sym}(\{0,1\}^n))$ is contained in the group generated by the conjugates of U_n by powers of T_0.*

Given a periodic p.m.p. transformation U and $k \in \mathbb{N}$, we say that V is a k**th root** of U when $\mathrm{supp}\, U = \mathrm{supp}\, V$ and $V^k = U$. The following lemma is well-known.

Lemma 4.3. *Whenever $\mathscr{R}$ is an ergodic equivalence relation, every periodic element in $[\mathscr{R}]$ admits a kth root in $[\mathscr{R}]$.*

Proof. Let us first prove that every n-cycle $U \in [\mathscr{R}]$ admits a kth root. To this end, pick a fundamental domain A for the restriction of U to its support. Since $\mathscr{R}$ is ergodic, we can pick a k-cycle $V \in [\mathscr{R}]$ supported on A. Let B be a fundamental domain for V, and put $C \coloneqq A \setminus B$. Then it is straightforward to check that $W \in [\mathscr{R}]$ defined as follows is a kth root of U:

$$W(x) \coloneqq \begin{cases} UU^iVU^{-i}(x) & \text{if } x \in U^i(B), \\ U^iVU^{-i}(x) & \text{if } x \in U^i(C), \\ x & \text{otherwise.} \end{cases}$$

In the general case, one glues together the kth roots obtained for every $n \in \mathbb{N}$ by considering the restrictions of U to U-orbits of cardinality n. □

Remark 4.4. The same proof works more generally for *aperiodic* p.m.p. equivalence relations.

Theorem 4.5. *The set of $V \in [\mathscr{R}_{T_0}]$ such that (T_0, V) is an evanescent pair of topological generators of $\mathscr{R}_{T_0}$ is a dense G_δ subset of $[\mathscr{R}_{T_0}]$.*

Proof. Denote by $\mathcal{P}$ the set of periodic elements of $[\mathscr{R}_{T_0}]$. It is a direct consequence of Rokhlin's lemma that $\mathcal{P}$ is dense in $[\mathscr{R}_{T_0}]$. And similarly the subset $\mathcal{P}' \subseteq \mathcal{P}$ of $V \in [\mathscr{R}_{T_0}]$ with finite order (or equivalently, with bounded orbit size) is dense in $[\mathscr{R}_{T_0}]$.

Writing $\mathcal{P}$ as the intersection (over the positive integers q) of the open sets $\{V \in [\mathscr{R}_{T_0}] \colon \exists p \in \mathbb{N},\ d(V^{p!}, \mathrm{id}_X) < 1/q\}$ shows that $\mathcal{P}$ is a G_δ subset of $[\mathscr{R}_{T_0}]$.

Denote by $\mathcal{E}$ the set of $V \in [\mathscr{R}_{T_0}]$ such that for every n, the group $[\mathscr{R}_{T_0}]$ is topologically generated by conjugates of V^n by powers of T_0. We want to show that $\mathcal{P} \cap \mathcal{E}$ is dense G_δ, and since $\mathcal{P}$ is dense G_δ it suffices (by the Baire category theorem in the Polish group $[\mathscr{R}_{T_0}]$) to show that $\mathcal{E}$ is dense G_δ.

For every $m, n \in \mathbb{N}$, set

$$\mathcal{E}_{m,n} \coloneqq \left\{ V \in [\mathscr{R}_{T_0}] \colon \alpha_n(\mathrm{Sym}(\{0,1\}^n)) \leq \overline{\langle T_0^k V^m T_0^{-k} \colon k \in \mathbb{Z} \rangle} \right\}.$$

The density of the union of the $\alpha_n(\mathrm{Sym}(\{0,1\}^n))$ in $[\mathscr{R}_{T_0}]$ recalled above implies that $\mathcal{E} = \bigcap_{m,n \in \mathbb{N}} \mathcal{E}_{m,n}$. So it suffices to show that each $\mathcal{E}_{m,n}$ is dense G_δ.

Let us first check that each $\mathcal{E}_{m,n}$ is G_δ. Denote by $\mathbf{W}$ the subgroup of $\mathbf{F}_2 = \langle a_1, a_2 \rangle$ generated by the conjugates of a_2 by powers of a_1. So for $w = w(a_1, a_2) \in \mathbf{W}$ and $V \in [\mathscr{R}_{T_0}]$, the element $w(T_0, V^m)$ is a product of conjugates of V^m by powers of T_0. By the definition of the closure we can write $\mathcal{E}_{m,n}$ as

$$\mathcal{E}_{m,n} = \bigcap_{p \in \mathbb{N}} \bigcap_{\sigma \in \mathrm{Sym}(\{0,1\}^n)} \bigcup_{w \in \mathbf{W}} \left\{ V \in [\mathscr{R}_{T_0}] \colon d_u(w(T_0, V^m), \sigma) < \frac{1}{p} \right\}.$$

Since the map $V \mapsto w(T_0, V)$ is continuous, each of the above right-hand sets is open, so their reunion over $w \in \mathbf{W}$ is also open, and we conclude that $\mathcal{E}_{m,n}$ is G_δ.

To check the density, it suffices to show that, for each m, n, one can approximate arbitrary elements of $\mathcal{P}'$ by elements of $\mathcal{E}_{m,n}$. So let $U \in \mathcal{P}'$ and let $\epsilon > 0$. Denote by K the order of U. Pick $p \geq n$ such that $2^{-p}K < \epsilon/2$. Let A be the U-saturation of the support of $U_p = \alpha_p(\upsilon_p)$ (defined at the beginning of the section). The measure of A is at most ϵ. Finally, let V be a (Km)th root of U_p and define

$$\tilde{U}(x) := \begin{cases} U(x) & \text{if } x \in X \setminus A, \\ V(x) & \text{if } x \in A. \end{cases}$$

By construction $d_u(U, \tilde{U}) \leq \mu(A) < \epsilon$. Observe that $\tilde{U}^{Km} = (\tilde{U}^m)^K = U_p$, thus Lemma 4.2 yields that $\tilde{U} \in \mathcal{E}_{Km,p} \subseteq \mathcal{E}_{m,p}$. Since $p \geq n$, $\tilde{U} \in \mathcal{E}_{m,p} \subseteq \mathcal{E}_{m,n}$, so we are done. □

Let us make a few comments on the above result. First, one can check that the pair (T_0, V) produced in the construction of [LM16, Thm. 4.2] provides an explicit example of an evanescent pair of topological generators of $\mathscr{R}_{T_0}$. Also, the above proof can be adapted to show that any *rank one* p.m.p. ergodic transformation [ORW82, Sect. 8] can be completed to form an evanescent pair of topological generators (see [LM14b, Thm. 5.28] for an explicit example of a pair which is evanescent). Proving these results is beyond the scope of this paper, so we leave it as an exercise for the interested reader.

It is unclear whether every p.m.p. ergodic transformations can be completed to form an evanescent pair of topological generators for its full group. Nevertheless, we can characterize the existence of an evanescent pair as follows.

Theorem 4.6. *Let $\mathscr{R}$ be an ergodic p.m.p. equivalence relation, then $\mathscr{R}$ admits an evanescent pair of topological generators if and only if $\mathscr{R}$ has cost* 1.

Proof. If $\mathscr{R}$ admits an evanescent pair (T, V), then since V is periodic we have $\mu(\operatorname{supp} V^{n!}) \to 0$. Since any set of topological generators for $[\mathscr{R}]$ generates the equivalence relation $\mathscr{R}$, we conclude that $\mathscr{R}$ has cost 1.

As for the converse, Theorem 4 and 5 from [Dye59] provide an ergodic hyperfinite subequivalence relation which is isomorphic to that of the odometer. So we can pick a conjugate of the odometer $T \in [\mathscr{R}]$. Repeating the proof of Theorem 4.5, we see that the set $\mathcal{E}_T$ of $V \in [\mathscr{R}]$ such that for every $n \in \mathbb{N}$, $[\mathscr{R}_T]$ is *contained* in $\overline{\langle T^j V^n T^{-j} : j \in \mathbb{Z}\rangle}$ is dense G_δ in $[\mathscr{R}]$.

Let us now consider the set $\mathcal{E}_{\mathscr{R}}$ of $V \in [\mathscr{R}]$ such that (T, V) is an evanescent pair of topological generators of $\mathscr{R}$, and for $n \in \mathbb{N}$ the set $\mathcal{E}_n$ of $V \in [\mathscr{R}]$ such that V

is periodic and $\overline{\langle T^j V^n T^{-j} : j \in \mathbb{Z}\rangle} = [\mathscr{R}]$. Each $\mathcal{E}_n$ is G_δ by the same argument as in the proof of Theorem 4.5. Since $\mathcal{E}_{\mathscr{R}} = \bigcap_n \mathcal{E}_n$, it suffices to show that each $\mathcal{E}_n$ is dense in order to apply the Baire category theorem and finish the proof.

Let us fix $n \in \mathbb{N}$. Since $\mathcal{E}_T$ is dense in $[\mathscr{R}]$, we only need to approximate elements of $\mathcal{E}_T$ by elements of $\mathcal{E}_n$. Moreover, the set of $V \in [\mathscr{R}]$ such that $\mu(\operatorname{supp} V) < 1$ is open and dense, so we only need to approximate every $V \in \mathcal{E}_T$ with $\mu(\operatorname{supp} V) < 1$ by elements of $\mathcal{E}_n$.

So let $V \in \mathcal{E}_T$ with $\mu(\operatorname{supp} V) < 1$, and take $\epsilon > 0$.

Since $\mathscr{R}$ has cost 1, Lemma III.5 from [Gab00] yields a graphing Φ of cost $< \frac{1}{3}\min(\epsilon, \mu(X \setminus \operatorname{supp} V))$ such that $\{T\} \cup \Phi$ generates $\mathscr{R}$. Conjugating by elements of $[\mathscr{R}_T]$ and pasting the elements of Φ, we may as well assume that $\Phi = \{\varphi\}$ where $\mu(\operatorname{dom} \varphi) < \epsilon/3$ and φ is a pre-cycle of length 2 whose support disjoint from $\operatorname{supp} V$. We then pick $\psi \in [[\mathscr{R}_T]]$ such that $\varphi \sqcup \psi$ is a pre-cycle of length 3 of support disjoint from $\operatorname{supp} V$, and denote by U_1 the associated 3-cycle.

Now let U_2 be an nth root of U_1, let $V_2 \coloneqq VU_2$, then $d_u(V_2, V) < \epsilon$ and we claim that V_2 belongs to $\mathcal{E}_n$. In order to prove this, let us denote by G the closed group generated by the conjugates of V_2^n by powers of T.

Since U_2 and V have disjoint support, they commute, and so $V_2^n = U_2^n V^n = U_1 V^n$. So $(V_2^n)^3 = V^{3n}$, and since $V \in \mathcal{E}_T$, we have that $[\mathscr{R}_T] \leq G$. In particular $[\mathscr{R}_\psi] \leq G$, and conjugating by V_2^n (which acts as U_1 on $\operatorname{supp} U_1$), we get that $[\mathscr{R}_\varphi] \leq G$ (see also Lemma 3.7). Since $\mathscr{R} = \mathscr{R}_T \vee \mathscr{R}_\psi$, we conclude by Theorem 3.1 that G contains $[\mathscr{R}]$ as wanted. □

5 Proof of the main theorem

As shown in Proposition 2.1, the perfect kernel of $\mathrm{Sub}(\mathbf{F}_r)$, $1 < r < \infty$, is the space of infinite index subgroups. We will construct a p.m.p. action of $\mathbf{F}_r$ for which almost every Schreier graph contains all possible balls of Schreier graphs of transitive $\mathbf{F}_r$-actions on infinite sets.

Step 1: using a smaller subset. We start with a p.m.p. ergodic equivalence relation $\mathscr{R}$ on (X, μ) of cost $< r$. By the induction formula [Gab00, Proposition II.6], there is a subset $Y \subseteq X$ such that $1/2 < \mu(Y) < 1$ and such that the (normalized) cost of the restriction $\mathscr{R}_{\restriction Y}$ is still $< r$.

Using results of Dye [Dye59, Thm. 4 and 5] as in the proof of Theorem 4.6, one can pick a conjugate of the odometer $T \in [\mathscr{R}_{\restriction Y}]$. We view T as an element of $[\mathscr{R}_{\updownarrow Y}]$.

Now we apply Proposition 3.8 (where $\mathscr{R}_0 = \mathscr{R}_T$) to obtain pre-cycles $\varphi_2, \dots, \varphi_r \in [[\mathscr{R}_{\restriction Y}]]$ whose supports have measure $< \mu(Y)$. For $i \leq r$, we let U_i be the closing cycle of φ_i as defined after Definition 3.5. Set $\eta \coloneqq \mu(Y \setminus \operatorname{supp} U_2) > 0$. Let m_0 be a positive integer such that $\mu(X \setminus Y)/m_0 < \eta/2$.

Step 2: preparing the finite actions. Let $(G_n)_{n\geq 1}$ be an enumeration of the (finite radius) balls of the Schreier graphs of all the transitive $\mathbf{F}_r$-actions over an infinite set, up to labeled graph isomorphism, and for which the number of vertices satisfies $|G_n| \geq m_0$.

Since G_n comes from a transitive action over an infinite set, we can choose some $\ell \in \{1, ..., r\}$ and some $\zeta_n \in G_n$ such that there is no a_ℓ-labeled edge whose source is equal to ζ_n.

Pick $\delta_n, \xi_n \notin G_n$, set $G'_n \coloneqq G_n \sqcup \{\delta_n, \xi_n\}$ and add an a_ℓ-edge from ζ_n to δ_n, an a_1-edge from δ_n to ξ_n and an a_2-edge from ξ_n to itself.

In this way we obtain a finite *partial Schreier graph* and this can be extended to a genuine Schreier graph of an $\mathbf{F}_r = \langle a_1, ..., a_r\rangle$-action on the same set as follows: for each $i \in \{1, 2, \cdots, r\}$, we consider the connected components of the subgraph obtained by keeping only the edges labeled a_i. These are either cycles (we don't modify them) or oriented segments (possibly reduced to a single vertex), in which case we add one edge labelled a_i from the end to the beginning of the segment.

Therefore we obtain an *action ρ_n of $\mathbf{F}_r$ on the finite set G'_n and a special point $\xi_n \in G'_n \setminus G_n$ such that $\rho_n(a_2)\xi_n = \xi_n$.*

Step 3: defining the action. Set $C \coloneqq X \setminus Y$. Consider a partition $C = \sqcup_{n\geq 1} C_n$ where $\mu(C_n) > 0$ for every n. We are going to define an amplified version of the action ρ_n on C_n as follows.

For each $n \geq 1$, we take a measurable partition $C_n = \sqcup_{g\in G'_n} B_n^g$ such that $\mu(B_n^g)\,|G'_n| = \mu(C_n)$ for every $g \in G'_n$. Set $B_n \coloneqq B_n^{\xi_n}$. Using ergodicity of $\mathscr{R}$, for every $g \in G'_n \setminus \{\xi_n\}$ we choose $\psi_g : B_n \to B_n^g$ in the pseudo full group $[[\mathscr{R}]]$ of $\mathscr{R}$. In this way we obtain an action α_n of $\mathbf{F}_r$ defined on C_n by the formula

$$\text{if } x \in B_n^{g_0} \text{ and } \rho_n(\gamma)g_0 = g_1 \text{ then } \alpha_n(\gamma)x \coloneqq \psi_{g_1}\psi_{g_0}^{-1}(x),$$

and trivial on $X \setminus C_n$. Thus $\alpha_n(\mathbf{F}_r) \leq [\mathscr{R}_{\upharpoonright C_n}]$.

Gluing all the α_n together, we obtain an action α_∞ of $\mathbf{F}_r$ on X such that $\alpha_\infty(\mathbf{F}_r) \leq [\mathscr{R}_{\upharpoonright C}]$ and such that α_∞ restricted to C_n is α_n.

Let $T \in [\mathscr{R}_{\upharpoonright Y}]$ be the conjugate of the odometer introduced in Step 1. Theorem 4.5 states that the set of $V \in [\mathscr{R}_T]$ such that (T, V) is an evanescent pair of generators for $\mathscr{R}_T$ is dense so we can choose such a V with $\mu(\operatorname{supp} V) < \eta/2$. Let $W \in [\mathscr{R}_T]$ be such that $\mu(\operatorname{supp}(WU_2W^{-1}) \cap \operatorname{supp} V) = 0$. Set

- $B \coloneqq \cup_n B_n$ and remark that $\mu(B) \leq \mu(C)/m_0 < \eta/2$;

- $D \coloneqq \operatorname{supp}(WU_2W^{-1}) \cup \operatorname{supp} V$ and observe that $\mu(Y \setminus D) > \eta/2$.

Therefore there exists a subset $A \subseteq Y \setminus D$ of measure $\mu(A) = \mu(B)$. Let $I \in [\mathscr{R}]$ be an involution with support $A \cup B$ and which exchanges A and B.

We finally define the action α of $\mathbf{F}_r$, by setting

$$\begin{aligned} \alpha(a_1) &\coloneqq T\alpha_\infty(a_1), \\ \alpha(a_2) &\coloneqq V(WU_2W^{-1})(I\alpha_\infty(a_2)), \\ \alpha(a_i) &\coloneqq U_i\alpha_\infty(a_i) \qquad \text{for } i \geq 3. \end{aligned}$$

Step 4: density.

(a) We claim that $\overline{\alpha(\mathbf{F}_r)} \geq [\mathscr{R}_T]$.

Indeed let $S \in [\mathscr{R}_T]$ and let us fix $\epsilon > 0$. There exists n_0, such that if we set $C_{>n_0} \coloneqq \cup_{n>n_0} C_n$ then $\mu(C_{>n_0}) < \epsilon/2$. The elements U_2, I, $\alpha_1(a_2), \dots, \alpha_{n_0}(a_2)$ have uniformly bounded orbits. So we can pick $k \in \mathbb{N}$ such that U_2^k, I^k and $\alpha_1(a_2)^k, \dots, \alpha_{n_0}(a_2)^k$ are the identity.

By construction V, WU_2W^{-1}, I and $\alpha_\infty(a_2)$ have mutually disjoint supports and hence commute. Therefore $\rho(a_2)^k = V^k\alpha_\infty(a_2)^k$. Moreover for every $m \geq 1$, we have that

$$\operatorname{supp}(\alpha(a_1)^m\alpha(a_2)^k\alpha(a_1)^{-m}) \subseteq Y \cup C_{>n_0}.$$

The crucial assumption that (T, V) is an evanescent pair of generators now comes into play: there is a word $w(T, V^k)$ which is a product of conjugates of V^k by powers of T such that $d_u(w(T, V^k), S) < \epsilon/2$. Remark that $w(\alpha(a_1), \alpha(a_2^k))$ and $w(T, V^k)$ coincide on Y and can only differ on $C_{>n_0}$ which has measure less than $\epsilon/2$. Hence $d_u(w(\alpha(a_1), \alpha(a_2^k)), S) < \epsilon$ which implies that $\overline{\alpha(\mathbf{F}_r)} \geq [\mathscr{R}_T]$.

(b) We claim that $\overline{\alpha(\mathbf{F}_r)} \geq [\mathscr{R}_{\restriction Y}]$.

This follows from Proposition 3.8 and the fact that $W^{-1}\alpha(a_2)W, \alpha(a_3), \dots, \alpha(a_r)$ extend $\varphi_2, \dots, \varphi_r$ and they are all contained in $\overline{\alpha(\mathbf{F}_r)}$.

(c) We claim that $\overline{\alpha(\mathbf{F}_r)} \geq [\mathscr{R}]$.

This is a direct consequence of Corollary 3.4 granting that $\alpha(\Gamma)Y = X$, which we will now show.

Clearly $\alpha(a_2)Y \supset \alpha(a_2)A = B = \cup_n B_n$. For every n and $g \in G'_n \setminus \{\xi_n\}$, there exists $\gamma \in \Gamma$ of minimal length such that $\rho_n(\gamma)\xi_n = g$. Since $\rho_n(a_2)\xi_n = \xi_n$ and since $\alpha(a_2)_{\restriction C_n \setminus B_n} = \alpha_n(a_2)$ mimics the action of $\rho_n(a_2)$ on $G'_n \setminus \{\xi_n\}$, the minimality of the length of γ implies that $\alpha(\gamma)B_n = B_n^g$. Since this is true for every $g \in G'_n$ we get $\alpha(\Gamma)Y \supset C_n$; and this holds for every n. We thus have $\alpha(\Gamma)Y = X$ as wanted.

Step 5: totipotency. Consider a transitive action ρ of $\mathbf{F}_r$ on some infinite set. Let H be a Schreier ball such that $|H| \geq m_0$. Then by construction there exists n such that $H = G_n \subseteq G'_n$. Also remark that the restriction of the Schreier graph of the action α to $\cup_{g \in G_n} B_n^g \subseteq C_n$ mimics the partial Schreier graph H. Since ρ and H are arbitrary, the Schreier graph of α contains every sufficiently large Schreier ball of every transitive action of $\mathbf{F}_r$ and this finishes the proof of the main theorem.

□

Remark 5.1. The subspace $Y \subseteq X$ chosen in Step 1 of the above proof coincides with the subset where $\alpha(a_1)$ is aperiodic. So the event "no power of a_1 belongs to Λ" has measure $\mu(Y)$ in the IRS $\mathrm{Stab}^\alpha_* \mu$ associated with α. Observe that the measure $\mu(Y)$ can be chosen to take any value from the non-empty interval $\left(\max\left\{\frac{\mathcal{C}(R)-1}{r-1}, \frac{1}{2}\right\}, 1\right)$. Recalling that the density of $\alpha(\Gamma)$ implies Stab^α is essentially injective, then the following holds: *Every ergodic p.m.p. equivalence relation $\mathscr{R}$ of cost $< r$ can be realized (up to a null-set) by the action of $\mathbf{F}_r \curvearrowright \mathrm{Sub}(\mathbf{F}_r)$ for continuum many different totipotent IRS's of $\mathbf{F}_r$.*

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A.C., Institut für Algebra und Geometrie, KIT, 76128 Karlsruhe, Germany
alessandro.carderi@kit.edu

D.G., Université de Lyon, CNRS, ENS-Lyon, Unité de Mathématiques Pures et Appliquées, 69007 Lyon, France
damien.gaboriau@ens-lyon.fr

F.L.M., Université de Paris, Sorbonne Université, CNRS, Institut de Mathématiques de Jussieu-Paris Rive Gauche, F-75013 Paris, France
francois.le-maitre@imj-prg.fr